Exercise 41 (without Matlab except for e) and f))

We aim at solving the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{pmatrix} \qquad \text{et} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

with the Gauss elimination method.

a) Verify that the Gauss algorithm can not be executed till the end.

b) Find a matrix of permutations P such that the matrix PA can be facotrized. Write the linear system equivalent to $A\mathbf{x} = \mathbf{b}$ (i.e. with the same solution $\mathbf{x}$) which has PA as associated matrix.

c) Apply the Gauss algorithm to the matrix PA, and compute the LU-factorization of PA.

d) Compute $\mathbf{x}$ by solving the equivalent linear system of point b), starting from the obtained factorization and using the algorithms of forward and back forward substitution.

e) (**To solve with Matlab**) Compute the LU-factorization of A with Matlab, and verify that Matlab has used a permutations of lines of A. (*Hint: visualize the permutation matrix P computed by the command lu*).

f) (**To solve with Matlab**) Compute with Matlab the solution of $A\mathbf{x} = \mathbf{b}$ starting from the factorization obtained at point e). In particular, solve the triangular linear systems with the Matlab command \.

Exercise 42

Consider a set of many linear systems $A\mathbf{x}_1 = \mathbf{b}_1$, $A\mathbf{x}_2 = \mathbf{b}_2$, $A\mathbf{x}_3 = \mathbf{b}_3$, $\ldots$, with the same matrix A,

$$A = \begin{pmatrix} 5 & 0 & 1 & 3 \\ 0 & 4 & -2 & 7 \\ -1 & 2 & -3 & 0 \\ 2 & 9 & -9 & -5 \end{pmatrix}.$$

Since all the systems have the same matrix, it is possible to compute only once the LU-factorization of A. In particular, if we solve the linear systems with

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

we can build the inverse matrix of A,

$$A^{-1} = \left[\begin{array}{c|c|c|c} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{array} \right].$$

The Matlab command `inv(A)` follows the same procedure to compute the inverse of a matrix $A \in \mathbb{R}^{n \times n}$.

a) Using Matlab, compute the inverse of the matrix A before using the described procedure and then with the Matlab command `inv`, and verify that the same result is obtained. Use the Matlab command \ to solve the linear systems.

b) Let $N = 80$. Consider the matrix A of size 6400×6400 obtained with the command `gallery('poisson',N)`, and the vector $\mathbf{b}$ obtained with the command `rand(N*N,1)`. Solve the linear system $A\mathbf{x} = \mathbf{b}$, with the command $A \backslash b$ and the command `inv(A)*b`. Which is the faster command and why? (*Hint: the execution time of a set of commands can be measured with the commands* `tic` *and* `toc`).

Exercise 43

Consider the linear system $A\mathbf{x} = \mathbf{f}$ where the matrix A is tri-diagonal and invertible:

$$A = \begin{pmatrix} a_1 & c_1 & 0 & \cdots & 0 \\ b_2 & a_2 & c_2 & \ddots & \vdots \\ 0 & b_3 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix}.$$

a) Show that the matrices L and U of the LU-factorization of the matrix A are bi-diagonal matrices of the form,

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \beta_2 & 1 & 0 & \ddots & \vdots \\ 0 & \beta_3 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_n & 1 \end{pmatrix}, \quad U = \begin{pmatrix} \alpha_1 & \gamma_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \gamma_2 & \ddots & \vdots \\ 0 & 0 & \alpha_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_{n-1} \\ 0 & \cdots & 0 & 0 & \alpha_n \end{pmatrix},$$

and give the expressions of the coefficients α_i, β_i and γ_i as a function of the coefficients of A.

b) Compute the coefficients of the vector $\mathbf{x}$, solution of the linear system $A\mathbf{x} = \mathbf{f}$, with $\mathbf{f} = (f_i)_{i=1}^n \in \mathbb{R}^n$, as a function of the coefficients $\alpha_i, \beta_i, \gamma_i$ et f_i.

Exercise 44

Consider the matrix of size $n \times n$ built up with the command `A=gallery('binomial',n)`.

a) Visualize the non-zero elements of the matrix for $n = 400$, with the command `spy(A)`.

b) Compute the LU-factorization and check that permutations of lines have been done. (*Hint: visualize the matrix P with* `spy(P)`).

c) Visualize the non-zero elements of L and U. Use the command `nnz` to count the number of non-zero elements of the matrices A, L and U. What can be observed?

d) Estimate the time needed by Matlab to compute the LU-factorization of A, using the commands `tic` and `toc`.

e) Repeat points c) and d) for $n = 20^2, 21^2, \ldots, 35^2$. Visualize the computation time and the number of non-zero elements of the matrices L and U as a function of n on logarithm plots. Comment the results. How do the computation time and the memory storage grow as a function of n?

Exercise 45

Consider the Vandermonde matrix

$$A = \begin{pmatrix} x_1^{n-1} & \cdots & x_1^2 & x_1 & 1 \\ \vdots & \vdots & x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n^{n-1} & \cdots & x_n^2 & x_n & 1 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

where the points $x_1, \ldots, x_n$ are equi-spaces in the interval $[0, 1]$. We want to solve the linear system $A\mathbf{x} = \mathbf{b}$ where

$$b = \begin{pmatrix} 1 + x_1^2 \\ \vdots \\ 1 + x_n^2 \end{pmatrix}.$$

The exact solution is $\mathbf{x} = (0, \ldots, 0, 0, 1, 0, 1)^T$.

a) Solve the linear system with Matlab, using the command \. Let $\mathbf{x_c}$ be the obtained solution. Evaluate the relative error $\varepsilon_n = \frac{\|\mathbf{x_c}-\mathbf{x}\|}{\|\mathbf{x}\|}$ for $n = 4$.

b) An upper bound for the relative error is given by

$$\eta_n = \kappa(A)\, \texttt{eps}$$

where $\kappa(A)$ is the condition number of the matrix A (in Matlab can be computed with the command `cond`). Compare the relative error ε_n computed at the previous point with the upper bound η_n for $n = 4$.

c) Repeat points a) and b) for $n = 4, 6, 8, \ldots, 20$. Visualize the error ε_n, the upper bound η_n and the normalized residual $r_n = \|\mathbf{b} - A\mathbf{x_c}\|/\|\mathbf{b}\|$ as a function of n, on both a logarithmic and a semilogarithmic plot. Which kind of error convergence can be observed? Is the residual r_n a good indicator of the error ε_n?

d) Repeat point c) for the matrix

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

and the vector $b = (2, 2, \cdots, 2)^T$, with $n = 5, 10, \ldots, 100$. The exact solution is

$$\mathbf{x} = \begin{pmatrix} 1 \cdot (n) \\ 2 \cdot (n-1) \\ 3 \cdot (n-2) \\ \vdots \\ n \cdot 1 \end{pmatrix}.$$

Comment the obtained results.