61. For $w \in \mathbb{R}^n$ and $\|w\|_2 = 1$ the Housholder matrix P is defined by

$$P := \mathbb{I} - 2ww^T.$$

Determine all eigenvalues of P and their eigenvectors. What is the determinant of P?

62. Let $\mathbb{C}_n[x]$ ($n \in \mathbb{N}$) be the space of polynomials of degree n with complex coefficients. A basis of this space is given by $\{1, x, x^2, \ldots, x^n\}$, the so-called monomials. For $p(x) = \sum_{i=0}^n a_i x^i$ ($a_i \in \mathbb{C}$) we denote by $p'(x)$ its derivative. We are in interested in $p'(x)$ with respect to the monomial basis. Show that

$$D : \mathbb{C}_n[x] \to \mathbb{C}_n[x] : p(x) \mapsto p'(x)$$

is linear. Define the corresponding matrix A_D.

63. Given are two bases of the space of the polynomials $\mathbb{C}_3[x]$, namely

$$\mu = \{1, x, x^2, x^3\} \qquad \text{(Monomial-Basis)} \qquad \text{and}$$
$$\eta = \{1, x, \frac{1}{2}(3x^2 - 1), \frac{1}{2}(5x^3 - 3x)\} \quad \text{(Legendre-Polynom-Basis).}$$

Therefore, we can represent a polynomial of degree 3 by

$$p(x) = \sum_{i=0}^{3} a_i \mu_i = \sum_{i=0}^{3} b_i \eta_i.$$

(a) Determine the transformation matrix A, which transforms a polynomial of degree 3 from monomial basis representation to the Legendre-Basis representation; i.e., $A \cdot a = b$.

(b) Transform $p(x) = 1 + 3x - 3x^2 + x^3$ to the Legendre-Basis representation.

(c) Let us assume we have a polynomial of degree 3 in the Legendre-Basis representation. Determine the matrix which calculates the derivative in the Legendre-Basis representation. Use (obligatory) exercise 62 (the matrix A_D for $n = 3$!).

64. Exercise without Matlab. Let be given the following data:

$$x_1 = 0 \quad y_1 = 12$$
$$x_2 = 1 \quad y_2 = 18$$
$$x_3 = 2 \quad y_3 = 6$$

a) Find the three polynomials of the Lagrange basis corresponding to the points $x_1 = 0, x_2 = 1$ and $x_3 = 2$.

b) Using the Lagrangian basis, find the polynomial of order 2 which interpolates the data. Compute then the value of the interpolating polynomial in $x = \frac{1}{2}$.

65. Let be given the data (x_i, y_i), $i = 1, \ldots, n+1$ where x_i are $n+1$ equidistributed knots in the interval $[a, b] = [-5, 5]$ and y_i are the evaluations of the function

$$f(x) = \frac{1}{1 + \exp(4x)}$$

without measurement error, i.e. $y_i = f(x_i)$. We want to reconstruct the function f starting from the data (x_i, y_i) and using polynomial interpolation.

a) Consider $n = 6$. Using the Matlab commands `polyfit` and `polyval` (see `help polyfit`, `help polyval`), compute the polynomial p_n of degree $n = 6$ which interpolates the data (x_i, y_i). Graphically compare the obtained polynomial with the "real" function f where the data come from. *Hint: evaluate both the polynomial and the function f on a fine grid, with much more points than $n + 1$.*

b) Repeat point a) with $n = 14$, and make the plot of the new polynomial interpolant on the same plot obtained at point a). Is the new approximation better the first one?

c) Repeat points a) and b) for the function

$$g(x) = \sin^2(x)$$

What can be observed here?

d) Repeat points a), b) and c) using the Tchebychev knots instead of the equidistributed ones. The formula to compute k Tchebychev knots is

$$t_j = -\cos\left(\frac{j\pi}{2k}\right), \quad j = 1, 3, 5, \ldots, 2k-1$$

$$x_j = \frac{t_j + 1}{2}(b-a) + a$$

Are the results obtained the same as for the equidistributed knots?