FWF-Research Project P10472-MAT

Nonlinear Transformation Groups and Distributions

Final Report

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Summary

The research project P10472-MAT "Nonlinear Transformation Groups and Distributions" was successfully completed in 1999. All project aims were reached. Moreover, beyond the initial scope of the project, new lines of research were opened and persued and several national and international cooperations were initiated as a result.

Contents

I. PROJECT GOALS AND CHOSEN METHODOLOGY

The aim of the project was to develop the theory of continuous transformation groups (as founded by S. Lie) in the framework of algebras of generalized functions in the sense of J. F. Colombeau in order to be able to formulate and study nonlinear group actions on generalized function solutions to partial differential equations. Lie's theory of transformation groups is essentially geometric: one-parameter subgroups of a given Lie group generate a flow on the space of independent and dependent variables; this flow acts on the graph of functions, thereby inducing an action of the group on the space of functions. In trying to generalize this theory to the distributional setting, the following questions/problems arise:

- Invariance of generalized functions under groups of transformations.
- Group invariant generalized function solutions to partial differential equations.
- Calculation of symmetry groups in the distributional or Colombeau framework.
- Conservation laws.
- Underlying geometry of generalized functions.

The classical theory of transformation groups presupposes either the function spaces on which the Lie groups of symmetries are to act to consist of smooth functions or, in the purely distributional setting, necessitates the group actions to be linear in the dependent variables. This in particular excludes weak solutions of nonlinear partial differential equations which have been a main focus of research in PDEs over the past decades. A general theory capable of simultaneously addressing nonlinear differential equations and generalized functions was developed by J. F. Colombeau starting from the 1980ies. It was the main objective of this research project to put the theory of transformation groups into the Colombeau framework in order to study nonlinear or even generalized group actions on generalized functions in the context of nonlinear partial differential equations.
The methods employed resulted from the construction of Colombeau algebras. Elements of these differential algebras are equivalence classes of nets of smooth functions where the construction induces certain growth restrictions with respect to a regularization parameter. The new theory therefore employs the classical setting for fixed parameter. Additionally, growth estimates on the resulting sequences had to be established and new methods had to be created in order to handle the process of forming equivalence classes of such sequences. In particular, a point value characterization of Colombeau generalized functions had to be developed in order to handle these problems.
The project started in October 1995. It was interrupted two times: once in 1996/1997 due to the civil service of M. Kunzinger and once in 1998 when he held a full position at the department of Mathematics at Vienna university. Thus the total time period of the project was 33 months which could be realized without additional costs.

II. RESULTS AND APPLICATIONS

To our great satisfaction not only were all project aims reached in their entirety but moreover the foundation of a geometric theory of generalized functions in the sense of Colombeau was laid. This line of research emerged from the results achieved in the second half of the project and currently is the main focus of interest for a highly active research group around M. Grosser and M. Kunzinger in Vienna entertaining a number of international contacts and cooperations. In more details the achieved results can be broken down to the following four branches (the lists of publications have been updated 2003):

Lie theory in the Colombeau framework

This central topic of the project was resolved completely. The methods of group analysis were put into the Colombeau framework. The determination of symmetries of generalized solutions to nonlinear partial differential equations was thereby made possible. In particular, distributions can now be transformed under nonlinear symmetry transformations. The theory required the development of a theory of generalized point values as well as a solution theory of ordinary differential equations in the Colombeau setting (achieved in the first year of the project). The following publications resulted from this stage of the project:

1. M. Kunzinger, *Lie-Transformation Groups in Colombeau Algebras*, PhD-thesis, Universität Wien, 1996.
2. M. Kunzinger, M. Oberguggenberger, *Symmetries of Differential Equations in Colombeau Algebras*, in: N. H. Ibragimov, F. M. Mahomed (Eds.), Modern Group Analysis VI, New Age Int. Publ., 9-20, (1997).

3. M. Oberguggenberger, M. Kunzinger, Characterization of Colombeau Generalized Functions by their Pointvalues, Math. Nachr. **203** (1999), 147-157.
4. R. Hermann, M. Oberguggenberger, *Ordinary Differential Equations and Generalized Functions.* In: M. Grosser, G. Hörmann, M. Kunzinger, M. Oberguggenberger (Eds.), Nonlinear Theory of Generalized Functions. Chapman & Hall/CRC Research Notes in Mathematics Vol. **401**, Boca Raton 1999, 85 - 98.
5. M. Kunzinger, *Lie Symmetries of Differential Equations in a Generalized Functions Setting.* In: M. Grosser, G. Hörmann, M. Kunzinger, M. Oberguggenberger (Eds.), Nonlinear Theory of Generalized Functions. Chapman & Hall/CRC Research Notes in Mathematics Vol. 401, Boca Raton 1999, 241 - 250.
6. M. Kunzinger, M. Oberguggenberger: *Group Analysis of Differential Equations and Generalized Functions*, SIAM J. Math. Anal. **310** (2000) no. 6, 1192-1213.

Singularities in physical field theories

As a side result of the project it turned out that the methods developed so far could successfully be employed to study distributional solutions to field equations of physical theories. This is documented by the following publications:

7. G. Hörmann, M. Kunzinger, *Nonlinearity and Self-Interaction in Physical Field Theories with Singularities*, Integral Transf. Special Funct. **6**(1997), 190-199.
8. G. Hörmann, M. Kunzinger, *Regularized derivatives in a 2-dimensional model of self-interacting fields with singular data*, Zeitschr. Anal. u. Anw. **19**(2000), 147 - 158.

Geodesics of distributional metrics in General Relatvity

The geodesic flow of a generalized space-time metric can be viewed as a generalized group action, hence can be treated by the methods of the project. This application resulted in the following papers:

9. M. Kunzinger, R. Steinbauer: A rigorous solution concept for geodesic and geodesic deviation equations in impulsive gravitational waves. J. Math. Phys. **40**(3)(1999), 1479 – 1489.
10. M. Kunzinger, R. Steinbauer: A note on the Penrose junction conditions. Class. Quantum Gravity **16**(1999), 1255 – 1264.

Geometric Theory of Generalized Functions

As was already pointed out above, the second phase of the project led in a natural way to the question of developing a theory not only of Lie group actions but, more generally, of global analysis on differentiable manifolds in the Colombeau setting. This branch of research tuned out to be most important and promising. The project therefore laid the foundation to a very active new field of research for years to come, namely the geometric theory of generalized functions. First results are already available; a book publication containing the results of this project as well as the foundations of the geometric theory of generalized functions is under way. The material laid out in the preprints finished in the final stages of the projects will already be part of this book. The publications referred to above are:

11. M. Grosser, M. Kunzinger, R. Steinbauer, H. Urbantke, J. Vickers: *Diffeomorphism invariant construction of nonlinear generalized functions*, Ann. Phys., vol **9**, SI, 2000.
12. M. Grosser, E. Farkas, M. Kunzinger, R. Steinbauer: On the Foundations of Nonlinear Generalized Functions I, II, Mem. Amer. Math. Soc. **153** (729), 2001.
13. M. Grosser, M. Kunzinger, R. Steinbauer, J. Vickers: *A Global Theory of Algebras of Generalized Functions*, Adv. Math. **166**(1), 179–206, 2002.
14. M. Grosser, M. Kunzinger, M. Oberguggenberger und R. Steinbauer, *Geometric Theory of Generalized Functions*, vol. **537** of Mathematics and its applications, Kluwer, 2001.

III. INTERNATIONAL COOPERATIONS

In the course of this project an intensive cooperation with J. Vickers and J. Wilson (University of Southampton) was initiated which resulted in several joint publications. There was continuous exchange of ideas with E. E. Rosinger (Pretoria). Moreover, there emerged cooperations concerning applications of generalized transformation groups to hyperbolic conservation laws with the group around S. Pilipović (Novi Sad). Finally, fruitful discussions with P. Olver (Minnesota) positively influenced the course of the project.

IV. SURPRISES, SUCCESSES AND OBSTACLES

As was already pointed out, the project was carried out successfully: all project goals were reached completely. There were no insurmountable obstacles, the solution of the problems encountered took the time we had initially scheduled, the level of difficulty was roughly as expected. It came rather as a surprise that the ideas developed with a view to the original project outlay unfolded such fruitful synergies in connection with problems from general relativity. It was also unexpected that from the geometric theory of Colombeau generalized functions there originated a long-term research field which in the framework of international cooperations will be the main focus of our research group in the years to come. Funding of the research position of M. Kunzinger by the FWF has led to establishing and confounding our research group and will have a far-reaching effect for the coming years.