Weighted Energy Decay for 1D Wave Equation

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Abstract

We obtain a dispersive long-time decay in weighted energy norms for solutions to the 1D wave equation with generic potential. The decay extends the results obtained by Murata for the 1D Schrödinger equation.

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1 Introduction

In this paper, we establish a dispersive long time decay for the solutions to 1D wave equation

$$\ddot{\psi}(x,t) = -H\psi(x,t) := \Big(\frac{d^2}{dx^2} + V(x)\Big)\psi(x,t), \quad x \in \mathbb{R} \tag{1.1}$$

in weighted energy norms. In vectorial form, equation (1.1) reads

$$i\dot{\Psi}(t) = \mathcal{H}\Psi(t), \tag{1.2}$$

where

$$\Psi(t) = \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} 0 & i \\ i(\frac{d^2}{dx^2} + V) & 0 \end{pmatrix}. \tag{1.3}$$

For $s, \sigma \in \mathbb{R}$, let us denote by $H^s_\sigma = H^s_\sigma(\mathbb{R}^3)$ the weighted Sobolev spaces introduced by Agmon, [1], with the finite norms

$$\|\psi\|_{H^s_\sigma} = \|(1+|x|^2)^{\sigma/2}(1+|\frac{d}{dx}|^2)^{s/2}\psi\|_{L^2} < \infty.$$

We assume that $V(x)$ is a real function, and

$$|V(x)| + |V'(x)| \le C(1+|x|)^{-\beta}, \qquad x \in \mathbb{R} \tag{1.4}$$

for some $\beta > 4$. Then the multiplication by $V(x)$ is bounded operator $H^1_s \to H^1_{s+\beta}$ for any $s \in \mathbb{R}$.

We restrict ourselves to the following"regular case" in the terminology of [10] (or "nonsingular case" in [13])

$$\textit{The point } \lambda = 0 \textit{ is neither eigenvalue nor resonance for the operator } H \tag{1.5}$$

Then the truncated resolvent of the Schrödinger operator $H = -\frac{d^2}{dx^2} - V(x)$ is bounded at the end point $\lambda = 0$ of the continuous spectrum. It is known that the spectral condition holds for *generic potentials* [10], [13].

Definition 1.1. *i) $\mathcal{F}$ is the complex Hilbert space $\dot{H}^1 \oplus L^2$ of vector-functions $\Psi = (\psi, \pi)$ with the norm*

$$\|\,\Psi\|_{\mathcal{F}} = \|\nabla\psi\|_{L^2} + \|\pi\|_{L^2} < \infty.$$

ii) $\mathcal{F}_\sigma$ is the complex Hilbert space $H^1_\sigma \oplus H^0_\sigma$ of vector-functions $\Psi = (\psi, \pi)$ with the norm

$$\|\,\Psi\|_{\mathcal{F}_\sigma} = \|\psi\|_{H^1_\sigma} + \|\pi\|_{H^0_\sigma} < \infty.$$

Definition 1.2. *For real $\alpha > 1$ denote by $\langle\alpha\rangle$ the number from $\mathbb{N}$ such that*

$$\langle\alpha\rangle < \alpha \le 1 + \langle\alpha\rangle$$

Our main result is the following long time decay of the solutions to (1.2): in the "regular case" for initial data $\Psi_0 = \Psi(0) \in \mathcal{F}_\sigma$ with $\sigma > 2$ we have

$$\|\mathcal{P}_c\Psi(t)\|_{\mathcal{F}_{-\sigma}} = \mathcal{O}(|t|^{-\gamma}), \quad \gamma = \min\{\langle\sigma - 1/2\rangle, \sigma - 1, \langle\beta/2 - 1/2\rangle, \beta/2 - 1\}, \quad t \to \pm\infty \quad (1.6)$$

Here $\mathcal{P}_c$ is a Riesz projector onto the continuous spectrum of the operator $\mathcal{H}$. The decay is desirable for the study of asymptotic stability and scattering for the solutions to nonlinear hyperbolic equations.

Let us comment on previous results in this direction. Local energy decay has been established first in the scattering theory for linear Schrödinger equation developed since 50' by Birman, Kato, Simon, and others. For wave equations with compactly supported potentials, and similar hyperbolic PDEs, Vainberg [21] established the decay in local energy norms for solutions with compactly supported initial data.

However, applications to asymptotic stability of solutions to the nonlinear equations also require an exact characterization of the decay for the corresponding linearized equations in weighted norms (see e.g. [3, 4, 5, 19]).

The decay of type (1.6) in weighted norms has been established first by Jensen and Kato [10] for the Schrödinger equation in the dimension $n = 3$. The result has been extended to all other dimensions by Jensen and Nenciu [8, 9, 11], and to more general PDEs of the Schrödinger type by Murata [13]. The survey of the results can be found in [16].

For the "free" wave equations with $V(x) = 0$ some estimates in weighted L^p-norms have been established in [2, 6].

In [12] the decay of type (1.6) in the weighted energy norms has been proved for the wave equation in the dimension $n = 3$. The approach develops the Jensen-Kato techniques to make it applicable to the reativistic equations. Namely, the decay of the low energy component of the solution follows by the Jensen-Kato techniques while the decay for the high energy component requires novel robust ideas. This problem has been resolved with a modified approach based on the Born series and convolution. Let us note that the decay rate in (1.6) corresponds to the spatial decay of the initial function $\Psi(0)$ and potential $V(x)$ in contrast to the Schrödinger case [10], where the decay rate is $t^{-3/2}$. This difference is related to the presence of the lacuna for the free 3D wave equation.

Here we extend our approach [12] to the dimension $n = 1$. The extension is not straightforward since the decay (1.6) violates for the free 1D wave equation corresponding to $V(x) = 0$ when the solutions does not decay. Hence, the decay (1.6) cannot be deduced by pertrubation arguments from the corresponding estimate for the free equation. This difficulty is well known, and it is caused by the "zero resonance function" $\psi(x) = \text{const}$ corresponding to the end point $\lambda = 0$ of the continuous spectrum of the free 1D Schrödinger operator $-d^2/dx^2$.

Main idea of our approach to $n = 1$ is a spectral analysis of the "bad" term, without decay. Namely, we show that the bad term does not contribute to the high energy component. Therefore, the decay $\sim t^{-\gamma}$ for the high energy component follows. On the other hand, for the low energy component, the decay $\sim t^{-\gamma}$ holds for the "generic" potentials by methods [10, 13]. This decay implies the asymptotic completeness since $\gamma > 1$.

Our paper is organized as follows. In Section 2 we obtain the time decay for the solution to the free wave equation and state the spectral properties of the free resolvent. In Section 3 we obtain spectral properties of the perturbed resolvent and prove the decay (1.6).

2 Free wave equation

First, we consider the free wave equation:

$$\ddot{\psi}(x,t) = \psi''(x,t), \quad x \in \mathbb{R}, \quad t \in \mathbb{R} \tag{2.1}$$

In vectorial form equation (2.1) reads

$$i\dot{\Psi}(t) = \mathcal{H}_0 \Psi(t), \tag{2.2}$$

where

$$\Psi(t) = \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix}, \quad \mathcal{H}_0 = \begin{pmatrix} 0 & i \\ i\frac{d^2}{dx^2} & 0 \end{pmatrix} \tag{2.3}$$

2.1 Spectral properties

We state spectral properties of the free wave dynamical group $\mathcal{G}(t)$. For $t > 0$ and $\Psi_0 = \Psi(0) \in \mathcal{F}$, there exist a unique solution $\Psi(t) \in C_b(\mathbb{R}, \mathcal{F})$ to the free wave equation (2.2). Hence, $\Psi(t)$ admits the spectral Fourier-Laplace representation

$$\theta(t)\Psi(t) = \frac{1}{2\pi i} \int\limits_{\mathbb{R}} e^{-i(\omega+i\varepsilon)t} \mathcal{R}_0(\omega + i\varepsilon)\Psi_0 \, d\omega, \quad t \in \mathbb{R} \tag{2.4}$$

with any $\varepsilon > 0$ where $\theta(t)$ is the Heavyside function, $\mathcal{R}_0(\omega) = (\mathcal{H}_0 - \omega)^{-1}$ for $\omega \in \mathbb{C}^+ := \{\text{Im}\,\omega > 0\}$ is the resolvent of the operator $\mathcal{H}_0$. The representation follows from the stationary equation $\omega\tilde{\Psi}^+(\omega) = \mathcal{H}_0\tilde{\Psi}^+(\omega) + i\Psi_0$ for the Fourier-Laplace transform $\tilde{\Psi}^+(\omega) := \int_{\mathbb{R}} \theta(t)e^{i\omega t}\Psi(t)dt$, $\omega \in \mathbb{C}^+$. The solution $\Psi(t)$ is continuous bounded function of $t \in \mathbb{R}$ with the values in $\mathcal{F}$ by the energy conservation for the free wave equation (2.2). Hence, $\tilde{\Psi}^+(\omega) = -i\mathcal{R}_0(\omega)\Psi_0$ is analytic function of $\omega \in \mathbb{C}^+$ with the values in $\mathcal{F}$, and bounded for $\omega \in \mathbb{R} + i\varepsilon$. Therefore, the integral (2.4) converges in the sense of distributions of $t \in \mathbb{R}$ with the values in $\mathcal{F}$. Similarly to (2.4),

$$\theta(-t)\Psi(t) = -\frac{1}{2\pi i} \int\limits_{\mathbb{R}} e^{-i(\omega-i\varepsilon)t} \mathcal{R}_0(\omega - i\varepsilon)\Psi_0 \, d\omega, \quad t \in \mathbb{R} \tag{2.5}$$

For the resolvent $\mathcal{R}_0(\omega)$ the following matrix representation holds

$$\mathcal{R}_0(\omega) = \begin{pmatrix} \omega R_0(\omega^2) & iR_0(\omega^2) \\ -i(1 + \omega^2 R_0(\omega^2)) & \omega R_0(\omega^2) \end{pmatrix} \tag{2.6}$$

where $R_0(\zeta)$ stands for the free Schrödinger resolvent

$$R_0(\zeta, x - y) = (-\frac{d^2}{dx^2} - \zeta)^{-1} = -\frac{\exp(i\sqrt{\zeta}|x - y|)}{2i\sqrt{\zeta}}, \quad \zeta \in \mathbb{C}^+, \quad \text{Im}\,\zeta^{1/2} > 0 \tag{2.7}$$

Definition 2.1. *Denote by $\mathcal{L}(B_1, B_2)$ the Banach space of bounded linear operators from a Banach space B_1 to a Banach space B_2.*

The explicit formula (2.7) implies the properties of $R_0(\zeta)$ which are obtained in [1, 13]:
i) $R_0(\zeta)$ is strongly analytic function of $\zeta\in\mathbb{C}\setminus[0,\infty)$ with the values in $\mathcal{L}(H^{-1}_0,H^1_0)$;
ii) For $\zeta>0$, the convergence holds $R_0(\zeta\pm i\varepsilon)\to R_0(\zeta\pm i0)$ as $\varepsilon\to 0+$ in $\mathcal{L}(H^{-1}_\sigma,H^1_{-\sigma})$ with $\sigma>1/2$, uniformly in $\zeta\ge r$ for any $r>0$.
iii) For any $M\ge 0$ the following asymptotic expansion holds

$$R_0(\zeta)=\sum_{k=-1}^{M}A_k\zeta^{k/2}+\mathcal{O}(\zeta^{(M+1)/2}),\quad \zeta\to 0,\quad \zeta\in\mathbb{C}\setminus[0,\infty)\tag{2.8}$$

in the norm of $\mathcal{L}(H^{-1}_\sigma;H^1_{-\sigma})$ with $\sigma>3/2+M+1$. Here

$$A_{-1}=\mathrm{Op}\Big[\frac{i}{2}\Big],\quad A_0=\mathrm{Op}\Big[-\frac{1}{2}|x-y|\Big],\tag{2.9}$$

and $A_k\in\mathcal{L}(H^{-1}_\sigma;H^1_{-\sigma})$ with $\sigma>3/2+k$ for $k=-1,0,1,....$
iv) The asymptotics (2.8) can be differentiated $M+2$ times: for $1\le r\le M+2$,

$$\partial^r_\zeta R_0(\zeta)=\partial^r_\zeta\Big(\sum_{k=-1}^{M}A_k\zeta^{k/2}\Big)+\mathcal{O}(\zeta^{\frac{M+1}{2}-r}),\quad \zeta\to 0,\quad \zeta\in\mathbb{C}\setminus[0,\infty)\tag{2.10}$$

in the norm of $\mathcal{L}(H^{-1}_\sigma;H^1_{-\sigma})$ with $\sigma>3/2+M+1$.

Let $\mathcal{A}_{-1}$ be the operator with the integral kernel

$$\mathcal{A}_{-1}(x-y)=\begin{pmatrix}0&0\\-i\delta(x-y)&0\end{pmatrix}+\begin{pmatrix}0&-1/2\\0&0\end{pmatrix}.\tag{2.11}$$

Then the properties i) – iv) and (2.6) imply the following lemma.

Lemma 2.2. *i) The resolvent $\mathcal{R}_0(\omega)$ is strongly analytic function of $\omega\in\mathbb{C}\setminus\mathbb{R}$ with the values in $\mathcal{L}(\mathcal{F}_0,\mathcal{F}_0)$.*
ii) For $\omega\ne 0$, the convergence holds $\mathcal{R}_0(\omega\pm i\varepsilon)\to\mathcal{R}_0(\omega\pm i0)$ as $\varepsilon\to 0+$ in $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>1/2$, uniformly in $|\omega|\ge r$ for any $r>0$.
iii) For any $M\ge 0$, the following asymptotics hold

$$\mathcal{R}_0(\omega)=\sum_{k=-1}^{M}\omega^k\mathcal{A}_k+\mathcal{O}(|\omega|^{M+1}),\quad \omega\to 0\tag{2.12}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})$ with $\sigma>3/2+M+1$. Here $\mathcal{A}_k\in\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})$ for $k=-1,0,1,...$ and $\sigma>3/2+k$;
iv) The asymptotics (2.12) can be differentiated $M+1$ times: for $1\le r\le M+1$,

$$\partial^r_\omega\mathcal{R}_0(\omega)=\partial^r_\omega\Big(\sum_{k=-1}^{M}\omega^k\mathcal{A}_k\Big)+\mathcal{O}(|\omega|^{M+1-r}),\quad \omega\to 0\tag{2.13}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>3/2+M+1$.

Finally, we state the asymptotics of $\mathcal{R}_0(\omega)$ for large ω which follow from known Agmon-Jensen-Kato decay [1, (A.2')] and [10, Theorem 8.1] of the resolvent R_0

Proposition 2.3. *For any $r > 0$ the following bounds hold for $m = 0, 1$ and $l = -1, 0, 1$,*

$$\|R_0^{(k)}(\zeta)\|_{\mathcal{L}(H^m_\sigma, H^{m+l}_{-\sigma})} \le C(r,k)|\zeta|^{-\frac{1-l+k}{2}}, \qquad \zeta \in \mathbb{C} \setminus (0,\infty), \qquad |\zeta| \ge r \tag{2.14}$$

with $\sigma > 1/2 + k$ for any $k = 0, 1, 2, ...$.

Then for $\mathcal{R}_0(\omega)$ we obtain

Corollary 2.4. *For any $r > 0$ the bounds hold*

$$\|\mathcal{R}_0^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} \le C(r,k) < \infty, \qquad \omega \in \mathbb{C} \setminus \mathbb{R}, \qquad |\omega| \ge r \tag{2.15}$$

with $\sigma > 1/2 + k$ for $k = 0, 1, 2, ...$.

Proof. The bounds follow from representation (2.6) for $\mathcal{R}_0(\omega)$ and asymptotics (2.14) for $R_0(\zeta)$ with $\zeta = \omega^2$. □

Corollary 2.5. *For $t \in \mathbb{R}$ and $\Psi_0 \in \mathcal{F}_\sigma$ with $\sigma > 1/2$, the group $\mathcal{G}(t)$ admits the integral representation*

$$\mathcal{G}(t)\Psi_0 = \frac{1}{2\pi i} \int\limits_{\Gamma} e^{-i\omega t} \Big[\mathcal{R}_0(\omega + i0) - \mathcal{R}_0(\omega - i0)\Big] \Psi_0 \, d\omega \tag{2.16}$$

where the integral converges in the sense of distributions of $t \in \mathbb{R}$ with the values in $\mathcal{F}_{-\sigma}$.

Proof. Summing up the representations (2.4) and (2.5), and sending $\varepsilon \to 0+$, we obtain (2.16) by the Cauchy theorem, Lemma 2.2 and Corollary 2.4. □

2.2 Time decay

The estimates (2.15) do not allow obtain the decay of $\mathcal{G}(t)$ by partial integration in (2.16). We deduce the decay from explicit formulas. The matrix kernel of the dynamical group $\mathcal{G}(t)$ can be written as $\mathcal{G}(t, x - y)$, where

$$\mathcal{G}(t,z) = \begin{pmatrix} \dot G(t,z) & G(t,z) \\ \ddot G(t,z) & \dot G(t,z) \end{pmatrix}, \quad z \in \mathbb{R} \tag{2.17}$$

and

$$G(t,z) = \frac{1}{2}\theta(t - |z|) \tag{2.18}$$

Let us represent $\mathcal{G}(t,z)$ as $\mathcal{G}(t,z) = \mathcal{G}_0 + \mathcal{G}_r(t,z)$, where

$$\mathcal{G}_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.19}$$

Evidently, the free wave group $\mathcal{G}(t)$ does not decays which not correspond to (1.6). On the other hand, $\mathcal{G}_0$ is only nondecreasing term. More exactly, in the next section we will prove the following basic proposition

Proposition 2.6. *For the operator $\mathcal{G}_r(t)$ with the kernel $\mathcal{G}_r(t,x-y)$, the following asymptotics holds*

$$\mathcal{G}_r(t)=\mathcal{O}(t^{-\sigma+1}),\quad t\to\infty \tag{2.20}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})$ with $\sigma>1$.

The following key observation is that the "bad term" $\mathcal{G}_0$ does not contribute to the high energy component of the total group $\mathcal{G}(t)$ since (2.19) contains just one zero frequency This suggests that the high energy component of the group $\mathcal{G}(t)$ decays like $t^{-\sigma+1}$.

More precisely, let us introduce the following *low energy* and *high energy* components of $\mathcal{G}(t)$:

$$\mathcal{G}_l(t)=\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega t}l(\omega)\Big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\Big]\,d\omega \tag{2.21}$$

$$\mathcal{G}_h(t)=\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega t}h(\omega)\Big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\Big]\,d\omega \tag{2.22}$$

where $l(\omega)\in C_0^\infty(\mathbb{R})$ is an even function, supp $l\in[-2,2]$, $l(\omega)=1$ if $|\omega|\le 1$, and $h(\omega)=1-l(\omega)$.

Theorem 2.7. *Let $\sigma>1$. Then the following asymptotics hold*

$$\mathcal{G}_h(t)=\mathcal{O}(t^{-\sigma+1}),\quad t\to\infty \tag{2.23}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})$.

Proof. We deduce asymptotics (2.23) from Proposition 2.6.
Step i) Let $\Psi(0)\in\mathcal{F}_\sigma$. Denote

$$\Psi^+(t)=\theta(t)\mathcal{G}(t)\Psi(0),\ \Psi_0^+(t)=\theta(t)\mathcal{G}_0(t)\Psi(0),\ \Psi_h^+(t)=\theta(t)\mathcal{G}_h(t)\Psi(0),\ \Psi_r^+(t)=\theta(t)\mathcal{G}_r(t)\Psi(0)$$

Then

$$\begin{aligned}\Psi_h^+(t) &= \frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega t}h(\omega)\mathcal{R}_0(\omega+i0)\Psi(0)d\omega\\ &= \frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega t}h(\omega)\tilde\Psi^+(\omega)d\omega=\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega t}h(\omega)\Big[\tilde\Psi_0^+(\omega)+\tilde\Psi_r^+(\omega)\Big]\,d\omega\\ &= \Psi_r^+(t)+\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega t}h(\omega)\tilde\Psi_0^+(\omega)d\omega-\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega t}l(\omega)\tilde\Psi_r^+(\omega)d\omega\end{aligned} \tag{2.24}$$

where the first term $\Psi_r^+(t)$ decays like (2.23) by (2.20).
Step ii) Let us consider the second summand in the RHS of (2.24). By (2.19) the matrix function $\tilde\Psi_0^+(\omega)$ is a smooth function for $|\omega|>1$, and $\partial_\omega^k\tilde\Psi_0^+(\omega)=\mathcal{O}(\omega^{-1-k})$, $k=0,1,2...$, $\omega\to\infty$. Hence partial integration implies that

$$\Big\|\int\limits_{\mathbb{R}} e^{-i\omega t}h(\omega)\tilde\Psi_0^+(\omega)d\omega\Big\|_{\mathcal{F}_{-\sigma}}=\mathcal{O}(t^{-N}),\quad\forall N\in\mathbb{N} \tag{2.25}$$

Step iii) Finally, let us consider the third summand in the RHS of (2.24).

$$\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega t} l(\omega)\tilde\Psi_r^+(\omega)d\omega = [L\star\Psi_r^+](t) = \mathcal{O}(t^{-\sigma+1}),\quad \tilde L = l \tag{2.26}$$

in the norm of $\mathcal{F}_\sigma$, since $L(t)=\mathcal{O}(t^{-N})$, $t\to\infty$ for any $N\in\mathbb{N}$, and $\|\Psi_r(t)\|_{\mathcal{F}_{-\sigma}}=\mathcal{O}(t^{-\sigma+1})$ by (2.20). Finally, (2.24)- (2.26) imply (2.23). □

2.3 Proof of Proposition 2.6

Proof. We consider an arbitrary $t\ge 1$. Let us split the initial function Ψ_0 in two terms, $\Psi_0=\Psi'_{0,t}+\Psi''_{0,t}$ such that

$$\Psi'_{0,t}(x)=0 \ \text{ for } |x|>t/2,\qquad \text{and}\qquad \Psi''_{0,t}(x)=0 \ \text{ for } |x|<t/3 \tag{2.27}$$

and

$$\|\Psi'_{0,t}\|_{\mathcal{F}_\sigma}+\|\Psi''_{0,t}\|_{\mathcal{F}_\sigma}\le C\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\ge 1 \tag{2.28}$$

We estimate $\mathcal{G}_r(t)\Psi'_{0,t}$ and $\mathcal{G}_r(t)\Psi''_{0,t}$ separately.

Step i) Let us consider $\mathcal{G}_r(t)\Psi''_{0,t}=\mathcal{G}(t)\Psi''_{0,t}-\mathcal{G}_0\Psi''_{0,t}$. First we estimate $\mathcal{G}(t)\Psi''_{0,t}=(g_1(\cdot,t),g_2(\cdot,t))$. Using energy conservation for the wave equation and properties (2.27) and (2.28) we obtain

$$\|g_1'(\cdot,t)\|_{H^0_{-\sigma}}+\|g_2(\cdot,t)\|_{H^0_{-\sigma}}\le\|\mathcal{G}(t)\Psi''_{0,t}\|_{\mathcal{F}}=\|\Psi''_{0,t}\|_{\mathcal{F}}\le Ct^{-\sigma}\|\Psi''_{0,t}\|_{\mathcal{F}_\sigma}\le Ct^{-\sigma}\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\ge 1 \tag{2.29}$$

Further, the Hölder inequality, energy conservation and properties (2.28)-(2.27) imply

$$\begin{aligned}
\|g_1(\cdot,t)\|^2_{H^0_{-\sigma}} &= \int(1+x^2)^{-\sigma}g_1^2(x,t)dx=\int(1+x^2)^{-\sigma}\Big(\int\limits_0^t\dot g_1(x,s)ds-g_1(x,0)\Big)^2dx\\
&\le 2\int(1+x^2)^{-\sigma}g_1^2(x,0)dx+2t\int(1+x^2)^{-\sigma}\Big(\int\limits_0^t\dot g_1^2(x,s)ds\Big)dx \qquad (2.30)\\
&= 2\|(\Psi''_{0,t})_1\|^2_{H^0_{-\sigma}}+2t\int\limits_0^t\|(\mathcal{G}(s)\Psi''_{0,t})_2\|^2_{H^0_{-\sigma}}ds\le C\Big(t^{-4\sigma}\|(\Psi''_{0,t})_1\|^2_{H^0_\sigma}+t\int\limits_0^t\|\mathcal{G}(s)\Psi''_{0,t}\|^2_{\mathcal{F}}ds\Big)\\
&\le C\Big(t^{-4\sigma}\|\Psi_0\|^2_{\mathcal{F}_\sigma}+t\int\limits_0^t\|\Psi''_{0,t}\|^2_{\mathcal{F}}ds\Big)\le C\Big(t^{-4\sigma}\|\Psi_0\|^2_{\mathcal{F}_\sigma}+t^{2-2\sigma}\|\Psi_0\|^2_{\mathcal{F}_\sigma}\Big)\le Ct^{2-2\sigma}\|\Psi_0\|^2_{\mathcal{F}_\sigma}
\end{aligned}$$

Hence, (2.29) and (2.30) imply

$$\|\mathcal{G}(t)\Psi''_{0,t}\|_{\mathcal{F}_{-\sigma}}\le Ct^{-\sigma+1}\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\ge 1 \tag{2.31}$$

Second we estimate $\mathcal{G}_0\Psi''_{0,t}$. By Cauchy inequality

$$\begin{aligned}
|\mathcal{G}_0^{12}\pi''_{0,t}| &= \Big|\frac12\int\pi''_{0,t}(x)dx\Big|\le C\Big(\int|\pi''_{0,t}(x)|^2(1+x^2)^\sigma dx\Big)^{1/2}\Big(\int\limits_{t/3}^\infty\frac{dx}{(1+x^2)^\sigma}\Big)^{1/2}\\
&\le Ct^{-\sigma+1/2}\|\pi''_{0,t}\|_{H^0_\sigma}\le Ct^{-\sigma+1/2}\|\Psi_0\|_{\mathcal{F}_\sigma}
\end{aligned}$$

where $\pi''_{0,t}$ is the second component of $\Psi''_{0,t}$. Therefore,

$$\|\mathcal{G}_0\Psi''_{0,t}\|_{\mathcal{F}_{-\sigma}} = \|\mathcal{G}_0^{12}\pi''_{0,t}\|_{H^0_{-\sigma}} \leq Ct^{-\sigma+1/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geq 1 \tag{2.32}$$

Finally, (2.31)- (2.32) imply that

$$\|\mathcal{G}_r(t)\Psi''_{0,t}\|_{\mathcal{F}_{-\sigma}} \leq Ct^{-\sigma+1}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geq 1 \tag{2.33}$$

Step ii) Now we consider $\mathcal{G}_r(t)\Psi'_{0,t} = \mathcal{G}(t)\Psi'_{0,t} - \mathcal{G}_0\Psi'_{0,t}$. Formulas (2.17)-(2.19), and (2.27) imply that

$$[\mathcal{G}_r(t)\Psi'_{0,t}](x) = 0, \quad |x| < t/2.$$

Denote by $\chi(x)$ the characteristic function of the domain $|x| > t/2$. Then

$$\begin{aligned}\|\mathcal{G}_r(t)\Psi'_{0,t}\|_{\mathcal{F}_{-\sigma}} &= \|\chi(x)(\mathcal{G}(t)\Psi'_{0,t} - \mathcal{G}_0(t)\Psi'_{0,t})\|_{\mathcal{F}_{-\sigma}} \leq \|\mathcal{G}(t)\Psi'_{0,t}\|_{\mathcal{F}_{-\sigma}} + \|\chi(x)\mathcal{G}_0(t)\Psi'_{0,t}\|_{\mathcal{F}_{-\sigma}} \\ &\leq Ct^{-\sigma}\big(\|\mathcal{G}(t)\Psi'_{0,t}\|_{\mathcal{F}} + \|(\mathcal{G}(t)\Psi'_{0,t})_1\|_{L^2}\big) + \|\chi(x)\mathcal{G}_0(t)\Psi'_{0,t}\|_{\mathcal{F}_{-\sigma}} \end{aligned} \tag{2.34}$$

By energy conservation and (2.28), we obtain

$$\|\mathcal{G}(t)\Psi'_{0,t}\|_{\mathcal{F}} = \|\Psi'_{0,t}\|_{\mathcal{F}} \leq \|\Psi'_{0,t}\|_{\mathcal{F}_\sigma} \leq C\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geq 1 \tag{2.35}$$

Further, similarly (2.30), using energy conservation we obtain

$$\begin{aligned}\|(\mathcal{G}(t)\Psi'_{0,t})_1\|^2_{L^2} &\leq 2\|(\Psi'_{0,t})_1\|^2_{L^2} + 2t\int\limits_0^t \|(\mathcal{G}(s)\Psi'_{0,t})_2\|^2_{L^2}ds \leq C\Big(\|\Psi'_{0,t}\|^2_{\mathcal{F}_\sigma} + t\int\limits_0^t \|\Psi'_{0,t}\|^2_{\mathcal{F}}ds\Big) \\ &\leq C\Big(\|\Psi_0\|^2_{\mathcal{F}_\sigma} + t^2\|\Psi'_{0,t}\|^2_{\mathcal{F}}\Big) \leq Ct^2\|\Psi_0\|^2_{\mathcal{F}_\sigma}\end{aligned} \tag{2.36}$$

Finaly, we estimate the last summand in the RHS of (2.34). Denote $\pi'_{0,t}$ the second component of $\Psi'_{0,t}$. By Cauchy inequality

$$|\mathcal{G}_0^{12}\pi'_{0,t}| = \Big|\frac{1}{2}\int \pi'_{0,t}(x)dx\Big| \leq C\|\pi'_{0,t}\|_{H^0_\sigma} \leq C\|\Psi'_{0,t}\|_{\mathcal{F}_\sigma}$$

since $\sigma > 1$. Hence

$$\|\chi(x)\mathcal{G}_0(t)\Psi'_{0,t}\|^2_{\mathcal{F}_{-\sigma}} = \int\limits_{|x|>t/2} (1+x^2)^{-\sigma}|\mathcal{G}_0^{12}\pi'_{0,t}|^2dx \leq Ct^{-2\sigma+1}\|\Psi'_{0,t}\|^2_{\mathcal{F}_\sigma}$$

Finally, the last estimate and (2.34)-(2.36)

$$\|\mathcal{G}_r(t)\Psi'_{0,t}\|_{\mathcal{F}_{-\sigma}} \leq Ct^{-\sigma+1}\|\Psi'_{0,t}\|_{\mathcal{F}_\sigma}, \quad t \geq 1$$

□

3 Perturbed wave equation

To prove the long time decay for the perturbed wave equation, we first establish the spectral properties of the generator.

3.1 Spectral properties

Let us collect the properties of the perturbed Schrödinger resolvent $R(\zeta)=(H-\zeta)^{-1}$ obtained in [1, 10, 13] under conditions (1.4) and (1.5). Note, that in [10] is considered 3D case, but corresponding properties can be proved in 1D case similarly.

R1. $R(\zeta)$ is strongly meromorphic function of $\zeta\in\mathbb{C}\setminus[0,\infty)$ with the values in $\mathcal{L}(H_0^{-1},H_0^1)$; the poles of $R(\zeta)$ are located at a finite set of eigenvalues $\zeta_j<0$, $j=1,...,N$, of the operator H with the corresponding eigenfunctions $\psi_j(x)\in H_s^2$ with any $s\in\mathbb{R}$.
R2. For $\zeta>0$, the convergence holds $R(\zeta\pm i\varepsilon)\to R(\zeta\pm i0)$ as $\varepsilon\to 0+$ in $\mathcal{L}(H_\sigma^{-1},H_{-\sigma}^1)$ with $\sigma>1/2$, uniformly in $\zeta\ge r$ for any $r>0$.
R3. Assume $\beta>3+2M$, $M=0,1,2....$ The expansion holds:

$$R(\zeta)=\sum_{j=0}^{M}B_j\zeta^{j/2}+\mathcal{O}(\zeta^{(M+1)/2}),\quad \zeta\to 0,\quad \zeta\in\mathbb{C}\setminus(0,\infty) \tag{3.1}$$

in the norm of $\mathcal{L}(H_\sigma^{-1},H_{-\sigma}^1)$ with $\sigma>3/2+M$. The expansion (3.1) can be differentiated $M+1$ times.
R4. Assume $\beta>k+1$, $k=0,1,2,....$ For any $r,\delta>0$, the bounds hold for $m=0,1$ and $l=-1,0,1$

$$\|R^{(k)}(\zeta)\|_{\mathcal{L}(H_\sigma^m;H_{-\sigma}^{m+l})}\le C(r,\delta,k)|\zeta|^{-(1-l+k)/2},\quad \zeta\in\mathbb{C}\setminus(0,\infty),\quad |\zeta|>r,\quad |\arg\zeta|\le\pi-\delta \tag{3.2}$$

with $\sigma>1/2+k$. The resolvent $\mathcal{R}(\omega)=(\mathcal{H}-\omega)^{-1}$ can be expressed similarly to (2.6):

$$\mathcal{R}(\omega)=\begin{pmatrix}\omega R(\omega^2) & iR(\omega^2)\\ -i(1+\omega^2R(\omega^2)) & \omega R(\omega^2)\end{pmatrix}. \tag{3.3}$$

Hence, the properties **R1** – **R4** imply the corresponding properties of $\mathcal{R}(\omega)$:

Lemma 3.1. *Let the potential V satisfy conditions (1.4) and (1.5). Then*
i) $\mathcal{R}(\omega)$ is strongly meromorphic function of $\omega\in\mathbb{C}\setminus\mathbb{R}$ with the values in $\mathcal{L}(\mathcal{F}_0,\mathcal{F}_0)$;
ii) The poles of $\mathcal{R}(\omega)$ are located at a finite set of imaginary axe

$$\Sigma=\{\omega_j^\pm=\pm\sqrt{\zeta_j},\ j=1,...,N\}$$

of eigenvalues of the operator $\mathcal{H}$ with the corresponding eigenfunctions $\begin{pmatrix}\psi_j(x)\\ \omega_j^\pm\psi_j(x)\end{pmatrix}$;
iii) For $\omega\in\mathbb{R}$, the convergence holds $\mathcal{R}(\omega\pm i\varepsilon)\to\mathcal{R}(\omega\pm i0)$ as $\varepsilon\to 0+$ in $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>3/2$;
iv) Assume $\beta>1+2k$, $k=1,2,...,$ and $r<\mathrm{dist}(\Sigma,0)$.
Then the bounds hold

$$\|\mathcal{R}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}\le C(r,k)<\infty,\quad 0<|\mathrm{Im}\,\omega|\le r,\quad |\mathrm{Re}\,\omega|\le 2 \tag{3.4}$$

with $\sigma>1/2+k$;
v) Assume $\beta>k+1$, $k=0,1,2,...$ Then the bounds hold

$$\|\mathcal{R}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}\le C(k)<\infty,\quad |\mathrm{Re}\,\omega|\ge 1 \tag{3.5}$$

with $\sigma>1/2+k$.

Finally, let us denote by $\mathcal{V}$ the matrix

$$\mathcal{V} = \begin{pmatrix} 0 & 0 \\ iV & 0 \end{pmatrix} \tag{3.6}$$

Then the vectorial equation (1.2) reads

$$i\dot{\Psi}(t) = (\mathcal{H}_0 + \mathcal{V})\Psi(t)$$

The resolvents $\mathcal{R}(\omega)$ and $\mathcal{R}_0(\omega)$ are related by the Born perturbation series

$$\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega) + \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega), \quad \omega \in \mathbb{C} \setminus [\mathbb{R} \cup \Sigma] \tag{3.7}$$

which follows by iteration of $\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)$. An important role in (3.7) plays the product $\mathcal{W}(\omega) := \mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}$. We obtain the asymptotics of $\mathcal{W}(\omega)$ for large ω.

Lemma 3.2. *Let the potential V satisfy (1.4) with $\beta > 1/2 + k + \sigma$ for $k = 0, 1, 2, ...$, with some $\sigma > 0$. Then bounds hold*

$$\|\mathcal{W}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_{-\sigma},\mathcal{F}_{\sigma})} \le C(k)|\omega|^{-2}, \quad \omega \in \mathbb{C} \setminus \mathbb{R}, \quad |\omega| > 1 \tag{3.8}$$

Proof. Bounds (3.8) follow from the algebraic structure of the matrix

$$\mathcal{W}^{(k)}(\omega) = \mathcal{V}\mathcal{R}_0^{(k)}(\omega)\mathcal{V} = \begin{pmatrix} 0 & 0 \\ -iVR_0^{(k)}(\omega^2)V & 0 \end{pmatrix}, \tag{3.9}$$

since (2.14) implies that for $\omega \in \mathbb{C} \setminus \mathbb{R}$, $|\omega| > 1$

$$\|VR_0^{(k)}(\omega^2)Vf\|_{H^0_\sigma} \le C\|R_0^{(k)}(\omega^2)Vf\|_{H^0_{\sigma-\beta}} \le C(k)|\omega|^{-2}\|Vf\|_{H^1_{\beta-\sigma}} \le C(k)|\omega|^{-2}\|f\|_{H^1_{-\sigma}} \tag{3.10}$$

with $1/2 + k < \beta - \sigma$ for $k = 0, 1, 2, ...$ □

3.2 Time decay

In this section we combine the spectral properties of the perturbed resolvent and time decay for the unperturbed dynamics using the (finite) Born perturbation series. Our main result is the following.

Theorem 3.3. *Let conditions (1.4) and (1.5) hold. Then for $\sigma > 2$*

$$\|e^{-it\mathcal{H}} - \sum_{\omega_J \in \Sigma} e^{-i\omega_j t}P_j\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(|t|^{-\gamma}), \quad \gamma = \min\{\langle\sigma - 1/2\rangle, \sigma - 1, \langle\beta/2 - 1/2\rangle, \beta/2 - 1\} \tag{3.11}$$

as $t \to \pm\infty$. Here P_j are the Riesz projectors onto the corresponding eigenspaces.

Proof. Lemma 3.1 and bounds (3.5) with $k = 0$ imply similarly to (2.16), that

$$\Psi(t) - \sum_{\omega_j \in \Sigma} e^{-i\omega_j t}P_j\Psi_0 = \frac{1}{2\pi i}\int e^{-i\omega t}\Big[\mathcal{R}(\omega + i0) - \mathcal{R}(\omega - i0)\Big]\Psi_0\, d\omega = \Psi_l(t) + \Psi_h(t) \tag{3.12}$$

where P_j stands for the corresponding Riesz projector

$$P_j\Psi_0 := -\frac{1}{2\pi i}\int_{|\omega-\omega_j|=\delta}\mathcal{R}(\omega)\Psi_0 d\omega$$

with a small $\delta > 0$, and

$$\Psi_l(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} l(\omega)e^{-i\omega t}\Big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\Big]\Psi_0\; d\omega \tag{3.13}$$

$$\Psi_h(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} h(\omega)e^{-i\omega t}\Big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\Big]\Psi_0\; d\omega \tag{3.14}$$

where $l(\omega)$ and $h(\omega)$ are defined in Section 2.2. Further we analyze $\Psi_l(t)$ and $\Psi_h(t)$ separately.

3.2.1 Time decay of $\Psi_l(t)$

Let $\sigma > 3/2$ and $\beta > 3$. By Lemma 3.1 iv)-v) we apply integration by parts γ times, with $\gamma = \min\{\langle\sigma - 1/2\rangle, \langle\beta/2 - 1/2\rangle\}$ and obtain

$$\|\Psi_l(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-\gamma}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t\in\mathbb{R}. \tag{3.15}$$

3.2.2 Time decay of Ψ_h

Let us substitute the series (3.7) into the spectral representation (3.14) for $\Psi_h(t)$:

$$\begin{aligned}\Psi_h(t) &= \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}h(\omega)\Big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\Big]\Psi_0\; d\omega \qquad (3.16)\\ &+ \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}h(\omega)\Big[\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\; d\omega\\ &+ \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}h(\omega)\Big[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega+i0)-\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega-i0)\Big]\Psi_0\; d\omega\\ &= \Psi_{h1}(t)+\Psi_{h2}(t)+\Psi_{h3}(t), \qquad t\in\mathbb{R}\end{aligned}$$

Further we analyze each term Ψ_{hk}, $k = 1, 2, 3$ separately.
Step i) The first term $\Psi_{h1}(t) = \mathcal{G}_h(t)\Psi_0$ by (2.22). Hence, Theorem 2.7 implies that

$$\|\Psi_{h1}(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-\sigma+1}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t\in\mathbb{R} \tag{3.17}$$

Step ii) Now we consider the second term $\Psi_{h2}(t)$. Denote $h_1(\omega) = \sqrt{h(\omega)}$ and let

$$\Phi_{h1} = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}h_1(\omega)\Big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\Big]\Psi_0\; d\omega$$

It is obvious that for Φ_{h1} the inequality (3.17) also holds. Namely,

$$\|\Phi_{h1}(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-\sigma+1}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t\in\mathbb{R} \tag{3.18}$$

Now the second term $\Psi_{h2}(t)$ can be rewritten as a convolution.

Lemma 3.4. *The convolution representation holds*

$$\Psi_{h2}(t) = i\int\limits_0^t \mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\,d\tau, \qquad t\in\mathbb{R} \tag{3.19}$$

where the integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma > 2$.

Proof. Then the term $\Psi_{h2}(t)$ can be rewritten as

$$\Psi_{h2}(t) = \frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega t}h_1^2(\omega)\Big[\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0) - \mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\,d\omega \tag{3.20}$$

Let us integrate the first term in the right hand side of (3.20), denoting

$$\mathcal{G}^{\pm}_{h1}(t) := \theta(\pm t)\mathcal{G}_{h1}(t), \quad \Phi^{\pm}_{h1}(t) := \theta(\pm t)\Phi_{h1}(t), \qquad t\in\mathbb{R}$$

We know that $h_1(\omega)\mathcal{R}_0(\omega+i0)\Psi_0 = i\tilde{\Phi}^{+}_{h1}(\omega)$, hence integrating the first term in the right hand side of (3.20), we obtain that

$$\begin{aligned}\Psi^{+}_{h2}(t) &= \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i\omega t}h_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\tilde{\Phi}^{+}_{h1}(\omega)\,d\omega \\ &= \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i\omega t}h_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int_{\mathbb{R}} e^{i\omega\tau}\Phi^{+}_{h1}(\tau)d\tau\Big]d\omega \\ &= \frac{1}{2\pi}(i\partial_t+i)^2\int\limits_{\mathbb{R}} \frac{e^{-i\omega t}}{(\omega+i)^2}\,h_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int_{\mathbb{R}} e^{i\omega\tau}\Phi^{+}_{h1}(\tau)d\tau\Big]d\omega \end{aligned} \tag{3.21}$$

The last double integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma > 2$ by (3.18), Lemma 2.2 ii), and (2.15) with $k = 0$. Hence, we can change the order of integration by the Fubini theorem. Then we obtain that

$$\Psi^{+}_{h2}(t) = i\int_{\mathbb{R}} \mathcal{G}^{+}_{h1}(t-\tau)\mathcal{V}\Phi^{+}_{h1}(\tau)d\tau = \begin{cases} i\displaystyle\int_0^t \mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)d\tau &, \ t>0 \\ 0 &, \ t<0\end{cases} \tag{3.22}$$

since

$$\begin{aligned}\mathcal{G}^{+}_{h1}(t-\tau) &= \frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i\omega(t-\tau)}h_1(\omega)\mathcal{R}_0(\omega+i0)\,d\omega \\ &= \frac{1}{2\pi i}(i\partial_t+i)^2\int\limits_{\mathbb{R}} \frac{e^{-i\omega(t-\tau)}}{(\omega+i)^2}\,h_1(\omega)\mathcal{R}_0(\omega+i0)\,d\omega\end{aligned}$$

by (2.4). Similarly, integrating the second term in the right hand side of (3.20), we obtain

$$\Psi^{-}_{h2}(t) = i\int_{\mathbb{R}} \mathcal{G}^{-}_{h1}(t-\tau)\mathcal{V}\Phi^{-}_{h1}(\tau)d\tau = \begin{cases} 0 &, \ t>0 \\ i\displaystyle\int_0^t \mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)d\tau &, \ t<0\end{cases} \tag{3.23}$$

Now (3.19) follows since $\Psi_{h2}(t)$ is the sum of two expressions (3.22) and (3.23). □

Lemma 3.5.

$$\|\Psi_{h2}(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-\gamma}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad \gamma=\min\{\sigma-1, \frac{\beta}{2}-1\}, \quad t\in\mathbb{R}. \tag{3.24}$$

Proof. We apply Theorem 2.7 with h_1 instead h to the integrand in (3.19). For $2<\sigma<\beta/2$ we obtain

$$\|\mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_{-\sigma}} \le \frac{C\|\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{\sigma-1}} \le \frac{C\|\Phi_{h1}(\tau)\|_{\mathcal{F}_{-\sigma}}}{(1+|t-\tau|)^{\sigma-1}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{\sigma-1}(1+|\tau|)^{\sigma-1}},$$

and for $\sigma>\beta/2$

$$\|\mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_{-\sigma}} \le \|\mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_{-\beta/2}} \le \frac{C\|\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_{\beta/2}}}{(1+|t-\tau|)^{\frac{\beta}{2}-1}}$$

$$\le \frac{C\|\Phi_{h1}(\tau)\|_{\mathcal{F}_{-\beta/2}}}{(1+|t-\tau|)^{\frac{\beta}{2}-1}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_{\beta/2}}}{(1+|t-\tau|)^{\frac{\beta}{2}-1}(1+|\tau|)^{\frac{\beta}{2}-1}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{\frac{\beta}{2}-1}(1+|\tau|)^{\frac{\beta}{2}-1}}$$

Hence (3.19) implies (3.24). □

Step iii) Finally, let us rewrite the last term Ψ_{h3} as

$$\Psi_{h3}(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}h(\omega)\mathcal{N}(\omega)\Psi_0\, d\omega, \tag{3.25}$$

where $\mathcal{N}(\omega) := \mathcal{M}(\omega+i0)-\mathcal{M}(\omega-i0)$ for $\omega\in\mathbb{R}$, and

$$\mathcal{M}(\omega) := \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega) = \mathcal{R}_0(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega), \quad \omega\in\mathbb{C}\setminus\mathbb{R}. \tag{3.26}$$

Now we obtain the asymptotics of $\mathcal{N}$ and its derivatives for large ω.

Lemma 3.6. *For* $0\le k<\min\{\beta-3/2, \sigma-1/2\}$ *the bounds hold*

$$\|\mathcal{N}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} \le C(k)|\omega|^{-2}, \quad \omega\in\mathbb{R}, \quad |\omega|\ge 1 \tag{3.27}$$

Proof. We have

$$\mathcal{M}^{(k)} = \sum_{k_1+k_2+k_3=k}\frac{k!}{k_1!k_2!k_3!}\mathcal{R}_0^{(k_1)}\mathcal{W}^{(k_2)}\mathcal{R}^{(k_3)} \tag{3.28}$$

Lemma 3.2 and bounds (2.15), (3.5) imply

$$\|\mathcal{R}_0^{(k_1)}\mathcal{W}^{(k_2)}\mathcal{R}^{(k_3)}(\omega)f\|_{\mathcal{F}_{-\sigma}} \le \|\mathcal{R}_0^{(k_1)}\mathcal{W}^{(k_2)}\mathcal{R}^{(k_3)}(\omega)f\|_{\mathcal{F}_{-\sigma_1}} \le C(k_1)\|\mathcal{W}^{(k_2)}\mathcal{R}^{(k_3)}(\omega)f\|_{\mathcal{F}_{\sigma_1}}$$

$$\le \frac{C(k_1,k_2)}{|\omega|^2}\|\mathcal{R}^{(k_3)}(\omega)f\|_{\mathcal{F}_{-\sigma_1}} \le \frac{C(r,k_1,k_2,k_3)}{|\omega|^2}\|f\|_{\mathcal{F}_{\sigma_1}} \le \frac{C(r,k_1,k_2,k_3)}{|\omega|^2}\|f\|_{\mathcal{F}_\sigma}, \quad |\omega|\ge 1$$

under the conditions

$$\sigma>\sigma_1>1/2+\max\{k_1,k_3\}, \quad \beta>1/2+k_2+\sigma_1, \quad \beta>1+k_3$$

All these inequalities hold if $\sigma>1/2+k$, $\beta>1+k$, and

$$1/2+\max\{k_1,k_3\}<\sigma_1<\min\{\sigma,\beta-1/2-k_2\}$$

□

Now we prove the desired decay of $\Psi_{h3}(t)$ from (3.25).

Lemma 3.7.

$$\|\Psi_{h3}(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-\gamma}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad \gamma = \min\{\langle\sigma-1/2\rangle, \sigma-1, \langle\beta/2-1/2\rangle, \beta/2-1\}, \quad t\in\mathbb{R}. \tag{3.29}$$

Proof. First, in the case $2<\sigma<\beta/2$ there exists $k\ge 1$ such that $1/2+k<\sigma\le 3/2+k$. Then $\beta>1+2k>1+k$, and by Lemma 3.6

$$\mathcal{N}^{(k)}(\omega)) \in L^1([1,\infty];\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})). \tag{3.30}$$

Then we can apply k times integration by parts in (3.25) to obtain

$$\|\Psi_{h3}(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-k}\|\Psi_0\|_{\mathcal{F}_\sigma} = C(1+|t|)^{\langle\sigma-1/2\rangle}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t\in\mathbb{R}.$$

Since $k=\langle\sigma-1/2\rangle$ by definition 1.2.
Second, in the case $4<\beta<2\sigma$ there exists $k\ge1$ such that $k+1/2<\beta/2\le k+3/2$. Then $\sigma>1/2+k$ and $\beta>2k+1>1+k$. Hence (3.30) holds by Lemma 3.6 and using k times integration by parts we obtain

$$\|\Psi_3(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{\langle\beta/2-1/2\rangle}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t\in\mathbb{R}.$$

This completes the proof of the lemma and Theorem 3.3. □

Corollary 3.8. *The asymptotics* (3.11) *imply* (1.6) *with the projector*

$$\mathcal{P}_c = 1 - \sum_{\omega_j\in\Sigma} P_j \tag{3.31}$$

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