# Weighted Energy Decay for 3D Wave Equation

E. A. Kopylova [1]
*Institute for Information Transmission Problems RAS*
*B.Karetnyi 19, Moscow 101447,GSP-4, Russia*
e-mail: elena.kopylova@univie.ac.at

**Abstract**

We obtain a dispersive long-time decay in weighted energy norms for solutions to the 3D wave equation with generic potential. The decay extends the results obtained by Jensen and Kato for the 3D Schrödinger equation.

*Keywords*: dispersion, wave equation, resolvent, spectral representation, weighted spaces, continuous spectrum, Born series, convolution, long-time asymptotics, asymptotic completeness.

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# 1 Introduction

In this paper, we establish a dispersive long time decay for the solutions to 3D wave equation

$$\ddot{\psi}(x,t) = \Delta\psi(x,t) + V(x)\psi(x,t), \quad x \in \mathbb{R}^3 \tag{1.1}$$

in weighted energy norms. In vectorial form, equation (1.1) reads

$$i\dot{\Psi}(t) = \mathcal{H}\Psi(t), \tag{1.2}$$

where

$$\Psi(t) = \left( \begin{array}{c} \psi(t) \\ \dot{\psi}(t) \end{array} \right), \qquad \mathcal{H} = \left( \begin{array}{cc} 0 & i \\ i(\Delta+V) & 0 \end{array} \right). \tag{1.3}$$

For $s, \sigma \in \mathbb{R}$, let us denote by $H^s_\sigma = H^s_\sigma(\mathbb{R}^3)$ the weighted Sobolev spaces introduced by Agmon, [1], with the finite norms

$$\|\psi\|_{H^s_\sigma} = \|(1+|x|^2)^{\sigma/2}(1+|\nabla|^2)^{s/2}\psi\|_{L^2} < \infty.$$

We assume that $V(x)$ is a real function, and

$$|V(x)| + |\nabla V(x)| \le C(1+|x|)^{-\beta}, \qquad x \in \mathbb{R}^3 \tag{1.4}$$

for some $\beta > 4$. Then the multiplication by $V(x)$ is bounded operator $H^1_s \to H^1_{s+\beta}$ for any $s \in \mathbb{R}$.

We restrict ourselves to the "regular case" in the terminology of [13] (or "nonsingular case" in [19]), where the truncated resolvent of the Schrödinger operator $H = -\Delta + V(x)$ is bounded at the end point $\lambda = 0$ of the continuous spectrum. In other words, the point $\lambda = 0$ is neither eigenvalue nor resonance for the operator $H$; this holds for *generic potentials*.

Let $\dot{H}^1$ denote the completion of the complex space $C_0^\infty(\mathbb{R}^3)$ with the norm $\|\nabla\psi(x)\|_{L^2}$. Equivalently, using Sobolev's embedding theorem, $\dot{H}^1 = \{\psi(x) \in L^6(\mathbb{R}^3) : \quad |\nabla\psi(x)| \in L^2\}$, and

$$\|\psi\|_{L^6} \le C\|\psi\|_{\dot{H}^1}. \tag{1.5}$$

**Definition 1.1.** *i) $\mathcal{F}$ is the complex Hilbert space $\dot{H}^1 \oplus L^2$ of vector-functions $\Psi = (\psi, \pi)$ with the norm*

$$\|\,\Psi\|_{\mathcal{F}} = \|\nabla\psi\|_{L^2} + \|\pi\|_{L^2} < \infty.$$

*ii) $\mathcal{F}_\sigma$ is the complex Hilbert space $H^1_\sigma \oplus H^0_\sigma$ of vector-functions $\Psi = (\psi, \pi)$ with the norm*

$$\|\,\Psi\|_{\mathcal{F}_\sigma} = \|\psi\|_{H^1_\sigma} + \|\pi\|_{H^0_\sigma} < \infty.$$

**Definition 1.2.** *For real $\alpha > 1$ denote by $\langle\alpha\rangle$ the number from $\mathbb{N}$ such that*

$$\langle\alpha\rangle < \alpha \le 1 + \langle\alpha\rangle$$

Our main result is the following long time decay of the solutions to (1.2): in the "regular case" for initial data $\Psi_0 = \Psi(0) \in \mathcal{F}_\sigma$ with $\sigma > 2$ we have

$$\|\mathcal{P}_c\Psi(t)\|_{\mathcal{F}_{-\sigma}} = \mathcal{O}(|t|^{-\gamma}), \quad \gamma = \min\{\langle\sigma - 1/2\rangle, \sigma - 1, \langle\beta/2 - 1/2\rangle, \beta/2 - 1\}, \quad t \to \pm\infty \tag{1.6}$$

Here $\mathcal{P}_c$ is a Riesz projector onto the continuous spectrum of the operator $\mathcal{H}$. The decay is desirable for the study of asymptotic stability and scattering for the solutions to nonlinear hyperbolic equations. The study has been started in 90' for nonlinear Schrödinger equation, [5, 20, 21, 23, 24], and continued last decade [6, 7, 16]. The study has been extended to the Klein-Gordon equation in [10, 25]. Further extension need more information on the decay for the corresponding linearized equations that stipulated our investigation.

Let us comment on previous results in this direction. Local energy decay has been established first in the scattering theory for linear Schrödinger equation developed since 50' by Birman, Kato, Simon, and others. For wave equations with compactly supported potentials, and similar hyperbolic PDEs, Vainberg [27] established the decay in local energy norms for solutions with compactly supported initial data. The decay in the $L^p$ norms for wave and Klein-Gordon equations was obtained in [3, 4, 8, 15, 18, 28, 29].

However, applications to asymptotic stability of solutions to the nonlinear equations also require an exact characterization of the decay for the corresponding linearized equations in weighted norms (see e.g. [5, 6, 7, 25]).

The decay of type (1.6) in weighted norms has been established first by Jensen and Kato [13] for the Schrödinger equation in the dimension $n = 3$. The result has been extended to all other dimensions by Jensen and Nenciu [11, 12, 14], and to more general PDEs of the Schrödinger type by Murata [19]. The survey of the results can be found in [22].

For the "free" wave equations with $V(x) = 0$ some estimates in weighted $L^p$-norms have been established in [2, 9]. For the 3D wave equation (1.1), the decay (1.6) in the weighted energy norms was not proved until now. Let us note that the decay rate in (1.6) corresponds to the spatial decay of the initial function $\Psi(0)$ and potential $V(x)$ in contrast to the Schrödinger case [13], where the decay rate is $t^{-3/2}$. This difference is related to the presence of the lacuna for the free 3D wave equation.

Now let us comment on our approach. The problem was that the Jensen-Kato [13] approach is not applicable directly to the wave equations. The approach relies on the spectral Fourier-Laplace representation

$$P_c\Psi(t) = \frac{1}{2\pi i}\int\limits_0^\infty e^{-i\omega t}\Big[R(\omega + i0) - R(\omega - i0)\Big]\Psi_0 d\omega, \qquad t \in \mathbb{R} \tag{1.7}$$

where $R(\omega)$ is the resolvent of the Schrödinger operator $H = -\Delta + V$, and $P_c$ is the corresponding projector onto the continuous spectrum of $H$. Integration by parts implies the time decay of type (1.6) since the resolvent $R(\omega)$ is sufficiently smooth and its derivatives $\partial_\omega^k R(\omega)$ have a good decay at $|\omega| \to \infty$ for large $k$ in the weighted norms. On the other hand, in the case of the wave equation, the derivatives do not decay.

Let us illustrate this difference in the case of the corresponding free 3D equations:

i) the resolvent of the free Schrödinger equation is the integral operator with the kernel

$$R_{\rm S}(\omega, x - y) = \frac{e^{i\sqrt{\omega}|x-y|}}{4\pi|x - y|},$$

ii) the resolvent of the free wave equation is the integral operator with the matrix kernel

$$R_{\rm W}(\omega, x - y) = \begin{pmatrix} 0 & 0 \\ -i\delta(x - y) & 0 \end{pmatrix} + \frac{e^{i\omega|x-y|}}{4\pi|x - y|}\begin{pmatrix} \omega & i \\ -i\omega^2 & \omega \end{pmatrix} \tag{1.8}$$

and the region of integration in the corresponding formula (1.7) is changed to $(-\infty, \infty)$. Leading singularity of the Schrödinger resolvent is $\sqrt{\omega}$ at $\omega = 0$. Hence, the contribution of law frequencies into the integral (1.7) decays like $t^{-3/2}$ in the Schrödinger case. In the case of the wave equation, the resolvent is smooth function of $\omega$, hence the contribution of the law frequencies decays like $\sim t^{-N}$ with any $N > 0$.

Now let us discuss the contribution of high frequencies into the integral (1.7). For the Schrödinger case, the contribution decays like $\sim t^{-N}$ with any $N > 0$. This follows by partial integration since the derivatives $\partial_\omega^k R_{\rm S}(\omega, x-y)$ decay like $|\omega|^{-k/2}$ as $\omega \to \infty$.

On the other hand, the kernel $R_{\rm W}(\omega, x-y)$ does not decay for large $|\omega|$, and differentiation in $\omega$ does not improve the decay (cf. the bounds (2.17) and (3.4)). Hence, for the wave equation the integration by parts does not provide the long time decay.

This difference is not only technical. It reflects the fact that the multiplication by $t^N$, with large $N$, improves the smoothness of the solutions to the Schrödinger equation in contrast to the wave equation. This corresponds to distinct nature of the wave propagation in relativistic and nonrelativistic wave equations:
i) for a solution $\psi(x,t)$ to the Schrödinger equation, main singularity is concentrated at $t = 0$ and disappears at infinity for $t \neq 0$ due to infinite speed of propagation.
ii) for a solution $\psi(x,t)$ to the wave equation, the singularities move with bounded speed, thus they are present forever in the space.

Thus, the proof of the decay for the high energy component of the solution requires novel robust ideas. This problem is resolved at present paper with a modified approach based on the Born series and convolution. Namely, the resolvent $\mathcal{R}(\omega)$ of the operator $\mathcal{H}$ admits the finite Born expansion

$$\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega) + \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega) \tag{1.9}$$

where $\mathcal{R}_0(\omega)$ stands for the free resolvent with the integral kernel (1.8) corresponding to $V = 0$, and $\mathcal{V} = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}$. Taking the inverse Fourier-Laplace transform, we obtain the corresponding expansion for the dynamical group $\mathcal{U}(t)$ of the wave equation (1.2),

$$\mathcal{U}(t) = \mathcal{U}_0(t) + i\int_0^t \mathcal{U}_0(t-s)\mathcal{V}\mathcal{U}_0(s)ds - iF^{-1}_{\omega\to t}\Big[\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)\Big] \tag{1.10}$$

where $\mathcal{U}_0(t)$ stands for the free dynamical group corresponding to $V = 0$. The expansion corresponds to iterative procedure in solving the perturbed wave (1.2). Further we consider separately each term in the right hand side of (1.10):

I. As we noted above, for the first term $\mathcal{U}_0(t)$ we cannot deduce the time decay (1.6) from the spectral representation of type (1.7). We establish the decay using an analog of the strong Huygens principle extending Vainberg's trick [27].

II. For the second term we also cannot deduce the time decay from the spectral representation. However, the decay follows by standard estimates for the convolution using the decay of the first term and the condition (1.4) on the potential.

III. Finally, the time decay for the last term follows from the spectral representation by the Jensen-Kato technique since $\|\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\| \sim |\omega|^{-2}$ as $|\omega| \to \infty$ that follows from the (expected) lucky structure of the matrix $\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}$ (see (3.7)).

Our paper is organized as follows. In Section 2 we obtain the time decay for the solution to the free wave equation and state the spectral properties of the free resolvent. In Section 3 we obtain spectral properties of the perturbed resolvent and prove the decay (1.6).

# 2 Free wave equation

## 2.1 Time decay

First, we prove the time decay (1.6) for the free wave equation:

$$\ddot{\psi}(x,t) = \Delta\psi(x,t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}. \tag{2.1}$$

In vectorial form equation (2.1) reads

$$i\dot{\Psi}(t) = \mathcal{H}_0\Psi(t), \tag{2.2}$$

where

$$\Psi(t) = \left(\begin{array}{c} \psi(t) \\ \dot{\psi}(t) \end{array}\right), \qquad \mathcal{H}_0 = \left(\begin{array}{cc} 0 & i \\ i\Delta & 0 \end{array}\right). \tag{2.3}$$

Denote by $\mathcal{U}_0(t)$ the dynamical group of the equation (2.2). It is strongly continuous group in the Hilbert space $\mathcal{F}_0$. The group is unitary after a suitable modification of the norm that follows from the energy conservation.

**Proposition 2.1.** *Let $\sigma > 1$. Then for $\Psi_0 \in \mathcal{F}_\sigma$*

$$\|\mathcal{U}_0(t)\Psi_0\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-\sigma+1}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}. \tag{2.4}$$

*Proof. Step i)* It suffices to consider $t > 0$. In this case the matrix kernel of the dynamical group $\mathcal{U}_0(t)$ can be written as $\mathcal{U}_0(x-y,t)$ where

$$\mathcal{U}_0(z,t) = \left(\begin{array}{cc} \dot{U}(z,t) & U(z,t) \\ \ddot{U}(z,t) & \dot{U}(z,t) \end{array}\right), \quad z \in \mathbb{R}^3, \tag{2.5}$$

and

$$U(z,t) = \frac{\delta(t-|z|)}{4\pi t}, \quad t > 0. \tag{2.6}$$

*Step ii)* We consider an arbitrary $t \ge 1$. Let us split the initial function $\Psi_0$ in two terms, $\Psi_0 = \Psi'_{0,t} + \Psi''_{0,t}$ such that

$$\|\Psi'_{0,t}\|_{\mathcal{F}_\sigma} + \|\Psi''_{0,t}\|_{\mathcal{F}_\sigma} \le C\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \ge 1, \tag{2.7}$$

and

$$\Psi'_{0,t}(x) = 0 \ \text{ for } |x| > t/2, \qquad \text{and} \qquad \Psi''_{0,t}(x) = 0 \ \text{ for } |x| < t/3. \tag{2.8}$$

First we obtain the estimate (2.4) for $\mathcal{U}_0(t)\Psi''_{0,t}$. For any $f \in \dot{H}^1$ the Hölder inequality and inequality (1.5) imply

$$\|f\|_{L^2_{-\sigma}} = \Big(\int (1+|x|^2)^{-\sigma} f^2(x)dx\Big)^{1/2} \le C\|f\|_{L^6}\Big(\int (1+|x|^2)^{-3\sigma/2}dx\Big)^{1/3} \le C\|f\|_{\dot{H}^1} \tag{2.9}$$

since $3\sigma > 3$. Hence the estimate (2.4) for $\mathcal{U}_0(t)\Psi''_{0,t}$ follows by energy conservation for the wave equation, (2.8) and (2.7):

$$\|\mathcal{U}_0(t)\Psi''_{0,t}\|_{\mathcal{F}_{-\sigma}} \le C\|\mathcal{U}_0(t)\Psi''_{0,t}\|_{\mathcal{F}} \le C\|\Psi''_{0,t}\|_{\mathcal{F}} \le Ct^{-\sigma}\|\Psi''_{0,t}\|_{\mathcal{F}_\sigma} \le Ct^{-\sigma}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \ge 1. \quad (2.10)$$

*Step iii)* It remains to estimate $\mathcal{U}_0(t)\Psi'_{0,t}$ for $t \ge 1$. Formulas (2.5)-(2.6), and (2.8) imply

$$[\mathcal{U}_0(t)\Psi'_{0,t}](x) = 0, \quad |x| < t/2.$$

Similar to (2.9) for $f \in \dot{H}^1$ such that $f(x) = 0$ for $r = |x| < t/2$ we obtain

$$\|f\|_{L^2_{-\sigma}} \le C\Big(\int\limits_{r>t/2}(1+r^2)^{-3\sigma/2}r^2dr\Big)^{1/3}\|f\|_{\dot{H}^1} \le Ct^{-\sigma+1}\|f\|_{\dot{H}^1},$$

Hence, by energy conservation

$$\|\mathcal{U}_0(t)\Psi'_{0,t}\|_{\mathcal{F}_{-\sigma}} \le Ct^{-\sigma+1}\|\mathcal{U}_0(t)\Psi'_{0,t}\|_{\mathcal{F}} \le Ct^{-\sigma+1}\|\Psi'_{0,t}\|_{\mathcal{F}} \le Ct^{-\sigma+1}\|\Psi'_{0,t}\|_{\mathcal{F}_\sigma} \le Ct^{-\sigma+1}\|\Psi_0\|_{\mathcal{F}_\sigma}.$$

Finally, the last estimate and (2.10) imply (2.4). □

## 2.2 Spectral properties

We state spectral properties of the free wave dynamical group $\mathcal{U}_0(t)$. For $t > 0$ and $\Psi_0 = \Psi(0) \in \mathcal{F}$, there exist a unique solution $\Psi(t) \in C_b(\mathbb{R}, \mathcal{F})$ to the free wave equation (2.2). Hence, $\Psi(t)$ admits the spectral Fourier-Laplace representation

$$\theta(t)\Psi(t) = \frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i(\omega+i\varepsilon)t}\mathcal{R}_0(\omega+i\varepsilon)\Psi_0\, d\omega, \quad t \in \mathbb{R} \quad (2.11)$$

with any $\varepsilon > 0$ where $\theta(t)$ is the Heavyside function, $\mathcal{R}_0(\omega) = (\mathcal{H}_0 - \omega)^{-1}$ for $\omega \in \mathbb{C}^+ := \{\omega \in \mathbb{C} : \mathrm{Im}\ \omega > 0\}$ is the resolvent of the operator $\mathcal{H}_0$. The representation follows from the stationary equation $\omega\tilde{\Psi}^+(\omega) = \mathcal{H}_0\tilde{\Psi}^+(\omega) + i\Psi_0$ for the Fourier-Laplace transform $\tilde{\Psi}^+(\omega) := \int_{\mathbb{R}}\theta(t)e^{i\omega t}\Psi(t)dt$, $\omega \in \mathbb{C}^+$. The solution $\Psi(t)$ is continuous bounded function of $t \in \mathbb{R}$ with the values in $\mathcal{F}$ by the energy conservation for the free wave equation (2.2). Hence, $\tilde{\Psi}^+(\omega) = -i\mathcal{R}(\omega)\Psi_0$ is analytic function of $\omega \in \mathbb{C}^+$ with the values in $\mathcal{F}$, and bounded for $\omega \in \mathbb{R} + i\varepsilon$. Therefore, the integral (2.11) converges in the sense of distributions of $t \in \mathbb{R}$ with the values in $\mathcal{F}$. Similarly to (2.11),

$$\theta(-t)\Psi(t) = -\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i(\omega-i\varepsilon)t}\mathcal{R}_0(\omega-i\varepsilon)\Psi_0\, d\omega, \quad t \in \mathbb{R}. \quad (2.12)$$

For the resolvent $\mathcal{R}_0(\omega)$ the following matrix representation holds

$$\mathcal{R}_0(\omega) = \left(\begin{array}{cc} \omega R_0(\omega^2) & iR_0(\omega^2) \\ -i(1+\omega^2R_0(\omega^2)) & \omega R_0(\omega^2) \end{array}\right) \quad (2.13)$$

where

$$R_0(\zeta, x-y) = \frac{e^{i\sqrt{\zeta}|x-y|}}{4\pi|x-y|}, \quad \zeta \in \mathbb{C}^+, \quad \mathrm{Im}\,\zeta^{1/2} > 0. \quad (2.14)$$

**Definition 2.2.** *Denote by $\mathcal{L}(B_1, B_2)$ the Banach space of bounded linear operators from a Banach space $B_1$ to a Banach space $B_2$.*

The explicit formula (2.14) implies the properties of Shrödinger resolvent $R_0(\zeta)$ which are obtained in [13, Lemmas 2.1 and 2.2]:

i) $R_0(\zeta)$ is strongly analytic function of $\zeta \in \mathbb{C} \setminus [0, \infty)$ with the values in $\mathcal{L}(H_0^{-1}, H_0^1)$;
ii) For $\zeta \geq 0$, the convergence holds $R_0(\zeta \pm i\varepsilon) \to R_0(\zeta \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)$ with $\sigma, \sigma' > 1/2$, $\sigma + \sigma' > 2$ (see Appendix A).

In [17] we revise the Agmon-Jensen-Kato decay of the Shrödinger resolvent [1, (A.2')], [13, (8.1)]:

**Proposition 2.3.** *(cf.[17, Proposition A.1]) For any $r > 0$ the bounds hold for $m = 0, 1$ and $l = -1, 0, 1$,*

$$\|R_0^{(k)}(\zeta)\|_{\mathcal{L}(H_\sigma^m, H_{-\sigma}^{m+l})} \leq C(k, r)|\zeta|^{-(1-l+k)/2}, \quad \omega \in \mathbb{C} \setminus \mathbb{R}, \qquad |\omega| \geq r \tag{2.15}$$

*with $\sigma > 1/2 + k$ for any $k = 0, 1, 2, \ldots$.*

Since $R_0(\omega^2)$ is smooth function in the neighborhood of $\omega = 0$ then Proposition 2.3 implies

**Corollary 2.4.** *For $m = 0, 1$ and $l = -1, 0, 1$ the bounds hold*

$$\|R_0^{(k)}(\omega^2)\|_{\mathcal{L}(H_\sigma^m, H_{-\sigma}^{m+l})} \leq C(k)(1 + |\omega|)^{-(1-l)/2}, \quad \omega \in \mathbb{C} \setminus \mathbb{R}. \tag{2.16}$$

*with $\sigma > 1$ for $k = 0$, and $\sigma > 1/2 + k$ for $k = 1, 2, \ldots$*

The formulas (2.13)-(2.14) and the bounds (2.16) imply the properties of $\mathcal{R}_0(\omega)$:

i) $\mathcal{R}_0(\omega)$ is strongly analytic function of $\omega \in \mathbb{C} \setminus \mathbb{R}$ with the values in $\mathcal{L}(\mathcal{F}_0, \mathcal{F}_0))$;
ii) The convergence holds $\mathcal{R}_0(\omega \pm i\varepsilon) \to \mathcal{R}_0(\omega \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma'})$ with $\sigma, \sigma' > 1/2$, $\sigma + \sigma' > 2$.
iii) The bounds hold

$$\|\mathcal{R}_0^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} \leq C(k) < \infty, \quad k = 0, 1, 2, \ldots, \quad \omega \in \mathbb{C} \setminus \mathbb{R} \tag{2.17}$$

with $\sigma > 1$ for $k = 0$, and $\sigma > 1/2 + k$ for $k = 1, 2, \ldots$

**Corollary 2.5.** *For $t \in \mathbb{R}$ and $\Psi_0 \in \mathcal{F}_\sigma$ with $\sigma > 1$, the group $\mathcal{U}_0(t)$ admits the integral representation*

$$\mathcal{U}_0(t)\Psi_0 = \frac{1}{2\pi i} \int e^{-i\omega t} \Big[\mathcal{R}_0(\omega + i0) - \mathcal{R}_0(\omega - i0)\Big] \Psi_0 \, d\omega \tag{2.18}$$

*where the integral converges in the sense of distributions of $t \in \mathbb{R}$ with the values in $\mathcal{F}_{-\sigma}$.*

*Proof.* Summing up the representations (2.11) and (2.12), and sending $\varepsilon \to 0+$, we obtain (2.18) by the Cauchy theorem and the bounds (2.17). $\square$

**Remark 2.6.** *The estimates (2.17) do not allow obtain the decay (2.4) by partial integration in (2.18). This is why we deduce the decay in Section 2.1 from explicit formulas (2.5) and (2.6).*

# 3 Perturbed wave equation

To prove the long time decay for the perturbed wave equation, we first establish the spectral properties of the generator.

## 3.1 Spectral properties

According [13, p. 589] and [19, formula (3.1)], let us introduce a generalized eigenspace $\mathbf{M}$ for the perturbed Schrödinger operator $H = -\Delta + V$:

$$\mathbf{M} = \{\psi \in H^1_{-1/2-0} : \ (1 + A_0 V)\psi = 0\},$$

where $A_0$ is the operator with the kernel $A_0(x, y) = \dfrac{1}{4\pi|x-y|}$. Below we assume that

$$\mathbf{M} = 0. \tag{3.1}$$

In [13, p. 591] the point $\lambda = 0$ is called then "regular point" for the Schrödinger operator $H$ (it corresponds to the "nonsingular case" in [19, Section 7]). The condition holds for *generic potentials* $V$ satisfying (1.4) (see [13, p. 589]).

Denote by $R(\zeta) = (H - \zeta)^{-1}$, $\zeta \in \mathbb{C} \setminus \mathbb{R}$, the resolvent of the Schrödinger operator $H$.

**Remark 3.1.** i) By [19, Theorem 7.2], the condition (3.1) is equivalent to the boundedness of the resolvent $R(\zeta)$ at $\zeta = 0$ in the norm of $\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})$ with a suitable $\sigma > 0$.
ii) By Lemma 3.2 in [13], the condition (3.1) is equivalent to absence of nonzero solutions $\psi \in H^1_{-\sigma}$, with $\sigma \le 3/2$, to the equation $H\psi = 0$.
iii) $N(H) \subset \mathbf{M}$ where $N(H)$ is the zero eigenspace of the operator $H$. The imbedding is obtained in [13, Theorem 3.6]. The functions from $\mathbf{M} \setminus N(H)$ are called *zero resonance functions*. Hence, the condition (3.1) means that $\lambda = 0$ is neither eigenvalue nor resonance for the operator $H$.

Let us collect the properties of $R(\zeta)$ obtained in [1, 13, 19] under conditions (1.4) and (3.1):

**R1.** $R(\zeta)$ is strongly meromorphic function of $\zeta \in \mathbb{C} \setminus [0, \infty)$ with the values in $\mathcal{L}(H^{-1}_0, H^1_0)$; the poles of $R(\zeta)$ are located at a finite set of eigenvalues $\zeta_j < 0$, $j = 1, ..., N$, of the operator $H$ with the corresponding eigenfunctions $\psi^1_j(x), ..., \psi^{\kappa_j}_j(x) \in H^2_s$ with any $s \in \mathbb{R}$, where $\kappa_j$ is the multiplicity of $\zeta_j$.
**R2.** For $\zeta > 0$, the convergence holds $R(\zeta \pm i\varepsilon) \to R(\zeta \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})$ with $\sigma > 1/2$, uniformly in $\zeta \ge \rho$ for any $\rho > 0$ ([13, Lemma 9.1]).
**R3.** The operator $I + R_0(\zeta)V$ is invertible in $\mathcal{L}(H^1_{-\sigma}; H^1_{-\sigma})$, with $\sigma > 1$ for sufficiently small $\zeta \in \mathbb{C} \setminus (0, \infty)$. Hence, formula (2.14) and the identities

$$R = (I + R_0 V)^{-1} R_0 \quad R' = (1 - RV)R_0'(1 - VR), \quad R'' = \Big[(1 - RV)R_0'' - 2R'VR_0'\Big](1 - VR)...$$

imply that there exists $\varepsilon > 0$ such that for $m = 0, 1$ and $l = -1, 0, 1$ the bounds hold

$$\|R^{(k)}(\zeta)\|_{\mathcal{L}(H^m_\sigma; H^{m+l}_{-\sigma})} \le C(\varepsilon, k), \quad |\zeta| < \varepsilon$$

with $\sigma > 1$ for $k = 0$ and $\sigma > 1/2 + k$ for $k = 1, 2, ....$

**R5.** (cf.[17]) For any $r, \delta > 0$, the bounds hold for $m = 0, 1$ and $l = -1, 0, 1$

$$\|R^{(k)}(\zeta)\|_{\mathcal{L}(H^m_\sigma;H^{m+l}_{-\sigma})} \le C(r, \delta, k)|\zeta|^{-(1-l+k)/2}, \quad \zeta \in \mathbb{C} \setminus (0, \infty), \quad |\zeta| > r, \quad |\arg \zeta| \le \pi - \delta \quad (3.2)$$

with $\sigma > 1/2 + k$ for any $k = 0, 1, 2, ....$

The resolvent $\mathcal{R}(\omega) = (\mathcal{H} - \omega)^{-1}$ can be expressed similarly to (2.13):

$$\mathcal{R}(\omega) = \left( \begin{array}{cc} \omega R(\omega^2) & iR(\omega^2) \\ -i(1 + \omega^2 R(\omega^2)) & \omega R(\omega^2) \end{array} \right). \quad (3.3)$$

Hence, the properties **R1** – **R5** imply the corresponding properties of $\mathcal{R}(\omega)$:

**Lemma 3.2.** *Let the potential $V$ satisfy conditions (*1.4*) and (*3.1*). Then*
*i) $\mathcal{R}(\omega)$ is strongly meromorphic function of $\omega \in \mathbb{C} \setminus \mathbb{R}$ with the values in $\mathcal{L}(\mathcal{F}_0, \mathcal{F}_0)$;*
*ii) The poles of $\mathcal{R}(\omega)$ are located at a finite set of imaginary axe*

$$\Sigma = \{\omega_j^\pm = \pm\sqrt{\zeta_j},\ j = 1, ..., N\}$$

*of eigenvalues of the operator $\mathcal{H}$ with the corresponding eigenfunctions $\left( \begin{array}{c} \psi_j^\kappa(x) \\ \omega_j^\pm \psi_j^\kappa(x) \end{array} \right)$, $\kappa = 1, ..., \kappa_j$;*
*iii) For $\omega \in \mathbb{R}$, the convergence holds $\mathcal{R}(\omega \pm i\varepsilon) \to \mathcal{R}(\omega \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ with $\sigma > 1$.*
*v) Let $r < \mathrm{dist}(\Sigma, 0)$. Then the bounds hold*

$$\|\mathcal{R}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} \le C(r, k) < \infty, \quad 0 < |\mathrm{Im}\,\omega| \le r \quad (3.4)$$

*with $\sigma > 1$ for $k = 0$, and with $\sigma > 1/2 + k$ for any $k = 0, 1, 2, ....$*

Finally, let us denote by $\mathcal{V}$ the matrix

$$\mathcal{V} = \left( \begin{array}{cc} 0 & 0 \\ V & 0 \end{array} \right). \quad (3.5)$$

and let $\mathcal{W}(\omega) := \mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}$. We obtain the asymptotics of $\mathcal{W}(\omega)$ for small and large $\omega$.

**Lemma 3.3.** *Let the potential $V$ satisfy (*1.4*) with $\beta > 1 + \sigma$ for $k = 0$ and $\beta > 1/2 + k + \sigma$ for $k = 1, 2, ...$, with some $\sigma > 0$. Then the bounds hold*

$$\|\mathcal{W}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_{-\sigma},\mathcal{F}_\sigma)} \le C(k)(1 + |\omega|)^{-2}. \quad (3.6)$$

*Proof.* Bounds (3.6) follow from the algebraic structure of the matrix

$$\mathcal{W}^{(k)}(\omega) = \mathcal{V}\mathcal{R}_0^{(k)}(\omega)\mathcal{V} = \left( \begin{array}{cc} 0 & 0 \\ iVR_0^{(k)}(\omega^2)V & 0 \end{array} \right), \quad (3.7)$$

since (2.16) implies that for $\omega \in \mathbb{C} \setminus \mathbb{R}$,

$$\|VR_0^{(k)}(\omega^2)Vf\|_{H^0_\sigma} \le C\|R_0^{(k)}(\omega^2)Vf\|_{H^0_{\sigma-\beta}} \le C(k)(1+|\omega|)^{-2}\|Vf\|_{H^1_{\beta-\sigma}} \le C(k)(1+|\omega|)^{-2}\|f\|_{H^1_{-\sigma}} \quad (3.8)$$

with $1 < \beta - \sigma$ for $k = 0$ and with $1/2 + k < \beta - \sigma$ for $k = 1, 2, ....$ $\square$

## 3.2 Time decay

In this section we combine the spectral properties of the perturbed resolvent and time decay for the unperturbed dynamics using the (finite) Born perturbation series. Our main result is the following.

**Theorem 3.4.** *Let conditions* (1.4) *and* (3.1) *hold. Then for* $\sigma > 2$

$$\|e^{-it\mathcal{H}}-\sum_{\omega_J\in\Sigma}e^{-i\omega_J t}P_J\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}=\mathcal{O}(|t|^{-\gamma}),\quad \gamma=\min\{[\sigma-1/2],\sigma-1,[\beta/2-1/2],\beta/2-1\} \tag{3.9}$$

*as* $t\to\pm\infty$. *Here* $P_J$ *are the Riesz projectors onto the corresponding eigenspaces.*

*Proof.* We use the Born perturbation series

$$\mathcal{R}(\omega)=\mathcal{R}_0(\omega)-\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)+\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega),\quad \omega\in\mathbb{C}\setminus[\Gamma\cup\Sigma] \tag{3.10}$$

which follows by iteration of $\mathcal{R}(\omega)=\mathcal{R}_0(\omega)-\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)$. Let us substitute the series into the spectral representation of type (2.11) for the solution to (1.1) with $\Psi(0)=\Psi_0\in\mathcal{F}_\sigma$ where $\sigma>3/2$. Then Lemma 3.2 and bounds (3.4) with $k=0$ imply similarly to (2.18), that

$$\begin{aligned}
\Psi(t)\ &-\ \sum_{\omega_J\in\Sigma}e^{-i\omega_J t}P_J\Psi_0=\frac{1}{2\pi i}\int e^{-i\omega t}\Big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\Big]\Psi_0\,d\omega \qquad (3.11)\\
&=\ \frac{1}{2\pi i}\int e^{-i\omega t}\Big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\Big]\Psi_0\,d\omega\\
&+\ \frac{1}{2\pi i}\int e^{-i\omega t}\Big[\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\,d\omega\\
&+\ \frac{1}{2\pi i}\int e^{-i\omega t}\Big[[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}](\omega+i0)-[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}](\omega-i0)\Big]\Psi_0\,d\omega\\
&=\ \Psi_1(t)+\Psi_2(t)+\Psi_3(t),\qquad t\in\mathbb{R}
\end{aligned}$$

where $P_J$ stands for the corresponding Riesz projector

$$P_J\Psi_0:=-\frac{1}{2\pi i}\int_{|\omega-\omega_J|=\delta}\mathcal{R}(\omega)\Psi_0 d\omega$$

with a small $\delta>0$. Further we analyze each term $\Psi_k$ separately.

### 3.2.1 Time decay of $\Psi_1$

The first term $\Psi_1(t)=\mathcal{U}_0(t)\Psi_0$ by (2.18). Hence, Proposition 2.1 implies that

$$\|\Psi_1(t)\|_{\mathcal{F}_{-\sigma}}\le C(1+|t|)^{-\sigma+1}\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\in\mathbb{R}. \tag{3.12}$$

### 3.2.2 Time decay of $\Psi_2$

The second term $\Psi_2(t)$ can be rewritten as a convolution.

**Lemma 3.5.** *The convolution representation holds*

$$\Psi_2(t) = i\int\limits_0^t \mathcal{U}_0(t-\tau)\mathcal{V}\Psi_1(\tau)\ d\tau, \qquad t\in\mathbb{R} \tag{3.13}$$

*where the integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma>2$.*

*Proof.* The term $\Psi_2(t)$ can be rewritten as

$$\Psi_2(t) = \frac{1}{2\pi i}\int\limits_{\mathbb{R}} \Big[e^{-i\omega t}\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0) - e^{-i\omega t}\mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\ d\omega. \tag{3.14}$$

Let us integrate the first term in the right hand side of (3.14) denoting

$$\mathcal{U}_0^{\pm}(t) := \theta(\pm t)\mathcal{U}_0(t), \quad \Psi_1^{\pm}(t) := \theta(\pm t)\Psi_1(t), \qquad t\in\mathbb{R}.$$

We know that $\mathcal{R}_0(\omega+i0)\Psi_0 = i\tilde{\Psi}_1^{+}(\omega)$, hence integrating the first term in the right hand side of (3.14), we obtain that

$$\begin{aligned}\Psi_{21}(t) &= \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i\omega t}\mathcal{R}_0(\omega+i0)\mathcal{V}\tilde{\Psi}_1^{+}(\omega)\ d\omega \\ &= \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{-i\omega t}\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int_{\mathbb{R}} e^{i\omega\tau}\Psi_1^{+}(\tau)d\tau\Big]d\omega \\ &= \frac{1}{2\pi}(i\partial_t+i)^2\int\limits_{\mathbb{R}} \frac{e^{-i\omega t}}{(\omega+i)^2}\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int_{\mathbb{R}} e^{i\omega\tau}\Psi_1^{+}(\tau)d\tau\Big]d\omega. \end{aligned} \tag{3.15}$$

The last double integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma>2$ by (3.12), and (2.17) with $k=0$. Hence, we can change the order of integration by the Fubini theorem. Then we obtain that

$$\Psi_{21}(t) = i\int_{\mathbb{R}} \mathcal{U}_0^{+}(t-\tau)\mathcal{V}\Psi_1^{+}(\tau)d\tau = \begin{cases} i\displaystyle\int_0^t \mathcal{U}_0(t-\tau)\mathcal{V}\Psi_1(\tau)d\tau &, \ t>0 \\ 0 &, \ t<0 \end{cases} \tag{3.16}$$

since

$$\mathcal{U}_0^{+}(t-\tau) = \frac{1}{2\pi i}(i\partial_t+i)^2\int\limits_{\mathbb{R}}\frac{e^{-i\omega(t-\tau)}}{(\omega+i)^2}\mathcal{R}_0(\omega+i0)\ d\omega$$

by (2.11). Similarly, integrating the second term in the right hand side of (3.14), we obtain

$$\Psi_{22}(t) = i\int_{\mathbb{R}} \mathcal{U}_0^{-}(t-\tau)\mathcal{V}\Psi_1^{-}(\tau)d\tau = \begin{cases} 0 &, \ t>0 \\ i\displaystyle\int_0^t \mathcal{U}_0(t-\tau)\mathcal{V}\Psi_1(\tau)d\tau &, \ t<0 \end{cases} \tag{3.17}$$

Now (3.13) follows since $\Psi_2(t)$ is the sum of two expressions (3.16) and (3.17). □

**Lemma 3.6.**

$$\|\Psi_2(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-\gamma}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad \gamma = \min\{\sigma-1, \frac{\beta}{2}-1\}, \quad t\in\mathbb{R}. \tag{3.18}$$

*Proof.* We apply Proposition 2.1 to the integrand in (3.13). For $2<\sigma<\beta/2$ we obtain

$$\|\mathcal{U}_0(t-\tau)\mathcal{V}\Psi_1(\tau)\|_{\mathcal{F}_{-\sigma}} \le \frac{C\|\mathcal{V}\Psi_1(\tau)\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{\sigma-1}} \le \frac{C\|\Psi_1(\tau)\|_{\mathcal{F}_{-\sigma}}}{(1+|t-\tau|)^{\sigma-1}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{\sigma-1}(1+|\tau|)^{\sigma-1}},$$

and for $\sigma>\beta/2$

$$\|\mathcal{U}_0(t-\tau)\mathcal{V}\Psi_1(\tau)\|_{\mathcal{F}_{-\sigma}} \le \|\mathcal{U}_0(t-\tau)\mathcal{V}\Psi_1(\tau)\|_{\mathcal{F}_{-\beta/2}} \le \frac{C\|\mathcal{V}\Psi_1(\tau)\|_{\mathcal{F}_{\beta/2}}}{(1+|t-\tau|)^{\frac{\beta}{2}-1}}$$

$$\le \frac{C\|\Psi_1(\tau)\|_{\mathcal{F}_{-\beta/2}}}{(1+|t-\tau|)^{\frac{\beta}{2}-1}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_{\beta/2}}}{(1+|t-\tau|)^{\frac{\beta}{2}-1}(1+|\tau|)^{\frac{\beta}{2}-1}} \le \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{\frac{\beta}{2}-1}(1+|\tau|)^{\frac{\beta}{2}-1}}$$

Hence (3.13) implies (3.18). □

### 3.2.3 Time decay of $\Psi_3$

Let us rewrite the last term in (3.11) as

$$\Psi_3(t) = \frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t}\mathcal{N}(\omega)\Psi_0\,d\omega, \tag{3.19}$$

where $\mathcal{N}(\omega) := \mathcal{M}(\omega+i0)-\mathcal{M}(\omega-i0)$ for $\omega\in\mathbb{R}$, and

$$\mathcal{M}(\omega) := \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega) = \mathcal{R}_0(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega), \quad \omega\in\mathbb{C}\setminus\mathbb{R}. \tag{3.20}$$

Let us obtain the properties of the operator function $\mathcal{N}$.

**Lemma 3.7.** *For $0\le k<\min\{\beta-3/2, \sigma-1/2\}$ the bounds hold*

$$\|\mathcal{N}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} \le C(k)(1+|\omega|)^{-2}, \quad \omega\in\mathbb{R}. \tag{3.21}$$

*Proof.* We have

$$\mathcal{M}^{(k)} = \sum_{k_1+k_2+k_3=k}\frac{k!}{k_1!k_2!k_3!}\mathcal{R}_0^{(k_1)}\mathcal{W}^{(k_2)}\mathcal{R}^{(k_3)} \tag{3.22}$$

and Lemma 3.3 and bounds (2.17), (3.4) imply for $0<|\mathrm{Im}\,\omega|<r$ that

$$\|\mathcal{R}_0^{(k_1)}\mathcal{W}^{(k_2)}\mathcal{R}^{(k_3)}(\omega)f\|_{\mathcal{F}_{-\sigma}} \le \|\mathcal{R}_0^{(k_1)}\mathcal{W}^{(k_2)}\mathcal{R}^{(k_3)}(\omega)f\|_{\mathcal{F}_{-\sigma_1}} \le C(k_1)\|\mathcal{W}^{(k_2)}\mathcal{R}^{(k_3)}(\omega)f\|_{\mathcal{F}_{\sigma_1}}$$

$$\le \frac{C(k_1,k_2)}{(1+|\omega|)^2}\|\mathcal{R}^{(k_3)}(\omega)f\|_{\mathcal{F}_{-\sigma_1}} \le \frac{C(r,k_1,k_2,k_3)}{(1+|\omega|)^2}\|f\|_{\mathcal{F}_{\sigma_1}} \le \frac{C(r,k_1,k_2,k_3)}{(1+|\omega|)^2}\|f\|_{\mathcal{F}_\sigma}$$

if we choose $\sigma$, $\sigma_1$, $\beta$ such that

$$\sigma>\sigma_1>\left\{\begin{array}{c}1,\ \text{if } k_1=k_3=0\\ 1/2+\max\{k_1,k_3\},\ \text{if } k_1>0 \text{ or } k_3>0\end{array}\right., \quad \beta>\left\{\begin{array}{c}1+\sigma_1,\ \text{if } k_2=0\\ 1/2+k_2+\sigma_1,\ \text{if } k_2>0\end{array}\right.$$

All these inequalities hold if we choose $\sigma > \max\{1, 1/2+k\}$, $\beta > 3/2+k$, and

$$1 < \sigma_1 < \min\{\sigma, \beta - 1/2 - k\}, \quad \text{if} \quad k_1 = k_3 = 0,$$

$$1/2 + \max\{k_1, k_3\} < \sigma_1 < \min\{\sigma, \beta - 1\}, \quad \text{if} \quad k_2 = 0,$$

$$1/2 + \max\{k_1, k_3\} < \sigma_1 < \min\{\sigma, \beta - 1/2 - k_2\}, \quad \text{if} \quad k_2 > 0 \text{ and } \max\{k_1, k_3\} > 0.$$

□

Now we prove the desired decay of $\Psi_3(t)$ from (3.19).

**Lemma 3.8.**

$$\|\Psi_3(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-\gamma}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad \gamma = \min\{\langle\sigma - 1/2\rangle, \sigma - 1, \langle\beta/2 - 1/2\rangle, \beta/2 - 1\}, \quad t \in \mathbb{R}. \tag{3.23}$$

*Proof.* First, in the case $2 < \sigma < \beta/2$ there exists $k \ge 1$ such that $1/2 + k < \sigma \le 1/2 + 1 + k$ Then $\beta > 1 + 2k > 3/2 + k$, and by Lemma 3.7

$$\mathcal{N}^{(k)}(\omega)) \in L^1(\mathbb{R}; \mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})).$$

Then we can apply $k$ times integration by parts in (3.19) to obtain

$$\|\Psi_3(t)\|_{\mathcal{F}_{-\sigma}} \le C(k)(1+|t|)^{-k}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}. \tag{3.24}$$

If, additionally, $1/2 + k < \sigma \le 1 + k$, then

$$\|\Psi_3(t)\|_{\mathcal{F}_{-\sigma}} \le C(k)(1+|t|)^{-\sigma+1}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}. \tag{3.25}$$

Since $k = \langle\sigma - 1/2\rangle$ (see definition 1.2), then

$$\|\Psi_3(t)\|_{\mathcal{F}_{-\sigma}} \le C(k)(1+|t|)^{-\min\{\langle\sigma-1/2\rangle, \sigma-1\}}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}.$$

Second, in the case $4 < \beta < 2\sigma$ there exists $k \ge 1$ such that $2k+1 < \beta \le 2k+3$. Then $1/2 + k < \sigma$ and $3/2 + k < 2k + 1 < \beta$. Hence by Lemma 3.7 again we can apply $k$ times integration by parts to obtain (3.24).
If, additionally, $2k + 1 < \beta \le 2k + 2$, then

$$\|\Psi_3(t)\|_{\mathcal{F}_{-\sigma}} \le C(k)(1+|t|)^{-\beta/2+1}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}. \tag{3.26}$$

Hence,

$$\|\Psi_3(t)\|_{\mathcal{F}_{-\sigma}} \le C(k)(1+|t|)^{-\min\{\langle\beta/2-1/2\rangle, \beta/2+1\}}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}.$$

This completes the proof of the lemma and Theorem 3.4. □

**Corollary 3.9.** *The asymptotics* (3.9) *imply* (1.6) *with the projector*

$$\mathcal{P}_c = 1 - \sum_{\omega_J \in \Sigma} P_J \tag{3.27}$$

# A Appendix: Weighted estimates

**Lemma A.1.** *The operator $K(x,y)$ with the integral kernel $K(x,y) = \dfrac{1}{|x-y|}$ belongs to $\mathcal{L}(L^2_{\sigma_1}, L^2_{-\sigma_2})$ with $\sigma_1, \sigma_2 > 1/2$, $\sigma_1 + \sigma_2 > 2$.*

*Proof.* We must prove that for any $\psi_i \in L^2_{\sigma_i}$ , $i = 1, 2$

$$|\langle K\psi_1, \psi_2\rangle| \leq C\|\psi_1\|_{L^2_{\sigma_1}}\|\psi_2\|_{L^2_{\sigma_2}}. \tag{A.1}$$

We have

$$\langle K\psi_1, \psi_2\rangle = \int \hat{\psi}_1(k)\frac{1}{k^2}\overline{\hat{\psi}_2}(k)dk = \int\limits_{|k|\leq 1} ... + \int\limits_{|k|\geq 1} ... = I_1 + I_2$$

Evidently, $I_2 \leq C\|\psi_1\|_{L^2}\|\psi_2\|_{L^2}$. It remains to estimate $I_1$. By the Hölder inequality

$$|I_1| \leq C\Big(\int |\hat{\psi}_1|^p\,|\overline{\hat{\psi}_2}|^p\,dk\Big)^{1/p}\Big(\int\limits_{|k|\leq 1}\frac{1}{k^{2q}}\,dk\Big)^{1/q} \leq C\Big(\int |\hat{\psi}_1|^p\,|\overline{\hat{\psi}_2}|^p\,dk\Big)^{1/p}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $1 < q < 3/2$, and $p > 3$. Furthermore, again by the Hölder inequality

$$|I_1| \leq C\|\hat{\psi}_1\|_{L^{p_1}}\|\hat{\psi}_2\|_{L^{p_2}} \leq C\|\psi_1\|_{L^{r_1}}\|\psi_2\|_{L^{r_2}}\,, \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \quad \frac{1}{p_i} + \frac{1}{r_i} = 1, \quad i = 1, 2.$$

with $p_i > 3$, and $1 < r_i < 3/2$, $i = 1, 2$. Next we obtain

$$\|\psi_i\|_{L^{r_i}} = \|(1+|x|)^{-\sigma_i}(1+|x|)^{\sigma_i}\psi_i\|_{L^{r_i}} \leq C\|(1+|x|)^{-\sigma_i}\|_{L^{s_i}}\|\psi_i\|_{L^2_{\sigma_i}} \leq C\|\psi_i\|_{L^2_{\sigma_i}}, \ \frac{1}{s_i} + \frac{1}{2} = \frac{1}{r_i}, \ s_i < 6$$

if $\sigma_i s_i > 3$. Let us verify that such constants exist. We have $\sigma_i > 1/2$ and

$$s_i > \frac{3}{\sigma_i}, \quad \frac{1}{s_i} < \frac{\sigma_i}{3}, \quad \frac{1}{r_i} < \frac{\sigma_i}{3} + \frac{1}{2}, \quad \frac{1}{p_i} > \frac{1}{2} - \frac{\sigma_i}{3}, \quad \frac{1}{p} > 1 - \frac{\sigma_1 + \sigma_2}{3}, \quad p < \frac{3}{3 - (\sigma_1 + \sigma_2)}$$

Finally, let us choose $\sigma_1$, $\sigma_2 > 1/2$, and $\sigma_1 + \sigma_2 > 2$. Then we take

$$3 < p < \frac{3}{3 - (\sigma_1 + \sigma_2)}.$$

□

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