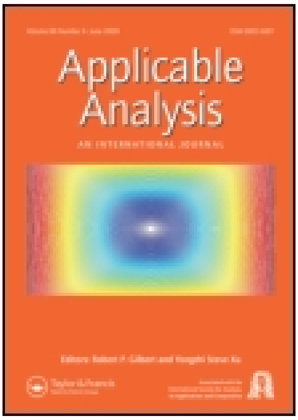

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On asymptotic stability of solitary waves in discrete Klein–Gordon equation coupled to a nonlinear oscillator

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The long-time asymptotics is analysed for finite energy solutions of the 1D discrete Klein–Gordon equation coupled to a nonlinear oscillator. The coupled system is invariant with respect to the phase rotation group $U(1)$. For initial states close to a solitary wave, the solution converges to a sum of another solitary wave and dispersive wave which is a solution to the free Klein–Gordon equation. The proofs develop the strategy of Buslaev–Perelman: the linearization of the dynamics on the solitary manifold, the symplectic orthogonal projection, method of majorants, etc.

Keywords: long-time asymptotics; discrete Klein–Gordon equation; nonlinear oscillator; solitary wave

AMS Subject Classifications: 39A14; 39A30

1. Introduction

Our main goal is the study of the distinguished dynamical role of the 'quantum stationary states' for a model $U(1)$-invariant nonlinear discrete Klein–Gordon equation

$$\ddot{\psi}(x,t) = \Delta_L\,\psi(x,t) - m^2\psi(x,t) + \delta(x)F(\psi(0,t)), \quad m>0, \quad x\in\mathbb{Z}. \tag{1.1}$$

Here F is a continuous function, $\delta(x)=\delta_{0x}$ and Δ_L stands for the difference Laplacian in $\mathbb{Z}$, defined by

$$\Delta_L\psi(x) = \psi(x+1) - 2\psi(x) + \psi(x-1), \quad x\in\mathbb{Z}$$

for functions $\psi:\mathbb{Z}\to\mathbb{C}$. Physically, Equation (1.1) describes the system of the free discrete Klein–Gordon equation coupled to a nonlinear oscillator attached at the point $x=0$: F is a nonlinear 'oscillator force'. In vectorial form, Equation (1.1) reads

$$\dot{\Psi}(t) = \begin{bmatrix} 0 & 1 \\ \Delta_L - m^2 & 0 \end{bmatrix}\Psi(t) + \delta(x)\begin{bmatrix} 0 \\ F(\psi) \end{bmatrix}, \quad \Psi(t) = \begin{bmatrix} \psi(t) \\ \dot{\psi}(t) \end{bmatrix}. \tag{1.2}$$

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We identify a complex number $\psi = \psi_1 + i\psi_2 \in \mathbb{C}$ with the real two-dimensional vector $\psi = (\psi_1, \psi_2) \in \mathbb{R}^2$ and assume that the vector version $\mathbf{F}$ of the oscillator force admits a real-valued potential

$$\mathbf{F}(\psi) = -\nabla U(\psi), \quad \psi \in \mathbb{C}, \quad U \in C^2(\mathbb{C}), \tag{1.3}$$

where the gradient is taken with respect to $\mathrm{Re}\,\psi$ and $\mathrm{Im}\,\psi$. Then (1.2) can be formally written as the Hamiltonian system

$$\dot{\Psi}(t) = JD\mathcal{H}(\Psi), \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{1.4}$$

where $D\mathcal{H}$ is the variational derivative of the Hamilton functional

$$\mathcal{H}(\Psi) = \frac{1}{2}\sum_{\mathbb{Z}} \left(|\pi(x)|^2 + |\nabla_L \psi(x)|^2 + m^2 |\psi(x)|^2 \right) + U(\psi(0)), \quad \Psi = \begin{bmatrix} \psi \\ \pi \end{bmatrix}, \tag{1.5}$$

with $\nabla_L \psi(x) = \psi(x+1) - \psi(x)$. We assume that $U(\psi) = u(|\psi|^2)$ with $u \in C^2(\mathbb{R})$. Therefore, by (1.3),

$$F(\psi) = a(|\psi|^2)\psi, \quad \psi \in \mathbb{C}, \quad a \in C^1(\mathbb{R}), \tag{1.6}$$

where $a(\cdot) = -2u'(\cdot) \in C^1(\mathbb{R})$ is real. Then $F(e^{i\theta}\psi) = e^{i\theta}F(\psi)$, $\theta \in [0, 2\pi]$ and $F(0) = 0$ for continuous F. Hence, $e^{i\theta}\psi(x,t)$ is a solution to (1.1) if $\psi(x,t)$ is. Therefore, Equation (1.1) is $U(1)$-invariant in the sense of [1].

The main subject of this article is an analysis of the special role of 'quantum stationary states', or *solitary waves* in the sense of [1], which are finite energy solutions of the form

$$\Psi(x,t) = \Psi_\omega(x)e^{i\omega t}, \quad \omega \in \mathbb{R}, \quad \Psi_\omega = \begin{bmatrix} \psi_\omega \\ i\omega\psi_\omega \end{bmatrix}, \quad \psi_\omega \in l^2(\mathbb{Z}). \tag{1.7}$$

The frequency ω and the amplitude $\psi_\omega(x)$ solve the following *nonlinear eigenvalue problem*:

$$-\omega^2\psi_\omega(x) = \Delta_L \psi_\omega(x) - m^2\psi_\omega(x) + \delta(x)F(\psi_\omega(0)), \quad x \in \mathbb{Z}, \tag{1.8}$$

which follows directly from (1.2) and (1.6) since $\omega \in \mathbb{R}$. We prove the asymptotics of type

$$\Psi(\cdot, t) \sim \Psi_{\omega_\pm} e^{i\omega_\pm t} + W(t)\Phi_\pm, \quad t \to \pm\infty, \tag{1.9}$$

where $W(t)$ is the dynamical group of the free Klein–Gordon equation, $\Phi_\pm \in l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$ are the corresponding asymptotic states, and the remainder converges to zero as $\mathcal{O}(|t|^{-1/2})$ in the global norm of $l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$. The asymptotics hold for the solutions with initial states close to the *stable part* of the solitary manifold, extending the results of [2–7] to Equation (1.1).

For the first time, the asymptotics of type (1.9) were established by Soffer and Weinstein [8,9] (see also [10]) for nonlinear $U(1)$-invariant continuous Schrödinger equation with small initial states if the nonlinear coupling constant is sufficiently small. The next result was obtained by Buslaev and Perelman [3], who proved that the solitary manifold attracts finite-energy solutions of a 1D nonlinear

$U(1)$-invariant translation invariant Schrödinger equation with initial states sufficiently close to the *stable part* of the solitary manifold. For a detailed discussion of our motivation, and of previous results [4,6–11], we refer the reader to the introduction of [2].

The existence of discrete breathers are obtained in [12–17]. For discrete Schrödinger equation coupled to a nonlinear oscillator, the asymptotics of type (1.9) are proved in [18]. The asymptotic stability of standing waves in the discrete nonlinear Schrödinger equation are proved in [19].

Let us comment on the general strategy of our proofs. We develop the approach [2,3,20,21] for our problem. First, we apply the *symplectic projection* onto the solitary manifold to separate the motion along the solitary manifold and in transversal direction. Second, we obtain the *modulation equations* for the parameters of the symplectic projection, and linearize the transversal dynamics at the projection. The linearized equation is nonautonomous, and this is one of the fundamental difficulties in the proof. This difficulty is handled by 'freezing the coefficients' and using the modulation equations to estimate the arising error terms. Principal role in the rest of the proof is played by the uniform decay of the frozen linearized dynamics in the continuous spectral space, and the method of majorants.

This article is organized as follows. In Section 2, we prove the existence of global solution to Equation (1.2). In Section 3, we describe all nonzero solitary waves and formulate the main theorem. In Section 4, we collect main properties of the linearized equation. In Section 5, we establish the time decay for the linearized equation in the continuous spectrum. In Section 6, the modulation equations for the parameters of the soliton are displayed. The decay of the transversal component is proved in Sections 7 and 8. In Section 9, we obtain the soliton asymptotics (1.9). In Appendix we study the resolvent of linearized equation.

2. Global well-posedness

Existence of global solution to (1.2) is guaranteed by the following theorem, whose proof is similar to the proof of Theorem 2.1 from [22]. To have *a priori* estimates available for the proof of the global well-posedness, we assume that

$$U(\psi) \geq A - B|\psi|^2, \quad \psi \in \mathbb{C}, \quad \text{where } A \in \mathbb{R} \quad \text{and} \quad 0 \leq B < m^2/2. \tag{2.1}$$

THEOREM 2.1

(i) *Let conditions* (1.3), (1.6) *and* (2.1) *hold. Then for any* $\Psi_0 = (\psi_0, \pi_0) \in l^2 \oplus l^2$, *there exists a unique solution* $\Psi \in C_b(\mathbb{R}, l^2 \oplus l^2)$ *to Equation* (1.2) *with initial condition* $\Psi(x, 0) = \Psi_0(x)$.

(ii) *The energy conserved is,*

$$\mathcal{H}(\Psi(t)) = \mathcal{H}(\Psi_0), \quad t \in \mathbb{R}. \tag{2.2}$$

Let us outline the proof. The energy conservation (2.2) implies *a priori* bound for the solution. Hence, it suffices to prove the theorem assuming that U is uniformly bounded together with its derivatives since *a priori* bounds imply that the nonlinearity $F(z)$ can be modified for large values of $|z|$. Then we establish the existence and uniqueness of the solution $\psi \in C_b([0, \tau], l^2)$ for some $\tau > 0$ by the contraction mapping method. We then use the energy conservation to extend the solution $\psi(x, t)$ for $t \in \mathbb{R}$ and prove that $\psi \in C_b(\mathbb{R}, l^2)$.

Lemma 2.2 *Suppose that Theorem* 2.1 *is true for the nonlinearities U that satisfy the following additional condition*:

$$U_k := \sup_{z\in\mathbb{C}} |\nabla^k U(z)| < \infty, \quad k = 0, 1, 2. \tag{2.3}$$

Then Theorem 2.1 *is valid without this additional condition.*

Proof Define

$$\Lambda(\Psi_0) = \sqrt{\frac{\mathcal{H}(\Psi_0) - A}{\frac{1}{2}m^2 - B}}, \tag{2.4}$$

where A and B are constants from (2.1). Let U do not satisfy (2.3). We may pick a modified potential function $\tilde{U}(z) \in C^2(\mathbb{C})$, $\tilde{U}(z) = \tilde{U}(|z|)$ so that

(i) $\tilde{U}(z) = U(z) \quad \text{for } |z| \le \Lambda(\Psi_0), \quad z \in \mathbb{C},$ (2.5)

(ii) $\tilde{U}(z)$ satisfies (2.1) with the same constants A and B as $U(\psi)$ does, and

$$\sup_{z\in\mathbb{C}} |\nabla^k \tilde{U}(z)| < \infty, \quad k = 0, 1, 2.$$

By the assumption of Lemma 2.2, Theorem 2.1 holds for the nonlinearity $\tilde{F} = -\nabla\tilde{U}$ instead of $F = -\nabla U$. Hence, there is a unique solution $\Psi(x, t) \in C_b(\mathbb{R}, l^2 \oplus l^2)$ to the equation

$$\dot{\Psi}(t) = \begin{bmatrix} 0 & 1 \\ \Delta_L - m^2 & 0 \end{bmatrix} \Psi(t) + \delta(x) \begin{bmatrix} 0 \\ \tilde{F}(\psi) \end{bmatrix},$$

with $\Psi(x, 0) = \Psi_0$. This is a Hamilton system, with the Hamiltonian functional

$$\tilde{\mathcal{H}}(\Psi) = \frac{1}{2}\sum_{\mathbb{Z}} \left(|\pi(x)|^2 + |\nabla_L \psi(x)|^2 + m^2|\psi(x)|^2 \right) + \tilde{U}(\psi(0)), \quad \Psi = \begin{bmatrix} \psi \\ \pi \end{bmatrix}. \tag{2.6}$$

We have $|\psi_0(0)|^2 \le \Lambda^2(\Psi_0)$ since

$$A + \left(\tfrac{1}{2}m^2 - B\right)|\psi_0(0)|^2 \le \tfrac{1}{2}m^2|\psi_0(0)|^2 + U(\psi_0(0) \le \mathcal{H}(\Psi_0).$$

Thus, according to the choice of $\tilde{U}$ (equality (2.5)), $\tilde{U}(\psi_0(0)) = U(\psi_0(0))$, hence $\tilde{\mathcal{H}}(\Psi_0) = \mathcal{H}(\Psi_0)$. Further,

$$|\psi(0, t)|^2 \le \frac{\tilde{\mathcal{H}}(\Psi(t)) - A}{\frac{1}{2}m^2 - B} = \frac{\tilde{\mathcal{H}}(\Psi_0) - A}{\frac{1}{2}m^2 - B} = \frac{\mathcal{H}(\Psi_0) - A}{\frac{1}{2}m^2 - B} = \Lambda^2(\Psi_0).$$

Therefore, $\tilde{F}(\psi(0, t)) = F(\psi(0, t))$ for all $t \ge 0$, and $\Psi(x, t)$ is also a solution to (1.2) with the nonlinearity $F = -\nabla U$. ■

From now on, we shall assume in the proof of Theorem 2.1 that the bounds (2.3) hold.

LEMMA 2.3

(i) *Let $\Psi_0 \in l^2 \oplus l^2$. There is a $\tau > 0$ that depends only on U_2 in* (2.3) *and for which there is a unique solution $\Psi \in C_b([0, \tau], l^2 \oplus l^2)$ to Equation* (1.2) *with the initial data Ψ_0.*

(ii) *The energy functional $\mathcal{H}$ is conserved in time.*

Proof Denote the dynamical group for the free Klein–Gordon equation by $W(t)$. The discrete Fourier transform of $u : \mathbb{Z} \to \mathbb{C}$ is defined by the formula

$$\hat{u}(\theta) = \sum_{x \in \mathbb{Z}} u(x) e^{i\theta x}, \quad \theta \in T := \mathbb{R}/2\pi\mathbb{Z}.$$

After taking the Fourier transform, the operator Δ_L becomes the operator of multiplication by $\phi(\theta) = 2\cos\theta - 2$:

$$\widehat{\Delta_L u}(\theta) = \phi(\theta)\widehat{u}(\theta).$$

In the Fourier transform, we have

$$[\widehat{W(t)}Y](\theta) = \begin{bmatrix} \cos\sqrt{2+m^2-2\cos\theta}\, t & \dfrac{\sin\sqrt{2+m^2-2\cos\theta}\, t}{\sqrt{2+m^2-2\cos\theta}} \\ \sqrt{2+m^2-2\cos\theta}\sin\sqrt{2+m^2-2\cos\theta}\, t & \cos\sqrt{2+m^2-2\cos\theta}\, t \end{bmatrix} \times \widehat{Y}(\theta), \quad \theta \in T. \tag{2.7}$$

Then the solution ψ to (1.2) with the initial data $\Psi(x, 0) = \Psi_0(x)$ admits the Duhamel representation

$$\Psi(x, t) = W(t)\Psi_0(x) + Z[\psi(x, t)], \tag{2.8}$$

where

$$Z[\psi(x, t)] = \int_0^t W(s)\begin{bmatrix} 0 \\ \delta F(\psi(0, t-s)) \end{bmatrix} \mathrm{d}s, \quad \delta := \delta(x).$$

By the Parseval identity,

$$\left\| W(s)\begin{pmatrix} 0 \\ \delta \end{pmatrix} \right\|_{l^2 \oplus l^2} = \left\| \begin{pmatrix} \dfrac{\sin\sqrt{2+m^2-2\cos\theta}\, t}{\sqrt{2+m^2-2\cos\theta}} \\ \cos\sqrt{2+m^2-2\cos\theta}\, t \end{pmatrix} \right\|_{L^2(T)\oplus L^2(T)} \le C < \infty.$$

Hence, for $\psi_1, \psi_2 \in C_b([0, \tau], l^2)$,

$$\begin{aligned} \|Z[\psi_2(\cdot, t)] - Z[\psi_1(\cdot, t)]\|_{l^2\oplus l^2} &= \left\| \int_0^t W(s)\begin{bmatrix} 0 \\ \delta\Big(F(\psi_2(0, t-s)) - F(\psi_2(0, t-s))\Big) \end{bmatrix} \mathrm{d}s \right\|_{l^2\oplus l^2} \\ &\le \int_0^t \left\| W(s)\begin{pmatrix} 0 \\ \delta \end{pmatrix} \right\|_{l^2\oplus l^2} |F(\psi_2(0, t-s)) - F(\psi_2(0, t-s))| \mathrm{d}s \\ &\le C\int_0^t |F(\psi_2(0, t-s)) - F(\psi_2(0, t-s))| \mathrm{d}s \\ &\le CU_2\, t \sup_{0\le s\le t} |\psi_2(s) - \psi_1(s)|, \end{aligned} \tag{2.9}$$

where we have used (2.3) with $k=2$. Then the mapping $\Psi \mapsto W(t)\Psi_0 + Z[\psi]$ is a contraction in the space $C_b([0, \tau], l^2 \oplus l^2)$ if $\tau = 1/(2U_2)$. This proves the part (i) of the lemma. The part (ii) of the lemma also follows by contraction. The energy conservation follows from the Hamiltonian structure (1.4):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\Psi(t)) = \langle D\mathcal{H}(\Psi(t)), \dot{\Psi}(t)\rangle = \langle D\mathcal{H}(\Psi(t)), JD\mathcal{H}(\Psi(t))\rangle = 0, \quad t \in [0, \tau].$$

■

Proof of Theorem 2.1 The solution $\Psi \in C_b([0, \tau], l^2 \oplus l^2)$ constructed in Lemma 2.2 exists for $0 \le t \le \tau$, where the time span τ depends only on U_2. Hence, the bound (2.2) at $t = \tau$ allows us to extend the solution Ψ to the time interval $[\tau, 2\tau]$. We proceed by induction. This complete the proof of Theorem 2.1. ■

3. Solitary waves and statement of the main theorem

There exist two different sets of solitary waves. The first set $\mathcal{S}_+$ corresponds to $|\omega| < m$, and the second set $\mathcal{S}_-$ corresponds to $|\omega| > \sqrt{m^2+4}$. Denote by $k(\omega)$ the positive solution of the equation

$$\cosh k(\omega) = |m^2 - \omega^2 + 2|/2, \quad |\omega| < m \quad \text{or} \quad |\omega| > \sqrt{m^2+4}. \tag{3.1}$$

Lemma 3.1 *The sets of all nonzero solitary waves are given by*

$$\mathcal{S}_+ = \left\{\Psi_\omega e^{i\Theta} = \begin{bmatrix} \psi_\omega \\ i\omega\psi_\omega \end{bmatrix} e^{i\Theta}, \quad \psi_\omega = Ce^{-k(\omega)|x|} : \ |\omega| < m, \quad \Theta \in [0, 2\pi]\right\},$$

$$\mathcal{S}_- = \left\{\Psi_\omega e^{i\Theta} = \begin{bmatrix} \psi_\omega \\ i\omega\psi_\omega \end{bmatrix} e^{i\Theta}, \quad \psi_\omega = C(-1)^{|x|}e^{-k(\omega)|x|} : \ |\omega| > \sqrt{m^2+4}, \quad \Theta \in [0, 2\pi]\right\},$$

where $C > 0$ satisfies the following relations:

$$\sinh k(\omega) = \pm a(C^2)/2. \tag{3.2}$$

Proof Let us calculate the solitary waves (1.7). After the Fourier transform, Equation (1.8) becomes

$$(2 - 2\cos\theta + m^2 - \omega^2)\widehat{\psi}_\omega = F(C), \tag{3.3}$$

where $C = \psi_\omega(0)$. Therefore

$$\psi_\omega(x) = \frac{F(C)}{2\pi}\int_T \frac{e^{-i\theta x}}{\phi(\theta) + m^2 - \omega^2}\mathrm{d}\theta, \quad m^2 - \omega^2 \in \mathbb{C} \setminus [-4, 0].$$

Using [23, Lemma 2.1], we obtain that

$$\psi_\omega(x) = i\,\frac{F(C)e^{i\theta(\omega)|x|}}{2\sin\theta(\omega)}, \quad x \in \mathbb{Z}, \quad m^2 - \omega^2 \in \mathbb{C} \setminus [-4, 0], \tag{3.4}$$

where $\theta(\omega)$ is the unique root of the equation

$$2\cos\theta - 2 = m^2 - \omega^2 \tag{3.5}$$

in the domain $D := \{-\pi \le \mathrm{Re}\,\theta \le \pi,\ \mathrm{Im}\ \theta > 0\}$. Further, we consider separately two cases.

(I) First, let us consider the case $|\omega| < m$. Then $\theta(\omega) = ik(\omega)$, where $k(\omega) > 0$ is defined in (3.1), and $\sin\,\theta(\omega) = \sin ik(\omega) = i \sinh k(\omega)$. Therefore (3.4) reads

$$\psi_\omega(x) = \frac{F(C)e^{-k(\omega)|x|}}{2\sinh k(\omega)}, \quad x \in \mathbb{Z}.$$

Hence,

$$C = \psi_\omega(0) = \frac{F(C)}{2\sinh k(\omega)}.$$

This implies the equation

$$\sinh k(\omega) = \frac{F(C)}{2C} = \frac{a(C^2)}{2}. \tag{3.6}$$

(II) Second, consider the case $|\omega| > \sqrt{m^2+4}$. Then $\theta(\omega) = \pi + ik(\omega)$, $k(\omega) > 0$, and $\sin\theta(\omega) = -\sin ik(\omega) = -i\ \sinh k(\omega)$. Therefore (3.4) reads

$$\psi_\omega(x) = -\frac{F(C)(-1)^{|x|}e^{-k(\omega)|x|}}{2\sinh k(\omega)}, \quad x \in \mathbb{Z}.$$

Hence,

$$C = \psi_\omega(0) = -\frac{F(C)}{2\sinh k(\omega)},$$

and

$$\sinh k(\omega) = -\frac{F(C)}{2C} = -\frac{a(C^2)}{2} > 0. \tag{3.7}$$

■

Corollary 3.2 *The set $\mathcal{S}_+$ resp. $\mathcal{S}_-$ is a smooth manifold with the co-ordinates $\theta \in \mathbb{R}$ mod 2π and $C > 0$ such that $a(C^2) > 0$ resp. $a(C^2) < 0$.*

Remark 3.3 We will analyse only the solitary waves with $a'(C^2) \neq 0$. On the manifolds $\mathcal{S}_\pm$ we have $\omega^2 = m^2 + 2 \pm 2\cosh k$ with $\sinh k = \pm a(C^2)/2$ according to (3.6) and (3.7). Hence, the parameters θ, ω locally are also smooth coordinates on $\mathcal{S}_\pm$ at the points with $a'(C^2) \neq 0$ since $\omega\omega' = \pm k' \sinh k = \frac{a'(C^2)C\sinh k}{\cosh k} \neq 0$.

Example 3.4 Let us consider the potential $U(C) = C^2/4 - C^2/2$. Then $F(C) = (-C^2+1)C = -C^3+C$. The value $C \in (0,1)$ corresponds to $a(C^2) = -C^2+1 > 0$, and $C \in (1,\ \infty)$ corresponds to $a(C^2) < 0$ (Figure 1). If $C \in (0,1)$ then the equation $2\sinh k(\omega)C = F(C)$ has unique solution if $\sinh k \in (0, 1/2)$, since $\cosh k \in (1, \sqrt{5}/2)$, and nonzero solitary waves exist for $m^2 - \sqrt{5} + 2 < \omega^2 < m^2$. If $C \in (1,\infty)$ then the equation $2\sinh k(\omega)C = -F(C)$ has the unique solution for $\sinh k \in (0,\infty)$, since $\cosh k \in (1,\infty)$, and nonzero solitary waves exist for $\omega^2 > m^2 + 4$.

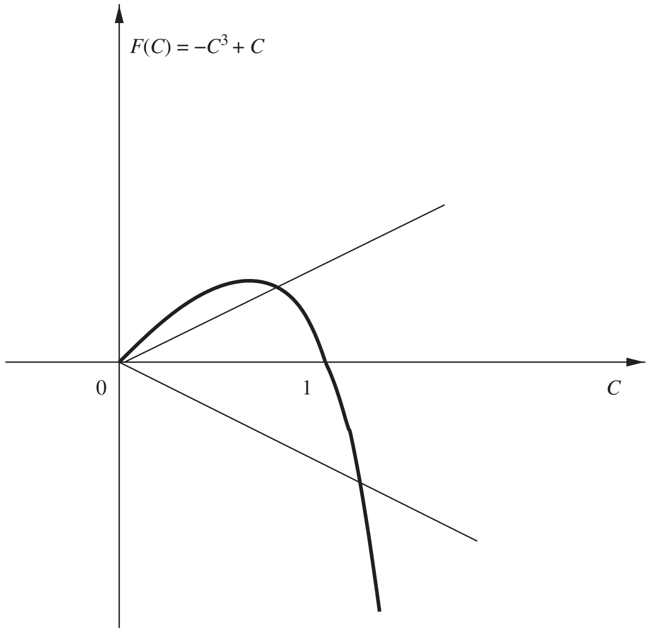

Figure 1. Solution of Equation (3.6).

The soliton solution is a trajectory $\Psi_{\omega(t)}(x)e^{i\Theta(t)}$, where the parameters satisfy the equation $\dot{\Theta} = \omega$, $\dot{\omega} = 0$. The real form of the solitary wave is $Y_\omega e^{j\Theta}$ where $Y_\omega = (\psi_\omega, 0, 0, \omega\psi_\omega)$, and j is the matrix

$$j = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.8}$$

Linearization at the solitary wave $Y_\omega e^{j\Theta}$ leads to the operator (cf. [2,4])

$$\mathbf{C} := \begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ \mathbf{D}_1 & 0 & 0 & \omega \\ 0 & \mathbf{D}_2 & -\omega & 0 \end{pmatrix}, \tag{3.9}$$

where

$$\mathbf{D}_1 = \Delta_L - m^2 + \delta(x)[a + 2a'C^2], \quad \mathbf{D}_2 = \Delta_L - m^2 + \delta(x)a, \quad a = a(C^2),\ a' = a'(C^2).$$

Note that $\mathbf{C} = q\mathbf{B}$, where

$$q = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} D_1 & 0 & 0 & \omega \\ 0 & D_2 & -\omega & 0 \\ 0 & -\omega & -1 & 0 \\ \omega & 0 & 0 & -1 \end{pmatrix} \tag{3.10}$$

$q^2 = -I$, and B is selfadjoint operator. We will show in Appendix A that the continuous spectrum of $\mathbf{C}$ coincides with $\mathcal{C}_- \cup \mathcal{C}_+$, where

$$\mathcal{C}_+ := \left[-i\left(\sqrt{m^2+4}-\omega\right), -i(m-\omega)\right] \cup \left[i(m+\omega), i\left(\sqrt{m^2+4}+\omega\right)\right],$$

$$\mathcal{C}_- := \left[-i\left(\sqrt{m^2+4}+\omega\right), -i(m+\omega)\right] \cup \left[i(m-\omega), i\left(\sqrt{m^2+4}-\omega\right)\right].$$

The point 0 belongs to the discrete spectrum, and the dimension of its invariant subspace is at least 2. In Appendix A we will show that if

$$a'(a^2(2+m^2-\omega^2)+8\omega^2) \neq 2a\omega^2(4+a^2)/C^2, \tag{3.11}$$

then the invariant subspace associated to the eigenvalue $\lambda = 0$ is of dimension exactly 2. We will analyse that only the solitary waves satisfies (3.11) and the condition $a' \neq 0$. We also assume more specific condition.

Definition 3.5 The solitary wave, $\Psi_\omega(x)e^{i\Theta}$, satisfies the spectral condition **SP** if

(i) $a' \neq 0$ and (3.11) holds.
(ii) there is no eigenvalue of $\mathbf{C}$ except $\lambda = 0$.

The functional spaces we are going to consider are the weighted Banach spaces $l^p_\beta = l^p_\beta(\mathbb{Z})$, $p \in [1, \infty)$, $\beta \in \mathbb{R}$ of complex-valued functions with the norm

$$\|u\|_{l^p_\beta} = \|(1+|x|)^\beta u(x)\|_{l^p}. \tag{3.12}$$

Our main theorem is the following theorem.

THEOREM 3.6 *Let conditions* (1.3) *and* (1.6) *and* (2.1) *hold*, $\beta \geq 2$, *and* $\Psi(x,t) \in C(\mathbb{R}, l^2 \oplus l^2)$ *be the solution to Equation* (1.1) *with initial value* $\Psi_0(x) = \Psi(x,0) \in (l^2 \cap l^1_\beta) \oplus (l^2 \cap l^1_\beta)$ *which is close to a solitary wave* $\Psi_{\omega 0}e^{i\Theta_0}$

$$d := \|\Psi_0 - \Psi_{\omega_0}e^{i\Theta_0}\|_{(l^2\cap l^1_\beta)\oplus(l^2\cap l^1_\beta)} \ll 1. \tag{3.13}$$

Further assume that the spectral condition **SP** *holds for the solitary wave with* $\omega = \omega_0$. *Then for d sufficiently small the solution admits the following asymptotics*:

$$\Psi(\cdot,t) = \Psi_{\omega_\pm}e^{i(\omega_\pm t+\gamma_\pm)} + W(t)\Phi_\pm + r_\pm(t), \quad t \to \pm\infty, \tag{3.14}$$

where $\Phi_\pm \in l^2 \oplus l^2$ *are the corresponding asymptotic states*, $\omega_\pm$, $\gamma_\pm$ *are some constants and*

$$\|r_\pm(t)\|_{l^2\oplus l^2} = \mathcal{O}(|t|^{-1/2}), \quad t \to \pm\infty. \tag{3.15}$$

4. Linearized dynamics

In this section we summarize the properties of the linearized dynamics. The proofs can be found in Appendix A and [2]. The linearized equation reads

$$\dot{X}(x,t) = \mathbf{C}X(x,t) \tag{4.1}$$

Theorem 2.1 generalizes to Equation (4.1): the equation admits unique solution $X(x,t) \in C_b(\mathbb{R},\ (l^2)^4)$ for every initial function $X(x,0) = X_0 \in (l^2)^4$. The resolvent $\mathbf{R}(\lambda) := (\mathbf{C} - \lambda)^{-1}$ is an operator with matrix-valued integral kernel (Appendix A.1)

$$\mathbf{R}(\lambda, x, y) = \Gamma(\lambda, x, y) + P(\lambda, x, y), \tag{4.2}$$

where

$$\Gamma(\lambda, x, y) = \frac{e^{i\theta_+|x-y|} - e^{i\theta_+(|x|+|y|)}}{4\sin\theta_+}\mathcal{A}_+ + \frac{e^{i\theta_-|x-y|} - e^{i\theta_-(|x|+|y|)}}{4\sin\theta_-}\mathcal{A}_-, \tag{4.3}$$

$$P(\lambda, x, y) = \tfrac{1}{2D}\Big((i\alpha - 2\sin\theta_-)e^{i\theta_+(|x|+|y|)}\mathcal{A}_+ + (i\alpha - 2\sin\theta_+)e^{i\theta_-(|x|+|y|)}\mathcal{A}_- + \beta e^{i(\theta_+|x|+\theta_-|y|)}\mathcal{B}_+ + \beta e^{i(\theta_-|x|+\theta_+|y|)}\mathcal{B}_-\Big), \tag{4.4}$$

$$\mathcal{A}_+ = \begin{pmatrix} \nu_+\sigma_1 & \sigma_1 \\ -\nu_+^2\sigma_1 & \nu_+\sigma_1 \end{pmatrix}, \quad \mathcal{A}_- = \begin{pmatrix} \nu_-\sigma_2 & \sigma_2 \\ -\nu_-^2\sigma_2 & \nu_-\sigma_2 \end{pmatrix},$$

$$\mathcal{B}_+ = \begin{pmatrix} \nu_-\sigma_3 & \sigma_3 \\ \nu_+\nu_-\sigma_3 & \nu_+\sigma_3 \end{pmatrix}, \quad \mathcal{B}_- = \begin{pmatrix} \nu_+\sigma_4 & \sigma_4 \\ \nu_+\nu_-\sigma_4 & \nu_-\sigma_4 \end{pmatrix},$$

$$\sigma_1 = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix},$$

with $\nu_+ = -i\omega + \lambda$, $\nu_- = i\omega + \lambda$. Here $\theta_\pm(\lambda)$ is the root of $2\cos\theta_\pm = \omega + 2 \pm i\lambda$ defined and analytic in $\mathbb{C} \setminus \mathcal{C}_\pm$, and $\operatorname{Im}\theta_\pm(\lambda) > 0$ for $\lambda \in \mathbb{C} \setminus \mathcal{C}_\pm$. The constants α, β and the determinant $D = D(\lambda)$ are given by the formulas

$$\alpha = a + a'C^2,\ \beta = a'C^2,\ D = 2i\alpha(\sin\theta_+ + \sin\theta_-) - 4\sin\theta_+\sin\theta_- + \alpha^2 - \beta^2.$$

The poles of the resolvent correspond to the roots of the determinant $D(\lambda)$. If spectral condition **SP** holds, then the determinant has the only root $\lambda = 0$ with the multiplicity 2 (Appendix A.2).

Observe that the solitary wave (1.8) and their derivatives in ω are related by the following identities:

$$\mathbf{D}_2\psi_\omega = -\omega^2\psi_\omega \quad \mathbf{D}_1(\partial_\omega\psi_\omega) = -\omega^2\partial_\omega\psi_\omega - 2\omega\psi_\omega. \tag{4.5}$$

These formulae imply that the vectors jY_ω and $\partial_\omega Y_\omega$ lie in the generalized two-dimensional null space $\mathcal{X}^0$ of the non-selfadjoint operator $\mathbf{C}$ defined in (4.1), and

$$\mathbf{C}jY_\omega = 0, \quad \mathbf{C}\partial_\omega Y_\omega = jY_\omega. \tag{4.6}$$

The symplectic form Ω for the real vectors X and Z is defined by

$$\Omega(X, Z) = \langle X, qZ\rangle, \tag{4.7}$$

where q is the matrix from (3.10).

Lemma 4.1 *Under the condition* (3.11),

$$\Omega(jY_\omega, \partial_\omega Y_\omega) = -\partial_\omega \sum \omega\psi_\omega^2 \neq 0. \tag{4.8}$$

We will prove Lemma 4.1 in Appendix B.

Hence, the symplectic form Ω is nondegenerate on $\mathcal{X}^0$, i.e. $\mathcal{X}^0$ is a symplectic subspace. Therefore, there exists a symplectic projection operator $\mathbf{P}^0$ from $(l^2)^4$ to $\mathcal{X}^0$ given by the formula

$$\mathbf{P}^0 Z = \frac{1}{\Omega(jY_\omega, \partial_\omega Y_\omega)}[\Omega(Z, \partial_\omega Y_\omega)jY_\omega + \Omega(Z, jY_\omega)\partial_\omega Y_\omega]. \tag{4.9}$$

Denote by $\mathbf{P}^c = 1 - \mathbf{P}^0$ the symplectic projector onto the continuous spectral subspace.

Remark 4.2 On the generalized null space we have $\boldsymbol{C}^2 = 0$ by (4.6), and so the semigroup $e^{t\mathbf{C}}$ reduces to $1 + \boldsymbol{C}t$ as usual for the exponential of the nilpotent part of an operator.

5. Time decay in continuous spectrum

Denote by $e^{\mathbf{C}t}$ the dynamical group of Equation (4.1) acting in the space $(l^2)^4$. Due to Remark 4.2, the solutions $X(t) = e^{\mathbf{C}t}X_0$ of the linearized Equation (4.1) do not decay as $t \to \infty$ if $\mathbf{P}^0 X_0 \neq 0$. On the other hand, we expect the time decay of $\mathbf{P}^c X(t)$, as a consequence of the Laplace representation for $\mathbf{P}^c e^{\mathbf{C}t}$

$$\mathbf{P}^c e^{\mathbf{C}t} = -\frac{1}{2\pi i}\int_{\mathcal{C}_+ \cup \mathcal{C}_-} e^{\lambda t}(\mathbf{R}(\lambda + 0) - \mathbf{R}(\lambda - 0))\mathrm{d}\lambda. \tag{5.1}$$

The decay for the oscillatory integral is obtained from the analytic properties of the resolvent $\mathbf{R}(\lambda)$ for $\lambda \in \mathcal{C}_+ \cup \mathcal{C}_-$.

Clearly, in order to understand the decay of $\mathbf{P}^c e^{t\mathbf{C}}$, it is crucial to study the behaviour of $\mathbf{R}(\lambda, x, y)$ near the branch points $\lambda = \pm i(\sqrt{m^2 + 4} \pm \omega)$ and $\lambda = \pm i(m \pm \omega)$ (where $\sin\theta_\pm$ vanish).

We deduce time decay for the group $\mathbf{P}^c e^{t\mathbf{C}}$ by the following version of Lemma 10.2 from [24], which is itself based on Zygmund's lemma [25, p. 45]. Let $\mathcal{F} : [a, b] \to \mathbf{B}$ be a C^2 function with values in a Banach space $\mathbf{B}$. Let us consider the Fourier integral

$$I(t) = \int_a^b e^{-it\nu}\mathcal{F}(\nu)\mathrm{d}\nu.$$

Lemma 5.1 *Suppose that* $\mathcal{F}(a) = \mathcal{F}(b) = 0$, $\mathcal{F}'' \in L^1(a + \delta, b; \mathbf{B})$ *for any* $\delta > 0$, *and*

$$\mathcal{F}''(a + \zeta) = \mathcal{O}(\zeta^{-3/2}), \quad \zeta \downarrow 0$$

in the norm of $\mathbf{B}$. *Then*

$$I(t) = \mathcal{O}(t^{-3/2}) \quad \text{as } t \to \infty \text{ in the norm of } \mathbf{B}.$$

We will apply Lemma 5.1 to the function $\mathcal{F}(\lambda) = \mathbf{R}(\lambda + 0) - \mathbf{R}(\lambda - 0)$ with values in the Banach space $\mathcal{B} = B((l^1_\beta)^4, (l^\infty_{-\beta})^4)$, the space of continuous linear maps $(l^1_\beta)^4 \to (l^\infty_{-\beta})^4$ for any $\beta \geq 2$.

Theorem 5.2 *Assume that the spectral condition* **SP** *holds so that* $\lambda = 0$ *is the only point in the discrete spectrum of the operator* $\mathbf{C} = \mathbf{C}(\omega)$. *Then for* $\beta \geq 2$,

$$\|\mathbf{P}^c e^{\mathbf{C}t}\|_{\mathcal{B}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \tag{5.2}$$

First, we use the formulas (5.1) and (4.2) to obtain

$$-2\pi i\mathbf{P}^c e^{\mathbf{C}t} = \int_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}(\Gamma(\lambda+0)-\Gamma(\lambda-0))\mathrm{d}\lambda + \int_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}(P(\lambda+0)-P(\lambda-0))\mathrm{d}\lambda. \tag{5.3}$$

Theorem 5.2 follows from the two lemmas where we apply Lemma 5.1 to each summand in the RHS of (5.3) separately.

LEMMA 5.3 *If the assumption of Theorem* 5.2 *holds*, *then*

$$\int_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}(\Gamma(\lambda+0)-\Gamma(\lambda-0))\mathrm{d}\lambda = \mathcal{O}(t^{-3/2}), \quad t\to\infty \tag{5.4}$$

in the norm $\mathcal{B}$.

Proof We consider only the integral over $\mathcal{C}_+$ near the vicinity of the branch point $i(m+\omega)$. The other branch points can be handled in the same way. Consider, for example, the integral of the first term in (4.3). The expression (4.3) implies for $y>0$ that

$$\Gamma_{11}^{+}(\lambda,x,y) = -(\omega+i\lambda)\begin{cases} 0, & x\le 0,\\ \dfrac{e^{i\theta_+ y}(e^{-i\theta_+ x}-e^{i\theta_+ x})}{4\sin\theta_+}, & 0\le x\le y,\\ \dfrac{e^{i\theta_+ x}(e^{-i\theta_+ y}-e^{i\theta_+ y})}{4\sin\theta_+}, & x\ge y.\end{cases}$$

For $\lambda\in\mathcal{C}_+$, the root θ_+ is real, and $\theta_+(\lambda+0)=-\theta_+(\lambda-0)$. Then, for $y>0$,

$$\Gamma_{11}^{+}(\lambda+0,x,y)-\Gamma_{11}^{+}(\lambda-0,x,y) = (\omega+i\lambda)\Theta(x)\frac{\sin\theta_+|x|\sin\theta_+|y|}{\sin\theta_+}, \tag{5.5}$$

where $\Theta(x)=1$ for $x>0$ and zero otherwise. Equality (A.5) implies that

$$\begin{aligned}\cos\theta_+ &= \frac{(2+m^2-(\omega+i\lambda)^2)}{2},\\ \sin\theta_+ &= \frac{\sqrt{(\omega+m+i\lambda)(\omega-m+i\lambda)(4+m^2-(\omega+i\lambda)^2)}}{2}.\end{aligned} \tag{5.6}$$

Hence

$$\sin\theta_+ \sim \sqrt{m+\omega+i\lambda}, \quad \lambda\to i(\omega+m). \tag{5.7}$$

The second derivative of function $f(\lambda)=(\omega+i\lambda)\frac{\sin\theta_+|x|\sin\theta_+|y|}{\sin\theta_+}$ admits the bound

$$|f''(\lambda)| \le \frac{C(1+|x|^2)(1+|y|^2)}{(m+\omega+i\lambda)^{3/2}}, \quad i(\omega+m)<\lambda<i(\omega+m+\varepsilon)$$

for sufficiently small $\varepsilon>0$. For $y<0$, the identical calculation leads to the same bound. We choose $\zeta(\lambda)\in C_0^\infty$ such that supp $\zeta\subset(i(\omega+m-1),\ i(\omega+m+\varepsilon)$ and $\zeta(\lambda)=1$ for $\lambda\in(i(\omega+m),\ i(\omega+m+\varepsilon/2))$. Then the operator-valued function

$\mathcal{F}_1(\lambda)=\zeta(\lambda)(\Gamma^+_{11}(\lambda+0)-\Gamma^+_{11}(\lambda-0))$ satisfies the conditions of Lemma 5.1 with $a=i(\omega+m)$, $b=i(\omega+m+\varepsilon)$ and $\mathbf{B}=\mathcal{B}$. ■

Next, we consider the second summand in the RHS of (5.3).

LEMMA 5.4 *In the situation of Theorem* 5.2

$$\int_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}(P(\lambda+0)-P(\lambda-0))\mathrm{d}\lambda=\mathcal{O}(t^{-3/2}), \tag{5.8}$$

in the norm $\mathcal{B}$.

Proof We consider only the integral in the vicinity of the points $i(m+\omega)$. For example, consider the integral of

$$P_{11}(\lambda,x,y)=\frac{(2i\sin\theta_-+\alpha)\nu_+e^{i\theta_+(|x|+|y|)}+(2i\sin\theta_++\alpha)\nu_-e^{i\theta_-(|x|+|y|)}-\beta(\nu_+e^{i\theta_+|y|+i\theta_-|x|}-\nu_-e^{i\theta_-|y|+i\theta_+|x|})}{2i\alpha(\sin\theta_++\sin\theta_-)-4\sin\theta_+\sin\theta_-+\alpha^2-\beta^2}.$$

The Puiseux expansion in $z=\omega+m+i\lambda$ as $z\to0$, Im $z\geq0$ implies

$$P_{11}(\lambda,x,y)=A_0+A_1(x,y)z^{1/2}+A_2(x,y)\mathcal{O}(z),$$

where $|A_j(x,y)|\leq C_j(1+|x|^j)(1+|y|^j)$, $j=1,2$. Hence,

$$\mathcal{F}_2(\lambda)=\zeta(\lambda)(P_{11}(\lambda+0)-P_{11}(\lambda-0))=\mathcal{O}(z^{1/2}),\quad z\to0$$

in the norm of $\mathcal{B}$. Similarly, differentiating twice the function $P_{11}(\lambda,x,y)$ in λ, we obtain that

$$\mathcal{F}_2''(\lambda)=-P_{11}''(\lambda+0)+P_{11}''(\lambda-0)=\mathcal{O}(z^{-3/2}),\quad z\to0$$

in the norm of $\mathcal{B}$. Therefore, the function $\mathcal{F}_2(\lambda)$, satisfies the conditions of Lemma 5.1. ■

6. Modulation equations

In this section, we obtain the modulation equations which allow to construct the solutions $\Psi(x,t)$ of Equation (1.2) close at each time t to a soliton (i.e. to one of the functions $\Psi_\omega(x)$ in the set $\mathcal{S}_+\cup\mathcal{S}_-$ described in Section 3) with time varying ('modulating') parameters $(\omega,\Theta)=(\omega(t),\Theta(t))$. Let us rewrite Equation (1.2) in the real form

$$\dot\Psi(x,t)=\begin{pmatrix}0&0&1&0\\0&0&0&1\\\Delta_L-m^2&0&0&0\\0&\Delta_L-m^2&0&0\end{pmatrix}\Psi(x,t)+\delta(x)\mathbf{F}(\Psi(0,t)), \tag{6.1}$$

where $\Psi(x,t)\in\mathbb{R}^4$ with $\mathbf{F}(\Psi)\in\mathbb{R}^4$ which is the real vector version of $F(\Psi)\in\mathbb{C}^2$. Then $\Psi(x,t)=e^{j\Theta(t)}Y_{\omega(t)}(x)$ is a solution of (6.1) if and only if $\dot\Theta=\omega$ and $\dot\omega=0$.

We look for a solution to (6.1) in the form

$$\Psi(x,t) = e^{j\Theta(t)}\big(Y_{\omega(t)}(x) + Z(x,t)\big) = e^{j\Theta(t)}Y(x,t), \quad Y(x,t) = Y_{\omega(t)}(x) + Z(x,t). \quad (6.2)$$

Since this is a solution of (6.1) as long as $Z \equiv 0$ and $\dot{\Theta} = \omega$ and $\dot{\omega} = 0$ it is natural to look for solutions in which Z is small and

$$\Theta(t) = \int_0^t \omega(s)\mathrm{d}s + \gamma(t),$$

with γ treated perturbatively. Observe that so far this representation is under-determined since for any $(\omega(t), \Theta(t))$ it just amounts to a definition of Z; it is made unique by restricting Z to lie in the continuous spectrum $\mathbf{P}_t^c = \mathbf{P}^c(\omega(t))$ or equivalently that

$$\mathbf{P}_t^0 Z(t) = 0, \quad \mathbf{P}_t^0 = \mathbf{P}^0(\omega(t)) = I - \mathbf{P}^c(\omega(t)). \quad (6.3)$$

Now we give the *modulation equations* for $\omega(t)$, $\gamma(t)$ which ensure that the conditions (6.3) are preserved by the time evolution.

LEMMA 6.1

(i) *Assume that given a solution of* (6.1) *with regularity as described in Theorem* 2.1, *which can be written in the form* (6.2) and (6.3) *with continuously differentiable* $\omega(t)$, $\theta(t)$. *Then*

$$\dot{Z} = \mathbf{C}Z - \dot{\omega}\partial_\omega Y_\omega - \dot{\gamma}j(Y_\omega + Z) + \mathbf{Q}, \quad (6.4)$$

where $\mathbf{Q}(Z,\omega) = \delta(x)(\mathbf{F}(Y_\omega + Z) - \mathbf{F}(Y_\omega) - \mathbf{F}'(Y_\omega)Z)$, *and*

$$\dot{\omega} = \frac{\langle \mathbf{P}^0\mathbf{Q}, qjY\rangle}{\langle \partial_\omega Y_\omega - \partial_\omega \mathbf{P}^0 Z, qjY\rangle} \quad (6.5)$$

$$\dot{\gamma} = \frac{\langle j\mathbf{P}^0(\partial_\omega Y_\omega - \partial_\omega \mathbf{P}^0 Z), \mathbf{P}^0\mathbf{Q}\rangle}{\langle \partial_\omega Y_\omega - \partial_\omega \mathbf{P}^0 Z, qjY\rangle}, \quad (6.6)$$

where $\mathbf{P}^0 = \mathbf{P}^0(\omega(t))$ *is the projection operator defined in* (4.9), *and* $\partial_\omega \mathbf{P}^0 = \partial_\omega \mathbf{P}^0(\omega(t))$.

(ii) *Conversely, given* ψ *a solution of* (6.1) *as in Theorem* 2.1 *and continuously differentiable functions* $\omega(t)$, $\theta(t)$ *which satisfy* (6.5) and (6.6), *then* Z *defined by* (6.2) *satisfies* (6.4) *and the condition* (6.3) *holds at all times if it holds initially*.

Proof Substituting (6.2) into (6.1), we obtain

$$\dot{\gamma}jY + \dot{Y} = CZ + Q, \quad (6.7)$$

which implies (6.4). Further, taking the scalar product of (6.7) with $(\mathbf{P}^0)^*qjY$ we obtain

$$\dot{\gamma}\langle jY, (\mathbf{P}^0)^* qjY\rangle + \langle \dot{Y}, (\mathbf{P}^0)^* qjY\rangle = \langle Q, (\mathbf{P}^0)^* qjY\rangle. \quad (6.8)$$

Since $q\mathbf{P}^0 = (\mathbf{P}^0)^* q$, and $(\mathbf{P}^0)^2 = \mathbf{P}^0$, then

$$\langle jY, (\mathbf{P}^0)^* qjY\rangle = \langle \mathbf{P}^0 jY, (\mathbf{P}^0)^* qjY\rangle = \langle \mathbf{P}^0 jY, q\mathbf{P}^0 jY\rangle = 0.$$

Then (6.8) becomes

$$\langle \mathbf{P}^0 \dot{Y}, qjY \rangle = \langle \mathbf{P}^0 Q, qjY \rangle.$$

Notice also that $\mathbf{P}^0 \dot{Z} = -\dot{\omega}\partial_\omega \mathbf{P}^0 Z$, $\mathbf{P}^0 \dot{Y}_\omega = \dot{\omega}\partial_\omega Y_\omega$, hence

$$\mathbf{P}^0 \dot{Y} = (\partial_\omega Y_\omega - \partial_\omega \mathbf{P}^0 Z)\dot{\omega}. \tag{6.9}$$

This immediately implies (6.5). Finally, taking the scalar product of (6.7) with $q\mathbf{P}^0 \dot{Y}$, we obtain

$$\dot{\gamma}\langle jY, q\mathbf{P}^0 \dot{Y} \rangle = \langle Q, q\mathbf{P}^0 \dot{Y} \rangle.$$

Substituting (6.9) in the above equation leads to (6.6). ■

It remains to show, for appropriate initial data close to a soliton, that there exist solutions to (6.5) and (6.6), at least locally. To achieve this, observe that by Lemma 4.1 the denominator appearing on the right-hand side (RHS) of (6.5) and (6.6) does not vanish for small $\|X\|_{(l^1_\beta)^4}$. This is because

$$\langle \partial_\omega \psi_\omega, \psi_\omega \rangle = \frac{1}{2}\partial_\omega \int |\psi_\omega|^2 \mathrm{d}x \neq 0. \tag{6.10}$$

This has the consequence that the orthogonality conditions can really be satisfied for small X because they are equivalent to a locally well-posed set of ordinary differential equations for $t \to (\theta(t), \omega(t))$. This implies the following corollary.

Corollary 6.2

(i) *In the situation of* (i) *in Lemma* 6.4, *assume that* (6.10) *holds. If* $\|Z\|_{(l^p_\beta)^4}$ *is sufficiently small for some* p, β, *RHSs of* (6.5) *and* (6.6) *are smooth in* Θ, ω *and there exists a continuous* $\mathcal{R} = \mathcal{R}(\omega, Z)$ *such that*

$$|\dot{\gamma}(t)| \leq \mathcal{R}|Z(0,t)|^2, \quad |\dot{\omega}(t)| \leq \mathcal{R}|Z(0,t)|^2.$$

(ii) *Assume that given* Ψ, *a solution of* (6.1) *as in Theorem* 2.1. *If* ω_0 *satisfies* (6.10) *and* $Z(x, 0) = e^{-j\Theta_0}\Psi(x, 0) - Y_{\omega_0}(x)$ *is small in some* $(l^p_\beta)^4$ *norm and satisfies* (6.3) *there is a time interval on which there exist* C^1 *functions* $t \mapsto (\omega(t), \gamma(t))$ *which satisfy* (6.5) and (6.6).

7. Time decay for the transversal dynamics

Let us represent the initial data Ψ_0 in a convenient form for the application of modulation equations: Lemma 7.1 will allow us to assume that (6.3) holds initially without loss of generality.

Lemma 7.1 *In the situation of Theorem* 3.6, *there exists a solitary wave* $\Psi_{\tilde{\omega}_0}$ *satisfying the spectral condition* **SP** *such that in vector form*

$$\Psi_0(x) = e^{j\tilde{\Theta}_0}(Y_{\tilde{\omega}_0}(x) + Z_0(x)), \quad Y_{\tilde{\omega}_0} = (\psi_{\tilde{\omega}_0}, 0, 0, \omega\psi_{\tilde{\omega}_0}),$$

and for $Z_0(x)$ *we have*

$$\mathbf{P}^0(\tilde{\omega}_0)(Z_0) = 0, \tag{7.1}$$

and

$$\|Z_0\|_{(l^1_\beta)^4\cap(l^2)^4} = \tilde{d} = O(d) \quad \text{as } d \to 0.$$

Proof This can be proved as in [2, Lemma 10.1] by a standard application of the implicit function theorem. ■

In Section 9, we will show that our main theorem (Theorem 3.6) can be derived from the following time decay of the transversal component $Z(t)$.

THEOREM 7.2 *Let all the assumptions of Theorem* 3.6 *hold. For d sufficiently small, there exist* C^1 *functions* $t \mapsto (\omega(t), \gamma(t))$ *defined for* $t \geq 0$ *such that the solution* $\Psi(x, t)$ *of* (6.1) *can be written as in* (6.2)–(6.3) *with* (6.5)–(6.6) *satisfied, and there exists a number* $\overline{M} > 0$, *depending only on the initial data, such that*

$$M(T) = \sup_{0\leq t\leq T} [(1+t)^{3/2}\|Z(t)\|_{(l^\infty_{-\beta})^4} + (1+t)^3\left(|\dot\gamma| + |\dot\omega|\right)|] \leq \overline{M}, \tag{7.2}$$

uniformly in $T>0$, *and* $\overline{M} = O(d)$ *as* $d \to 0$.

Remarks 7.3

- (i) This theorem will be deduced from Proposition 8.1 in the next section.
- (ii) Theorem 2.1 implies that the norms in the definition of M are continuous functions of time (and so M is also).
- (iii) The result holds also for negative time.
- (iv) The result implies in particular that $t^3|\dot\Theta - \omega| + t^3|\dot\omega| \leq C$, hence $\omega(t)$ and $\Theta(t) - t\omega_+$ *should converge as* $t \to \infty$ while $\Psi(x, t) - e^{j\Theta(t)}Y_{\omega(t)}(x)$ have limit zero in $(l^\infty_{-\beta})^4$.

8. Proof of transversal decay

8.1. *Inductive argument (proof of Theorem 7.2)*

Let us write the initial data in the form

$$\Psi_0(x) = e^{j\Theta_0}(Y_{\omega_0}(x) + Z_0(x)), \tag{8.1}$$

with $d = \|Z_0\|_{(l^1_\beta)^4\cap(l^2)^4}$ sufficiently small. By Lemma 7.1 we can assume that $\mathbf{P}^0(\omega_0)(Z_0) = 0$ without loss of generality. Then the local existence asserted in Corollary 6.2 implies the existence of an interval $[0, t_1]$ on which are defined C^1 functions $t \mapsto (\omega(t), \gamma(t))$ satisfying (6.5) and (6.6) and such that $M(t_1) = \rho$ for some $t_1 > 0$ and $\rho > 0$. By continuity we can make ρ as small as we like by making d and t_1 small. The following proposition is proved in Section 8.4.

PROPOSTION 8.1 *In the situation of Theorem* 7.2 *let* $M(t_1) \leq \rho$ *for some* $t_1 > 0$ *and* $\rho > 0$. *Then there exist numbers* d_1 *and* ρ_1, *independent of* t_1, *such that*

$$M(t_1) \leq \rho/2 \tag{8.2}$$

if $d = \|Z_0\|_{(l^1_\beta)^4\cap(l^1)^4} < d_1$ *and* $\rho < \rho_1$.

Assuming the truth of Proposition 8.1 for now Theorem 7.2 will follow from the next argument.

Consider the set $\mathcal{T}$ of $t_1 \geq 0$ such that $(\omega(t), \gamma(t))$ are defined on $[0, t_1]$ and $M(t_1) \leq \rho$. This set is relatively closed by continuity. On the other hand, (8.2) and Corollary 6.2 with sufficiently small ρ and d imply that this set is also relatively open, and hence sup $\mathcal{T} = +\infty$, completing the proof of Theorem 7.2. ■

8.2. *Frozen linearized equation*

A crucial part of the proof of Proposition 8.1 is the estimation of the first term in M, for which purpose it is necessary to make use of the dispersive properties obtained in Sections 4 and 5. Rather than studying directly (6.4), whose linear part is non-autonomous, it is convenient (following [3,4]) to introduce a small modification of (6.2), which leads to an autonomous linearized equation. This new ansatz for the solution is

$$\Psi(x,t) = e^{j\Theta}(Y_\omega(x) + e^{-j(\Theta-\tilde{\Theta})}\tilde{Z}), \quad \text{where } \tilde{\Theta}(t) = \omega_1 t + \Theta_0, \\ \text{and} \quad \omega_1 = \omega(t_1) \tag{8.3}$$

so that, $\tilde{Z} = e^{j(\Theta-\tilde{\Theta})}Z$ and $Z = e^{-j(\Theta-\tilde{\Theta})}\tilde{Z}$. Since

$$\dot{Z} = e^{-j(\Theta-\tilde{\Theta})}\Big(\dot{\tilde{Z}} - j(\omega + \dot{\gamma} - \omega_1)\tilde{Z}\Big),$$

Equation (6.4) implies

$$\dot{\tilde{Z}} = j(\omega-\omega_1)\tilde{Z} + e^{j(\Theta-\tilde{\Theta})}\mathbf{C}\Big(e^{-j(\Theta-\tilde{\Theta})}\tilde{Z}\Big) - e^{j(\Theta-\tilde{\Theta})}\Big(j\dot{\gamma}Y_\omega + \dot{\omega}\partial_\omega Y_\omega - \mathbf{Q}[e^{-j(\Theta-\tilde{\Theta})}\tilde{Z}]\Big). \tag{8.4}$$

The matrices $\mathbf{C}$ and $e^{j\phi}$, where $\phi = \Theta - \tilde{\Theta}$, do not commute

$$\mathbf{C}e^{j\phi} - e^{j\phi}\mathbf{C} = \delta(x)b\sin\phi\sigma, \text{ where } \sigma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; b = 2a'C^2. \tag{8.5}$$

Using (8.5), we rewrite Equation (8.4) as

$$\dot{\tilde{Z}} = j(\omega-\omega_1)\tilde{Z} + \mathbf{C}\tilde{Z} - e^{j(\Theta-\tilde{\Theta})}\Big(\delta(x)b\sin(\Theta-\tilde{\Theta})\sigma\tilde{Z} + j\dot{\gamma}Y_\omega + \dot{\omega}\partial_\omega Y_\omega - \mathbf{Q}[e^{-j(\Theta-\tilde{\Theta})}\tilde{Z}]\Big).$$

To obtain a perturbed *autonomous* equation, we rewrite the first two terms on the RHS by freezing the coefficients at $t = t_1$. Note that

$$j(\omega - \omega_1) + \mathbf{C} = \mathbf{C_1} + j\delta(x)(V - V_1),$$

where $V = a + bP_1$, $V_1 = V(t_1)$, and $\mathbf{C_1} = \mathbf{C}(t_1)$. The equation for $\tilde{Z}$ now reads

$$\dot{\tilde{Z}} = \mathbf{C_1}\tilde{Z} + j\delta(x)(V - V_1)\tilde{Z} \\ - e^{j(\Theta-\tilde{\Theta})}\Big(\delta(x)b\sin(\Theta-\tilde{\Theta})\sigma\tilde{Z} + j\dot{\gamma}Y_\omega + \dot{\omega}\partial_\omega Y_\omega - \mathbf{Q}[e^{-j(\Theta-\tilde{\Theta})}\tilde{Z}]\Big). \tag{8.6}$$

The first term is now independent of t; the idea is that if there is sufficiently rapid convergence of $\omega(t)$ as $t \to \infty$, the other remaining terms are small *uniformly with respect to* t_1. Finally, Equation (8.6) can be written in the following *frozen form*:

$$\dot{\tilde{Z}} = \mathbf{C_1}\tilde{Z} + Z_R, \tag{8.7}$$

where

$$\begin{aligned} Z_R &= j\delta(x)(V - V_1)\tilde{Z} \\ &\quad - e^{j(\Theta-\tilde{\Theta})}\Big(\delta(x)b\sin(\Theta - \tilde{\Theta})\sigma\tilde{Z} + j\dot{\gamma}Y_\omega + \dot{\omega}\partial_\omega Y_\omega - \mathbf{Q}[e^{-j(\Theta-\tilde{\Theta})}\tilde{Z}]\Big). \end{aligned} \tag{8.8}$$

Remark 8.2 The advantage of (8.7) over (6.4) is that it can be treated as a perturbed autonomous linear equation, so that the estimates from Section 4 can be used directly. The additional terms in Z_R can be estimated as small uniformly in t_1 (Lemma 8.3). This is the reason for the introduction of the ansatz (8.3).

LEMMA 8.3 *In the situation of Proposition* 8.1 *there exists* $c > 0$, *independent of* t_1, *such that for* $0 \le t \le t_1$

$$|a(t) - a_1| + |b(t) - b_1| + |\Theta(t) - \tilde{\Theta}(t)| \le c\rho,$$

where

$$\rho := \sup_{0\le t\le t_1} (1 + t^3)(|\dot{\gamma}(t)| + |\dot{\omega}(t)|) \le M(t_1). \tag{8.9}$$

Proof This can be proved as in [2, Lemma 11.4]. ∎

8.3. *Projection onto discrete and continuous spectral spaces*

From Sections 4 and 5 we have information concerning $U(t) = e^{\mathbf{C}_1 t}$, in particular decay on the subspace orthogonal to the (two-dimensional) generalized null space. It is therefore necessary to introduce a further decomposition to take advantage of this. Recall, by comparing (6.2), (6.3) and (8.3) that

$$\tilde{Z} = e^{j(\Theta-\tilde{\Theta})}Z \quad \text{and} \quad \mathbf{P}^0_t Z(t) = 0. \tag{8.10}$$

Introduce the symplectic projections $\mathbf{P}^0_1 = \mathbf{P}^0_{t_1}$ and $\mathbf{P}^c_1 = \mathbf{P}^c_{t_1}$ onto the discrete and continuous spectral subspaces defined by the operator $\mathbf{C_1}$ and write, at each time $t \in [0, t_1]$,

$$\tilde{Z}(t) = G(t) + H(t) \tag{8.11}$$

with $G(t) = \mathbf{P}^0_1\tilde{Z}(t)$ and $H(t) = \mathbf{P}^c_1\tilde{Z}(t)$. The following lemma shows that it is only necessary to estimate $H(t)$.

LEMMA 8.4 *In the situation of Proposition* 8.1, *assume that*

$$\sup_{0\le t\le t_1} \big(|\omega(t) - \omega_1| + |\Theta(t) - \Theta_1(t)|\big) = \Delta$$

is sufficiently small. Then for $0 \le t \le t_1$ *there exists* $c(\Delta, \omega_1)$ *such that*

$$c(\Delta,\omega_1)^{-1}\|H\|_{(l^\infty_{-\beta})^4\cap(l^2)^4} \le \|\tilde{Z}\|_{(l^\infty_{-\beta})^4\cap(l^2)^4} \le c(\Delta,\omega_1)\|H\|_{(l^\infty_{-\beta})^4\cap(l^2)^4}. \tag{8.12}$$

Proof This can be proved as in [2, Lemma 11.5]. ∎

8.4. *Proof of Proposition 8.1*

To prove Proposition 8.1, we explain how to estimate both terms in M, (7.2), to be $\leq \rho/4$, uniformly in t_1.

Estimation of the second term in M. As in Corollary 6.2, we have

$$|\dot{\gamma}(t)| + |\dot{\omega}(t)| \leq c_0 |Z(0,t)|^2 \leq c_0 \frac{M(t)^2}{(1+|t|)^3}, \quad t \leq t_1$$

since $|Z(0,t)| \leq \|Z(t)\|_{(l^\infty_{-\beta})^4}$. Finally, let $\rho_1 < 1/(4c_0)$ to complete the estimate for the second term in M as $\leq \rho/4$.

Estimation of the first term in M. By Lemma 8.4 it is enough to estimate H. Let us apply the projection $\mathbf{P}_1^c$ to both sides of (8.7). Then the equation for H reads

$$\dot{H} = \mathbf{C}_1 H + \mathbf{P}_1^c Z_R. \tag{8.13}$$

Now to estimate H we use the Duhamel representation

$$H(t) = U(t)H(0) + \int_0^t U(t-s)\mathbf{P}_1^c Z_R(s)\mathrm{d}s, \quad t \leq t_1, \tag{8.14}$$

with $U(t) = e^{\mathbf{C}_1 t}$ the one parameter group just introduced. Recall that $\mathbf{P}_1^0 H(t) = 0$ for $t \in [0,\ t_1]$. Therefore

$$\|U(t)H(0)\|_{(l^\infty_{-\beta})^4} \leq c(1+t)^{-3/2}\|H(0)\|_{(l^1_\beta)^4 \cap (l^2)^4} \leq c(1+t)^{-3/2}\|\tilde{Z}(0)\|_{(l^1_\beta)^4 \cap (l^2)^4} \tag{8.15}$$

by Theorem 5.2 and inequality (8.12). Let us estimate the integrand on the RHS of (8.14). We use the representation (8.8) for Z_R and apply Theorem 5.2, Corollary 6.2 and Lemma 8.3 to obtain

$$\begin{aligned}\|U(t-s)\mathbf{P}_1^c Z_R\|_{(l^\infty_{-\beta})^4} &\leq c(1+t-s)^{-3/2}\|\mathbf{P}_1^c Z_R(t)\|_{(l^1_\beta)^4} \\ &\leq c(1+t-s)^{-3/2}\left(|\tilde{Z}(0,t)|^2 + \rho|\tilde{Z}(0,t)|\right) \\ &\leq c(1+t-s)^{-3/2}\left(\|\tilde{Z}(t)\|^2_{(l^\infty_{-\beta})^4} + \rho\|\tilde{Z}(t)\|_{(l^\infty_{-\beta})^4}\right), \quad t \leq t_1.\end{aligned} \tag{8.16}$$

Now (8.12), (8.14), (8.15) and (8.16) imply

$$\begin{aligned}\|\tilde{Z}(t)\|_{(l^\infty_{-\beta})^4} &\leq c(1+t)^{-3/2}\|\tilde{Z}(0)\|_{(l^1_\beta)^4 \cap (l^2)^4} \\ &\quad + c_1 \int_0^t \frac{\mathrm{d}s}{(1+t-s)^{3/2}}\left(\|\tilde{Z}(s)\|^2_{(l^\infty_{-\beta})^4} + \rho\|\tilde{Z}(s)\|_{(l^\infty_{-\beta})^4}\right).\end{aligned}$$

Multiply by $(1+t)^{3/2}$ to deduce

$$\begin{aligned}(1+t)^{3/2}\|\tilde{Z}(t)\|_{(l^\infty_{-\beta})^4} &\leq cd + c_1 \int_0^t \frac{(1+t)^{3/2}(1+s)^{-3}}{(1+t-s)^{3/2}}(1+s)^3\|\tilde{Z}(s)\|^2_{(l^\infty_{-\beta})^4}\mathrm{d}s \\ &\quad + c_1\rho \int_0^t \frac{(1+t)^{3/2}(1+s)^{-3/2}}{(1+t-s)^{3/2}}(1+s)^{3/2}\|\tilde{Z}(s)\|_{(l^\infty_{-\beta})^4}\mathrm{d}s\end{aligned} \tag{8.17}$$

since $\|\tilde{Z}(0)\|_{(l^1_\beta)^4\cap(l^2)^4} \le d$. Introduce the majorant

$$m(t) := \sup_{[0,t]}(1+s)^{3/2}\|\tilde{Z}(s)\|_{(l^\infty_{-\beta})^4}, \quad t \le t_1$$

and hence

$$m(t) \le cd + c_1 m^2(t)\int_0^t \frac{(1+t)^{3/2}(1+s)^{-3}}{(1+t-s)^{3/2}}\,\mathrm{d}s + \rho c_1 m(t)\int_0^t \frac{(1+t)^{3/2}(1+s)^{-3/2}}{(1+t-s)^{3/2}}\,\mathrm{d}s. \tag{8.18}$$

It is easy to see (by splitting up the integrals into $s<t/2$ and $s \ge t/2$) that both these integrals are bounded independent of t. Thus (8.18) implies that there exist c, c_2, c_3, independent of t_1, such that

$$m(t) \le cd + \rho c_2 m(t) + c_3 m^2(t), \quad t \le t_1.$$

Recall that $m(t_1) \le \rho \le \rho_1$ by assumption. Therefore this inequality implies that $m(t)$ is bounded for $t \le t_1$, and moreover,

$$m(t) \le c_4\mathrm{d}, \quad t \le t_1$$

if d and ρ are sufficiently small. The constant c_4 does not depend on t_1. We choose d in (3.13) small enough that $d<\rho/(4c_4)$. Therefore,

$$\sup_{[0,t_1]}(1+t)^{3/2}\|\tilde{Z}(t)\|_{l^\infty_{-\beta}} < \rho/4$$

if d and ρ are sufficiently small. This bounds the first term as $<\rho/4$ by (8.10) and hence $M(t_1)<\rho/2$, completing the proof of Proposition 8.1. ■

9. Soliton asymptotics

Here we prove our main Theorem 3.6 using the bounds (7.2). For the solution $\Psi(x, t)$ to (1.2) let us define the accompanying soliton as $S(x, t) = \Psi_{\omega(t)}(x)e^{i\Theta(t)}$, where $\dot{\Theta}(t) = \omega(t) + \dot{\gamma}(t)$. Then for the difference $Z(x, t) = \Psi(x, t) - S(x, t)$ we obtain easily from Equations (1.2) and (1.8)

$$\begin{aligned}\dot{Z}(x,t) = &\begin{bmatrix} 0 & 1 \\ \Delta_L - m^2 & 0 \end{bmatrix} Z(x,t) - i\dot{\gamma}S(x,t) - \dot{\omega}\partial_\omega S(x,t) \\ &+ \delta(x)(F(\Psi(0,t)) - F(S(0,t))).\end{aligned} \tag{9.1}$$

Then

$$Z(t) = W(t)Z(0) + \int_0^t W(t-\tau)R(\cdot,\tau)\mathrm{d}\tau, \tag{9.2}$$

where $R(x, t) = -i\dot{\gamma}S(x, t) - \dot{\omega}\partial_\omega S(x, t) + \delta(x)\Big(F(\psi(0, t)) - F(s(0, t))\Big)$, and $W(t)$ is the dynamical group of the free Klein–Gordon equation. Let us rewrite (9.2) as

$$\begin{aligned}Z(t) = &W(t)\Big(Z(0) + \int_0^\infty W(-\tau)R(\cdot,\tau)\mathrm{d}\tau\Big) \\ &- \int_t^\infty W(t-\tau)R(\cdot,\tau)\mathrm{d}\tau = W(t)\Phi_+ + r_+(t).\end{aligned} \tag{9.3}$$

Since $\gamma(t)-\gamma_+$, $\omega(t)-\omega_+=\mathcal{O}(t^{-2})$, and therefore $\Theta(t)-\omega_+t-\gamma_+=\mathcal{O}(t^{-1})$ for $t\to\infty$, to establish the asymptotic behaviour (3.14) it suffices to prove that

$$\Phi_+=Z(0)+\int_0^\infty W(-\tau)R(\cdot,\tau)\mathrm{d}\tau\in(l^2)^2 \quad \text{and} \quad \|r_+(t)\|_{(l^2)^2}=\mathcal{O}(t^{-1/2})),\ t\to\infty. \tag{9.4}$$

Let us recall that by (7.2)

$$|\dot{\omega}(t)|\le c(1+t)^{-3},\ |\dot{\gamma}(t)|\le c(1+t)^{-3},\ |F(\psi(0,t))-F(s(0,t))|$$
$$\le c|\chi(0,t)|\le c(1+t)^{-3/2}.$$

Hence, the 'unitarity' in $l^2\oplus l^2$ of the group $W(t)$ implies (9.4). ■

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Appendix A: The resolvent

A.1. *Calculation of the matrix kernel*

Here we will construct matrix kernel of the resolvent $\mathbf{R}(\lambda)$ which is the solution to the equation

$$(\mathbf{C}-\lambda)\mathbf{R}(\lambda,x,y)=\delta(x-y)I. \tag{A.1}$$

We calculate only the first column $R_I(\lambda)$ of the matrix $\mathbf{R}(\lambda)$. The other column can be calculated similarly. We have

$$(\mathbf{C}-\lambda)R_I(\lambda,x,y)=\delta(x-y)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}. \tag{A.2}$$

If $x\neq 0$ and $x\neq y$, (A.2) takes the form

$$\begin{pmatrix}-\lambda & \omega & 1 & 0\\ -\omega & -\lambda & 0 & 1\\ \Delta_L-m^2 & 0 & -\lambda & \omega\\ 0 & \Delta_L-m^2 & -\omega & -\lambda\end{pmatrix}R_I(\lambda,x,y)=0,\quad x\neq 0,\ x\neq y. \tag{A.3}$$

The general solution is a linear combination of exponential solutions of the type $e^{i\theta x}v$. Substituting this solution into (A.3), we get

$$\begin{pmatrix}-\lambda & \omega & 1 & 0\\ -\omega & -\lambda & 0 & 1\\ 2\cos\theta-2-m^2 & 0 & -\lambda & \omega\\ 0 & 2\cos\theta-2-m^2 & -\omega & -\lambda\end{pmatrix}v=0. \tag{A.4}$$

For nonzero vectors v, the determinant of the matrix vanishes if

$$2 - 2\cos\theta + m^2 = (\omega \pm i\lambda)^2.$$

Finally, we obtain four roots $\pm\theta_\pm(\lambda)$ in $D := -\pi \le \mathrm{Re}\,\theta \le \pi$ with

$$2 - 2\cos\theta_\pm(\lambda) = (\omega \pm i\lambda)^2 - m^2. \tag{A.5}$$

We choose the cuts in the complex plane λ:
the cut $\mathcal{C}_+ := [-i(\sqrt{m^2+4} - \omega),\ -i(m-\omega)] \cup [i(m+\omega), i(\sqrt{m^2+4}+\omega)]$ for $\theta_+(\lambda)$ and the cut $\mathcal{C}_- := [-i(\sqrt{m^2+4}+\omega),\ -i(m+\omega)] \cup [i(m-\omega), i(\sqrt{m^2+4}-\omega)]$ for $\theta_-(\lambda)$. Then

$$\mathrm{Im}\,\theta_\pm(\lambda) > 0, \quad \lambda \in \mathbb{C} \setminus \mathcal{C}_\pm. \tag{A.6}$$

Further, we obtain four linearly independent exponential solutions to (A.4)

$$v_+ e^{\pm i\theta_+ x} = \begin{pmatrix} 1 \\ i \\ \lambda - i\omega \\ \omega + i\lambda \end{pmatrix} e^{\pm i\theta_+ x}, \quad v_- e^{\pm i\theta_- x} = \begin{pmatrix} 1 \\ -i \\ \lambda + i\omega \\ \omega - i\lambda \end{pmatrix} e^{\pm i\theta_- x}.$$

Now we can solve Equation (A.2). First, we rewrite it using the representation (4.1) for the operator $\mathbf{C}$

$$\begin{pmatrix} -\lambda & \omega & 1 & 0 \\ -\omega & -\lambda & 0 & 1 \\ \Delta_L - m^2 & 0 & -\lambda & \omega \\ 0 & \Delta_L - m^2 & -\omega & -\lambda \end{pmatrix} \begin{pmatrix} R_{11} \\ R_{21} \\ R_{31} \\ R_{41} \end{pmatrix} = \delta(x-y) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \delta(x) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a+b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix} \begin{pmatrix} R_{11} \\ R_{21} \\ R_{31} \\ R_{41} \end{pmatrix}. \tag{A.7}$$

Let us consider $y > 0$ for the concreteness. Then the RHS vanishes in the open intervals $(-\infty, 0)$, $(0, y)$ and (y, ∞). Hence, for the parameter λ outside the cuts $\mathcal{C}_\pm$, the solution admits the representation

$$R_I(\lambda, x, y) = \begin{pmatrix} 0 \\ 0 \\ \delta(x-y) \\ 0 \end{pmatrix} + \begin{cases} A_+ e^{-i\theta_+ x} v_+ + A_- e^{-i\theta_- x} v_-, & x < 0 \\ B_+^- e^{-i\theta_+ x} v_+ + B_-^- e^{-i\theta_- x} v_- + B_+^+ e^{i\theta_+ x} v_+ + B_-^+ e^{i\theta_- x} v_-, & 0 < x < y \\ C_+ e^{i\theta_+ x} v_+ + C_- e^{i\theta_- x} v_-, & x > y \end{cases} \tag{A.8}$$

since by (A.6), the exponent $e^{-i\theta_\pm x}$ decays for $x \to -\infty$, and similarly, $e^{i\theta_\pm x}$ decays for $x \to \infty$. Next we need eight equations to calculate the eight constants $A_+, \ldots, C_-$. We have two continuity equations and two jump conditions for the derivatives at the points $x = 0$ and $x = y$. These four vector equations give just eight scalar equations for the calculation.

Continuity at $x = y$: $R_I(y-0,\ y) = R_I(y+0,\ y)$, i.e.

$$B_+^- v_+/e_+ + B_-^- v_-/e_- + B_+^+ v_+ e_+ + B_-^+ v_- e_- = C_+ v_+ e_+ + C_- v_- e_-,$$

where $e_\pm := e^{i\theta_\pm y}$. It is equivalent to

$$\begin{cases} B_+^-/e_+ + B_+^+ e_+ = C_+ e_+, \\ B_-^-/e_- + B_-^+ e_- = C_- e_-. \end{cases} \tag{A.9}$$

Continuity at $x = 0$: $R_I(-0,\ y) = R_I(+0,\ y)$, i.e.

$$A_+ v_+ + A_- v_- = B_+^- v_+ + B_-^- v_- + B_+^+ v_+ + B_-^+ v_-,$$

that is equivalent to

$$\begin{cases} A_+ = B_+^- + B_+^+, \\ A_- = B_-^- + B_-^+. \end{cases} \tag{A.10}$$

'Jump' at $x=y$: Equation (A.7) implies

$$\begin{cases} B_+^- e^{i\theta_+}/e_+ + B_+^+ e^{-i\theta_+} e_+ + B_-^- e^{i\theta_-}/e_- + B_-^+ e^{-i\theta_-} e_- - C_+ e_+ e^{-i\theta_+} - C_- e_- e^{-i\theta_-} = \lambda, \\ B_+^- e^{i\theta_+}/e_+ + B_+^+ e^{-i\theta_+} e_+ - B_-^- e^{i\theta_-}/e_- - B_-^+ e^{-i\theta_-} e_- - C_+ e_+ e^{-i\theta_+} + C_- e_- e^{-i\theta_-} = -i\omega. \end{cases}$$

Hence,

$$\begin{cases} B_+^- e^{i\theta_+}/e_+ + B_+^+ e^{-i\theta_+} e_+ - C_+ e_+ e^{-i\theta_+} = \dfrac{\lambda - i\omega}{2}, \\ B_-^- e^{i\theta_-}/e_- + B_-^+ e^{-i\theta_-} e_- - C_- e_- e^{-i\theta_-} = \dfrac{\lambda + i\omega}{2}. \end{cases} \tag{A.11}$$

After substitution of $C_\pm$ from (A.9), the constants $B_\pm^+$ cancel and we get

$$\frac{B_+^-}{e_+}(e^{i\theta_+} - e^{-i\theta_+}) = \frac{\lambda - i\omega}{2}, \quad \frac{B_-^-}{e_-}(e^{i\theta_-} - e^{-i\theta_-}) = \frac{\lambda + i\omega}{2},$$

and then

$$B_+^- = -\frac{\omega + i\lambda}{4\sin\theta_+} e_+, \quad B_-^- = \frac{\omega - i\lambda}{4\sin\theta_-} e_-. \tag{A.12}$$

'Jump' at $x=0$: Equation (A.7) implies

$$\begin{cases} B_+^- e^{-i\theta_+} + B_+^+ e^{i\theta_+} + B_-^- e^{-i\theta_-} + B_-^+ e^{i\theta_-} - A_+ e^{-i\theta_+} - A_- e^{-i\theta_-} = -(a+b)(A_+ + A_-), \\ B_+^- e^{-i\theta_+} + B_+^+ e^{i\theta_+} - B_-^- e^{-i\theta_-} - B_-^+ e^{i\theta_-} - A_+ e^{-i\theta_+} + A_- e^{-i\theta_-} = -a(A_+ - A_-). \end{cases}$$

Hence,

$$\begin{cases} B_+^- e^{-i\theta_+} + B_+^+ e^{i\theta_+} - A_+ e^{-i\theta_+} = -\alpha A_+ - \beta A_-, \\ B_-^- e^{-i\theta_-} + B_-^+ e^{i\theta_-} - A_- e^{-i\theta_-} = -\beta A_+ - \alpha A_-. \end{cases}$$

where $\alpha = a + \frac{b}{2}$, $\beta = \frac{b}{2}$. Substituting here (A.10), we get after cancellations,

$$\begin{cases} (e^{i\theta_+} - e^{-i\theta_+} + \alpha) B_+^+ + \beta B_-^+ = -\alpha B_+^- - \beta B_-^-, \\ \beta B_+^+ + (e^{i\theta_-} - e^{-i\theta_-} + \alpha) B_-^+ = -\beta B_+^- - \alpha B_-^-. \end{cases}$$

Hence, the solution is given by

$$\begin{pmatrix} B_+^+ \\ B_-^+ \end{pmatrix} = -\frac{1}{D} \begin{pmatrix} 2i\sin\theta_- + \alpha & -\beta \\ -\beta & 2i\sin\theta_+ + \alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} B_+^- \\ B_-^- \end{pmatrix}, \tag{A.13}$$

where D is the determinant

$$D := (2i\sin\theta_+ + \alpha)(2i\sin\theta_- + \alpha) - \beta^2 \tag{A.14}$$

and B_+^-, B_-^- are given by (A.12). The formulas (A.12) and (A.13) imply

$$\begin{aligned} B_+^+ &= \frac{1}{2D}\left(\frac{2i\alpha\sin\theta_- + \alpha^2 - \beta^2}{2\sin\theta_+}(\omega + i\lambda)e_+ + i\beta(\omega - i\lambda)e_-\right), \\ B_-^+ &= \frac{1}{2D}\left(i\beta(\omega + i\lambda)e_+ - \frac{2i\alpha\sin\theta_+ + \alpha^2 - \beta^2}{2\sin\theta_-}(\omega - i\lambda)e_-\right). \end{aligned} \tag{A.15}$$

Using the identities

$$2i\alpha\sin\theta_- + \alpha^2 - \beta^2 = D - 2i\alpha\sin\theta_+ + 4\sin\theta_+\sin\theta_-,$$
$$2i\alpha\sin\theta_+ + \alpha^2 - \beta^2 = D - 2i\alpha\sin\theta_- + 4\sin\theta_+\sin\theta_-,$$

let us rewrite (A.15) as

$$B^+_+ = \frac{\omega+i\lambda}{4\sin\theta_+}e_+ - \frac{1}{2D}\big((i\alpha - 2\sin\theta_-)(\omega+i\lambda)e_+ + i\beta(\omega-i\lambda)e_-\big), \tag{A.16}$$
$$B^+_- = -\frac{\omega-i\lambda}{4\sin\theta_-}e_- + \frac{1}{2D}\big(i\beta(\omega+i\lambda)e_+ + (i\alpha-2\sin\theta_+)(\omega-i\lambda)e_-\big).$$

Finally, the formulas (A.8)–(A.10), (A.12) and (A.16) give the first column $R_I(\lambda,x,y)$ of the resolvent for $y>0$:

$$R_I(\lambda,x,y) = \Gamma_I(\lambda,x,y) + P_I(\lambda,x,y) + \delta(x-y)\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \tag{A.17}$$

where

$$\Gamma_I(\lambda,x,y) = -\frac{\omega+i\lambda}{4\sin\theta_+}(e^{i\theta_+|x-y|} - e^{i\theta_+(|x|+|y|)})v_+ + \frac{\omega-i\lambda}{4\sin\theta_-}(e^{i\theta_-|x-y|} - e^{i\theta_-(|x|+|y|)})v_-, \tag{A.18}$$

and

$$P_I(\lambda,x,y) = \frac{1}{2D}\Big[\Big(-(i\alpha-2\sin\theta_-)(\omega+i\lambda)e^{i\theta_+(|x|+|y|)} - i\beta(\omega-i\lambda)e^{i(\theta_+|x|+\theta_-|y|)}\Big)v_+$$
$$+\Big(i\beta(\omega+i\lambda)e^{i(\theta_-|x|+\theta_+|y|)} + (i\alpha-2\sin\theta_+)(\omega-i\lambda)e^{i\theta_-(|x|+|y|)}\Big)v_-\Big]. \tag{A.19}$$

Note that if $y<0$ we get the same formulas.

A.2. *The poles of the resolvent*

The poles of the resolvent correspond to the roots of the determinant (A.14)

$$D(\lambda) := \alpha^2 + 2i\alpha(\sin\theta_+ + \sin\theta_-) - 4\sin\theta_+\sin\theta_- - \beta^2 = 0, \tag{A.20}$$

with $\theta_\pm$ as in (A.5)–(A.6). Thus $D(\lambda)$ is an analytic function on $\mathbb{C}\setminus\mathcal{C}_\pm$. Since there are two possible values for the square roots in $\theta_\pm$, there is a corresponding four-sheeted function $\tilde{D}(\lambda)$ analytic on a four-sheeted cover of $\mathbb{C}$ which is branched over $\mathcal{C}_-$ and $\mathcal{C}_+$. We call the sheet defined by (A.6) the *physical sheet*.

Proposition A.1 *If* (3.11), *holds then* $\lambda=0$ *is a root of the determinant* $D(\lambda)$ *with multiplicity* 2.

Proof First, let us check that $\lambda=0$ is a root of $D(\lambda)$. For $\lambda=0$, we get

$$\theta_+ = \theta_- = \theta_0, \quad \sin\theta_0 = ia/2, \quad \cos\theta_0 = (2+m^2-\omega^2)/2. \tag{A.21}$$

Hence,

$$D(0) = \alpha^2 - \beta^2 + 2i\alpha ia + a^2 = (a+b/2)^2 - b^2/4 - 2(a+b/2)a + a^2 = 0.$$

Now let us compute $D'(\lambda)$

$$D' = 2i\alpha(\theta'_+\cos\theta_+ + \theta'_-\cos\theta_-) - 4\theta'_+\cos\theta_+\sin\theta_- - 4\theta'_-\cos\theta_-\sin\theta_+. \tag{A.22}$$

Differentiating (A.5) we obtain

$$\theta'_{\pm}\sin\theta_{\pm} = \pm i(\omega \pm i\lambda). \tag{A.23}$$

Hence, $\theta'_{+}(0) = -\theta'_{-}(0) = \pm 2\omega/a$ and then $D'(0) = 0$ by (A. 21). Therefore, $\lambda = 0$ is the root of $D(\lambda)$ of multiplicity at least 2. Differentiating (A.22) and (A.23), we obtain

$$D''(0) = 2i\alpha\Big(-2(\theta'_0)^2\sin\theta_0 + 2\theta''_0\cos\theta_0\Big) + (\theta'_0)^2 - 8\theta''_0\cos\theta_0\sin\theta_0, \quad (\theta'_0)^2 = (\theta'_{+}(0))^2 \tag{A.24}$$

$$\theta''_0 = \theta''_{+}(0) = -\frac{1 + (\theta'_0)^2\cos\theta_0}{\sin\theta_0}. \tag{A.25}$$

Substituting (A.25) into (A.24) and using (A.21), we get

$$D''(0) = 2i\Big(a + \frac{b}{2}\Big)\Big(-2\frac{\cos\theta_0}{\sin\theta_0} - 2\frac{(\theta'_0)^2}{\sin\theta_0}\Big) + 8(\theta'_0)^2(1 + \cos^2\theta_0) + 8\cos\theta_0$$

$$= 8(\theta'_0)^2\cos^2\theta_0 - \frac{4b}{a}(\cos\theta_{+} + (\theta'_0)^2) = 8\frac{4\omega^2}{a^2}\Big(1 + \frac{a^2}{4}\Big) - \frac{4b}{a}\Big(\frac{2 + m^2 - \omega^2}{2} + \frac{4\omega^2}{a^2}\Big) \neq 0$$

if (3.11) holds. ■

Appendix B: Proof of Lemma 4.1

We have

$$\sum_{x\in\mathbb{Z}}\psi_\omega^2(x) = C^2\Bigg(\sum_{x=0}^{\infty}e^{-2kx} + \sum_{x=1}^{\infty}e^{-2kx}\Bigg) = C^2\Big(\frac{1}{1 - e^{-2k}} + \frac{e^{-2k}}{1 - e^{-2k}}\Big) = C^2\frac{e^k + e^{-k}}{e^k - e^{-k}} = C^2\frac{\cosh k}{\sinh k}.$$

We now differentiate

$$\partial_\omega\sum\omega\psi_\omega^2(x) = C^2\frac{\cosh k}{\sinh k} + \omega\Big(2CC'\frac{\cosh k}{\sinh k} - C^2\frac{k'}{\sinh^2 k}\Big).$$

Differentiating the identity $2\cosh k = \pm(m^2 - \omega^2 + 2)$, we obtain $k' = \mp\dfrac{\omega}{\sinh k}$. Further, differentiating the identity (3.2), we obtain $k'\cosh k = \pm a'(C^2)CC'$. Then

$$C' = \pm\frac{k'\cosh k}{a'(C^2)C} = -\frac{\omega\cosh k}{a'(C^2)C\sinh k}.$$

Hence,

$$\partial_\omega\sum\omega\psi_\omega^2(x) = C^2\frac{\cosh k}{\sinh k} + \omega^2\Big(-\frac{2\cosh^2 k}{a'(C^2)\sinh^2 k} \pm \frac{C^2}{\sinh^3 k}\Big) \neq 0 \tag{A.26}$$

if

$$b(\cosh k\sinh^2 k \pm \omega^2) \neq 4\omega^2\cosh^2 k\sinh k,$$

which is equivalently (3.11) since

$$\sinh k = \pm\frac{a}{2}, \quad \cosh^2 k = 1 + \frac{a^2}{4}, \quad \cosh k = \pm\cos\theta = \pm\frac{2 + m^2 - \omega^2}{2}.$$