On the eigenfunction expansion for Hamilton operators

A. Komech

Faculty of Mathematics of Vienna University
and Institute for Information Transmission Problems RAS
e-mail: alexander.komech@univie.ac.at

E. Kopylova

Faculty of Mathematics of Vienna University
and Institute for Information Transmission Problems RAS
e-mail: elena.kopylova@univie.ac.at

Abstract

A spectral representation for solutions to linear Hamilton equations with nonnegative energy in Hilbert spaces is obtained. This paper continues our previous work on Hamilton equations with positive definite energy. Our approach is a special version of M. Krein's spectral theory of J-selfadjoint operators in Hilbert spaces with indefinite metric.

As a principal application of these results, we justify the eigenfunction expansion for linearized nonlinear relativistic Ginzburg–Landau equations.

1 Introduction

We consider complex linear Hamilton operators in a complex Hilbert space $\mathscr{X}$,

$$A = JB, \qquad \text{where} \qquad B^* = B, \qquad J^* = -J, \qquad J^2 = -1. \qquad (1.1)$$

In particular, the operator $J : \mathscr{X} \to \mathscr{X}$ is bounded. The selfadjoint operator B is defined on a dense domain $D(B) \subset \mathscr{X}$. Our aim is to prove the well-posedness of the Cauchy problem for the equation

$$\dot{X}(t) = AX(t), \qquad (1.2)$$

and obtain a spectral representation for solutions and the corresponding spectral resolution for A. For example, for $J = i$ the solutions are given by $X(t) = e^{iBt}X(0)$. A more general 'commutative case', when $JB = BJ$, reduces to $J = i$, since $JB = iB_1$, where $B_1 = -iJB$ is the selfadjoint operator. However, $JB \neq BJ$ for linearizations of $U(1)$-invariant nonlinear Schrödinger equations as shown in Appendix of [26].

We develop the theory in the case of nonnegative 'energy operators' B with spectral gap and finite 'degeneracy of the vacuum':

Condition I $\qquad\qquad \sigma(B) \subset \{0\} \cup [\delta, \infty), \quad \delta > 0 \qquad (1.3)$

Condition II $\qquad\qquad \dim \operatorname{Ker} B < \infty. \qquad (1.4)$

These conditions hold, in particular, for all equations considered in [20]–[23]. The motivation for the theory was discussed in [26], in which the simplest case $\sigma(B) \subset (\delta, \infty)$ (i.e., dim $\operatorname{Ker} B = 0$) was studied.

We reduce the problem to a selfadjoint generator developing a special version of M. Krein's spectral theory of J-selfadjoint operators in Hilbert spaces with indefinite metric [2], [27]. We apply this version for justification of the eigenfunction expansions for the linearization of relativistic nonlinear Ginzburg–Landau equation [22]. The generator of the linearization reads

$$A = \begin{pmatrix} 0 & 1 \\ -S & 0 \end{pmatrix}, \qquad (1.5)$$

where $S := -\frac{d^2}{dx^2} + m^2 + V_0(x)$. Our results are concerned with the following:

• The existence and uniqueness and formula for generalized solutions to (1.2) under conditions (1.3), (1.4) for all initial states X with finite energy $\langle BX, X \rangle$. Here, $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathscr{X}$.

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- The eigenfunction expansion

$$\begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix} = t\Phi_0 + \Psi_0 + \sum e^{-i\omega_k t} C_k a_k + \int_{|\omega| \geq m} e^{-i\omega t} C(\omega) a_\omega \, d\omega \quad (1.6)$$

for solutions to (1.2) with generator (1.5). Here, $\Phi_0 \in \mathrm{Ker}\, A$, and a_k are the eigenvectors of A, Ψ_0 is the associated eigenvector to Φ_0, while a_ω are generalized eigenfunctions of A.

Such eigenfunction expansions were used in [4, 5, 22] for the calculation of 'Fermi Golden Rule' (FGR) in the context of the nonlinear Schrödinger and Klein–Gordon equations. This is a nondegeneracy condition, which was introduced in [42] in the framework of nonlinear wave and Schrödinger equations. This condition means a strong coupling of discrete and continuous spectral components of solutions providing the radiation of energy to infinity and which results in the asymptotic stability of solitary waves. The calculation of FGR, as given in [4, 5, 26], relies on eigenfunction expansions of type (1.6). Our main Theorem 6.4 justifies the eigenfunction expansion [22, (5.14)], for which no detailed proof was given before. This justification was one of our main motivation for writing the present paper.

The eigenfunction expansion (1.6) extends our previous result [26], where the expansion was established only for odd solutions. In this framework we have $\mathrm{Ker}\, B = 0$ and $\Phi_0 = \Psi_0 = 0$. This framework was sufficient for the proof of asymptotic stability of standing solitons for the nonlinear relativistic Ginzburg–Landau equations under odd perturbations [22]. However, to establish the asymptotic stability under arbitrary perturbations we need the expansion (1.6) for solutions without antisymmetry.

Let us comment on our approach. First, we reduce the abstract problem (1.2) under conditions (1.3), (1.4) to a selfadjoint generator justifying the classical M. Krein transformation [10]. This reduction is a special version of spectral theory of J-selfadjoint operators in Hilbert spaces with indefinite metric [2, 27], extending our approach [26] to the case $\mathrm{Ker}\, B \neq 0$. This extension required new robust ideas i) to analyze the structure of spectrum of the reduced selfadjoint operator, and ii) to find the canonical form of the Hamilton operator. We provide a broad range of examples satisfying all the imposed conditions (1.3),(1.4), (2.12), and (3.1).

Second, we apply this abstract spectral theory to operator (1.5) and construct the eigenfunction expansion for the reduced selfadjoint operator

following the method of Section 5 from [26]. At last, we deduce (1.6) by extending our approach from [26], which relies on the methods of PDO.

One of our novelties is a vector-valued treatment of the convergence of the integral over the continuous spectrum in (1.6). Namely, we show that the integral is the limit of the corresponding integrals over $m \leq |\omega| \leq M$ as $M \to \infty$ in the Sobolev space $H^1(\mathbb{R})$. In its own turn, the integral over $m \leq |\omega| \leq M$ is absolutely converging in the weighted L^2-space with the weight $(1 + |x|)^{-s}$, where $s > 1$.

Finally, calculation of the symplectic normalization of the generalized eigenfunctions requires extra arguments pertaining to the nondegenerate case [26].

We now give some comments on the related works. Some spectral properties of the Hamilton non-selfadjoint operators were studied by V. Buslaev and G. Perelman [3, 4, 5], M. B. Erdogan and W. Schlag [8, 39], S. Cuccagna, D. Pelinovsky and V. Vougalter [7]. It is worth noting that the eigenfunction expansions of J-selfadjoint operators were not justified previously.

Spectral resolution of bounded J-selfadjoint nonnegative operators in Krein spaces was constructed by M. Krein, H. Langer and Yu. Shmul'yan [27, 28], and extended to unbounded *definitizable* operators by M. Krein, P. Jonas, H. Langer and others [13, 14, 16, 29, 30]. The corresponding unitary operators were examined by P. Jonas [15]. However, the spectral resolution alone is insufficient for justification of eigenfunction expansion. Our version of the theory under conditions (1.3), (1.4) allows us to justify the eigenfunction expansion (1.5).

The spectral theory of definitizable operators was applied to the Klein–Gordon equations with non-positive energy by P. Jonas, H. Langer, B. Najman and C. Tretter [17, 18, 31, 32, 33], where the existence and uniqueness of classical solutions were proved, and the existence of unstable eigenvalues (imaginary frequencies) was studied. The instability is related to the known *Klein paradox* in quantum mechanics [38].

The scattering theory for the Klein–Gordon equations with non-positive energy was developed by C. Gérard and T. Kako using the theory of definitizable operators in Krein spaces [9, 19].

The plan of our paper is as follows. In Section 2, we justify the M. Krein transformation under conditions (1.3), (1.4), and find the structure of spectrum of the corresponding selfadjoint generator. In Section 3, we construct

the spectral representation for solutions to (1.2) and deduce the canonical form of the Hamilton generator. In Section 4, we check all conditions (1.3), (1.4), (2.12), and (3.1) for operator (1.5). In Sections 5 and 6, we justify the eigenfunction expansion (1.6) by applying the methods of Sections 3–4. In Section 7, we calculate symplectic normalization of the generalized eigenfunctions. Finally, in the Appendix we construct examples of Hamilton equations satisfying all the imposed conditions.

2 Reduction to symmetric generator

In this section, we shall reduce (1.2) to an equation with selfadjoint generator.

2.1 Generalized solutions

Throughout the paper, $D(B)$ is a dense domain of the selfadjoint operator B. We set $\Lambda := B^{1/2} \geq 0$ and denote by $\mathscr{V} \subset \mathscr{X}$ the Hilbert space which is the domain of Λ endowed with the norm

$$\|X\|_{\mathscr{V}} := \|\Lambda X\|_{\mathscr{X}} + \|X\|_{\mathscr{X}}. \tag{2.1}$$

We have the continuous injections of Hilbert spaces $\mathscr{V} \subset \mathscr{X}$, and the operator

$$\Lambda : \mathscr{V} \to \mathscr{X} \tag{2.2}$$

is continuous. By definition (2.1),

$$\mathscr{K} \subset \mathscr{V}. \tag{2.3}$$

For example, $\mathscr{V}$ becomes the Sobolev space $H^1(\mathbb{R}^n)$ if $\mathscr{X} = L^2(\mathbb{R}^n)$ and $A = -i\Delta$.

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Since Λ and B are selfadjoint operators, we have

$$\mathscr{X} = \mathscr{K} \oplus \mathscr{R}, \quad \mathscr{K} := \operatorname{Ker}\Lambda = \operatorname{Ker}B, \quad \mathscr{R} := \operatorname{Ran}\Lambda = \operatorname{Ran}B = \mathscr{K}^\perp. \quad (2.4)$$

Further, we assume henceforth that $\mathscr{R}$ is endowed with the norm of $\mathscr{X}$. Then $\Lambda_+ := \Lambda|_{\mathscr{R}} : \mathscr{R} \cap \mathscr{V} \to \mathscr{R}$ is an invertible operator by (1.3); i.e.,

$$\Lambda_+^{-1} : \mathscr{R} \to \mathscr{V} \quad (2.5)$$

is the bounded operator. We will consider solutions

$$X(t) \in C(\mathbb{R}, \mathscr{V}) \quad (2.6)$$

to equation (1.2). The equation will be understood in the sense of *mild solutions* [6]

$$X(t) - X(0) = A \int_0^t X(s)ds, \qquad t \in \mathbb{R}, \quad (2.7)$$

where the Riemann integral converges in $\mathscr{V}$ by (2.6).

2.2 Krein substitution

Let us reduce equation (2.7) by the well-known substitution

$$Z(t) := \Lambda X(t) \in C(\mathbb{R}, \mathscr{R}) \quad (2.8)$$

used by M. Krein in the theory of parametric resonance: see formula (1.40) of [10, Chapter VI]. Applying Λ to both sides of equation (2.7), we obtain

$$Z(t) - Z(0) = \Lambda J \Lambda \int_0^t Z(s)ds, \qquad t \in \mathbb{R}. \quad (2.9)$$

Formally, (2.9) reads

$$i\dot{Z}(t) = HZ(t), \qquad t \in \mathbb{R}, \quad (2.10)$$

where H stands for the 'Schrödinger operator'

$$H = \Lambda i J \Lambda, \quad (2.11)$$

which is 'formally symmetric'.

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2.3 Equivalence of reduction

In order to prove the equivalence of equations (1.2) and (2.10) we introduce the following new condition.

Condition III $\qquad\qquad J\mathscr{K} \subset \mathscr{V}.$ $\qquad\qquad$ (2.12)

We denote by $\Pi_{\mathscr{K}} : \mathscr{X} \to \mathscr{K}$ the orthogonal projection, and set

$$P := \Pi_{\mathscr{K}} J\Lambda.$$

Lemma 2.1. *Let conditions* (1.4) *and* (2.12) *hold. Then the operator* $P :$ $\mathscr{X} \to \mathscr{V}$ *is continuous.*

Proof. It suffices to note that

$$\Pi_{\mathscr{K}} J\Lambda = \sum_1^N |Y_k\rangle \langle Y_k| J\Lambda = -\sum_1^N |Y_k\rangle \langle \Lambda J Y_k|, \qquad (2.13)$$

where $Y_k \in \mathscr{K} \subset \mathscr{V}$, $N = \dim \mathscr{K}$, and $\Lambda J Y_k \in \mathscr{X}$ by (2.12). $\qquad\blacksquare$

Equation (2.7) with $X(t) \in C(\mathbb{R}, \mathscr{V})$ can be written as

$$X(t) - X(0) = J\Lambda \int_0^t Z(s)ds, \qquad t \in \mathbb{R}. \qquad (2.14)$$

By Lemma 2.1 this equation implies the system

$$\left. \begin{aligned} X_{\mathscr{R}}(t) - X_{\mathscr{R}}(0) &= (1 - \Pi_{\mathscr{K}})J\Lambda \int_0^t Z(s)ds \\[2ex] X_{\mathscr{K}}(t) - X_{\mathscr{K}}(0) &= \Pi_{\mathscr{K}} J\Lambda \int_0^t Z(s)ds, \end{aligned} \right| \qquad (2.15)$$

where $X_{\mathscr{K}}(t) = \Pi_{\mathscr{K}} X(t)$ and $X_{\mathscr{R}}(t) = (1 - \Pi_{\mathscr{K}})X(t)$.

Lemma 2.2. *i) Let* $X(t) \in C(\mathbb{R}, \mathscr{V})$ *be a solution to* (1.2) *in the sense* (2.7). *Then* $Z(t) = \Lambda X(t) \in C(\mathbb{R}, \mathscr{R})$ *is the solution to* (2.10) *in the sense* (2.9).

ii) Let $Z(t) \in C(\mathbb{R}, \mathscr{R})$ *be a fixed solution to* (2.10) *in the sense* (2.9). *Then there exists a unique solution* $X(t) \in C(\mathbb{R}, \mathscr{V})$ *to* (1.2) *in the sense* (2.7) *satisfying* (2.8).

Proof. It suffices to prove ii). The uniqueness holds, because

$$X_{\mathscr{R}}(t) = \Lambda_+^{-1} Z(t), \qquad X_{\mathscr{K}}(t) - X_{\mathscr{K}}(0) = \Pi_{\mathscr{K}} J\Lambda \int_0^t Z(s)ds, \quad (2.16)$$

where the first equation follows from (2.8), and the second one, from the second equation of (2.15).

To prove the existence we define $X_{\mathscr{R}}(t)$ and $X_{\mathscr{K}}(t)$ by (2.16). Then (2.8) holds, and $X(t) = X_{\mathscr{R}}(t) + X_{\mathscr{K}}(t) \in C(\mathbb{R}, \mathscr{V})$. Hence, the first equation (2.16) together with (2.9) and (2.8) imply that

$$
\begin{aligned}
X_{\mathscr{R}}(t) - X_{\mathscr{R}}(0) &= \Lambda_+^{-1}[Z(t) - Z(0)] = \Lambda_+^{-1}\Lambda J\Lambda \int_0^t Z(s)ds \\
&= (1 - \Pi_{\mathscr{K}})J\Lambda^2 \int_0^t X(s)ds. \quad (2.17)
\end{aligned}
$$

Finally, the second equation (2.16) can be written as

$$X_{\mathscr{K}}(t) - X_{\mathscr{K}}(0) = \Pi_{\mathscr{K}} J\Lambda^2 \int_0^t X(s)ds \quad (2.18)$$

by (2.8). Summing up, we obtain (2.7). ∎

2.4 Symmetry and spectrum

The domain of H is equal to

$$D(H) = \{Z \in \mathscr{V} : J\Lambda Z \in \mathscr{V}\} = \Lambda_{\mathscr{R}}^{-1}(J\mathscr{V} \cap \mathscr{R}) + \mathscr{K}. \quad (2.19)$$

Obviously, the operator H is symmetric on $D(H)$, and hence, H is a closable operator in $\mathscr{X}$. However, we still do not know whether its domain is dense in $\mathscr{X}$. This is why we need our last condition

Condition IV $\qquad\qquad\qquad H^* = H. \qquad\qquad\qquad (2.21)$

A broad range of examples is provided by Lemma A.2. A concrete example is given by (4.11).

Theorem 2.3. *Let conditions* (1.3), (1.4), (2.12)*, and* (2.21) *hold. Then*

$$\sigma(H) \subset (-\infty, -\varepsilon] \cup 0 \cup [\varepsilon, \infty) \quad (2.22)$$

with some $\varepsilon > 0$.

Proof. Operator $\Lambda + \Pi_{\mathcal{K}} : \mathcal{V} \to \mathcal{X}$ is invertible by condition (1.3), since $(\Lambda + \Pi_{\mathcal{K}})|_{\mathcal{K}} = \Pi_{\mathcal{K}}$ and $(\Lambda + \Pi_{\mathcal{K}})|_{\mathcal{R}} = \Lambda_+$. Hence, the operator

$$H_+ := (\Lambda + \Pi_{\mathcal{K}})iJ(\Lambda + \Pi_{\mathcal{K}}) \tag{2.23}$$

is also invertible; i.e., its inverse

$$H_+^{-1} := (\Lambda + \Pi_{\mathcal{K}})^{-1}iJ(\Lambda + \Pi_{\mathcal{K}})^{-1} \tag{2.24}$$

is a bounded operator on $\mathcal{X}$. On the other hand, this operator is symmetric on $\mathcal{X}$, and hence it is selfadjoint. Moreover, H_+ is injective operator on $\mathcal{X}$. Hence, Theorem 13.11 (b) of [37] implies that H_+ is a selfadjoint operator with a dense domain $D(H_+)$. Further,

$$H_+ = H + \Pi_{\mathcal{K}}iJ(\Lambda + \Pi_{\mathcal{K}}) + (\Lambda + \Pi_{\mathcal{K}})iJ\Pi_{\mathcal{K}} + \Pi_{\mathcal{K}}iJ\Pi_{\mathcal{K}}$$

$$= H + T. \tag{2.25}$$

Here, $\Pi_{\mathcal{K}}iJ(\Lambda + \Pi_{\mathcal{K}})$ and $\Pi_{\mathcal{K}}iJ\Pi_{\mathcal{K}}$ are finite-range operators $\mathcal{V} \to \mathcal{V}$. On the other hand, (2.12) implies that $(\Lambda + \Pi_{\mathcal{K}})iJ\Pi_{\mathcal{K}}$ is also a finite-range operator from $\mathcal{V}$ to $\mathcal{V}$. Hence, $T : \mathcal{V} \to \mathcal{V}$ is the finite-range operator which is symmetric in $\mathcal{X}$. As the result, (2.25) implies that H is defined and symmetric on $D(H_+)$.

Further, the resolvent $(H_+ - \lambda)^{-1} : \mathcal{X} \to \mathcal{X}$ is bounded and analytic in a small complex neighborhood $\mathcal{O}$ of $\lambda = 0$, and

$$H - \lambda = H_+ - \lambda - T = [1 - T(H_+ - \lambda)^{-1}](H_+ - \lambda), \quad \lambda \in \mathcal{O}. \tag{2.26}$$

Here, the operator $H - \lambda$ is invertible for $\text{Im}\,\lambda \neq 0$ by (2.21), while $H_+ - \lambda$ is invertible in a small complex neighborhood $\mathcal{O}$ of $\lambda = 0$. Hence, $\text{Ker}\,[1 - T(H_+ - \lambda)^{-1}] = 0$ for $\lambda \in \mathcal{O}$ with $\text{Im}\,\lambda \neq 0$. Therefore, $1 - T(H_+ - \lambda)^{-1}$ is invertible for these λ by Fredholm's theorem, inasmuch as T is a finite-rank operator. Hence, it is also invertible in $\mathcal{O}$ outside a discrete set. Now (2.26) implies (2.22). ∎

Remark 2.4. *Let conditions (1.3), (1.4), and (2.12) hold. Then*

i) The domain of H is dense in $\mathcal{X}$, as is shown in the proof of Theorem 2.3.

ii) H admits selfadjoint extensions, because

$$N_+ = N_-, \qquad N_\pm := \dim[\text{Ran}\,(H \mp i)]^\perp. \tag{2.27}$$

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Indeed, $\mathrm{Ran}\,(H_+ - \lambda) = \mathcal{X}$ *for λ from a small complex neighborhood $\mathcal{O}$ of $\lambda = 0$. On the other hand, the dimension of $(\mathrm{Ran}\,[1 - T(H_+ - \lambda)^{-1}])^{\perp}$ is constant in $\mathcal{O}$ outside a discrete set, because T is a finite-rank operator. Therefore, (2.26) implies that $\dim[\mathrm{Ran}\,(H - \lambda)]^{\perp}$ is also constant in $\mathcal{O}$ outside a discrete set, verifying (2.27).* ∎

3 Dynamical group and canonical form

We construct spectral representation for solutions to (1.2) and deduce the canonical form of the Hamilton generator.

3.1 Spectral representation of solutions

We will construct solutions to (2.10), and afterwords, reconstruct the corresponding solutions to (1.2). The Spectral Theorem implies the following lemma.

Lemma 3.1. *Let conditions (1.3), (1.4), (2.12), (3.1) hold. Then, for any $Z(0) \in \mathcal{R}$, equation (2.10) admits a unique solution $Z(t) \in C(\mathbb{R}, \mathcal{R})$ in the sense (2.9). The solution is given by*

$$Z(t) = e^{-iHt}Z(0) \in C(\mathbb{R}, \mathcal{R}). \qquad (3.1)$$

Now we can reconstruct solutions to (1.2) using formulas (2.16):

$$X(t) = \Lambda_+^{-1}e^{-iHt}\Lambda X(0) + X_{\mathcal{K}}(0) + P\int_0^t e^{-iHs}\Lambda X(0)ds, \qquad (3.2)$$

where the operator $P : \mathcal{R} \to \mathcal{V}$ is bounded by Lemma 2.1. To evaluate the integral in (3.2), we denote by Π_0 and Π_R, respectively, the spectral projections of $\mathcal{X}$ onto $\mathrm{Ker}\,H \cap \mathcal{R}$ and $R := \mathrm{Ran}\,H \subset \mathcal{R}$. Obviously,

$$e^{-iHs} = \Pi_0 + e^{-iH_Rs}\Pi_R, \quad \int_0^t e^{-iHs}ds = t\Pi_0 + i(e^{-iH_Rt} - 1)H_R^{-1}\Pi_R, \qquad (3.3)$$

where $H_R := H|_{R \cap D(H)}$. Now (3.2) reads

$$X(t) = \Lambda_+^{-1}e^{-iHt}\Lambda X(0) + X_{\mathcal{K}}(0)$$

$$+ tP\Pi_0\Lambda X(0) + iP(e^{-iH_Rt} - 1)H_R^{-1}\Pi_R\Lambda X(0). \qquad (3.4)$$

Lemmas 3.1 and 2.2 imply the following proposition.

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Proposition 3.2. *Let conditions* (1.3), (1.4), (2.12), (3.1) *hold. Then, for any* $X(0) \in \mathscr{V}$,

i) Equation (1.2) *admits a unique solution* $X(t) \in C(\mathbb{R}, \mathscr{V})$.

ii) The solution admits the spectral representation (3.4).

3.2 Spectral resolution

Representation (3.4) can be written as

$$
\begin{aligned}
e^{At} &= \Lambda_+^{-1} \int_{\mathbb{R}} e^{-i\omega t} dE(\omega)\Lambda + \Pi_{\mathscr{K}} \\
&\quad + tP\Pi_0\Lambda + iP \int_{|\omega|\geq\varepsilon} \frac{e^{-i\omega t} - 1}{\omega} dE(\omega)\Lambda,
\end{aligned}
\tag{3.5}
$$

where $dE(\omega)$ denotes the spectral family of H, and $\varepsilon > 0$ is the number from (2.22). Formally,

$$
e^{At} = \int_{\mathbb{R}} e^{-i\omega t} dF(\omega),
\tag{3.6}
$$

where

$$
\begin{aligned}
dF(\omega) &= \left[\Lambda_+^{-1} + \frac{iP}{\omega}\right]\chi_\varepsilon(\omega)dE(\omega)\Lambda \\
&\quad + \left[\Pi_{\mathscr{K}} - iP \int_{|\omega|\geq\varepsilon} \frac{dE(\omega)}{\omega}\Lambda\right]\delta(\omega)d\omega \\
&\quad + \Lambda_+^{-1}\Pi_0\Lambda\delta(\omega)d\omega - iP\Pi_0\Lambda\delta'(\omega)d\omega,
\end{aligned}
\tag{3.7}
$$

and χ_ε is the indicator of the set $|\omega| \geq \varepsilon$. Setting $t = 0$ in both sides of (3.6) and in their derivatives, we formally obtain

$$
1 = \int_{\mathbb{R}} dF(\omega), \qquad A = -i \int_{\mathbb{R}} \omega \, dF(\omega).
\tag{3.8}
$$

3.3 Canonical form

First we will identify the eigenvectors and the associated eigenvectors of A *formally* relying on (3.8). Afterwords, we will prove the identifications rigorously.

The set $\mathscr{W} := \{X \in \mathscr{V} : \Lambda X \in D(H)\}$ is dense in $\mathscr{V}$ under our conditions (1.3) and (1.4). Let us apply the both sides of identities (3.8) to an arbitrary $X \in \mathscr{W}$. Using (3.7), we formally obtain

$$X = \int_{\mathbb{R}} dF(\omega)X = X_\varepsilon + X_0 + X_a,$$

$$AX = -i\int_{\mathbb{R}} \omega dF(\omega)X = AX_\varepsilon + AX_0 + AX_a, \qquad (3.9)$$

where

$$X_\varepsilon := \int_{|\omega|\geq\varepsilon} \left[\Lambda_+^{-1} + \frac{iP}{\omega}\right] dE(\omega)\Lambda X$$

$$AX_\varepsilon = -i\int_{|\omega|\geq\varepsilon} \left[\omega\Lambda_+^{-1} + iP\right] dE(\omega)\Lambda X \qquad (3.10)$$

$$X_0 := \left[\Pi_{\mathscr{K}} - iP\int_{|\omega|\geq\varepsilon} \frac{dE(\omega)}{\omega}\Lambda\right] X, \qquad AX_0 = 0, \qquad (3.11)$$

$$X_a := \Lambda_+^{-1}\Pi_0\Lambda X, \qquad AX_a = P\Pi_0\Lambda X. \qquad (3.12)$$

Here, (3.10) means the expansion over the eigenvectors with eigenvalues $-i\omega$, while (3.11), with the zero eigenvalue. Formula (3.12) means that X_a is the associated eigenvector to the eigenvector $P\Pi_0\Lambda X$, which corresponds to the zero eigenvalue. We justify the formal calculations (3.9)–(3.12) in the following lemma.

Lemma 3.3. *Formulas* (3.9)–(3.12) *hold for* $X \in \mathscr{W}$.

Proof. i) Formulas (3.9) and (3.11) are obvious.

ii) The last formula of (3.12) follows from the fact that $AX_a = J\Lambda\Pi_0\Lambda X = \Pi_{\mathscr{K}}J\Lambda\Pi_0\Lambda X$, since $\Lambda J\Lambda\Pi_0\Lambda X = -iH\Pi_0\Lambda X = 0$ by definition of Π_0.

iii) Finally, let us prove (3.10). The representation (3.4) implies that $\dot{X}(\cdot) \in C(\mathbb{R}, \mathscr{V})$ for $X(0) \in \mathscr{W}$ since

$$\dot{X}(t) = -i\Lambda_+^{-1}e^{-iHt}H\Lambda X(0) + P\Pi_0\Lambda X(0) + Pe^{-iH_Rt}\Pi_R\Lambda X(0) \qquad (3.13)$$

by Hille–Yosida's theorem [37, Theorem 13.35 (c)]. On the other hand, according to (2.7),

$$\frac{X(t+\Delta t) - X(t)}{\Delta t} = A\frac{\int_t^{t+\Delta t} X(s)ds}{\Delta t}. \qquad (3.14)$$

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Here the left hand side converges to $\dot{X}(t)$ in $\mathscr{V}$ as $\Delta t \to 0$, since $\dot{X}(\cdot) \in C(\mathbb{R}, \mathscr{V})$, and the quotient on the right converges to $X(t)$ in $\mathscr{V}$ by (2.6). Hence, making $\Delta t \to 0$ we obtain

$$\dot{X}(t) = AX(t), \qquad (3.15)$$

since the operator $A = JB$ is closed in $\mathscr{X}$. Setting $t = 0$ in (3.13) and (3.15), and writing X instead of $X(0)$, we obtain

$$AX = -i\Lambda_+^{-1}H\Lambda X + P\Pi_0\Lambda X + P\Pi_R\Lambda X$$

$$= -i\int_{|\omega|\geq\varepsilon}\left[\omega\Lambda_+^{-1} + iP\right]dE(\omega)\Lambda X + P\Pi_0\Lambda X, \quad X \in \mathscr{W}. \ (3.16)$$

On the other hand, $AX = AX_\varepsilon + AX_0 + AX_a = AX_\varepsilon + P\Pi_0\Lambda X$ by (3.11) and (3.12). Hence, (3.16) implies (3.10). ∎

Corollary 3.4. *The nontrivial Jordan blocks occur only for $\lambda = 0$; they are of size 2×2 (in accordance with* [29, Proposition 5.1])*, and their number is* $\dim \operatorname{Ker} H_R = \dim[\operatorname{Ker} H \cap \mathscr{R}]$ *where* $H_R := H|_{R \cap D(H)}$. *This number is finite by* (1.4).

Further, we set $\Pi_{\mathscr{R}} := 1 - \Pi_{\mathscr{K}}$ and introduce the 'Green operator'

$$G := \Lambda_+^{-1}\Pi_{\mathscr{R}} + iPH_R^{-1}\Pi_R. \qquad (3.17)$$

It is continuous from $\mathscr{X}$ to $\mathscr{V}$ by our conditions (1.3) and (2.12) according to Lemma 2.1. Therefore, formulas (3.10) can be rewritten as

$$X_\varepsilon = G\int_{|\omega|\geq\varepsilon}dE(\omega)\Lambda X, \qquad AX_\varepsilon = -iG\int_{|\omega|\geq\varepsilon}\omega dE(\omega)\Lambda X, \qquad X \in \mathscr{W},$$
$$(3.18)$$

since the both integrals converge in $\mathscr{X}$ and belong to $R \subset \mathscr{R}$.

Corollary 3.5. *Let $h_k \in \mathscr{X}$ be an eigenfunction of H corresponding to an eigenvalue $\omega_k \neq 0$. Then*

$$a_k := Gh_k \in \mathscr{V} \qquad (3.19)$$

is the eigenfunction of A corresponding to the eigenvalue $-i\omega_k$.

4 Application to eigenfunction expansion

We are going to apply our results to justify the eigenfunction expansion
(1.6) in the context of the system considered in [22]. We have used this
expansion for the calculation of the Fermi Golden Rule [22, (5.14)].

4.1 Linearization at the kink

In [22, 23] we studied the 1D relativistic Ginzburg–Landau equation

$$\ddot{\psi}(x,t) = \frac{d^2}{dx^2}\psi(x,t) + F(\psi(x,t)), \qquad x \in \mathbb{R} \tag{4.1}$$

for real solutions $\psi(x,t)$. Here, $F(\psi) = -U'(\psi)$, where $U(\psi)$ is similar to
the Ginzburg–Landau potential $U_{GL}(\psi) = (\psi^2 - 1)^2/4$, which corresponds
to the cubic equation with $F(\psi) = \psi - \psi^3$. Namely, $U(\psi)$ is a real smooth
even function satisfying the following conditions:

$$
\begin{aligned}
U(\psi) &> 0, \quad \psi \neq \pm a \\[2mm]
U(\psi) &= \tfrac{m^2}{2}(\psi \mp a)^2 + O(|\psi \mp a|^{14}), \ x \to \pm a.
\end{aligned}
\tag{4.2}
$$

where $a, m > 0$. The main goal of [22, 23] was to prove the asymptotic
stability of solitons (kinks) $\psi(x,t) = s_v(x - vt)$ that move with constant
velocity $|v| < 1$, and

$$s_v(x) \to \pm a, \qquad x \to \pm\infty. \tag{4.3}$$

Substituting $\psi(x,t) = s_v(x - vt)$ into (4.1), we obtain the corresponding
stationary equation

$$v^2 s_v''(x) = s_v''(x) + F(s_v(x)), \qquad x \in \mathbb{R}. \tag{4.4}$$

The linearization of (4.1) at the kink $s_v(x - vt)$ in the moving frame reads
as (1.2) with $X = (\psi, \dot{\psi}) \in L^2(\mathbb{R}) \otimes \mathbb{C}^2$ (for the corresponding complexifi-
cation) and with the generator [23, (4.6)]

$$A_v = \begin{pmatrix} v\frac{d}{dx} & 1 \\ \frac{d^2}{dx^2} - m^2 - V_v(x) & v\frac{d}{dx} \end{pmatrix}. \tag{4.5}$$

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Here, the potential

$$V_v(x) = -F'(s_v(x)) - m^2 \in C^\infty(\mathbb{R}). \tag{4.6}$$

The kink $s_v(x)$ is an odd monotone function in a suitable coordinate x, while $F'(\psi) = -U''(\psi)$ is an even function of ψ. Hence, the potential $V_v(x)$ is an even function of x. Moreover,

$$|V_v(x)| \le Ce^{-\kappa|x|}, \qquad x \in \mathbb{R}, \tag{4.7}$$

where $\kappa > 0$. The generator (4.5) has the form $A_v = JB_v$ with

$$B_v = \begin{pmatrix} S_v & -v\frac{d}{dx} \\ v\frac{d}{dx} & 1 \end{pmatrix}, \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.8}$$

where $S_v := -\frac{d^2}{dx^2} + m^2 + V_v(x)$. Obviously, $JB_v \ne B_vJ$. Differentiating (4.4), we obtain

$$\left[S_v + v^2 \frac{d^2}{dx^2} \right] s_v'(x) = 0. \tag{4.9}$$

4.2 Spectral conditions

Conditions (1.4), (2.12) hold for operators (4.8) on $\mathscr{X} := L^2(\mathbb{R}) \otimes \mathbb{C}^2$ by Lemma A.2. Condition (1.3) for all $|v| < 1$ follows from Lemmas A.1 and A.2 of [26]. Here, we check (1.3) in the case $v = 0$ for the completeness of the exposition. We will write A, B and S, respectively, instead of A_0, B_0 and S_0:

$$A = \begin{pmatrix} 0 & 1 \\ -S & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.10}$$

where $S := -\frac{d^2}{dx^2} + m^2 + V_0(x)$. The operators B and S are essentially self-adjoint in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ and $L^2(\mathbb{R})$, respectively, by (4.7) and Theorems X.7 and X.8 of [35]. We will consider the closures of B and S, which are both selfadjoint. In this case,

$$\Lambda := B^{1/2} = \begin{pmatrix} \sqrt{S} & 0 \\ 0 & 1 \end{pmatrix}, \qquad H := \Lambda iJ\Lambda = i \begin{pmatrix} 0 & \sqrt{S} \\ -\sqrt{S} & 0 \end{pmatrix} = iJ\sqrt{S}. \tag{4.11}$$

Hence, the operator H is also selfadjoint on the domain $D(\sqrt{S}) \oplus D(\sqrt{S})$. Thus, condition (2.21) holds in our case.

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Lemma 4.1. *Condition* (1.3) *holds for the operator B on* $\mathscr{X} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$.

Proof. Equation (4.9) with $v = 0$ means that $\lambda = 0 \in \sigma_{pp}(S)$. Moreover, $\lambda = 0$ is the minimal eigenvalue of S, since the corresponding eigenfunction $s_0'(x)$ does not vanish [23, (1.9)]. Hence,

$$\sigma(S) \subset [0,\infty), \qquad \operatorname{Ker} S = (s_0'(x)). \tag{4.12}$$

Further, the continuous spectrum of S lies in $[m^2,\infty)$, and hence (4.12) implies

$$\sigma(S) = \{\lambda_0,\dots,\lambda_N\} \cup [m^2,\infty), \tag{4.13}$$

where $0 = \lambda_0 < \dots < \lambda_N < m^2$. Finally, $\sigma(B) = \sigma(S) \cup \{1\}$, by (1.3). $\blacksquare$

We will assume below the following spectral condition (imposed in [22])) at the edge point of the continuous spectrum of S:

The point m^2 is neither an eigenvalue nor a resonance of S. (4.14)

This condition provides a regularity of the eigenvalue expansion (1.6) at the edge points $\pm m$ of the continuous spectrum.

5 Orthogonal eigenfunction expansion

We are going to apply Proposition 3.2 to the case of operators (4.10). First, (4.13) implies that

$$\sigma(H) = (-\infty, -m] \cup \{\omega_{-N},\dots,\omega_{-1},\omega_0,\omega_1,\dots,\omega_N\} \cup [m,\infty), \tag{5.1}$$

where $\omega_{\pm k}^2 = \lambda_k$ for $k = 0,\dots,N$. We denote by $\sigma_c = (-\infty, -m] \cup [m,\infty)$ the continuous spectrum of H, and

$$\Psi_0 = \Lambda_+^{-1}\Pi_0\Lambda X(0), \qquad \Phi_0 = P\Pi_0\Lambda X(0). \tag{5.2}$$

Then $A\Psi_0 = 0$ and $A\Psi_0 = \Phi_0$ by (3.12), and hence, $t\Phi_0 + \Psi_0$ is the solution to (1.2). Now formula (3.5) can be rewritten as

$$X(t) = e^{At}X(0) = t\Phi_0 + \Psi_0 + \sum_{-N}^{N} e^{-i\omega_k t}C_k a_k$$

$$+ \int_{\sigma_c}\left[\Lambda_{\mathscr{R}}^{-1} + \frac{iP}{\omega}\right]e^{-i\omega t}dE(\omega)\Lambda X(0). \tag{5.3}$$

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Here, $a_0 \in \mathcal{K}$ and

$$a_k = \left[\Lambda_{\mathcal{R}}^{-1} + \frac{iP}{\omega_k} \right] h_k = G h_k \in \mathcal{X}, \quad k \neq 0, \tag{5.4}$$

where $h_k \in \mathcal{R}$ are the eigenfunctions of H corresponding to the eigenvalues $\omega_k \neq 0$. By Corollary 3.5, a_k are the eigenfunctions of A corresponding to the eigenvalues $-i\omega_k$.

Let us denote by $X^c(t)$ the integral in (5.3):

$$X^c(t) = \int_{\sigma_c} \left[\Lambda_{\mathcal{R}}^{-1} + \frac{iP}{\omega} \right] e^{-i\omega t} dE(\omega) \Lambda X(0). \tag{5.5}$$

To prove (1.6), it remains to justify the eigenfunction expansion

$$X^c(t) = \int_{\sigma_c} e^{-i\omega t} C(\omega) a_\omega \, d\omega, \tag{5.6}$$

where a_ω are the generalized eigenfunctions of A corresponding to the eigenvalues $-i\omega$, and the meaning of the convergence of the integral will be specified later. Then (1.6) will follow from (5.3).

By (5.3), the function $X^c(t)$ is the solution to (1.2), and hence

$$Z^c(t) := \Lambda X^c(t) = \int_{\sigma_c} e^{-i\omega t} dE(\omega) \Lambda X(0) \tag{5.7}$$

is the solution to (2.10). We will deduce (5.6) from the corresponding representation

$$Z^c(t) = \int_{\sigma_c} e^{-i\omega t} C(\omega) h_\omega \, d\omega, \tag{5.8}$$

where h_ω are generalized eigenfunctions of H corresponding to the eigenvalues ω normalized by

$$\langle h_\omega, h_{\omega'} \rangle = 2\pi \, \delta(\omega - \omega'), \qquad \omega, \omega' \in \sigma_c. \tag{5.9}$$

The normalization means by definition, that

$$\left. \begin{aligned} \langle Z_1, Z_2 \rangle &= 2\pi \int_{m \leq |\omega| \leq M} C_1(\omega) \overline{C_2(\omega)} d\omega \\ \text{for } Z_k &= \int_{m \leq |\omega| \leq M} C_k(\omega) h_\omega d\omega \in \mathcal{X} \end{aligned} \right| \tag{5.10}$$

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For $\rho \in \mathbb{R}$ we denote by $L_\rho^2 = L_\rho^2(\mathbb{R})$ the weighted Hilbert space with the norm

$$\|\psi\|_{L_\rho^2}^2 := \int \langle x \rangle^{2\rho} |\psi(x)|^2 dx, \qquad \langle x \rangle := (1+x^2)^{1/2}. \qquad (5.11)$$

Theorem 5.1. *Let condition (4.14) hold and $s > 1$. Then, for $\omega \in \sigma_c$, there exists $h_\omega \in L_{-s}^2 \otimes \mathbb{C}^2$ such that:*

i) h_ω is a continuous function of $\omega \in \sigma_c$ with values in $L_{-s}^2 \otimes \mathbb{C}^2$.

ii) The normalization (5.9) holds.

iii) h_ω are the generalized eigenfunctions of H, i.e.,

$$HZ = \int_{\sigma_c} \omega C(\omega) h_\omega \, d\omega \qquad if \quad Z = \int_{\sigma_c} C(\omega) h_\omega \, d\omega \in D(H). \qquad (5.12)$$

iv) The eigenfunction expansion (5.8) holds in the following sense:

$$\left\| Z^c(t) - \int_{m \le |\omega| \le M} e^{-i\omega t} C(\omega) h_\omega \, d\omega \right\|_{L^2 \otimes \mathbb{C}^2} \to 0, \qquad M \to \infty. \qquad (5.13)$$

Proof. i) We construct the generalized eigenfunctions and the eigenfunction expansion (5.8) by solving equation (2.10) for $Z^c(t) = (Z_1^c(t), Z_2^c(t))$. By (4.11), the equation is equivalent to the system

$$\dot{Z}_1^c(t) = \sqrt{S} Z_2^c(t), \qquad \dot{Z}_2^c(t) = -\sqrt{S} Z_1^c(t). \qquad (5.14)$$

Eliminating $Z_2^c(t)$, we obtain

$$\ddot{Z}_1^c(t) = -S Z_1^c(t). \qquad (5.15)$$

Further we apply Theorem XI.41 of [36] and the arguments of [36, pp 114-115]. Namely, the rapid decay (4.7) and our spectral condition (4.14) imply the following Limiting Absorption Principle (LAP) [1, 24, 36]:

$$R(\lambda \pm i\varepsilon) \to R_\pm(\lambda), \qquad \varepsilon \to +0, \quad \lambda \in [m^2, \infty), \qquad (5.16)$$

where $R(z) := (S-z)^{-1}$ and the convergence holds in the strong topology of the space of continuous operators $L_s^2 \to L_{-s}^2$ with $s > 1$. Moreover, the traces of the resolvent $R_\pm(\lambda)$ are continuous functions of $\lambda \ge m^2$ with values in $L(L_s^2, L_{-s}^2)$. The continuity at $\lambda > 0$ has been established by Agmon, see [1, 24]. The continuity at $\lambda = 0$ under condition (4.14) is proved in

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[25, formulas (3.12)]. The LAP serves as the basis for the eigenfunction expansion

$$Z_1^c(t) = \int_{\sigma_c} d\mathscr{E}(\omega^2)[Z_1^c(0)\cos\omega t + Z_2^c(0)\sin\omega t]$$

$$= \int_{\sigma_c} e^{-i\omega t}C(\omega)\,e_\omega\,d\omega, \tag{5.17}$$

where $d\mathscr{E}(\lambda)$ is the spectral resolution of S, while $e_\omega \in L^2_{-s}$ are generalized eigenfunctions of S corresponding to the eigenvalues $\omega^2 \geq m^2$. Here the first identity follows by Spectral Theorem, while the second follows by Theorem XI.41 (e) of [36]. The eigenfunctions are defined by formulas of [36, pp 114-115]:

$$e_\omega = W^*(\omega)f_\omega, \quad W(\omega) := [1+VR_0(\omega^2+i0)]^{-1}, \ \omega \in \sigma_c. \tag{5.18}$$

where $f_\omega(x) := \sin|\omega|x$ and $R_0(\lambda) := (-\Delta+m^2-\lambda)^{-1}$.

The operator $W(\omega)$ is a continuous function of $\omega \in \sigma_c$ with values in $L(L^2_s, L^2_s)$ by the formula

$$[1+VR_0(\lambda)]^{-1} = 1-VR(\lambda) \tag{5.19}$$

and the decay (4.7). Respectively, the adjoint operator $W^*(\omega)$ is a continuous function of $\omega \in \sigma_c$ with values in $L(L^2_{-s}, L^2_{-s})$. As the result, e_ω is a continuous function of $\omega \in \sigma_c$ with values in L^2_{-s}. The normalization of e_ω coincides with the same of the 'free' generalized eigenfunctions f_ω:

$$\langle e_\omega, e_{\omega'}\rangle = \pi\,\delta(|\omega|-|\omega'|)\,, \quad \omega, \omega' \in \sigma_c, \tag{5.20}$$

which follows from the last formula on page 115 of [36]. Finally, Theorem XI.41 (e) of [36] implies that the last integral (5.17) converges in $L^2 = L^2(\mathbb{R})$:

$$\|Z_1^c(t) - \int_{m\leq|\omega|\leq M} e^{-i\omega t}C(\omega)\,e_\omega\,d\omega\|_{L^2} \to 0\,, \qquad M \to \infty. \tag{5.21}$$

Now (5.8) for $Z_1^c(t)$ follows from (5.17). For $Z_2^c(t)$ we use the first equation of (5.14), which implies

$$Z_2^c(t) = -i\int_{\sigma_c} \text{sgn}\,\omega\ e^{-i\omega t}C(\omega)\,e_\omega\,d\omega. \tag{5.22}$$

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Combining (5.17) and (5.22), we obtain (5.8) with

$$h_\omega := \begin{pmatrix} 1 \\ -i \ \text{sgn}\,\omega \end{pmatrix} e_\omega, \qquad (5.23)$$

which is the continuous function of $\omega \in \sigma_c$ with values in $L^2_{-s} \otimes \mathbb{C}^2$.

ii) Normalization (5.9) follows from (5.20).

iii) $Z^c(t) \in D(H)$ means that $Z^c_{1,2}(t) \in D(\sqrt{S})$. Furthermore,

$$HZ^c(t) = i\sqrt{S} \begin{pmatrix} Z^c_2(t) \\ -Z^c_1(t) \end{pmatrix}. \qquad (5.24)$$

Now (5.12) follows from the expansions (5.17) and (5.22) for $Z^c_{1,2}(t)$ by [36, Theorem XI.41 (c)], since e_ω are the generalized eigenfunctions of S with the eigenvalues ω^2, and *formally*,

$$i\sqrt{S} \begin{pmatrix} -i \ \text{sgn}\,\omega \\ -1 \end{pmatrix} e_\omega = \begin{pmatrix} \text{sgn}\,\omega \\ -i \end{pmatrix} |\omega| e_\omega = \omega h_\omega . \qquad (5.25)$$

iv) (5.13) follows from (5.21) and similar convergence for Z^c_2. ∎

6 Nonorthogonal eigenfunction expansion

Let us denote by $Z^c_M(t,x)$ the integral in (5.13). This integral is defined for almost all x; i.e.,

$$Z^c_M(t,x) := \int_{m \le |\omega| \le M} e^{-i\omega t} C(\omega) h_\omega(x) \, d\omega, \qquad \text{a.a.} \quad x \in \mathbb{R}. \qquad (6.1)$$

To justify (5.6) we should adjust the meaning of this integral relying on the following lemma, which is proved in [26].

Lemma 6.1 ([26, Lemma 5.1]). *Let condition* (4.14) *hold. Then*

i) The integral (6.1) *converges absolutely in* $L^2_{-s} \otimes \mathbb{C}^2$ *for every $s > 1$:*

$$\int_{m \le |\omega| \le M} \|C(\omega) h_\omega\|_{L^2_{-s} \otimes \mathbb{C}^2} \, d\omega < \infty, \qquad M > m. \qquad (6.2)$$

ii) The integral of these $L^2_{-s} \otimes \mathbb{C}^2$-valued functions over $m \le |\omega| \le M$ coincides with (6.1) *almost everywhere.*

Further, we express $X^c(t)$ in terms of $Z^c(t)$ and the Green operator (3.17), and prove the appropriate continuity of G, which allows us to deduce (5.6) from (5.13).

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6.1 Reconstruction via the Green operator

Similarly to (3.18), we use (5.7) to rewrite integral (5.5) as

$$X^c(t) = G \int_{\sigma_c} e^{-i\omega t} dE(\omega) \Lambda X(0) = GZ^c(t), \qquad (6.3)$$

taking into account that $Z^c(t) \in R \subset \mathscr{R}$ and that the Green operator $G : \mathscr{X} \to \mathscr{V}$ is continuous. Now (5.8) implies that

$$X^c(t) = G \int_{\sigma_c} e^{-i\omega t} C(\omega) h_\omega d\omega. \qquad (6.4)$$

Similarly to (3.18),

$$AX^c(t) = -iG \int_{\sigma_c} e^{-i\omega t} \omega \, dE(\omega) \Lambda X(0)$$

$$= -iGH \int_{\sigma_c} e^{-i\omega t} dE(\omega) \Lambda X(0) = -iGHZ^c(t), \quad X(0) \in \mathscr{W}. \quad (6.5)$$

Therefore, (5.12) gives

$$AX^c(t) = -iG \int_{\sigma_c} e^{-i\omega t} \omega \, C(\omega) h_\omega d\omega. \qquad (6.6)$$

6.2 Continuity of the Green operator

Now we are going to establish the continuity of the Green operator G in the weighted norms (5.11). We will simplify the form of G in the concrete case (4.11) by proving that

$$PH_R^{-1} = \Pi_{\mathscr{K}} J \Lambda H_R^{-1} = 0. \qquad (6.7)$$

First we note that

$$\operatorname{Ker} H = \operatorname{Ker} S \oplus \operatorname{Ker} S, \quad R = \operatorname{Ran} H = \operatorname{Ran} S \oplus \operatorname{Ran} S$$

by (4.11). Further, we set

$$S_+ := S|_{\operatorname{Ran} S} : \operatorname{Ran} S \cap D(S) \to \operatorname{Ran} S,$$

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and let P_0 denote the orthogonal projection of $L^2(\mathbb{R})$ onto $\operatorname{Ker} S$. Then $P_+ := 1 - P_0$ is the orthogonal projection of $L^2(\mathbb{R})$ onto $\operatorname{Ran} S$, and now (4.11) implies

$$\Pi_{\mathscr{K}} = \begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi_{\mathscr{R}} = \begin{pmatrix} P_+ & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Pi_0 = \begin{pmatrix} P_0 & 0 \\ 0 & P_0 \end{pmatrix}, \quad \Pi_R = \begin{pmatrix} P_+ & 0 \\ 0 & P_+ \end{pmatrix}.$$

Hence, finally,

$$H_R^{-1} = i \begin{pmatrix} 0 & S_+^{-1/2} \\ -S_+^{-1/2} & 0 \end{pmatrix} \tag{6.8}$$

by (4.11), and therefore,

$$
\begin{aligned}
J\Lambda H_R^{-1} &= i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} S_+^{1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & S_+^{-1/2} \\ -S_+^{-1/2} & 0 \end{pmatrix} \\
&= -i \begin{pmatrix} S_+^{-1/2} & 0 \\ 0 & P_+ \end{pmatrix}.
\end{aligned} \tag{6.9}
$$

Applying $\Pi_{\mathscr{K}}$, we get (6.7).

Now definition (3.17) implies that

$$G = \Lambda_+^{-1}\Pi_{\mathscr{R}}. \tag{6.10}$$

The following lemma is a generalization of [26, Lemma 5.2].

Lemma 6.2. *The operator* $G : L_\rho^2 \otimes \mathbb{C}^2 \to L_\rho^2 \otimes \mathbb{C}^2$ *is continuous for every* $\rho \in \mathbb{R}$.

Proof. Using the first formula of (4.11) and the formula for $\Pi_{\mathscr{R}}$, we get

$$\Lambda_{\mathscr{R}}^{-1}\Pi_{\mathscr{R}} = \begin{pmatrix} S_+^{-1/2}P_+ & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} QP_+ & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.11}$$

where $Q := (SP_+ + P_0)^{-1/2}$. Hence, it suffices to prove the continuity of the operator Q_+P_+ in L_ρ^2, which means the continuity of operator

$$\langle x \rangle^\rho Q P_+ \langle x \rangle^{-\rho} : \quad L^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{6.12}$$

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To prove this continuity, we note that Q is a PDO of the class $HG_1^{-1,-1}$, see Definition 25.2 in [40]. This fact follows from [12, Theorem 29.1.9] and also by an extension of [40, Theorem 11.2] to PDOs with nonempty continuous spectrum. It is important that operator Q is a PDO with the main symbol ξ^2, and

$$\xi^2 \notin (-\infty, 0], \quad \xi \neq 0; \qquad \sigma(S_+) \cap (-\infty, 0] = \emptyset$$

by (4.12). Hence, conditions (10.1) and (10.2) of [40] hold.

Now the continuity (6.12) follows by the Theorem of Composition of the PDO. ∎

Lemma 6.2 with $\rho = -s$ and Lemma 5.1 imply (cf. (5.4)) that

$$a_\omega := Gh_\omega \in L^2_{-s} \otimes \mathbb{C}^2, \qquad s > 1. \tag{6.13}$$

Now we can prove the following lemma.

Lemma 6.3. a_ω *are the generalized eigenfunctions of A corresponding to the eigenvalues $-i\omega$.*

Proof. Formulas (6.4) and (6.6) imply that

$$X^c(t) = \int_{\sigma_c} e^{-i\omega t} C(\omega) a_\omega d\omega, \quad AX^c(t) = \int_{\sigma_c} e^{-i\omega t} \omega C(\omega) a_\omega d\omega, \tag{6.14}$$

for $X^c(0) \in \mathcal{W}$ by definition (6.13), Lemma 6.1 and the last corollary with $\rho = -s < -1$. These identities mean that a_ω are the generalized eigenfunctions in the sense of [36, (80b)]. ∎

Finally, the main result of our paper is the following.

Theorem 6.4. *Let condition (4.14) hold, $X(0) \in \mathcal{V}$ and $s > 1$. Then the eigenfunction expansion (5.6) holds in the following sense (cf. (5.13)):*

$$\left\| X^c(t) - \int_{m \leq |\omega| \leq M} e^{-i\omega t} C(\omega) a_\omega \, d\omega \right\|_{\mathcal{V}} \to 0, \qquad M \to \infty, \tag{6.15}$$

where the integral converges in $L^2_{-s} \otimes \mathbb{C}^2$, and hence a.e. as in (6.1).

Proof. Formulas (6.3) and (6.13) imply that

$$X^c(t) - \int_{m \leq |\omega| \leq M} e^{-i\omega t} C(\omega) a_\omega \, d\omega$$

$$= G \left[Z^c(t) - \int_{m \leq |\omega| \leq M} e^{-i\omega t} C(\omega) h_\omega \, d\omega \right]. \tag{6.16}$$

Therefore, (6.15) follows from (5.13), because the Green operator $G : \mathcal{X} \to \mathcal{V}$ is continuous. ∎

7 Symplectic normalization

Now let us renormalize h_ω as follows:

$$\langle h_\omega, h_{\omega'} \rangle = |\omega| \delta(\omega - \omega'), \qquad \omega, \omega' \in \sigma_c. \tag{7.1}$$

This means that

$$
\begin{aligned}
\langle Z_1, Z_2 \rangle &= \int_{m \leq |\omega| \leq M} |\omega| C_1(\omega) \overline{C_2(\omega)} d\omega \\[2mm]
\text{for } Z_k &= \int_{m \leq |\omega| \leq M} C_k(\omega) h_\omega d\omega \in \mathscr{X}
\end{aligned}
\right\} \tag{7.2}
$$

similarly to (5.10). We will express these formulas in terms of $X_k := GZ_k \in \mathscr{V}$ and the eigenfunctions $a_\omega := Gh_\omega$. First,

$$X_k = \int_{m \leq |\omega| \leq M} C_k(\omega) a_\omega d\omega \tag{7.3}$$

by Lemma 6.2. Further, $Z_k \in R$, and so (5.12), (7.2) imply that

$$\langle H_R^{-1} Z_1, Z_2 \rangle = \int_{m \leq |\omega| \leq M} \operatorname{sgn} \omega \, C_1(\omega) \overline{C_2(\omega)} d\omega. \tag{7.4}$$

On the other hand, this scalar product can be expressed in X_k.

Lemma 7.1. *Let Z_1, Z_2 be defined as in (7.2). Then*

$$\langle H_R^{-1} Z_1, Z_2 \rangle = -i \langle X_1, JX_2 \rangle. \tag{7.5}$$

Proof. First, $Z_1, Z_2 \in R \subset \mathscr{R}$, and hence,

$$\Pi_{\mathscr{R}} Z_k = Z_k.$$

Now (6.10) implies (7.5):

$$
\begin{aligned}
\langle X_1, JX_2 \rangle &= \langle GZ_1, JGZ_2 \rangle = \langle \Lambda_{\mathscr{R}}^{-1} \Pi_{\mathscr{R}} Z_1, J\Lambda_{\mathscr{R}}^{-1} \Pi_{\mathscr{R}} Z_2 \rangle \\[2mm]
&= -\langle \Lambda_{\mathscr{R}}^{-1} \Pi_{\mathscr{R}} J \Lambda_{\mathscr{R}}^{-1} Z_1, Z_2 \rangle = i \langle H_R^{-1} Z_1, Z_2 \rangle,
\end{aligned}
$$

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since

$$\Lambda_{\mathscr{R}}^{-1}\Pi_{\mathscr{R}}J\Lambda_{\mathscr{R}}^{-1} = \begin{pmatrix} S_+^{-1/2}P_+ & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} S_+^{-1/2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & S_+^{-1/2} \\ -S_+^{-1/2} & 0 \end{pmatrix} = -iH_R^{-1}$$

by the first formula of (4.11) and by (6.11) and (6.8). ∎

Using this lemma and (7.4), we get

$$-i\langle X_1, JX_2 \rangle = \int_{m \le |\omega| \le M} \operatorname{sgn}\omega\, C_1(\omega)\overline{C_2(\omega)}d\omega. \qquad (7.6)$$

By definition, (7.3) and (7.6) mean that

$$\langle a_\omega, Ja_{\omega'} \rangle = i\operatorname{sgn}\omega\, \delta(\omega - \omega'), \qquad \omega, \omega' \in \sigma_c. \qquad (7.7)$$

Now expansion (5.6) coincides with [5, (2.1.13)], thereby justifying our calculation of the Fermi Golden Rule for all solutions without the anti-symmetry condition imposed in [22].

A Examples

Let us show that conditions (1.4), (2.12) and (3.1) hold for elliptic PDO

$$P\psi(x) = \int e^{-ix\xi}P(x,\xi)\hat{\psi}(\xi)d\xi, \qquad (A.1)$$

which are the main objects of the theory. We will use the classes $\mathscr{S}^m$ of PDO similar to the ones introduced in [11].

Definition A.1. *i)* $P \in \mathscr{S}^m$ *if, for any multiindices* α, β,

$$\sup_{x \in \mathbb{R}^n} |(1+|x|)^N \partial_\xi^\alpha \partial_x^\beta P(x,\xi)| \le C_{\alpha\beta N}(1+|\xi|)^{m-|\alpha|}, \qquad \xi \in \mathbb{R}^n \quad (A.2)$$

with $N = 0$ *for* $\beta = 0$ *and any* $N > 0$ *for* $\beta \ne 0$.

ii) $P \in \mathscr{S}_0^m$ *if (A.2) holds for any multiindices* α, β *and all* $N > 0$.

iii) $P \in \mathscr{S}^m$ *is elliptic of order m if* $P = P_m + R$, *where* $P_m \in \mathscr{S}^m$ *and*

$$|P_m(x,\xi)| \ge C(1+|\xi|)^m, \qquad x, \xi \in \mathbb{R}^n, \qquad (A.3)$$

while $R \in \mathscr{S}_0^\mu$ *with* $\mu < m$.

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Let $\mathscr{H}^s = \mathscr{H}^s(\mathbb{R}^n)$ denote the Sobolev spaces, and $\mathscr{X} = L^2(\mathbb{R}^n)$. Any operator $P \in \mathscr{S}^m$ is continuous $H^s \to H^{s-m}$ for $s \in \mathbb{R}$, see Theorem 3.1 of [11].

Lemma A.2. *Let $B \in \mathscr{S}^m$ be an elliptic PDO of order m which is symmetric on $C_0^\infty(\mathbb{R}^n)$, and let $J \in \mathscr{S}^0$ be an elliptic PDO of order 0 which is antisymmetric on $C_0^\infty(\mathbb{R}^n)$. Then*

i) B (respectively, J) is selfadjoint (respectively, skew selfadjoint) operator with domain

$$D(B) = \mathscr{H}^m, \qquad D(J) = \mathscr{X}. \tag{A.4}$$

ii) Condition (1.4) holds.

iii) Condition (2.12) holds.

iv) Condition (3.1) holds.

Proof. i) The Fredholm theory of elliptic PDO on $\mathbb{R}^n$ [40, Section 25.4] implies that $B\psi \in \mathscr{X}$ if and only if $\psi \in \mathscr{H}^m$, and the same is true for B^*. Hence, $D(B^*) = D(B)$, and therefore, $B^* = B$. Similarly, $J^* = -J$.

ii) The Fredholm theory of elliptic PDOs on $\mathbb{R}^n$ implies that the space $\mathscr{K} := \operatorname{Ker} B$ is finite dimensional and $\mathscr{K} \subset \mathscr{H}^s$ for any $s \in \mathbb{R}$. Hence, (1.4) holds.

iii) The operator $B_+ := B + \Pi_{\mathscr{K}}$ and its main symbol $B_+^m(x,\xi)$ satisfy

$$B_+^m(x,\xi) \notin (-\infty, 0], \quad \xi \neq 0; \qquad \sigma(B_+) \cap (-\infty, 0] = \emptyset$$

by (1.3). Therefore, conditions (10.1) and (10.2) of [40] hold for B_+, and hence, $\Lambda_+ := \sqrt{B_+} \geq 0$ is also an elliptic PDO of class $\mathscr{S}^{m/2}$. This follows similarly to Theorem 29.1.9 of [12] and also by an extension of Theorem 11.2 of [40] to PDO with nonempty continuous spectrum. Finally, $\Lambda_+ = \Lambda + \Pi_{\mathscr{K}}$. Therefore, $\mathscr{V} = \mathscr{H}^{m/2}$, and hence (2.12) holds, inasmuch as $J \in \mathscr{S}^0$.

iv) The operator H is elliptic PDO of class $\mathscr{S}^m$ by the theorem of composition. It is obviously symmetric on $C_0^\infty(\mathbb{R}^n)$, and hence H is selfadjoint on the domain $\mathscr{H}^m$ by the argument above. Thus (3.1) is established. ∎

Remark A.3. *i) An example of elliptic operators $B \in \mathscr{S}^2$ and $J \in \mathscr{S}^0$ satisfying all conditions (1.1), (1.3), (1.4), (2.12), (3.1) is provided in Lemma 4.1.*

ii) In the framework of Lemma A.2 *we should take m* ≥ 0 *to keep condition* (1.3).

iii) The last condition of (1.1) *implies that the order of J should be zero.*

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