ON DISPERSION DECAY FOR DISCRETE WAVE EQUATIONS

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ABSTRACT. We derive dispersion estimates for solutions of the one-dimensional discrete wave equations. In particular, we weaken the conditions on the potentials of previous works.

1. INTRODUCTION

We are concerned with the one-dimensional discrete wave equation

$$\ddot{u}(t) = -\mathrm{H}u, \quad \mathrm{H} := -\Delta_L + q, \quad t \in \mathbb{R} \tag{1.1}$$

with a real potential q. Here Δ_L is the discrete Laplacian given by

$$(\Delta_L u)_n = u_{n+1} - 2u_n + u_{n-1}, \quad n \in \mathbb{Z}.$$

In matrix form (1.1) reads

$$\mathrm{i}\dot{\mathbf{u}}(t) = \mathbf{H}\mathbf{u}(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where

$$\mathbf{u}_n(t) = \big(u_n(t), \dot{u}_n(t)\big), \quad \mathbf{H} = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{iH} & 0 \end{pmatrix}$$

We suppose that the potential q satisfies

$$|q_n| \le C(1 + |n|)^{-\beta}, \quad n \in \mathbb{Z} \tag{1.3}$$

with some $\beta > 3$. We will use the weighted spaces $l^2_\sigma = l^2_\sigma(\mathbb{Z})$ with the norm

$$\|u\|_{l^2_\sigma} = \|(1 + |n|)^\sigma u\|_{l^2}, \quad \sigma \in \mathbb{R}.$$

Denote

$$B(\sigma, \sigma') = \mathcal{L}(l^2_\sigma, l^2_{\sigma'}), \quad \mathbf{B}(\sigma, \sigma') = \mathcal{L}(l^2_\sigma \oplus l^2_\sigma, l^2_{\sigma'} \oplus l^2_{\sigma'})$$

the spaces of bounded linear operators from l^2_σ to $l^2_{\sigma'}$ and from $l^2_\sigma \oplus l^2_\sigma$ to $l^2_{\sigma'} \oplus l^2_{\sigma'}$, respectively. We restrict ourselves to the non-singular case, when the boundary points $\lambda = 0, 4$ of the spectrum are not resonances for the operator $\mathrm{H} = -\Delta_L + q$.

Our main results are as follows. In the non-singular case the following asymptotics hold

$$e^{-it\mathbf{H}} P_c = \mathcal{O}(t^{-3/2}), \quad t \to \infty \tag{1.4}$$

in $\mathbf{B}(\sigma, -\sigma)$ with $\sigma > 5/2$. Here P_c is the Riesz projection in $l^2 \oplus l^2$ onto the (absolutely) continuous spectrum of $\mathbf{H}$.

In this respect we recall that under the condition (1.3) it is well-known that the spectrum of H consists of a purely absolutely continuous part covering $[0, 4]$

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plus a finite number of eigenvalues located in $\mathbb{R}\setminus[0,4]$. In addition, there could be resonances at the boundary of the continuous spectrum.

The dispersion decay of type (1.4) has been obtained for the first time in [6] for discrete Schrödinger, wave and Klein–Gordon equations with compactly supported potentials (the discrete Klein–Gordon equation corresponds to $\mathrm{H}=-\Delta_L+m^2+q$ with $m>0$ in (1.1)). The result has been generalized in [8] to discrete Schrödinger equation with non-compactly supported potentials under the decay condition (1.3) with $\beta>5$. Recently in [2] the dispersion decay was obtained under condition $\sum_{\mathbb{Z}}|n|^2|q_n|<\infty$ for discrete Schrödinger and Klein–Gordon equations and under condition

$$\sum_{n\in\mathbb{Z}}|n|^3|q_n|<\infty \tag{1.5}$$

for discrete wave equation. The result of [2] is based on generalization of the van der Corput lemma together with the novel fact that the scattering data associated with H are in the Wiener algebra.

Here we improve the result [2] for the wave equation by reducing the decay rate (1.5) to (1.3) with $\beta=3$. We adapt to the discrete casethe approach of [7], which relies on the Puiseux expansions of the resolvent at the edge points of the continuous spectrum.

2. Free equation

Here we consider the free equation (1.2) with $q=0$:

$$\mathrm{i}\dot{\mathbf{u}}(t)=\mathbf{H}_0\mathbf{u}(t),\quad t\in\mathbb{R}, \tag{2.1}$$

where

$$\mathbf{H}_0=\begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i}\mathrm{H}_0 & 0\end{pmatrix},\quad \mathrm{H}_0=-\Delta_L.$$

It is well-known that H_0 is self-adjoint and the discrete Fourier transform

$$\hat{u}(\theta)=\sum_{n\in\mathbb{Z}}u_n\mathrm{e}^{\mathrm{i}\theta n},\quad \theta\in\mathbb{T}:=\mathbb{R}/2\pi\mathbb{Z}.$$

maps H_0 to the operator of multiplication by $\phi(\theta)=2-2\cos\theta$:

$$-\widehat{\Delta_L u}(\theta)=\phi(\theta)\widehat{u}(\theta).$$

In particular, the spectrum $\mathrm{Spec}(\mathrm{H}_0)=[0,4]$ is purely absolutely continuous.

We will use the notation $[K]_{n,k}$ for the kernel of an operator K, that is,

$$(Ku)_n=\sum_{k\in\mathbb{Z}}[K]_{n,k}u_k,\quad n\in\mathbb{Z},$$

The kernel of the resolvent $\mathrm{R}_0(\omega)=(\mathrm{H}_0-\omega)^{-1}$ is given by

$$[\mathrm{R}_0(\omega)]_{n,k}=\frac{1}{2\pi}\int\limits_{\mathbb{T}}\frac{\mathrm{e}^{-\mathrm{i}\theta(n-k)}}{\phi(\theta)-\omega}d\theta=\frac{\mathrm{e}^{-\mathrm{i}\theta(\omega)|n-k|}}{2\mathrm{i}\sin\theta(\omega)},\quad \omega\in\Xi:=\mathbb{C}\setminus[0,4], \tag{2.2}$$

$n,k\in\mathbb{Z}$. Here $\theta(\omega)$ is the unique solution of the equation

$$2-2\cos\theta=\omega,\quad \theta\in\Sigma:=\{-\pi\le\operatorname{Re}\theta\le\pi,\ \operatorname{Im}\theta<0\}/2\pi\mathbb{Z}. \tag{2.3}$$

Observe that $\theta\mapsto\omega=2-2\cos\omega$ is a biholomorphic map from $\Sigma\to\Xi$.

Next we collect some properties obtained in [6].

Lemma 2.1. *For* $\mathrm{R}_0(\omega)$ *the following properties hold:*
P1 *The resolvent* $\mathrm{R}_0(\omega)$ *is an analytic function with values in* $B(0,0)$ *for* $\omega\in\Xi$.
P2 *For* $\omega\in(0,4)$ *the limiting absorption principle holds, which is the convergence*

$$\mathrm{R}_0(\omega\pm\mathrm{i}\varepsilon)\to\mathrm{R}_0(\omega\pm\mathrm{i}0),\quad \varepsilon\to 0+ \tag{2.4}$$

in $B(\sigma,-\sigma)$ *with* $\sigma>1/2$.
P3 *At the edge points* $\mu_-=0$ *and* $\mu_+=4$ *the following asymptotics hold*

$$\mathrm{R}_0(\omega)=A_\pm(\omega-\mu_\pm)^{-1/2}+B_\pm+\mathcal{O}(|\omega-\mu_\pm|^{1/2}),\quad \omega\to\mu_\pm,\quad \omega\in\Xi \tag{2.5}$$

in $B(\sigma,\,-\sigma)$ *with* $\sigma>5/2$. *Here* $A_\pm$, $B_\pm$ *are the operators associated with the kernels*

$$[A_\pm]_{n,k}=\frac{\mathrm{i}}{2}(\mp1)^{n-k+1},\quad [B_\pm]_{n,k}=-\frac{1}{2}\,|n-k|(\mp1)^{n-k+1}, \tag{2.6}$$

respectively.
P4 *The asymptotics* (2.5) *can be differentiated twice with respect to* ω:

$$\begin{aligned}\mathrm{R}_0'(\omega)&=-\tfrac{1}{2}A_\pm\,(\omega-\mu_\pm)^{-3/2}+\mathcal{O}(|\omega-\mu_\pm|^{-1/2}),\\ \mathrm{R}_0''(\omega)&=\tfrac{3}{4}A_\pm\,(\omega-\mu_\pm)^{-5/2}+\mathcal{O}(|\omega-\mu_\pm|^{-3/2}),\end{aligned}\qquad \omega\to\mu_\pm,\quad\omega\in\Xi, \tag{2.7}$$

in $B(\sigma,\,-\sigma)$ *with* $\sigma>5/2$.

Now we turn to the free wave equation. The resolvent $\mathbf{R}_0(\lambda)=(\mathbf{H}_0-\lambda)^{-1}$ can be expressed in terms of R_0 (see [6]):

$$\mathbf{R}_0(\lambda)=\begin{pmatrix}\lambda\mathrm{R}_0(\lambda^2) & \mathrm{i}\mathrm{R}_0(\lambda^2)\\ -\mathrm{i}(1+\lambda^2\mathrm{R}_0(\lambda^2)) & \lambda\mathrm{R}_0(\lambda^2)\end{pmatrix},\quad \lambda\in\mathbb{C}\setminus[-2,2]. \tag{2.8}$$

Then properties **P1**–**P4** imply the corresponding properties of $\mathbf{R}_0$. In particular,

$$[\mathbf{R}_0]^{12}(\lambda)=iA_-\lambda^{-1}+iB_-+\mathcal{O}(\lambda),\quad \lambda\to0,\quad \lambda\in\mathbb{C}\setminus[-2,2]. \tag{2.9}$$

where $[\cdot]^{ij}$ denotes the ij entry of the corresponding matrix operator.

The continuous spectrum of $\mathbf{H}_0$ coincides with $[-2,2]$. For the kernel of the free propagator the following spectral representation holds

$$[\mathrm{e}^{-\mathrm{i}t\mathbf{H}_0}]_{n,k}=\frac{1}{2\pi\mathrm{i}}\int\limits_{(-2,0)\cup(0,2)}\mathrm{e}^{-\mathrm{i}t\lambda}[\mathbf{R}_0(\lambda+\mathrm{i}0)-\mathbf{R}_0(\lambda-\mathrm{i}0)]_{n,k}\,d\lambda. \tag{2.10}$$

Due to (2.9) $[\mathbf{R}_0]^{12}(\lambda+\mathrm{i}0)-[\mathbf{R}_0]^{12}(\lambda-\mathrm{i}0)\sim\lambda^{-1}$ and then the first component $u_n(t)$ of the solution of the free wave equation (2.1) does not decay as $t\to\pm\infty$.

Remark 2.2. *(see* [2]*). Note that the first component of the solution is given by*

$$u_n(t)=\sum_{m\in\mathbb{Z}}c_{n-m}(t)u_m(0)+s_{n-m}(t)\dot u_m(0), \tag{2.11}$$

where

$$c_n(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\sqrt{1-\cos\theta}\sqrt{2}t)\mathrm{e}^{\mathrm{i}\theta n}d\theta=J_{2|n|}(2t), \tag{2.12}$$

$$\begin{aligned}s_n(t)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(\sqrt{1-\cos\theta}\sqrt{2}t)}{\sqrt{1-\cos\theta}}\mathrm{e}^{\mathrm{i}\theta n}d\theta=\int_0^t c_n(s)ds\\ &=\frac{t^{2|n|+1}}{2^{|n|}(|n|+1)!}\,{}_1F_2\Big(\frac{2|n|+1}{2};(\frac{2|n|+3}{2},2|n|+1);-t^2\Big).\end{aligned} \tag{2.13}$$

Here $J_n(x)$, ${}_pF_q(\underline{u};\underline{v};x)$ denote the Bessel and generalized hypergeometric functions, respectively. In particular, while $c_n(t)=O(t^{-1/2})$ for fixed n, we have $s_n(t)=\frac{1}{2}+O(t^{-1/2})$ for fixed n.

3. Limiting absorption principle

First we recall a few facts from scattering theory. Under the assumption $q\in\ell^1_1$ there exists Jost solutions $f^{\pm}(\theta)$ to

$$\mathrm{H}f=\omega f,\quad \omega\in\overline{\Xi}$$

normalized as

$$f_n^{\pm}(\theta)\sim \mathrm{e}^{\mp in\theta},\quad n\to\pm\infty,$$

where $\theta=\theta(\omega)\in\overline{\Sigma}$ is the solution to $2-2\cos\theta=\omega$. For $q\in\ell^1$ the Jost solutions exist outside of the edges of continuous spectrum. In this case one can show as in [1] that

$$|f_n^{\pm}(\theta)|\le C(\theta)\mathrm{e}^{\pm\,\mathrm{Im}(\theta)n},\quad \theta\in\overline{\Sigma}\setminus\{0;\pm\pi\},\quad n\in\mathbb{Z},\tag{3.1}$$

where $C(\theta)$ can be chosen uniformly in compact subsets of $\overline{\Sigma}$ avoiding the band edges. If additionally $q\in\ell^1_1$ then

$$|f_n^{\pm}(\theta)|\le C\max(1,\mp n)\mathrm{e}^{\pm\,\mathrm{Im}(\theta)n},\quad \theta\in\overline{\Sigma}.\tag{3.2}$$

Denote by $W(\theta)$ the Wronskian of Jost solutions:

$$W(\theta):=W(f^+(\theta),f^-(\theta))=f_0^+(\theta)f_1^-(\theta)-f_1^+(\theta)f_0^-(\theta)\tag{3.3}$$

Then the kernel of the resolvent $\mathrm{R}(\omega)=(\mathrm{H}-\omega)^{-1}:\ell^2\to\ell^2$ reads (cf. [9, (1.99)])

$$[\mathrm{R}(\omega)]_{n,k}=\frac{1}{W(\theta(\omega))}\left\{\begin{array}{ll} f_n^+(\theta(\omega)f_k^-(\theta(\omega)) & \text{for }\ n\ge k,\\ f_k^+(\theta(\omega))f_n^-(\theta(\omega)) & \text{for }\ n\le k.\end{array}\right.,\quad \omega\in\Xi.\tag{3.4}$$

The representation (3.4), the fact that $W(\theta)$ does not vanish for $\omega\in(0,4)$, and the bound (3.1) imply the limiting absorption principle for the perturbed one-dimensional Schrödinger equation.

Lemma 3.1. *(see* [2, Lemma 3.3]*). Let $q\in\ell^1$. Then the convergence*

$$\mathrm{R}(\omega\pm\mathrm{i}\varepsilon)\to\mathrm{R}(\omega\pm\mathrm{i}0),\quad \varepsilon\to 0+,\quad \omega\in(0,4)\tag{3.5}$$

holds in $B(\sigma,-\sigma)$ with $\sigma>1/2$.

Proof. For any $\omega\in(0,4)$ and any $n,k\in\mathbb{Z}$, there exist the pointwise limit

$$[\mathrm{R}(\omega\pm\mathrm{i}\varepsilon)]_{n,k}\to[\mathrm{R}(\omega\pm\mathrm{i}0)]_{n,k},\quad \varepsilon\to 0.$$

Moreover, the bound (3.1) implies that $|[\mathrm{R}(\omega\pm\mathrm{i}\varepsilon)]_{n,k}|\le C(\omega)$. Hence, the Hilbert–Schmidt norm of the difference $\mathrm{R}(\omega\pm\mathrm{i}\varepsilon)-\mathrm{R}(\omega\pm\mathrm{i}0)$ converges to zero in $B(\sigma,-\sigma)$ with $\sigma>1/2$ by the Lebesgue dominated convergence theorem. □

Corollary 3.2. *For any $\omega\in(0,4)$ and any fixed $\sigma>1/2$, the operators $\mathrm{R}^{\pm}(\omega):=\mathrm{R}(\omega\pm i0):\ell^2_\sigma\to\ell^2_{-\sigma}$ have integral kernels given by*

$$[\mathrm{R}^{\pm}(\omega)]_{n,k}=\frac{1}{W(\theta_\pm)}\left\{\begin{array}{ll} f_n^+(\theta_\pm)f_k^-(\theta_\pm) & \textit{for }\ n\ge k\\ f_k^+(\theta_\pm)f_n^-(\theta_\pm) & \textit{for }\ n\le k\end{array}\right.\tag{3.6}$$

where $\theta_+(\omega)$, and $\theta_-(\omega)=-\theta_+(\omega)$ are the solution to $2-2\cos\theta=\omega$ from $[-\pi,0]$ and $[0,\pi]$, respectively.

The resolvent $\mathbf{R}(\omega) = (\mathbf{H}-\omega)^{-1}$ can be expressed in terms of $\mathrm{R}(\omega) = (\mathrm{H}-\omega)^{-1}$ (see [6]):

$$\mathbf{R}(\omega) = \begin{pmatrix} \omega\mathrm{R}(\omega^2) & \mathrm{i}\mathrm{R}(\omega^2) \\ -\mathrm{i}(1+\omega^2\mathrm{R}(\omega^2)) & \omega\mathrm{R}(\omega^2) \end{pmatrix}. \tag{3.7}$$

Representation (3.7) and Lemma 3.1 imply the limiting absorption principle for the perturbed resolvent:

Lemma 3.3. *Suppose $q \in \ell^1$. Then for $\lambda \in (-2,0) \cup (0,2)$ the convergence*

$$\mathbf{R}(\lambda \pm \mathrm{i}\varepsilon) \to \mathbf{R}(\lambda \pm \mathrm{i}0), \quad \varepsilon \to 0+,$$

holds in $\mathbf{B}(\sigma,-\sigma)$ with $\sigma > 1/2$.

4. Ruiseux expansion of resolvent

Now we consider $\mathrm{R}(\omega)$ near the edge points $\mu_- = 0$ and $\mu_+ = 4$

Definition 4.1. *Any nonzero function $u \in \ell^\infty(\mathbb{Z})$ satisfying the equation or$\mathrm{H}u = \mu_- u$ ($\mathrm{H}u = \mu_+ u$) is called a resonance function, and in this case the point μ_- (or μ_+) is called a resonance.*

Lemma 4.2. *(see [2, Lemma 3.6]) Let $q \in \ell^1_1$. Then $\mu_- = 0$ (or $\mu_+ = 4$) is a resonance if and only if $W(0) = 0$ (or $W(\pi) = 0$).*

Below we assume that

$$\textbf{Spectral condition}\text{: The points } \mu_\pm \text{ are no resonances.} \tag{4.1}$$

The condition is equivalent to the boundedness of the resolvent $\mathrm{R}(\omega)$ at the edge points of the continuous spectrum (see Corollary 4.4 below). This boundedness provides the asymptotics (1.4).

Lemma 4.3. *Let $q \in \ell^1_1$ and $\sigma > \frac{3}{2}$. If $\mu_- = 0$ is no resonance then $\mathrm{R}(\omega)$ is continuous in $B(\sigma,-\sigma)$ for ω in a neighborhood of $[0,4]$ away from $\mu_+ = 4$ with $\mathrm{R}(0) \neq 0$. If $\mu_- = 0$ is a resonance then $\tilde{R}(\omega) = \sqrt{\omega}R(\omega)$ is continuous in $B(\sigma,-\sigma)$ for ω in a neighborhood of $[0,4]$ away from μ_+ with $\tilde{R}(0) \neq 0$. Similarly near $\mu_+ = 4$.*

Proof. By (3.4), if $W(0) \neq 0$ the claim follows directly from the estimate (3.2) since the kernel $(1+|n|)^{-\sigma}[\mathrm{R}(\omega)]_{n,k}(1+|k|)^{-\sigma}$ is continuous in the Hilbert–Schmidt norm by dominated convergence. Otherwise we use additionally that $W(\theta) = W_0\theta + o(\theta)$ with $W_0 \neq 0$ and the claim again follows. □

Corollary 4.4. *Let $q \in \ell^1_1$. Then condition (4.1) is equivalent to the boundedness of the families*

$$\{\mathrm{R}(\omega),\ |\omega - \mu_\pm| \le \varepsilon,\ \omega \in \Xi\} \tag{4.2}$$

in $B(\sigma,-\sigma)$ with $\sigma > 3/2$ for sufficiently small $\varepsilon > 0$.

The Born decomposition formulas

$$\mathrm{R}(\omega) = (1+\mathrm{R}_0(\omega)q)^{-1}\mathrm{R}_0(\omega), \qquad \mathrm{R}(\omega) = \mathrm{R}_0(\omega)(1+q\mathrm{R}_0(\omega))^{-1} \tag{4.3}$$

imply

$$(1+\mathrm{R}_0(\omega)q)^{-1} = 1 - \mathrm{R}(\omega)q, \qquad (1+q\mathrm{R}_0(\omega))^{-1} = 1 - q\mathrm{R}(\omega). \tag{4.4}$$

Hence, since $q \in B(\sigma,\sigma+\beta)$, we obtain from the previous lemma that for any $\sigma \in (3/2, \beta - 3/2)$ the operators $(1+\mathrm{R}_0(\omega)q)^{-1}$ and $(1+q\mathrm{R}_0(\omega))^{-1}$ are bounded in $B(-\sigma,-\sigma)$ and $B(\sigma,\sigma)$, respectively. In particular, using the following formulas for the derivatives of R (cf. [4, 5]):

$$\mathrm{R}'=(1+\mathrm{R}_0q)^{-1}\mathrm{R}_0'(1+q\mathrm{R}_0)^{-1},\ \mathrm{R}''=\Big[(1+\mathrm{R}_0q)^{-1}\mathrm{R}_0''-2\mathrm{R}'q\mathrm{R}_0'\Big](1+q\mathrm{R}_0)^{-1}. \quad (4.5)$$

for $\beta>3$ we obtain

$$\mathrm{R}'(\omega\pm\mathrm{i}\varepsilon)\to\mathrm{R}'(\omega\pm\mathrm{i}0),\quad \mathrm{R}''(\omega\pm\mathrm{i}\varepsilon)\to\mathrm{R}''(\omega\pm\mathrm{i}0),\quad \varepsilon\to0+,\quad \omega\in(0,4), \quad (4.6)$$

in $B(\sigma,-\sigma)$ with $\sigma>\frac{5}{2}$. Our next task will be to obtain asymptotics of the resolvent $\mathrm{R}(\omega)$ at the edge points $\mu_\pm$. We start with the following lemma:

Lemma 4.5. *Assume* (4.1)*, suppose* (1.3) *holds for some* $\beta>3$*, and let* $\sigma\in(3/2,\beta-3/2)$*. Then*

$$\|(1+\mathrm{R}_0(\omega)q)^{-1}\alpha^\pm\|_{\ell^2_{-\sigma}}=\mathcal{O}(|\omega-\mu_\pm|^{1/2}),\ \omega\to\mu_\pm,\ \omega\in\Xi, \quad (4.7)$$

and

$$\sum_n\alpha_n^\pm[(1+q\mathrm{R}_0(\omega))^{-1}f]_n=\mathcal{O}(|\omega-\mu_\pm|^{1/2}),\ \omega\to\mu_\pm,\ \omega\in\Xi, \quad (4.8)$$

for any $f\in\ell^2_\sigma$*, where* $\alpha_n^\pm=(\mp1)^n$.

In particular,

$$(1+\mathrm{R}_0(\omega)q)^{-1}A_\pm(1+q\mathrm{R}_0(\omega))^{-1}=\mathcal{O}(|\omega-\mu_\pm|),\ \omega\to\mu_\pm,\ \omega\in\Xi, \quad (4.9)$$

in $B(\sigma,-\sigma)$*, where* $A_\pm$ *is given in* (2.6).

Proof. The asymptotics (2.5) imply

$$\mathrm{R}(\omega)=(1+\mathrm{R}_0(\omega)q)^{-1}\mathrm{R}_0(\omega)=(1+\mathrm{R}_0(\omega)q)^{-1}[A_\pm(\omega-\mu_\pm)^{-1/2}+\mathcal{O}(1)],$$

$$\mathrm{R}(\omega)=\mathrm{R}_0(\omega)(1+q\mathrm{R}_0(\omega))^{-1}=[A_\pm(\omega-\mu_\pm)^{-1/2}+\mathcal{O}(1)](1+q\mathrm{R}_0(\omega))^{-1}.$$

and the claim follows from the continuity of $\mathrm{R}(\omega)$, $(1+\mathrm{R}_0(\omega)q)^{-1}$, and $(1+q\mathrm{R}_0(\omega))^{-1}$ in $B(-\sigma,-\sigma)$ and $B(\sigma,\sigma)$, respectively. The last claim follows since $A_\pm=\frac{1}{2\mathrm{i}}\alpha^\pm\otimes\alpha^\pm$. □

Lemma 4.6. *Suppose* (1.3) *holds for some* $\beta>3$ *and* (4.1) *holds. Then we have the following asymptotics in* $B(\sigma,-\sigma)$ *with* $\sigma>5/2$

$$\left.\begin{aligned}\mathrm{R}(\omega)&=\mathrm{R}_\pm+\mathcal{O}(|\omega-\mu_\pm|^{1/2}),\\ \mathrm{R}'(\omega)&=\mathcal{O}(|\omega-\mu_\pm|^{-1/2}),\\ \mathrm{R}''(\omega)&=\mathcal{O}(|\omega-\mu_\pm|^{-3/2}),\end{aligned}\right.\qquad \omega\to\mu_\pm,\quad\omega\in\Xi. \quad (4.10)$$

Proof. Asymptotics (2.5), (4.7)–(4.9), and formulas (4.5) imply

$$\mathrm{R}'(\omega)=\mathcal{O}(|\omega-\mu_\pm|^{-1/2}),\quad \mathrm{R}''(\omega)=\mathcal{O}(|\omega-\mu_\pm|^{-3/2}),\quad \omega\to\mu_\pm,\quad \omega\in\Xi \quad (4.11)$$

in $B(\sigma,-\sigma)$ with $\sigma>5/2$. The asymptotics (4.11) coincide with the asymptotics (4.10) for the derivatives. Asymptotics (4.10) for $\mathrm{R}(\omega)$ can be obtained by integration of asymptotics (4.10) for the first derivative. □

Then representation (3.7) and Lemma 4.6 imply

Corollary 4.7. *Let conditions* (1.3) *and* (4.1) *hold. Then the following asymptotics hold*

$$\begin{aligned} &\mathbf{R}(\lambda) = \mathbf{R}_\pm + \mathcal{O}(|\lambda \mp 2|^{1/2}), \\ &\mathbf{R}'(\lambda) = \mathcal{O}(|\lambda \mp 2|^{-1/2}), \qquad \lambda \to \pm 2, \quad \lambda \in \mathbb{C} \setminus [-2,2] \\ &\mathbf{R}''(\lambda) = \mathcal{O}(|\lambda \mp 2|^{-3/2}), \end{aligned} \tag{4.12}$$

in $\mathbf{B}(\sigma, -\sigma)$ *with* $\sigma > 5/2$.

Corollary 4.8. *The resolvent* $\mathbf{R}(\omega)$ *is analytic function of* ω *in* $\{|\omega| \le \delta, \pm \mathrm{Im}\, \omega \ge 0\}$ *for some small* $\delta > 0$.

5. Dispersion decay

Theorem 5.1. *Let conditions* (1.3) *with* $\beta > 3$ *and* (4.1) *hold. Then asymptotics* (1.4) *hold, i.e.*

$$e^{-it\mathbf{H}} P_c = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \tag{5.1}$$

in $\mathbf{B}(\sigma, -\sigma)$ *with* $\sigma > 5/2$.

Proof. For the dynamical group associated with the perturbed wave equation (1.2) the spectral representation holds (cf. [6]):

$$e^{-it\mathbf{H}} P_c = \frac{1}{2\pi i} \int\limits_{[-2,2]} e^{-it\lambda} (\mathbf{R}(\lambda + i0) - \mathbf{R}(\lambda - i0))\, d\lambda = \int\limits_{[-2,2]} e^{-it\lambda} F(\lambda) d\lambda, \tag{5.2}$$

where $F(\lambda) = \dfrac{1}{\pi} \mathrm{Im}\, \mathbf{R}(\lambda + i0)$. The asymptotic expansion of $F(\lambda)$ at the points ± 2 can be deduced from (4.12). Thus we obtain

$$\begin{aligned} &F(\lambda) = \mathcal{O}(|\lambda \mp 2|^{1/2}), \\ &F'(\lambda) = \mathcal{O}(|\lambda \mp 2|^{-1/2}), \qquad \lambda \to \pm 2, \quad \lambda \in (-2,2). \\ &F''(\lambda) = \mathcal{O}(|\lambda \mp 2|^{-3/2}), \end{aligned}$$

Hence the desired decay for large t follows from Lemma 5.2 below. □

The following lemma is a special case of [4, Lemma 10.2].

Lemma 5.2 ([4])**.** *Assume* $\mathcal{B}$ *is a Banach space,* $a > 0$, *and* $F \in C(0, a; \mathcal{B})$ *satisfies* $F(0) = F(a) = 0$, $F'' \in L^1_{loc}(0, a; \mathcal{B})$, *as well as* $F''(\lambda) = \mathcal{O}(\lambda^{-3/2})$ *and* $F''(a - \lambda) = \mathcal{O}(\lambda^{-3/2})$ *as* $\lambda \to 0+$. *Then*

$$\int\limits_0^a e^{-it\lambda} F(\lambda) d\lambda = \mathcal{O}(t^{-3/2}), \quad t \to \infty.$$

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