Limiting absorption principle for 1D discrete Dirac equation

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Abstract

We prove the limiting absorption principle for discrete 1D Dirac equation. *Keywords*: difference Laplacian, lattice, Cauchy problem, Dirac equation.
2000 Mathematics Subject Classification: 39A11, 35L10.

1 Introduction

We consider the 1D discrete version of the Dirac equation:

$$i\dot{\mathbf{w}}(t) := \mathcal{D}\mathbf{w}(t) = (\mathcal{D}_0 + q)\mathbf{w}(t), \quad \mathbf{w}_n = (u_n, v_n) \in \mathbb{C}^2, \quad n \in \mathbb{Z}. \tag{1.1}$$

The discrete Dirac self-adjoin operator D_0 is defined by

$$\mathcal{D}_0 = \begin{pmatrix} m & d \\ d^* & -m \end{pmatrix}, \quad m > 0,$$

where $(du)_n = u_n - u_{n+1}$. We suppose that the matrix potential q satisfies

$$|q_n^{ij}| \le C(1 + |n|)^{-\rho}, \quad n \in \mathbb{Z} \tag{1.2}$$

with some $\rho > 1$. We are going to use the weighted Hilbert spaces $l^2_\sigma = l^2_\sigma(\mathbb{Z})$ with the norm

$$\|u\|_{l^2_\sigma} = \|(1 + |n|)^\sigma u\|_{l^2}, \quad \sigma \in \mathbb{R}.$$

Let us denote $\mathbf{l}^2_\sigma = l^2_\sigma \oplus l^2_\sigma$, and let

$$B(\sigma, \sigma') = \mathcal{L}(l^2_\sigma, l^2_{\sigma'}), \quad \mathbf{B}(\sigma, \sigma') = \mathcal{L}(\mathbf{l}^2_\sigma, \mathbf{l}^2_{\sigma'})$$

be the spaces of bounded linear operators from l^2_σ to $l^2_{\sigma'}$ and from $\mathbf{l}^2_\sigma$ to $\mathbf{l}^2_{\sigma'}$, respectively. The continuous spectrum of operator $\mathcal{D}$ coincides with $\overline{\Gamma}$, where

$$\Gamma = (-\sqrt{m^2+4}, -m) \cup (m, \sqrt{m^2+4}).$$

Our main results are as follows. First, we prove that under condition (1.2) there are no embedded eigenvalues into the continuous spectrum of $\mathcal{D}$, developing the method of Naboko and Yakovlev [6], where the similar result was obtained for the 1D discrete Schrödinger operator.

For our second result we suppose that the matrix potential q is Hermitian. Denote by

$$\mathcal{R}(\lambda) = (\mathcal{D} - \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \overline{\Gamma}$$

[1] Research supported by the Austrian Science Fund (FWF) under Grant No. Y330, and RFBR grants.

the resolvent of operator $\mathcal{D}$. We prove the existence and continuity of the resolvent in the continuous spectrum. Namely, for $\lambda \in \Gamma$ the convergence (limiting absorption principle) holds:

$$\mathcal{R}(\lambda \pm i\varepsilon) \to \mathcal{R}(\lambda \pm i0), \quad \varepsilon \to 0+ \tag{1.3}$$

in $\mathbf{B}(\sigma, -\sigma)$ with $\sigma > 1/2$.

For continuous Schrödinger operator convergence of type (1.3) in the weighted Sobolev norms was established by Agmon [1]. The proof relies on Kato's theorem on the absence of embedded eigenvalues and on the Agmon's theorem on the decay of the eigenfunctions. For discrete Schrödinger equation the limiting absorption principle has been obtained in [2, 4, 7, 8] under different decay conditions on the potential and recently in [3] under weakest assumptions $q \in l^1$. The methods [3] rely on the representation of the resolvent via the Jost functions. Similar representation for discrete Dirac equation is possible only in the case when $q_{12} = q_{21}$ is real function. In present paper we consider general Hermitian potential and develop the approach of [1].

2 Free Dirac equation

Denote by Δ the difference Laplacian:

$$(\Delta u)_n := u_{n+1} + u_{n-1} - 2u_n.$$

The resolvent $\mathcal{R}_0(\lambda) = (\mathcal{D}_0 - \lambda)^{-1}$ of the free Dirac operator $\mathcal{D}_0$ can be expressed in terms of the resolvent $R_\Delta(\lambda) = (-\Delta - \lambda)^{-1}$ of operator $-\Delta$. Namely, since

$$(\mathcal{D}_0 - \lambda)(\mathcal{D}_0 + \lambda) = -\Delta + m^2 - \lambda^2,$$

then we obtain

$$\mathcal{R}_0(\lambda) = (\mathcal{D}_0 + \lambda) R_\Delta(\lambda^2 - m^2), \tag{2.1}$$

The kernel of the resolvent $R_\Delta(\lambda)$ reads (see [4])

$$[R_\Delta(\lambda)]_{n,m} = -i\,\frac{e^{-i\theta(\lambda)|n-m|}}{2\sin\theta(\lambda)}, \quad \lambda \in \mathbb{C} \setminus [0,4], \quad n, m \in \mathbb{Z}, \tag{2.2}$$

where $\theta(\lambda)$ is the unique solution of the equation

$$2 - 2\cos\theta = \lambda, \quad \theta \in \Sigma := \{-\pi \le \operatorname{Re}\theta \le \pi,\ \operatorname{Im}\theta < 0\}. \tag{2.3}$$

Properties of R_Δ (see [4]) imply:
i) The resolvent $\mathcal{R}_0(\lambda)$ is an analytic function with values in $\mathbf{B}(0,0)$ for $\lambda \in \mathbb{C} \setminus \overline{\Gamma}$.
ii) For $\lambda \in \Gamma$ the convergence (limiting absorption principle) holds

$$\mathcal{R}_0(\lambda \pm i\varepsilon) \to \mathcal{R}_0(\lambda \pm i0), \quad \varepsilon \to 0+ \tag{2.4}$$

in $\mathbf{B}(\sigma, -\sigma)$ with $\sigma > 1/2$.

3 Absence of embedded eigenvalues

Here we prove that there are not embedded eigenvalues into the continuous spectrum of $\mathcal{D}$, For any fixed $\lambda \in \Gamma$ consider the equation

$$\mathcal{D}\mathbf{w} = \lambda\mathbf{w}, \quad \mathbf{w}_n = (u_n, v_n), \quad n \in \mathbb{Z}. \tag{3.1}$$

Theorem 3.1. *Let condition* (1.2) *with* $\rho > 1$ *holds and let* $\mathbf{w} \in \mathbf{l}^2$ *is the solution of* (3.1). *Then* $\mathbf{w} = 0$.

Proof. We develop the method of S. Naboko and S. Yakovlev [6]. First we consider (3.1) on the positive semiaxis i.e. for $n \in \mathbb{Z}_+$. In the matrix form (3.1) reads

$$(A + P_n)\mathbf{w}_{n+1} = (B + Q_n)\mathbf{w}_n, \quad n \in \mathbb{Z}_+, \tag{3.2}$$

where we denoting

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -(m+\lambda) \end{pmatrix}, \quad P_n = \begin{pmatrix} 0 & 0 \\ q^{21}_{n+1} & q^{22}_{n+1} \end{pmatrix}, \quad B = \begin{pmatrix} \lambda - m & -1 \\ 1 & 0 \end{pmatrix}, \quad Q_n = \begin{pmatrix} q^{11}_n & q^{12}_n \\ 0 & 0 \end{pmatrix}.$$

Multiplying both sides of (3.2) by A^{-1} we obtain

$$(1 + \tilde{P}_n)\mathbf{w}_{n+1} = (\tilde{B} + \tilde{Q}_n)\mathbf{w}_n, \tag{3.3}$$

where

$$\tilde{P}_n = \begin{pmatrix} q^{21}_{n+1} & q^{22}_{n+1} \\ 0 & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} m^2 - \lambda^2 + 1 & m + \lambda \\ m - \lambda & 1 \end{pmatrix}, \quad \tilde{Q}_n = \begin{pmatrix} -(m+\lambda)q^{11}_n & -(m+\lambda)q^{12}_n \\ -q^{11}_n & -q^{12}_n \end{pmatrix}.$$

Denote $\nu = m^2 - \lambda^2 + 2$, $|\nu| < 2$. It is easy to check that the matrix $\tilde{B}$ have two eigenvalues μ and $\overline{\mu}$, where

$$\mu = \nu/2 + i\sqrt{1 - (\nu/2)^2}, \quad |\mu| = 1.$$

Then the matrix $\tilde{B}$ can be represent as

$$\tilde{B} = VDV^{-1} \tag{3.4}$$

where

$$D = \begin{pmatrix} \mu & 0 \\ 0 & \overline{\mu} \end{pmatrix}, \quad V = \begin{pmatrix} 1-\mu & 1-\overline{\mu} \\ \lambda - m & \lambda - m \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} \lambda - m & \overline{\mu} - 1 \\ m - \lambda & 1 - \mu \end{pmatrix} \frac{1}{(\overline{\mu} - \mu)(\lambda - m)}.$$

Using representation (3.4) we get

$$\tilde{B} + \tilde{Q}_n = VDV^{-1} + \tilde{Q}_n = V[D + V^{-1}\tilde{Q}_n V]V^{-1}.$$

Denote

$$\hat{Q}_n = V^{-1}\tilde{Q}_n V, \quad \tilde{\mathbf{w}}_n = V^{-1}\mathbf{w}_n$$

Then (3.3) becomes

$$\tilde{\mathbf{w}}_{n+1} = V^{-1}(1 + \tilde{P}_n)^{-1}V(D + \hat{Q}_k)\tilde{\mathbf{w}}_n. \tag{3.5}$$

Further, we represent $(1 + \tilde{P}_n)^{-1}$ as

$$(1 + \tilde{P}_n)^{-1} = \frac{1}{\delta_n}(1 + R_n), \quad R_n = \begin{pmatrix} 0 & -q^{22}_{n+1} \\ 0 & q^{21}_{n+1} \end{pmatrix}, \quad \delta_n = 1 + q^{12}_{n+1}. \tag{3.6}$$

Due to (3.5) -(3.6) the vector $\tilde{\mathbf{w}}_n$ can be express via the vector $\tilde{\mathbf{w}}_0$ as follows

$$\tilde{\mathbf{w}}_n = \prod_{k=1}^{n} \frac{1}{\delta_k}(D + S_k)\tilde{\mathbf{w}}_0, \tag{3.7}$$

where

$$S_k = \hat{Q}_k + V^{-1}R_kV(D + Q_k).$$

Note, that D is unitary matrix, while

$$\|S_k\| \le C|\tilde{q}_k| \to 0, \quad k \to \infty,$$

where $|q_k| = \max\limits_{i,j=1,2;l=k,k+1} |q_l^{ij}|$.

Lemma 3.1. *For any vector* $\mathbf{e} \in \mathbb{C}^2$ *and sufficiently large* K

$$\|(D + S_k)\mathbf{e}\|^2 \ge (1 - C|q_k|)\|\mathbf{e}\|^2, \quad k \ge K. \tag{3.8}$$

Proof. Denote $T_k = I + D^*S_k$. The unitarity of D implies

$$\|(D + S_k)\mathbf{e}\|^2 = \|T_k\mathbf{e}\|^2 = (T_k^*T_k\mathbf{e}, \mathbf{e}) \ge \lambda_k\|\mathbf{e}\|^2,$$

where λ_k is the smallest eigenvalue of positive, selfajoint operator $T_k^*T_k$ with $\mathrm{rank}(T_k^*T_k) = 2$. Further,

$$T_k^*T_k = (1 + S_k^*D)(1 + D^*S_k) = 1 + S_k^*D + D^*S_k + S_k^*S_k$$

Hence, $T_k^*T_k = 1 + \mathcal{T}_k$, where $\mathcal{T}_k$ is selfajoint and $\|\mathcal{T}_k\| \le C|q_k| < 1$ for sufficiently large k by (1.2). Then eigenvalues τ_k of $\mathcal{T}_k$ satisfy $|\tau_k| \le C|q_k|$. Therefore, for the minimal eigenvalue of $T_k^*T_k$ we obtain

$$\lambda_k = 1 + \tau_k \ge 1 - C|q_k| > 0, \quad k \ge K.$$

□

Corollary 3.1. $\mathbf{w}_k = 0$ *for sufficiently large* $k \in \mathbb{N}$.

Proof. Bound 3.8 and formula (3.7) imply

$$\|\tilde{\mathbf{w}}_n\|^2_{\mathbb{C}^2} \ge \prod_{k=K}^{n} \frac{1}{|\delta_k|}(1 - C|q_k|)\|\tilde{\mathbf{w}}_{K-1}\|^2_{\mathbb{C}^2}, \quad n \ge K.$$

Note, that

$$|\delta_k| = |1 + q_{k+1}^{12}| \le 1 + |q_k|$$

Then for $\mathbf{w} = V\tilde{\mathbf{w}}$ we obtain

$$\|\mathbf{w}\|^2_{\mathbf{l}^2} \ge C_1 \sum_{n=K}^{\infty} \prod_{k=K}^{n} \frac{1 - C|q_k|}{1 + |q_k|}\|\mathbf{w}_{K-1}\|^2_{\mathbb{C}^2} = \infty$$

if $\mathbf{w}_{K-1} \ne 0$. Hence, $\mathbf{w}_k = 0$ for $k \ge K - 1$. □

Similarly, we obtain that $\mathbf{w}_k = 0$ for $k \le -(K - 1)$. However, recurrence relation (3.2) can not have a nonzero solution with finite support. Therefore we get $\mathbf{w} \equiv 0$. Theorem 3.1 is proved. □

4 Limiting absorption principle

Here we extend (2.4) to perturbed resolvent $\mathcal{R}(\lambda) = (\mathcal{D} - \lambda)^{-1}$. We adopt general strategy from [1], and start with the following proposition

Proposition 4.1. *Let condition (1.2) with $\rho > 1$ holds, and let $\mathbf{w} \in \mathbf{l}^2_{-1/2-0}$ satisfies the equation*

$$D\mathbf{w} = \lambda\mathbf{w}, \quad \lambda \in \Gamma. \tag{4.1}$$

Moreover, let $\mathbf{w}$ satisfies

$$\mathbf{w} = \mathcal{R}_0(\lambda + i0)\mathbf{f} \quad \text{or} \quad \mathbf{w} = \mathcal{R}_0(\lambda - i0)\mathbf{f}, \tag{4.2}$$

where $\mathbf{f} \in \mathbf{l}^2_{\sigma'}$ with some $\sigma' > 1/2$. Then $\mathbf{w} \in \mathbf{l}^2_s$ for all $s \in \mathbb{R}$.

Proof. For concreteness we consider the "+" case and $\lambda \in \Gamma_+ = (m, \sqrt{m^2+4})$. Due to (2.1) we have

$$\mathbf{w} = (\mathcal{D}_0 + \lambda) R_0(\lambda^2 - m^2 + i0)\mathbf{f}.$$

In the Fourier transform, the equation reads

$$\hat{\mathbf{w}}(\theta) = \frac{(\hat{\mathcal{D}}_0 + \lambda)\hat{\mathbf{f}}(\theta)}{4\sin^2\theta/2 - \lambda^2 + m^2 - i0}, \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}, \quad \lambda^2 - m^2 \in (0,4), \tag{4.3}$$

where

$$\hat{\mathcal{D}}_0 = \begin{pmatrix} m & 1 - e^{-i\theta} \\ 1 - e^{i\theta} & -m \end{pmatrix},$$

and $\hat{\mathbf{f}}$ is a function from the Sobolev space $H^{\sigma'}$. As the first step, we prove that distribution (4.3) is not singular at two points $\theta_\pm \in T := \mathbb{R}/2\pi\mathbb{Z}$ satisfies $2|\sin\frac{\theta_\pm}{2}| = \sqrt{\lambda^2 - m^2}$.

Lemma 4.1. *The following identity holds*

$$(\hat{\mathcal{D}}_0 + \lambda)\hat{\mathbf{f}}(\theta_\pm) = 0 \tag{4.4}$$

Proof. Note that $\hat{\mathbf{f}}$ is continuous by Sobolev embedding theorem. Define

$$\hat{\mathbf{w}}_\varepsilon(\theta) = \frac{(\hat{\mathcal{D}}_0 + \lambda)\hat{\mathbf{f}}(\theta)}{4\sin^2\theta/2 - \lambda^2 + m^2 - i\varepsilon}, \quad \varepsilon > 0.$$

Then both $\hat{\mathbf{f}}, \hat{\mathbf{w}}_\varepsilon \in L^2(T)$, hence the Parseval identity implies that

$$\langle \mathbf{w}_\varepsilon, \mathbf{f}\rangle = \langle \hat{\mathbf{w}}_\varepsilon, \hat{\mathbf{f}}\rangle = \int_{-\pi}^{\pi} \frac{\langle (\hat{\mathcal{D}}_0 + \lambda)\hat{\mathbf{f}}(\theta), \hat{\mathbf{f}}(\theta)\rangle d\theta}{4\sin^2\theta/2 - \lambda^2 + m^2 - i\varepsilon}$$

$$\longrightarrow \pm i\pi \frac{\langle (\hat{\mathcal{D}}_0 + \lambda)\hat{\mathbf{f}}(\theta), \hat{\mathbf{f}}(\theta)\rangle}{2\sin\theta_\pm} + \text{P.V.} \int_{-\pi}^{\pi} \frac{\langle (\hat{\mathcal{D}}_0 + \lambda)\hat{\mathbf{f}}(\theta), \hat{\mathbf{f}}(\theta)\rangle d\theta}{2\sin\theta_\pm}, \quad \varepsilon \to 0+ \tag{4.5}$$

by the Sokhotsky-Plemelj formula since $\hat{\mathbf{f}}$ is the Hölder continuous with the Hölder exponent $\alpha \in (0, \min[\sigma' - 1/2, 1])$. On the other hand,

$$\langle \mathbf{w}_\varepsilon, \mathbf{f}\rangle = \langle \mathcal{R}_0(\lambda + i\varepsilon)\mathbf{f}, \mathbf{f}\rangle \longrightarrow \langle \mathbf{w}, \mathbf{f}\rangle = -\langle \mathbf{w}, q\mathbf{w}\rangle, \quad \varepsilon \to 0+ \tag{4.6}$$

since $\mathcal{R}_0(\lambda + i\varepsilon)\mathbf{f} \to \mathbf{w} \in \mathbf{l}^2_{-\sigma'}$ by (2.4), while $\mathbf{f} \in L^2_{\sigma'}$. The operators $\hat{\mathcal{D}}_0 + \lambda$ and q is selfajoint, hence the scalar products $\langle (\hat{\mathcal{D}}_0 + \lambda)\hat{\mathbf{f}}(\theta), \hat{\mathbf{f}}(\theta)\rangle$ and $\langle \mathbf{w}, q\mathbf{w}\rangle$ are real. Comparing (4.5) and (4.6), we conclude that $(\hat{\mathcal{D}}_0 + \lambda)\hat{\mathbf{f}}(\theta_\pm) = 0$ i.e. (4.4) is proved. □

Corollary 4.1. *Relation (4.4) and the Hölder continuity imply that*

$$\hat{\mathbf{w}}(\theta) = \frac{(\hat{\mathcal{D}}_0 + \lambda)\hat{\mathbf{f}}(\theta)}{4\sin^2\theta/2 - \lambda^2 + m^2} \in L^1(T). \tag{4.7}$$

The next lemma is a typical "problem of division".

Lemma 4.2. *Let $\hat{\mathbf{f}} \in H^s(T)$ with some $s > 1/2$ and (4.7) holds. Then*

$$\|\hat{\mathbf{w}}\|_{H^{s-1}(T)} \le C\|\hat{\mathbf{f}}\|_{H^s(T)}. \tag{4.8}$$

Proof. Take any $\varepsilon \in (0, \sqrt{\lambda^2 - m^2}/2)$, and a cutoff functions

$$\zeta_\pm(\theta) \in C_0^\infty(R), \quad \zeta_\pm(\theta) = \left\{ \begin{array}{ll} 1, & |\theta - \theta_\pm| < \varepsilon \\ 0, & |\theta - \theta_\pm| > 2\varepsilon \end{array} \right.$$

By (4.7), we have

$$\|(1 - \zeta_\pm(\theta))\hat{\mathbf{w}}(\theta)\|_{H^s(T)} = \|\frac{1 - \zeta_\pm(\theta)}{4\sin^2\theta/2 - \lambda^2 + m^2}(\hat{\mathcal{D}}_0 + \lambda)\hat{\mathbf{f}}(\theta)\|_{H^s(T)} \le C\|\hat{\mathbf{f}}\|_{H^s(T)}$$

since the function $(1 - \zeta_\pm(\theta))/(4\sin^2\theta/2 - \lambda^2 + m^2)$ is a multiplier in any Sobolev space. Hence, it remains to estimate the norm of the function $\zeta_\pm(\theta)\hat{\mathbf{w}}(\theta)$. For concreteness, consider the "+" case. We may assume that in the $\operatorname{supp}\zeta_+$, there exist the diffeomorhism $\eta = 4\sin^2\theta/2 - \lambda^2 + m^2$. Then, the problem reduces to the estimate

$$\|\varphi(\eta)\|_{H^{s-1}(T)} \le C\|\eta\varphi(\eta)\|_{H^s(T)} \tag{4.9}$$

taking into account that $\varphi(\eta) \in L^1(T)$ by (4.7). Since the function $\varphi(\eta)$ supported near zero then (4.9) is equivalent to

$$\|\varphi(\eta)\|_{H^{s-1}(\mathbb{R})} \le C\|\eta\varphi(\eta)\|_{H^s(R)} \tag{4.10}$$

This estimate follows from the Hardy inequality (see [1]). □

Now (4.8) can be rewritten as

$$\|\mathbf{w}\|_{\mathbf{l}^2_{s-1}} \le C\|\mathbf{f}\|_{\mathbf{l}^2_s}, \quad s > 1/2. \tag{4.11}$$

Therefore,$\mathbf{w} \in \mathbf{l}^2_{\sigma'-1}$ since $\mathbf{f} \in \mathbf{l}^2_{\sigma'}$ with some $\sigma' > 1/2$. Second, $\mathbf{f} = -q\mathbf{w}$, and hence (4.11) and condition (1.2) imply

$$\|\mathbf{w}\|_{\mathbf{l}^2_{s-1}} \le C\|\mathbf{w}\|_{\mathbf{l}^2_{s-\beta}} = C\|\mathbf{w}\|_{\mathbf{l}^2_{s-1-\delta}}, \quad s > 1/2,$$

where $\delta := \beta - 1 > 0$ since $\beta > 1$. Applying the last inequality to $s = \sigma' + \delta > 1/2$, we obtain that $\mathbf{w} \in \mathbf{l}^2_{s-1}$ with $s - 1 = \sigma' - 1 + \delta > \sigma' - 1$. Hence, by induction, $\mathbf{w} \in \mathbf{l}^2_{s-1}$ with any $s \in \mathbb{R}$. □

Now we prove the limiting absorption principle for perturbed resolvent.

Theorem 4.1. *Let condition (1.2) with $\rho > 1$ hold. Then for $\lambda \in \Gamma$, the convergence holds*

$$\mathcal{R}(\lambda \pm i\varepsilon) \to \mathcal{R}(\lambda \pm i0), \quad \varepsilon \to 0+ \tag{4.12}$$

in $\mathbf{B}(\sigma, -\sigma)$ with $\sigma > 1/2$.

Proof. For the proof we will use the Born splitting

$$\mathcal{R}(\lambda) = [1 + \mathcal{R}_0(\lambda)q]^{-1}\mathcal{R}_0(\lambda), \quad \lambda \in \mathbb{C} \setminus \overline{\Gamma}, \tag{4.13}$$

where the operator function $[1 + \mathcal{R}_0(\lambda)q]^{-1}$ is meromorphic in $\mathbb{C} \setminus \overline{\Gamma}$ by the Gohberg-Bleher theorem. Theorem 4.1 will follow from (2.4) and the Born splitting (4.13) if

$$[1 + \mathcal{R}_0(\lambda \pm i\varepsilon)q]^{-1} \to [1 + \mathcal{R}_0(\lambda \pm i0)q]^{-1}, \quad \varepsilon \to +0, \quad \lambda \in \Gamma$$

in $\mathbf{B}(-\sigma, -\sigma)$ with $\sigma > 1/2$. The convergence holds if and only if the both limiting operators $1 + \mathcal{R}_0(\lambda \pm i0)q : \mathbf{l}^2_{-\sigma} \to \mathbf{l}^2_{-\sigma}$ are invertible for $\lambda \in \Gamma$. The operators are invertible according to the Fredholm theorem by the following two lemmas.

Lemma 4.3. *i) Let condition* (1.2) *with* $\rho > 1$ *hold. Then for* $\lambda \in \Gamma$*, the operators*

$$\mathcal{R}_0(\lambda \pm i0)q : \mathbf{l}^2_{-\sigma} \to \mathbf{l}^2_{-\sigma}, \quad q\mathcal{R}_0(\lambda \pm i0) : \mathbf{l}^2_{\sigma} \to \mathbf{l}^2_{\sigma}$$

are compact for $\sigma \in (1/2, \beta - 1/2)$.

Proof. Choose $\sigma' \in (1/2, \min(\sigma, \beta - \sigma))$. The operator $q : \mathbf{l}^2_{-\sigma} \to \mathbf{l}^2_{\sigma'}$ is continuous by (1.2) since $\sigma + \sigma' < \rho$. Further, $\mathcal{R}_0(\lambda \pm i0) : \mathbf{l}^2_{\sigma'} \to \mathbf{l}^2_{-\sigma'}$ is continuous by (2.4) and the embedding $\mathbf{l}^2_{-\sigma'} \to \mathbf{l}^2_{-\sigma}$ is compact. Hence, the operators $\mathcal{R}_0(\lambda \pm i0)q : \mathbf{l}^2_{-\sigma} \to \mathbf{l}^2_{-\sigma}$ are compact. The compactness of $q\mathcal{R}_0(\lambda \pm i0) : \mathbf{l}^2_{\sigma} \to \mathbf{l}^2_{\sigma}$ follows by duality. □

Lemma 4.4. *Let condition* (1.2) *with* $\rho > 1$ *holds. Then for* $\lambda \in \Gamma$ *the equations*

$$[1 + \mathcal{R}_0(\lambda \pm i0)q]\mathbf{w} = 0 \tag{4.14}$$

admit only the zero solution in $\mathbf{l}^2_{-1/2-0}$.

Proof. We consider the case $\lambda + i0$ for concreteness. First, equality (4.14) implies that

$$(\mathcal{D}_0 + q - \lambda)\mathbf{w} = (\mathcal{D}_0 - \lambda)(1 + \mathcal{R}_0(\lambda + i0)q)\mathbf{w} = 0. \tag{4.15}$$

Second, from (4.14) it follows that

$$\mathbf{w} = \mathcal{R}_0(\lambda + i0)\mathbf{f}, \quad \text{where} \quad \mathbf{f} = -q\mathbf{w}. \tag{4.16}$$

If $\mathbf{w} \in \mathbf{l}^2_{-1/2-0}$ then $\mathbf{f} \in \mathbf{l}^2_{\sigma'}$ with $\sigma' := \rho - 1/2 > 1/2$ by (1.2). This fact and (4.15), (4.16) imply that $\mathbf{w} \in \mathbf{l}^2_s$ with any $s \in \mathbb{R}$ by Proposition 4.1. As a corollary, we obtain $\mathbf{w} \in \mathbf{l}^2$. It means that $\mathbf{w}$ is the eigenfunction of $\mathcal{D}$ with eigenvalue $\lambda \in \Gamma$. However, the embedded eigenvalue is forbidden by Theorem 3.1. Hence $\mathbf{w} = 0$. Lemma 4.4 is proved. □

Theorem 4.1 is also proved. □

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