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Asymptotic stability of solitons for non-linear hyperbolic equations

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Abstract. Fundamental results on asymptotic stability of solitons are surveyed, methods for proving asymptotic stability are illustrated based on the example of a non-linear relativistic wave equation with Ginzburg–Landau potential. Asymptotic stability means that a solution of the equation with initial data close to one of the solitons can be asymptotically represented for large values of the time as a sum of a (possibly different) soliton and a dispersive wave solving the corresponding linear equation. The proof techniques depend on the spectral properties of the linearized equation and may be regarded as a modern extension of the Lyapunov stability theory. Examples of non-linear equations with prescribed spectral properties of the linearized dynamics are constructed.

Bibliography: 45 titles.

Keywords: non-linear hyperbolic equations, asymptotic stability, relativistic invariance, Hamiltonian structure, symplectic projection, invariant manifold, soliton, kink, Fermi's golden rule, scattering of solitons, asymptotic state.

Contents

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1. Introduction

The theory of asymptotic stability of soliton solutions for non-relativistic non-linear equations has been considerably advanced over the past ten years. Solitons are known to be fundamentally important in the study of evolution equations, mainly because they are often easily found numerically, and also because they generally emerge in the long-time asymptotics of solutions of these equations. The first results in this direction were obtained by numerical simulation in 1965 by Zabusky and Kruskal [45] for the Korteweg-de Vries (KdV) equation. In 1967 Gardner, Greene, Kruskal, and Miura [12] found that the inverse scattering transform can be used to solve the KdV equation analytically. It was seen that any solution of that equation with rapidly decaying, sufficiently smooth initial data converges to a finite sum of soliton solutions moving to the right and a dispersive wave moving to the left. See [10] for a complete survey of these studies. These results were extended to other *integrable equations* by Its, Khruslov, Shabat, Zakharov, and others (see [11]).

The study of the asymptotic stability of soliton solutions was inspired by the problem of the stability and effective dynamics of elementary particles, because the latter may be identified with solitons of non-linear field equations. Such an identification is in the spirit of Heisenberg's theory of elementary particles in the context of non-linear hyperbolic partial differential equations [13], [14].

According to recent numerical experiments [22], solutions of general non-integrable non-linear wave equations with finite-energy initial data can be decomposed, for large values of time, into a finite number of weakly interacting solitons and a decaying dispersive wave. The present paper is devoted to methods for proving similar asymptotics with one soliton for non-integrable equations in the case where the initial data is close to the solitary manifold.

1.1. Survey of the literature. Soffer and Weinstein [38], [39], [44] (see also [34]) were the first to prove the asymptotic stability of solitons for a non-linear $U(1)$-invariant Schrödinger equation with potential, for small initial data and small coefficient of the non-linear term.

Later, Buslaev and Perelman [5], [6] established this result in the more difficult instance of a translation-invariant one-dimensional non-linear $U(1)$-invariant Schrödinger equation in the case when the dynamics linearized at the soliton has no non-zero eigenvalues. Techniques similar to those of [5], [6] were developed by Miller, Pego, and Weinstein for one-dimensional modified KdV- and regularized long-wave(RLW)-equations [30], [32].

Buslaev and Sulem [7] (see also [43]) examined the one-dimensional non-linear Schrödinger equation with a more complicated non-zero discrete spectrum of the linearized dynamics. For other dimensions, see [8], [17], [37], [42].

For a three-dimensional non-linear Klein–Gordon equation with potential the asymptotic stability of solitons was proved in [40], and for field-particle systems in [15], [16].

Cuccagna [9] proved the asymptotic stability of the wavefront for a three-dimensional relativistic wave equation. By definition, a wavefront is a solution that depends only on one spatial variable. Since it has infinite energy, such a solution is not a soliton.

The asymptotic stability of standing solitons for the Dirac equation with potential was established by Boussaid [2] in the three-dimensional case, and for the Dirac equation without potential by Boussaid and Cuccagna [3]. The one-dimensional case was examined by Pelinovsky and Stefanov [33].

The asymptotic stability of solitons was also investigated in our papers [4], [18], [20], [21], [23], [25]. In their recent joint papers, the author and Komech [27], [28] were the first to prove the asymptotic stability of solitons (kinks) for the relativistic non-linear wave equation with Ginzburg–Landau potential. In all the papers mentioned above, the proof of the asymptotic stability rests primarily on the same basic strategy. However, this approach faced serious implementation difficulties in the relativistic case, and this has been an obstacle to the theoretical development over the past 20 years.

In the present paper we outline the general strategy of the papers [5]–[7], [16] and present new methods elaborated in [27], [28].

1.2. Statement of the problem. We shall be mostly concerned here with the one-dimensional non-linear wave equation

$$\ddot{\psi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \qquad x \in \mathbb{R}. \tag{1.1}$$

We write $F(\psi) = -U'(\psi)$, where $U(\psi)$ is a potential of the Ginzburg–Landau type, that is, $U(\psi)$ satisfies the following conditions (see Fig. 1).

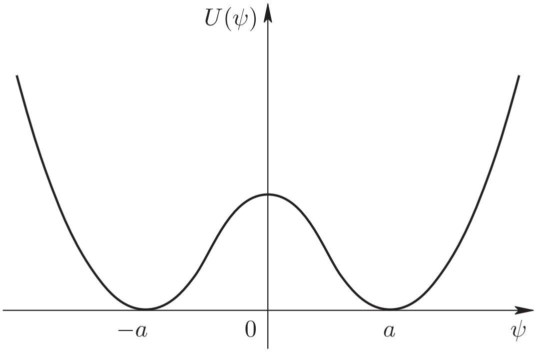

Figure 1. Potential $U(\psi)$

U1. *$U(\psi)$ is a smooth even function such that*

$$U(\psi) > 0 \quad \textit{for } \psi \neq a. \tag{1.2}$$

U2. *$U(\psi)$ is a parabola near the points $\pm a$,*

$$U(\psi) = \frac{m^2}{2}(\psi \mp a)^2, \qquad |\psi \mp a| < \delta, \tag{1.3}$$

for some $0 < \delta < a/2$ and $m > 0$.

The corresponding stationary equation is

$$s''(x) - U'(s(x)) = 0, \qquad x \in \mathbb{R}. \tag{1.4}$$

It has constant solutions $\psi(x) \equiv 0$ and $\psi(x) \equiv \pm a$. The non-constant solutions will be obtained using the 'energy integral'

$$\frac{(s')^2}{2} - U(s) = C,$$

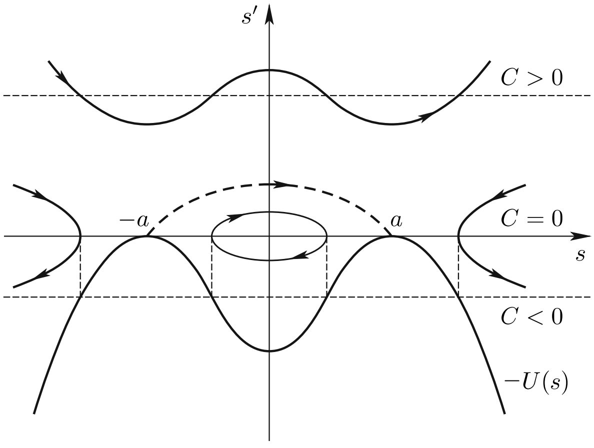

Figure 2. Phase portrait

where C is an arbitrary constant. The phase portrait of this equation is depicted in Fig. 2. It can be seen that for $C = 0$ there exists a so-called kink, namely, a finite-energy non-constant solution $s(x)$ of the stationary equation (1.4) such that $s(x) \to \pm a$ as $x \to \pm\infty$ (see Fig. 3). In addition, condition **U2** implies that $(s(x) \mp a)' = m^2(s(x) \mp a)$ as $x \to \infty$. Hence,

$$|s(x) \mp a| = Ce^{-m|x|}, \qquad x \to \pm\infty; \tag{1.5}$$

that is, the kink exponentially approaches its asymptotes $\pm a$. Equation (1.1) is relativistically invariant, so the solitons (or kinks) $s_{q,v}(x,t) = s(\gamma(x - vt - q))$, $q \in \mathbb{R}$, moving with velocity $|v| < 1$ are also solutions of equation (1.1). Here $\gamma = 1/\sqrt{1-v^2}$ is the Lorentz contraction. We linearize equation (1.1) at the kink $s(x)$. Substituting $\psi(x,t) = s(x) + \phi(x,t)$ into this equation, we formally obtain

$$\ddot{\phi}(x,t) = -H\phi(x,t) + \mathscr{O}(|\phi(x,t)|^2),$$

where $H := -d^2/dx^2 + m^2 + V(x)$ is the Schrödinger operator with potential $V(x) = -F'(s(x)) - m^2 = U''(s(x)) - m^2$. It is easily verified that the operator H has the following properties.

H1. The continuous spectrum of H coincides with the interval $[m^2, \infty)$.

H2. The point $\lambda_0 = 0$ belongs to the discrete spectrum, and $s'(x)$ is the corresponding eigenfunction.

H3. Since $s'(x) > 0$, λ_0 is the ground state, while the remaining points of the discrete spectrum (if there are any) lie in the interval $(0, m^2]$.

We additionally assume that the following condition holds.

U3. *The edge point $\lambda = m^2$ of the continuous spectrum of H is neither an eigenvalue nor a resonance.*

Remark 1.1. The precise definition of a resonance in the one-dimensional case may be found in [31] (see also [19] and [24]). Condition **U3** is equivalent to the boundedness of the resolvent of H at $\lambda = m^2$ (see [31], Theorem 7.2).

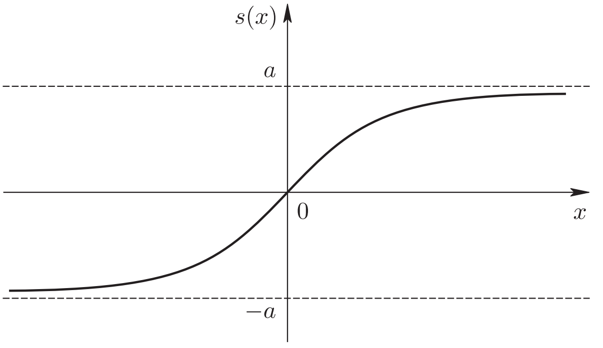

Figure 3. Kink

1.3. Results. Let $\psi_v = s(\gamma x)$, $\pi_v = -v\psi'_v(x)$. The main result for equation (1.1) is the soliton asymptotics

$$(\psi(x,t), \dot\psi(x,t)) \sim (\psi_{v_\pm}(x - v_\pm t - q_\pm), \pi_{v_\pm}(x - v_\pm t - q_\pm)) + W_0(t)\Phi_\pm, \qquad t \to \pm\infty, \tag{1.6}$$

for solutions with initial data close to some kink, where $W_0(t)$ is the dynamical group of the free Klein–Gordon equation, and $\Phi_\pm$ are the asymptotic scattering states. The terms $W_0(t)\Phi_\pm$ correspond to dispersive waves that transfer energy to infinity. The asymptotics (1.6) hold in the global energy norm of the Sobolev space $H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Remark 1.2. The asymptotics (1.6) can be interpreted as the interaction between the incoming soliton with trajectory $v_- t + q_-$ and the incoming dispersive wave $W_0(t)\Phi_-$, the result being the emergence of an outgoing soliton with new trajectory $v_+ t + q_+$ and a new outgoing dispersive wave $W_0(t)\Phi_+$. This interaction determines the (non-linear) *scattering operator* $\mathbf{S}\colon (v_-, q_-, \Phi_-) \mapsto (v_+, q_+, \Phi_+)$. However, the description of the domain (and the range) of this operator is still a matter for the future.

We shall prove the asymptotics (1.6) under two different forms of conditions on the discrete spectrum.

D1. *The discrete spectrum of the operator H is the single point $\lambda_0 = 0$.*

D2. *The discrete spectrum of H consists of two points $\lambda_0 = 0$ and $\lambda_1 \in (0, m^2)$, where*

$$4\lambda_1 > m^2. \tag{1.7}$$

In the second case we shall also assume the non-degeneracy condition also called the *Fermi golden rule*, which means an effective interaction between the non-linear term and the continuous spectrum. This interaction is responsible for the scattering of energy to infinity (see condition (10.0.11) in [7] or condition (1.11) in [28]). For equation (1.1), the Fermi golden rule is

F.
$$\int \varphi_{4\lambda_1}(x) F''(s(x)) \varphi^2_{\lambda_1}(x)\, dx \neq 0, \tag{1.8}$$

where φ_{λ_1} is the eigenfunction corresponding to the eigenvalue λ_1, and $\varphi_{4\lambda_1}$ is the odd continuous-spectrum eigenfunction corresponding to the point $4\lambda_1 \in (m^2, \infty)$.

The first case is addressed in Chapter I, and the second case in Chapter II. For simplicity of exposition, in the second case we examine only odd solutions of equation (1.1), establishing the asymptotic stability of a standing kink (that is, for $v = 0$ and $q = 0$). The asymptotic stability of moving kinks under condition **D2** can be obtained by combining the approaches of the two chapters.

In Chapter III we construct examples of non-linearities that satisfy our spectral conditions. We note that most works on the asymptotic stability of solitons also impose a number of conditions on the spectral properties of the corresponding linearized dynamics. However, almost everywhere these properties are only postulated, and in most cases no examples are known of non-linearities for which these properties are satisfied. We construct examples of potentials for which all the required spectral conditions hold: the properties of the discrete spectrum of the linearized equation, the absence of resonance, and the *Fermi golden rule*.

1.4. Methods. The proof of the asymptotic stability of solitons, as given in [5]–[9], [16], and [40], depends upon the characteristic general strategy of most studies in this direction. This approach is based on the methods of symplectic geometry for Hamiltonian systems in Hilbert space and methods of spectral theory for non-self-adjoint operators. Use is made, in particular, of symplectic projection onto the solitary manifold and onto symplectically orthogonal directions, separation of the dynamics along the solitary manifold and in the transversal direction, decay of the linearized transversal dynamics, modulation equations for soliton parameters, Poincaré normal forms, the method of majorants, and so on. Symplectic projection allows one to eliminate unstable directions corresponding to the zero discrete spectrum of the linearized dynamics. These methods may be regarded as a modern development of the Lyapunov theory of stability.

A similar strategy also applies to relativistic equations. However, the asymptotic stability of kinks for these equations was not established for a long time. One reason is that solutions of the one-dimensional linear Klein–Gordon equation with potential were not shown to decay sufficiently quickly, while the well-known decay of order $\sim t^{-1/2}$ which holds for the solutions of the free equation is not enough for the technique involved (see the discussion in the introduction to [9]). Accordingly, our first result in this direction was a proof of the rapid decay of order $\sim t^{-3/2}$ in weighted energy norms of the projection of the solution on the continuous spectrum, provided that there are no eigenvalues and resonances at the edge points of the continuous spectrum [19], [24], [26].

Furthermore, despite the availability of the general approach, a number of assertions and their proofs are significantly different due to the special nature of the relativistic equations, and some assertions are completely new, among which we mention the following estimates.

I. Estimates describing the growth rate of moments of solutions of the non-linear Klein–Gordon equation (see [27], Appendix A); these are the relativistic versions of the estimates (1.2.5) in [7] that were used there for the non-linear Schrödinger equation.

II. The relativistic version (4.27) of estimates of solutions in L^1-L^∞-norms.

Decay in weighted energy norms and the estimates I–II play a key role in obtaining the corresponding inequalities for majorants. The above properties can also be used to obtain the decay for the transversal component of the equation linearized at a soliton; this, in turn, guarantees the *radiation of energy to infinity*, providing for the asymptotic stability of the solitary manifold.

We remark that our papers [27], [28] were concerned with a slightly more general setting. Namely, instead of condition **U2** it was assumed there that

$$U(\psi) = \frac{m^2}{2}(\psi \mp a)^2 + \mathscr{O}(|\psi \mp a|^K), \qquad \psi \to \pm a, \tag{1.9}$$

with $K > 13$. The proof of the asymptotic stability of kinks runs along similar lines with minor technical modifications.

1.5. Open problems. It is easily verified that the well-known Ginzburg–Landau potential $U_{\rm GL}(\psi) = (\psi^2 - a^2)^2/(4a^2)$ satisfies condition (1.9) with $m^2 = 2$ and $K = 3$; it also satisfies the conditions **D2** and **F**. However, the edge point $\lambda = 2$ of the spectrum is a resonance for the corresponding linearized operator. This fact is the main reason why the asymptotic stability of kinks for an equation with potential $U_{\rm GL}$ has not been proved as yet.

For general non-linear hyperbolic equations and arbitrary finite-energy initial states, asymptotics of the form (1.6) and even more general ones

$$(\psi(x,t),\dot\psi(x,t)) \sim \sum_{k=1}^{N}(\psi_{v_\pm^k}(x-v_\pm^k t-q_\pm^k), \pi_{v_\pm^k}(x-v_\pm^k t-q_\pm^k)) + W_0(t)\Phi_\pm, \quad t \to \pm\infty,$$

have been observed in numerical simulations (Vinnichenko and coauthors [22]). However, the proof of these asymptotics is still a matter for the future. Such asymptotics are closely related to the problem of stability of elementary particles and the wave-particle duality in the context of Heisenberg's non-linear theory [13], [14].

1.6. Structure of the paper. This paper is organized as follows. Chapter I is concerned with methods for proving the asymptotic stability of moving kinks under condition **D1**, that is, in the case where there is no additional discrete spectrum. In § 2 we give the necessary definitions and formulate the main result. Section 3 is devoted to the symplectic structure of the solitary manifold, and § 4 to the linearization of the solution at a kink and to properties of the linearized equation. In § 5 we separate the dynamics into two components: along the solitary manifold and in the transversal direction. Section 6 is concerned with modulation equations for soliton parameters. The scheme of the proof of long-time decay of the transversal component is outlined in § 7. The soliton asymptotics (1.6) is established in § 8.

In Chapter II we examine the case when the operator H has an additional discrete spectrum satisfying condition **D2**. We shall be concerned only with odd solutions, and we prove the asymptotic stability of the standing kink corresponding to $v = q = 0$. Properties of the linearized equation are given in § 10. In § 11 we obtain the dynamical equations for the discrete and continuous components of the solution, and in § 12 we find the Poincaré normal forms for these equations. Section 13 is devoted to majorants and their estimates. The soliton asymptotics is derived in § 14.

In Chapter III we give examples of non-linear potentials that satisfy the spectral conditions of the first and second chapters.

Chapter I

Moving solitons

In this chapter we shall be concerned with the asymptotic stability of moving solitons in the case where there is no non-zero discrete spectrum of the linearized dynamics.

2. Main result

We write equation (1.1) as the system of two first-order equations

$$\begin{cases} \dot\psi(x,t) = \pi(x,t), \\ \dot\pi(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $F(\psi) = -U'(\psi)$, and all derivatives are understood in the sense of distributions. This is a Hamiltonian system, and the corresponding Hamiltonian is

$$\mathscr{H}(\psi,\pi) = \int \left[\frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + U(\psi(x)) \right] dx. \tag{2.2}$$

In vector form, the Cauchy problem for the system (2.1) is written as

$$\dot Y(t) = \mathscr{F}(Y(t)), \quad t \in \mathbb{R}, \qquad Y(0) = Y_0, \tag{2.3}$$

where $Y(t) = (\psi(t), \pi(t))$, $Y_0 = (\psi_0, \pi_0)$. We also use the vector form of the soliton solutions:

$$Y_{q,v}(t) = (\psi_v(x - vt - q), \pi_v(x - vt - q)), \qquad q \in \mathbb{R}, \quad v \in (-1,1), \tag{2.4}$$

where

$$\psi_v(x) = s(\gamma x), \qquad \pi_v(x) = -v\psi_v'(x). \tag{2.5}$$

Consider the soliton state $S(\sigma) := (\psi_v(x-b), \pi_v(x-b))$ with arbitrary parameters $\sigma := (b, v)$, where $b \in \mathbb{R}$ and $v \in (-1,1)$. Clearly, the soliton (2.4) can be written in the form $S(\sigma(t))$, where

$$\sigma(t) = (b(t), v(t)) = (vt + q, v). \tag{2.6}$$

The solitary manifold consists of all soliton states:

$$\mathscr{S} := \{S(\sigma) : \sigma \in \Sigma := \mathbb{R} \times (-1,1)\}. \tag{2.7}$$

We also define the phase space for the Cauchy problem (2.3). Given any $\alpha \in \mathbb{R}$, $p \geqslant 1$, and $k = 0, 1, 2, \ldots$, we let $W^{p,k}_\alpha$ denote the weighted Sobolev space of functions with finite norms

$$\|\psi\|_{W^{p,k}_\alpha} = \sum_{i=0}^{k} \|(1+|x|)^\alpha \psi^{(i)}\|_{L^p}.$$

We set $H^k_\alpha := W^{2,k}_\alpha$, $L^2_\alpha := H^0_\alpha$ and introduce the spaces $E_\alpha := H^1_\alpha \oplus L^2_\alpha$ and $W := W^{1,2}_0 \oplus W^{1,1}_0$ of vector functions $Y = (\psi, \pi)$ with finite norms

$$\|Y\|_{E_\alpha} = \|\psi\|_{H^1_\alpha} + \|\pi\|_{L^2_\alpha} \quad \text{and} \quad \|Y\|_W = \|\psi\|_{W^{2,1}_0} + \|\pi\|_{W^{1,1}_0}.$$

We shall work in the phase space $\mathscr{E} := E + \mathscr{S}$, where $E = E_0$, and $\mathscr{S}$ is given by (2.7). The metric in $\mathscr{E}$ is defined as follows:

$$\rho_{\mathscr{E}}(Y_1, Y_2) = \|Y_1 - Y_2\|_E, \qquad Y_1, Y_2 \in \mathscr{E}.$$

Clearly, the Hamiltonian (2.2) is continuous on the phase space $\mathscr{E}$. By adapting the techniques of [29], [35], [41] one can readily show that:

(i) *for any initial data $Y_0 \in \mathscr{E}$ there exists a unique solution $Y(t) \in C(\mathbb{R}, \mathscr{E})$ of problem* (2.3);

(ii) *the map $U(t)\colon Y_0 \mapsto Y(t)$ is continuous in $\mathscr{E}$ for any $t \in \mathbb{R}$*;

(iii) *the energy conservation law holds,*

$$\mathscr{H}(Y(t)) = \mathscr{H}(Y_0), \qquad t \in \mathbb{R}.$$

The main result of the first chapter is the following.

Theorem 2.1. *Assume that conditions* **U1–U3** *and* **D1** *are satisfied and assume that $Y(t)$ is the solution of the Cauchy problem* (2.3) *with initial data $Y_0 \in \mathscr{E}$ close to some kink $S(\sigma_0) = S_{q_0,v_0}$,*

$$Y_0 = S(\sigma_0) + X_0, \qquad d_0 := \|X_0\|_{E_\beta \cap W} \ll 1, \tag{2.8}$$

where $\beta > 5/2$. Then, for sufficiently small d_0, the following asymptotics holds:

$$Y(x,t) = \bigl(\psi_{v_\pm}(x - v_\pm t - q_\pm), \pi_{v_\pm}(x - v_\pm t - q_\pm)\bigr) + W_0(t)\Phi_\pm + r_\pm(x,t), \qquad t \to \pm\infty \tag{2.9}$$

with some constants $v_\pm$ and $q_\pm$. Here $W_0(t)$ is the dynamical group of the free Klein–Gordon equation and $\Phi_\pm \in E$ are the asymptotic scattering states. Moreover,

$$\|r_\pm(t)\|_E = \mathscr{O}(|t|^{-1/2}), \qquad t \to \pm\infty. \tag{2.10}$$

We note that it suffices to verify the asymptotics (2.9) as $t \to +\infty$, because the system (2.1) is time reversible.

3. Symplectic projection

3.1. Symplectic structure and the Hamiltonian form. The system (2.3) is a Hamiltonian system, that is, it can be written as

$$\dot{Y} = J\mathscr{D}\mathscr{H}(Y), \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.1}$$

where $\mathscr{D}\mathscr{H}$ is the Fréchet derivative of the Hamiltonian (2.2). At an arbitrary point we identify the tangent space to $\mathscr{E}$ with the space E and consider the symplectic form Ω given on E by

$$\Omega(Y_1, Y_2) = \langle Y_1, JY_2\rangle, \qquad Y_1, Y_2 \in E, \tag{3.2}$$

where $\langle Y_1, Y_2\rangle := \langle \psi_1, \psi_2\rangle + \langle \pi_1, \pi_2\rangle$ and $\langle \psi_1, \psi_2\rangle = \int \psi_1(x)\psi_2(x)\,dx$. Clearly, the form Ω is non-degenerate: $Y_1 = 0$ if $\Omega(Y_1, Y_2) = 0$ for any $Y_2 \in E$.

The expression $Y_1 \nmid Y_2$ means that the vectors $Y_1 \in E$ and $Y_2 \in E$ are symplectically orthogonal, that is, $\Omega(Y_1, Y_2) = 0$. A projection operator $\mathbf{P}\colon E \to E$ is called a symplectic orthogonal projection if $Y_1 \nmid Y_2$ for $Y_1 \in \operatorname{Ker}\mathbf{P}$ and $Y_2 \in \operatorname{Range}\mathbf{P}$.

3.2. Symplectic projection onto the solitary manifold. The tangent space $\mathscr{T}_{S(\sigma)}\mathscr{S}$ to the manifold $\mathscr{S}$ at a point $S(\sigma)$ is generated by the vectors

$$\begin{aligned}\tau_1 = \tau_1(v) &:= \partial_b S(\sigma) = (-\psi_v'(y), -\pi_v'(y)),\\ \tau_2 = \tau_2(v) &:= \partial_v S(\sigma) = (\partial_v\psi_v(y), \partial_v\pi_v(y)),\end{aligned} \tag{3.3}$$

which form a basis for the space $\mathscr{T}_{S(\sigma)}\mathscr{S}$ in the 'moving coordinate system' $y := x - b$. It is worth pointing out that the functions τ_j depend on y, rather than on x. From (2.5) it follows that, for any $\alpha \in \mathbb{R}$ and $v \in (-1, 1)$,

$$\tau_j(v) \in E_\alpha, \qquad j = 1, 2. \tag{3.4}$$

We claim that the symplectic form Ω is non-degenerate on the tangent space $\mathscr{T}_{S(\sigma)}\mathscr{S}$. To prove this we find explicit expressions for the vectors τ_1 and τ_2. By the definition (2.5) of the functions $\psi_v(y)$ and $\pi_v(y)$,

$$\tau_1 = \bigl(-\gamma s'(\gamma y), v\gamma^2 s''(\gamma y)\bigr), \qquad \tau_2 = \bigl(vy\gamma^3 s'(\gamma y), -\gamma^3 s'(\gamma y) - v^2 y\gamma^4 s''(\gamma y)\bigr).$$

Hence,

$$\Omega(\tau_1, \tau_2) = \langle \tau_1^1, \tau_2^2\rangle - \langle \tau_1^2, \tau_2^1\rangle = \gamma^4\langle s'(\gamma y), s'(\gamma y)\rangle > 0. \tag{3.5}$$

This means that $\mathscr{T}_{S(\sigma)}\mathscr{S}$ is a symplectic subspace. The 'symplectic orthogonal projection' onto $\mathscr{S}$ is defined in a small neighbourhood of the solitary manifold $\mathscr{S}$. For a detailed proof of this rather simple result we refer the reader to [16], [27], and [28]. The precise formulation is as follows.

Lemma 3.1. *The following assertions hold for any* $\alpha \in \mathbb{R}$.

(i) *There exist a neighbourhood* $\mathscr{O}_\alpha(\mathscr{S})$ *of the manifold* $\mathscr{S}$ *in the space* E_α *and a map* $\mathbf{\Pi}\colon \mathscr{O}_\alpha(\mathscr{S}) \to \mathscr{S}$ *such that* $\mathbf{\Pi}$ *is uniformly continuous on* $\mathscr{O}_\alpha(\mathscr{S})$ *in the metric of* E_α. *Moreover,*

$$\mathbf{\Pi}Y = Y \quad \text{for } Y \in \mathscr{S}, \qquad Y - S \nmid \mathscr{T}_S\mathscr{S}, \quad \text{where } S = \mathbf{\Pi}Y. \tag{3.6}$$

(ii) *For any* $q \in \mathbb{R}$, *the neighbourhood* $\mathscr{O}_\alpha(\mathscr{S})$ *is invariant under the translations*

$$T_q\colon (\psi(x), \pi(x)) \mapsto (\psi(x+q), \pi(x+q)), \tag{3.7}$$

and moreover, $\mathbf{\Pi}T_qY = T_q\mathbf{\Pi}$ *if* $Y \in \mathscr{O}_\alpha(\mathscr{S})$.

(iii) *For any* $\overline{v} < 1$, *there exists an* $r_\alpha(\overline{v}) > 0$ *such that* $S(\sigma) + X \in \mathscr{O}_\alpha(\mathscr{S})$ *for all* $b \in \mathbb{R}$ *if* $|v| \leqslant \overline{v}$ *and* $\|X\|_{E_\alpha} < r_\alpha(\overline{v})$.

The map $\mathbf{\Pi}$ is called the symplectic orthogonal projection onto $\mathscr{S}$.

4. Linearization on the solitary manifold

We shall seek a solution of the system (2.1) in the form of a sum

$$Y(t) = S(\sigma(t)) + X(t), \tag{4.1}$$

where $S(\sigma(t))$ is the soliton with parameters $\sigma(t) = (b(t), v(t))$, with $b(t) \in \mathbb{R}$ and $v(t) \in (-1,1)$ some smooth functions of the variable $t \in \mathbb{R}$. In terms of the components of the vector functions $Y = (\psi, \pi)$ and $X = (\Psi, \Pi)$, equation (4.1) can be written as

$$\begin{cases} \psi(x,t) = \psi_{v(t)}(x - b(t)) + \Psi(x - b(t), t), \\ \pi(x,t) = \pi_{v(t)}(x - b(t)) + \Pi(x - b(t), t). \end{cases} \tag{4.2}$$

Substituting these equations into the system (2.1), we obtain the following equations in the 'moving coordinate system' $y = x - b(t)$:

$$\begin{aligned} \dot\psi &= \dot v \partial_v \psi_v(y) - \dot b \psi_v'(y) + \dot\Psi(y,t) - \dot b \Psi'(y,t) = \pi_v(y) + \Pi(y,t), \\ \dot\pi &= \dot v \partial_v \pi_v(y) - \dot b \pi_v'(y) + \dot\Pi(y,t) - \dot b \Pi'(y,t) \\ &= \psi_v''(y) + \Psi''(y,t) + F(\psi_v(y) + \Psi(y,t)). \end{aligned} \tag{4.3}$$

The soliton equation with respect to the variable $y = x - b(t)$ assumes the form

$$-v\psi_v'(y) = \pi_v(y), \qquad -v\pi_v'(y) = \psi_v''(y) + F(\psi_v(y)), \tag{4.4}$$

and hence by (4.3) this immediately yields the equations for the functions $\Psi(t)$ and $\Pi(t)$:

$$\begin{aligned} \dot\Psi(y,t) &= \Pi(y,t) + \dot b \Psi'(y,t) + (\dot b - v)\psi_v'(y) - \dot v \partial_v \psi_v(y), \\ \dot\Pi(y,t) &= \Psi''(y,t) + \dot b \Pi'(y,t) + (\dot b - v)\pi_v'(y) - \dot v \partial_v \pi_v(y) \\ &\qquad + F(\psi_v(y) + \Psi(y,t)) - F(\psi_v(y)). \end{aligned} \tag{4.5}$$

We can rewrite equations (4.5) as

$$\dot X(t) = A(t)X(t) + T(t) + \mathscr{N}(t), \qquad t \in \mathbb{R}, \tag{4.6}$$

where the term $T(t)$ is independent of X and the term $\mathscr{N}(t)$ is at least quadratic in X. The linear operator $A(t) = A_{v,w}(t)$ depends on two parameters $v = v(t)$ and $w = \dot b(t)$, and can be written in the form

$$A_{v,w}\begin{pmatrix}\Psi \\ \Pi\end{pmatrix} := \begin{pmatrix} w\nabla & 1 \\ \Delta + F'(\psi_v) & w\nabla \end{pmatrix}\begin{pmatrix}\Psi \\ \Pi\end{pmatrix} = \begin{pmatrix} w\nabla & 1 \\ \Delta - m^2 - V_v(y) & w\nabla \end{pmatrix}\begin{pmatrix}\Psi \\ \Pi\end{pmatrix}, \tag{4.7}$$

where $\nabla = d/dx$, $\Delta = d^2/dx^2$, and the potential $V_v(y)$ is defined by

$$V_v(y) = -F'(\psi_v) - m^2. \tag{4.8}$$

Further, $T(t)$ and $\mathscr{N}(t) = \mathscr{N}(\sigma, X)$ are given by

$$T = \begin{pmatrix} (w - v)\psi_v' - \dot v \partial_v \psi_v \\ (w - v)\pi_v' - \dot v \partial_v \pi_v \end{pmatrix}, \qquad \mathscr{N}(\sigma, X) = \begin{pmatrix} 0 \\ N(v, \Psi) \end{pmatrix}, \tag{4.9}$$

where $v = v(t)$, $w = w(t)$, $\sigma = \sigma(t) = (b(t), v(t))$, $X = X(t) = (\Psi(t), \Pi(t))$, and

$$N(v, \Psi) = F(\psi_v + \Psi) - F(\psi_v) - F'(\psi_v)\Psi. \tag{4.10}$$

Note that the term $A(t)X(t)$ on the right-hand side of (4.6) is linear in $X(t)$, and $\mathscr{N}(t)$ is a *term of higher order* in $X(t)$. On the other hand, $T(t)$ is a zero-order term and does not vanish at $X(t) = 0$ since $S(\sigma(t))$ is not in general a kink if (2.6) does not hold. Also, by (3.3) and (4.9),

$$T(t) = -(w - v)\tau_1 - \dot{v}\tau_2. \tag{4.11}$$

Hence, $T(t) \in \mathscr{T}_{S(\sigma(t))}\mathscr{S}$ for all $t \in \mathbb{R}$. This implies the unstable nature of the non-linear dynamics *along the solitary manifold.*

4.1. Hamiltonian structure and spectrum. Our aim here is to study the spectral properties of the operator $A_{v,w}$. Let us examine in more detail the linear equation

$$\dot{X}(t) = A_{v,w}X(t), \qquad t \in \mathbb{R}, \tag{4.12}$$

for some fixed $v \in (-1, 1)$ and $w \in \mathbb{R}$. Consider the space $E^+ := H^2(\mathbb{R}) \oplus H^1(\mathbb{R})$. The Hamiltonian properties of equation (4.12) are established in the following lemma.

Lemma 4.1. (i) *For any* $v \in (-1, 1)$ *and* $w \in \mathbb{R}$, *equation* (4.12) *can be represented in the Hamiltonian form*

$$\dot{X}(t) = J\mathscr{D}\mathscr{H}_{v,w}(X(t)), \qquad t \in \mathbb{R},$$

where $\mathscr{D}\mathscr{H}_{v,w}$ *is the Fréchet derivative of the Hamiltonian*

$$\mathscr{H}_{v,w}(X) = \frac{1}{2}\int \left[|\Pi|^2 + |\Psi'|^2 + (m^2 + V_v)|\Psi|^2\right] dy + \int \Pi w \Psi' \, dy.$$

(ii) *The energy conservation law holds for the solutions* $X(t) \in C^1(\mathbb{R}, E^+)$:

$$\mathscr{H}_{v,w}(X(t)) = \text{const}, \qquad t \in \mathbb{R}.$$

(iii) *The skew-symmetry relation holds*:

$$\Omega(A_{v,w}X_1, X_2) = -\Omega(X_1, A_{v,w}X_2), \qquad X_1, X_2 \in E. \tag{4.13}$$

For the proof of Lemma 4.1, see [27].

Consider the action of the operator $A_{v,w}$ on tangent vectors $\tau = \tau_j(v)$ to the solitary manifold. Differentiation of (4.4) with respect to b and v gives

$$\begin{aligned} -v\psi_v'' &= \pi_v', & -v\pi_v'' &= \psi_v''' + F'(\psi_v)\psi_v', \\ -\psi_v' - v\partial_v\psi_v' &= \partial_v\pi_v, & -\pi_v' - v\partial_v\pi_v' &= \partial_v\psi_v'' + F'(\psi_v)\partial_v\psi_v. \end{aligned}$$

Hence,

$$A_{v,w}\begin{pmatrix} -\psi_v' \\ -\pi_v' \end{pmatrix} = \begin{pmatrix} (v-w)\psi_v'' \\ (v-w)\pi_v'' \end{pmatrix}, \qquad A_{v,w}\begin{pmatrix} \partial_v\psi_v \\ \partial_v\pi_v \end{pmatrix} = \begin{pmatrix} (w-v)\partial_v\psi_v' \\ (w-v)\partial_v\pi_v' \end{pmatrix} + \begin{pmatrix} -\psi_v' \\ -\pi_v' \end{pmatrix}.$$

As a result,

$$A_{v,w}[\tau_1] = (w - v)\tau_1', \qquad A_{v,w}[\tau_2] = (w - v)\tau_2' + \tau_1. \tag{4.14}$$

Now we examine the spectral properties of the operator $A_v = A_{v,v}$ corresponding to $w = v$:

$$A_v := \begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 - V_v(y) & v\nabla \end{pmatrix}. \tag{4.15}$$

The continuous spectrum of A_v coincides with the interval $\Gamma := (-i\infty, -im/\gamma] \cup [im/\gamma, i\infty)$. From (4.14) it follows that the tangent vector $\tau_1(v)$ is the eigenvector of A_v corresponding to the zero eigenvalue, and the tangent vector $\tau_2(v)$ is a generalized eigenvector (root vector), that is,

$$A_v[\tau_1(v)] = 0, \qquad A_v[\tau_2(v)] = \tau_1(v). \tag{4.16}$$

We claim that the root space of the operator A_v corresponding to the zero eigenvalue is two-dimensional. For this it suffices to check that the equation $A_v[u] = \tau_2$ has no non-zero solutions in the space $L^2 \oplus L^2$. Let us consider this equation in more detail:

$$\begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 - V_v(y) & v\nabla \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v\gamma^3 ys'(\gamma y) \\ -\gamma^3 s'(\gamma y) - v^2\gamma^4 ys''(\gamma y) \end{pmatrix}.$$

Using the first equation, we obtain $u_2 = v\gamma^3 ys'(\gamma y) - v\nabla u_1$. Substitution of u_2 into the second equation gives

$$H_v u_1 = -\gamma^3(1 + v^2)s'(\gamma y) - 2v^2\gamma^4 ys''(\gamma y), \tag{4.17}$$

where $H_v = -(1/\gamma^2)d^2/dy^2 + m^2 + V_v(y)$ is a modified Schrödinger operator. Setting $u_1 = -(1/2)v^2\gamma^5 y^2 s'(\gamma y) + \widetilde{u}_1$, we transform the last equation into

$$H_v \widetilde{u}_1 = -\gamma^2 \psi_v'. \tag{4.18}$$

Remark 4.2. The spectral properties of the operators H_v are identical for all $v \in (-1, 1)$, since the equality $V_v(x) = V_0(\gamma x)$ implies that

$$H_v = \mathscr{I}_v^{-1} H_0 \mathscr{I}_v, \quad \text{where} \quad \mathscr{I}_v \colon \psi(x) \mapsto \psi\left(\frac{x}{\gamma}\right). \tag{4.19}$$

This similarity of operators is related to the relativistic invariance of the initial equation (1.1). In particular, H_v has the properties **H1**–**H3** (with the eigenfunction ψ_v' instead of s'), as well as the properties **U3** and **D1**.

The point $\lambda_0 = 0$ lies in the discrete spectrum of H_v, and ψ_v' is the corresponding eigenfunction, so it follows from (4.18) that $\widetilde{u}_1$ is a root function of H_v. But this is impossible because this operator is self-adjoint. Thus, we have shown that **the root space of A_v corresponding to the zero eigenvalue is a two-dimensional Jordan block**.

We claim that A_v has no eigenvalues other than $\lambda = 0$. To this end we consider the spectral equation

$$\begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 - V_v(y) & v\nabla \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

From the first equation, we find that $u_2 = -(v\nabla - \lambda)u_1$ and substitute this into the second equation. This gives

$$(H_v + \lambda^2 - 2v\lambda\nabla)u_1 = 0. \tag{4.20}$$

Since in view of condition **D1** the operator $H_0 = H$ corresponding to $v = 0$ has only the zero eigenvalue, A_0 also has only the zero eigenvalue. If $v \neq 0$, then scalar multiplication of both sides of (4.20) by u_1 gives

$$\langle H_v u_1, u_1\rangle + \lambda^2\langle u_1, u_1\rangle = 0.$$

The operator H_v is self-adjoint, so λ^2 is a real number. A non-zero eigenvalue of A_v might occur as a bifurcation either from the point $\lambda = 0$ or from the edge points $\pm im/\gamma$ of the continuous spectrum as the parameter v varies continuously. We consider these cases separately.

(i) No bifurcation from $\lambda = 0$ is possible, since this is an isolated point of the discrete spectrum, and we already know that the corresponding root subspace is two-dimensional for any $v \in (-1, 1)$.

(ii) Bifurcation from an edge point is also impossible. Indeed, the eigenvalues λ generated by edge points must necessarily be purely imaginary because λ^2 is real. Let $\lambda \in (-im/\gamma, im/\gamma)$ be an eigenvalue of A_v. Then $A_v[u] = \lambda u$, where $u = (u_1, u_2) \in L^2 \oplus L^2$ is the corresponding eigenfunction. Consider the new function $p(x) = e^{\gamma^2 v\lambda x}u(x)$. Clearly, $p = (p_1, p_2)$ also lies in the space $L^2 \oplus L^2$. Equation (4.20) for p_1 can be rewritten as $(H_v + \gamma^2\lambda^2)p_1 = 0$, where $-\gamma^2\lambda^2 \in (0, m^2)$. In view of condition **D1**, this equation has no non-zero solutions in L^2.

This being so, **the operator A_v has only the one eigenvalue** $\lambda = 0$.

4.2. Decay of the transversal linearized dynamics. We consider the linearized equation

$$\dot X(t) = A_v X(t), \qquad t \in \mathbb{R}. \tag{4.21}$$

Let $\mathbf{P}_v^{\mathrm{d}}$ denote the symplectic orthogonal projection from E onto the tangent space $\mathscr{T}_{S(\sigma)}\mathscr{S}$. By the linearity,

$$\mathbf{P}_v^{\mathrm{d}} X = \sum p_{jl}(v)\tau_j(v)\Omega(\tau_l(v), X), \quad X \in E, \tag{4.22}$$

where the $p_{jl}(v)$ are smooth coefficients. Note that in the variables $y = x - b$ the projection $\mathbf{P}_v^{\mathrm{d}}$ is independent of b. We set $\mathbf{P}_v^{\mathrm{c}} := \mathbf{I} - \mathbf{P}_v^{\mathrm{d}}$. One of the key steps in the proof of the asymptotic stability of solitons is to establish the long-time decay of solutions of the transversal linearized equation. For $v = 0$, the following result is contained in [19], [24], and for $v \neq 0$ in [26].

Proposition 4.3. *Assume that conditions* **U1–U3** *are satisfied. Let $\beta > 5/2$. Then for any $X \in E_\beta$ the following long-time decay estimate in weighted norms holds*:

$$\|e^{A_v t}\mathbf{P}_v^{\mathrm{c}} X\|_{E_{-\beta}} \leqslant C(v)(1 + |t|)^{-3/2}\|X\|_{E_\beta}, \qquad t \in \mathbb{R}. \tag{4.23}$$

Here and in what follows, $e^{A_v t}$ denotes the dynamical group of equation (4.21). The decay estimate (4.23) readily implies uniform decay with respect to x, that is, decay in the norm of L^∞ holding for any $X \in E_\beta \cap W$. In order to prove this, we apply the projection $\mathbf{P}_v^{\mathrm{c}}$ to both sides of (4.21):

$$\mathbf{P}_v^{\mathrm{c}} \dot{X} = A_v \mathbf{P}_v^{\mathrm{c}} X = A_v^0 \mathbf{P}_v^{\mathrm{c}} X + \mathscr{V}_v \mathbf{P}_v^{\mathrm{c}} X, \tag{4.24}$$

where

$$A_v^0 = \begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 & v\nabla \end{pmatrix}, \qquad \mathscr{V}_v = \begin{pmatrix} 0 & 0 \\ -V_v & 0 \end{pmatrix}. \tag{4.25}$$

We set $Y = \mathbf{P}_v^{\mathrm{c}} X$ and invoke the Duhamel representation for the solution of (4.24):

$$e^{A_v t} Y = e^{A_v^0 t} Y + \int_0^t e^{A_v^0 (t-\tau)} \mathscr{V}_v e^{A_v \tau} Y \, d\tau, \qquad t \in \mathbb{R}.$$

Note that $e^{A_v^0 t} Z = e^{A_0^0 t} T_{vt} Z$, where the translation operator T_{vt} is defined in (3.7). This gives

$$e^{A_v t} Y = e^{A_0^0 t} T_{vt} Y + \int_0^t e^{A_0^0 (t-\tau)} T_{vt} [\mathscr{V}_v e^{A_v \tau} Y] \, d\tau, \qquad t \in \mathbb{R}. \tag{4.26}$$

The potential V_v is compactly supported, and hence by using the inequality (265) in [36], the Hölder inequality, and the inequality (4.23) we arrive at the following estimate for the first component of the vector function $e^{A_v t} Y$:

$$\begin{aligned}
\|(e^{A_v t} Y)_1\|_{L^\infty} &\leqslant C(1+|t|)^{-1/2} \|Y\|_W + C \int_0^t (1+|t-\tau|)^{-1/2} \|\mathscr{V}_v (e^{A_v \tau} Y)_1\|_{W_0^{1,1}} \, d\tau \\
&\leqslant C(1+|t|)^{-1/2} \|X\|_W + C \int_0^t (1+|t-\tau|)^{-1/2} \|e^{A_v \tau} \mathbf{P}_v^{\mathrm{c}} X\|_{E_{-\beta}} \, d\tau \\
&\leqslant C(1+|t|)^{-1/2} \|X\|_W + C \int_0^t (1+|t-\tau|)^{-1/2} (1+|\tau|)^{-3/2} \|X\|_{E_\beta} \, d\tau \\
&\leqslant C(1+|t|)^{-1/2} (\|X\|_W + \|X\|_{E_\beta}).
\end{aligned}$$

Thus, for any $\beta > 5/2$ and $X \in E_\beta \cap W$,

$$\|(e^{A_v t} \mathbf{P}_v^{\mathrm{c}} X)_1\|_{L^\infty} \leqslant C(v)(1+|t|)^{-1/2} (\|X\|_W + \|X\|_{E_\beta}), \qquad t \in \mathbb{R}. \tag{4.27}$$

4.3. Estimates for the non-linear term. We derive estimates for the non-linear term $N(v, \Psi)$ defined by (4.10). Let $\mathscr{R}(a)$ denote a positive function that is bounded for sufficiently small a. Using Cauchy's formula for the remainder gives

$$N = \frac{\Psi^2}{2} \int_0^1 (1-\rho) F''(\psi_v + \rho\Psi) \, d\rho. \tag{4.28}$$

By condition **U2**, the function $F''(\psi)$ vanishes in some neighbourhood of the points $\pm a$. Applying the Cauchy–Schwarz inequality, we get that

$$\|N\|_{L^1} = \mathscr{R}(\|\Psi\|_{L^\infty}) \|\Psi\|_{L^\infty} \|\Psi\|_{L^2_{-\beta}} = \mathscr{R}(\|\Psi\|_{L^\infty}) \|\Psi\|_{L^\infty} \|X\|_{E_{-\beta}}.$$

Further, differentiating (4.28) with respect to y, we have

$$N' = \frac{\Psi^2}{2} \int_0^1 (1-\rho)(\psi_v' + \rho\Psi') F'''(\psi_v + \rho\Psi) \, d\rho + \Psi\Psi' \int_0^1 (1-\rho) F''(\psi_v + \rho\Psi) \, d\rho.$$

Hence,

$$\begin{aligned}\|N'\|_{L^1} &= \mathscr{R}(\|\Psi\|_{L^\infty})\big[\|\Psi\|_{L^\infty}\|\Psi\|_{L^2_{-\beta}} + \|\Psi\|_{L^\infty}\|\Psi'\|_{L^2_{-\beta}}\big] \\ &= \mathscr{R}(\|\Psi\|_{L^\infty})\|\Psi\|_{L^\infty}\|X\|_{E_{-\beta}}.\end{aligned}$$

Thus, the non-linear term $N(v,\Psi)$ is estimated in the $W_0^{1,1}$-norm as follows:

$$\|N\|_{W_0^{1,1}} = \mathscr{R}(\|\Psi\|_{L^\infty})\|\Psi\|_{L^\infty}\|X\|_{E_{-\beta}}\,.$$

Moreover, (4.28) yields a similar estimate for $N(v,\Psi)$ in the norm of L^2_β. As a result, we have the estimate

$$\|N\|_{L^2_\beta\cap W_0^{1,1}} = \mathscr{R}(\|\Psi\|_{L^\infty})\|\Psi\|_{L^\infty}\|X\|_{E_{-\beta}}. \tag{4.29}$$

5. Symplectic decomposition of the dynamics

Equation (4.6) was obtained without any additional assumptions on the parameters $\sigma(t) = (v(t), b(t))$ in (4.1). Now we assume that $S(\sigma(t)) := \mathbf{\Pi} Y(t)$. This can be done provided that, for all $t \geqslant 0$,

$$Y(t) \in \mathscr{O}_\alpha(\mathscr{S}), \tag{5.1}$$

where $\mathscr{O}_\alpha(\mathscr{S})$ is the neighbourhood defined in Lemma 3.1. Condition (5.1) is satisfied for $t=0$ in view of the assumption (2.8). Consequently, the quantities $S(\sigma(0)) = \mathbf{\Pi} Y(0)$ and $X(0) = Y(0) - S(\sigma(0))$ are defined. Below we shall show that condition (5.1) with $\alpha = -\beta$ is satisfied for all $t \geqslant 0$ if d_0 in (2.8) is sufficiently small. We choose any $\overline{v} < 1$ such that

$$|v(0)| < \overline{v}, \tag{5.2}$$

and we let $r_{-\beta}(\overline{v})$ denote the positive number defined in Lemma 3.1, (iii) corresponding to $\alpha = -\beta$. Then $S(\sigma) + X \in \mathscr{O}_{-\beta}(\mathscr{S})$ if $|v| < \overline{v}$ and $\|X\|_{E_{-\beta}} < r_{-\beta}(\overline{v})$. Hence, $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ and the function $X(t) = Y(t) - S(\sigma(t))$ is defined for any $t \geqslant 0$ such that

$$|v(t)| < \overline{v} \quad \text{and} \quad \|X(t)\|_{E_{-\beta}} < r_{-\beta}(\overline{v}). \tag{5.3}$$

That condition (5.3) holds for all $t \geqslant 0$ is proved using the standard concept of the 'exit time'. We define the *majorants*

$$m_1(t) := \sup_{s\in[0,t]} (1+s)^{3/2}\|X(s)\|_{E_{-\beta}}, \qquad m_2(t) := \sup_{s\in[0,t]} (1+s)^{1/2}\|\Psi(s)\|_{L^\infty}\,, \tag{5.4}$$

where $X = (\Psi, \Pi)$. We set $\nu = \overline{v} - |v(0)|$ and consider some fixed $\varepsilon \in (0, r_{-\beta}(\overline{v}))$ to be specified below.

Definition 5.1. The exit time t_* is defined as follows:

$$t_* = \sup\{t : |v(t) - v(0)| < \nu,\ m_j(t) < \varepsilon,\ j = 1, 2\}. \tag{5.5}$$

Note that $m_j(0) < \varepsilon$ if $d_0 \ll 1$. Our purpose here is to show that $t_* = \infty$ if d_0 is sufficiently small. To do so it suffices to show that, for small d_0,

$$|v(t) - v(0)| < \frac{\nu}{2}\,, \quad m_j(t) < \frac{\varepsilon}{2}\,, \qquad 0 \leqslant t < t_*. \tag{5.6}$$

6. Modulation equations

In this section we derive equations for the soliton parameters $\sigma(t) = (b(t), v(t))$ of the symplectic projection $S(\sigma(t))$ of the solution $Y(t)$ of (2.3). Namely, we shall seek the solution of (2.3) in the form $Y(t) = S(\sigma(t)) + X(t)$, where $S(\sigma(t)) = \mathbf{\Pi} Y(t)$. In other words, we shall assume that the following symplectic orthogonality condition holds:

$$X(t) \nmid \mathscr{T}_{S(\sigma(t))}\mathscr{S}, \qquad t < t_*. \tag{6.1}$$

In view of Lemma 3.1, (iii), the projection $\mathbf{\Pi} Y(t)$ is defined for $t < t_*$. We can rewrite condition (6.1) in the form

$$\Omega(X(t), \tau_j(t)) = 0, \qquad j = 1, 2, \tag{6.2}$$

where the vectors $\tau_j(t) = \tau_j(\sigma(t))$ defined in (3.3) generate the tangent space $\mathscr{T}_{S(\sigma(t))}\mathscr{S}$ to the manifold $\mathscr{S}$ at the point $S(\sigma(t))$. For convenience, instead of the parameters (b, v) we shall use the parameters (c, v), where

$$c(t) = b(t) - \int_0^t v(\tau)\,d\tau, \qquad \dot c(t) = \dot b(t) - v(t) = w(t) - v(t). \tag{6.3}$$

We show how to obtain the modulation equations for the parameters $c(t)$ and $v(t)$ from the orthogonality conditions (6.2). To do this we differentiate (6.2) with respect to t:

$$0 = \Omega(\dot X, \tau_j) + \Omega(X, \dot\tau_j) = \Omega(A_{v,w}X + T + \mathscr{N}, \tau_j) + \Omega(X, \dot\tau_j), \qquad j = 1, 2. \tag{6.4}$$

From the skew-symmetry relation (4.13) and the equalities (4.14) it follows that

$$\Omega(A_{v,w}X, \tau_1) = -\Omega(X, A_{v,w}[\tau_1]) = -\dot c\,\Omega(X, \tau_1'), \tag{6.5}$$

$$\begin{aligned}\Omega(A_{v,w}X, \tau_2) &= -\Omega(X, A_{v,w}[\tau_2]) = -\Omega(X, \dot c\,\tau_2' - \tau_1)\\ &= -\dot c\,\Omega(X, \tau_2') - \Omega(X, \tau_1) = -\dot c\,\Omega(X, \tau_2'),\end{aligned} \tag{6.6}$$

because $\Omega(X, \tau_1) = 0$. Further, by (4.11),

$$\Omega(T, \tau_1) = -\dot v\,\Omega(\tau_2, \tau_1) = \dot v\,\Omega(\tau_1, \tau_2), \qquad \Omega(T, \tau_2) = -\dot c\,\Omega(\tau_1, \tau_2). \tag{6.7}$$

Using the equalities (6.5)–(6.7), we rewrite (6.4) as follows:

$$\begin{cases} 0 = -\dot c\,\Omega(X, \tau_1') + \dot v\big(\Omega(\tau_1, \tau_2) + \Omega(X, \partial_v \tau_1)\big) + \Omega(\mathscr{N}, \tau_1),\\ 0 = -\dot c\big(\Omega(X, \tau_2') + \Omega(\tau_1, \tau_2)\big) + \dot v\,\Omega(X, \partial_v \tau_2) + \Omega(\mathscr{N}, \tau_2).\end{cases} \tag{6.8}$$

Since $\tau_2' = -\partial_v \tau_1$, the determinant of this system equals

$$D = \Omega^2(\tau_1, \tau_2) - \Omega(X, \tau_1')\Omega(X, \partial_v \tau_2) = \Omega^2(\tau_1, \tau_2) + \mathscr{O}(\|X\|^2_{E_{-\beta}}).$$

Recall that $\Omega(\tau_1, \tau_2) \neq 0$ by (3.5). Hence, the determinant D is not zero for small $\|X\|_{E_{-\beta}}$. Solving the system (6.8) gives the required modulation equations:

$$\dot c = \frac{\Omega(\tau_1, \tau_2)\Omega(\mathscr{N}, \tau_2) + \Omega(X, \partial_v \tau_1)\Omega(\mathscr{N}, \tau_2) - \Omega(X, \partial_v \tau_2)\Omega(\mathscr{N}, \tau_1)}{D}, \tag{6.9}$$

$$\dot v = \frac{-\Omega(\tau_1, \tau_2)\Omega(\mathscr{N}, \tau_1) - \Omega(X, \tau_2')\Omega(\mathscr{N}, \tau_1) + \Omega(X, \tau_1')\Omega(\mathscr{N}, \tau_2)}{D}. \tag{6.10}$$

Using these equations, we readily obtain estimates for $\dot{c}$ and $\dot{v}$:

$$|\dot{v}(t)|, |\dot{c}(t)| \leqslant C_0(\overline{v}) \|X(t)\|^2_{E_{-\beta}}, \qquad 0 \leqslant t < t_*, \tag{6.11}$$

where $C_0(\overline{v})$ is some constant.

7. Decay of the transversal dynamics

Here we prove the main result characterizing the decay rate of the transversal component $X(t)$.

Proposition 7.1. *Under the hypotheses of Theorem 2.1, $t_* = \infty$ and*

$$\|X(t)\|_{E_{-\beta}} \leqslant \frac{\varepsilon}{(1+|t|)^{3/2}}, \quad \|\Psi(t)\|_{L^\infty} \leqslant \frac{\varepsilon}{(1+|t|)^{1/2}}, \qquad t \geqslant 0, \tag{7.1}$$

where ε is defined in Definition 5.1.

To derive these estimates we shall employ equation (4.6) for the transversal component $X(t)$, with consideration of the orthogonality condition (6.1).

Establishing the decay estimate (7.1) encounters two main difficulties common to all problems of this kind (see, for example, [7]). First, the linear part of equation (4.6) is non-autonomous, and hence methods of scattering theory cannot be applied directly. Following the approach of [7], we first examine the *frozen* linear equation

$$\dot{X}(t) = A_{v_1} X(t), \qquad 0 \leqslant t \leqslant t_1, \quad v_1 = v(t_1), \tag{7.2}$$

where the operator A_v is defined in (4.15) and t_1 is some fixed number in the interval $[0, t_*)$. The resulting errors are then estimated. Second, even for the frozen equation (7.2) a decay of type (7.1) for an arbitrary solution is impossible without the orthogonality condition (6.1). Indeed, in view of the equalities (4.16), equation (7.2) has *secular solutions*

$$X(t) = C_1 \tau_1(v) + C_2[\tau_1(v)t + \tau_2(v)], \tag{7.3}$$

which arise when the soliton (2.4) is differentiated with respect to the parameters q and v in the moving coordinate system $y = x - v_1 t$. The solutions (7.3) lie in the tangent space $\mathscr{T}_{S(\sigma_1)}\mathscr{S}$, where $\sigma_1 = (b_1, v_1)$ (with arbitrary $b_1 \in \mathbb{R}$), implying the unstable nature of the non-linear dynamics *along the solitary manifold*. In order to exclude the secular solutions, we assume that the symplectic orthogonality condition (6.1) is fulfilled. It is this condition that eliminates the increasing solutions (7.3).

We let $\mathscr{X}_v = \mathbf{P}^{\mathrm{c}}_v E$ denote the space that is symplectically orthogonal to the space $\mathscr{T}_{S(\sigma)}\mathscr{S}$. Now we have at our disposal the symplectically orthogonal decomposition

$$\mathscr{T}_{S(\sigma)}\mathscr{E} = \mathscr{T}_{S(\sigma)}\mathscr{S} + \mathscr{X}_v, \qquad \sigma = (b, v), \tag{7.4}$$

so the symplectic orthogonality condition (6.1) can be written in the following equivalent form:

$$\mathbf{P}^{\mathrm{d}}_{v(t)} X(t) = 0, \quad \mathbf{P}^{\mathrm{c}}_{v(t)} X(t) = X(t), \qquad 0 \leqslant t < t_*. \tag{7.5}$$

Since in view of (4.16) the tangent space $\mathscr{T}_{S(\sigma)}\mathscr{S}$ is invariant under the operator A_v, it follows from (4.13) that the space $\mathscr{X}_v$ is also invariant, that is, $A_v X \in \mathscr{X}_v$ for a dense set of $X \in \mathscr{X}_v$.

7.1. Frozen transversal dynamics. We fix an arbitrary $t_1 \in [0, t_*)$ and rewrite equation (4.6) in the 'frozen' form

$$\dot{X}(t) = A_1 X(t) + (A(t) - A_1) X(t) + T(t) + \mathscr{N}(t), \qquad 0 \leqslant t \leqslant t_1, \tag{7.6}$$

where $A_1 = A_{v_1}$ and $v_1 = v(t_1)$. Using the inequalities (6.11), we have

$$\|T(t)\|_{E_\beta \cap W} \leqslant C(\overline{v}) \|X\|^2_{E_{-\beta}}, \qquad 0 \leqslant t \leqslant t_1, \tag{7.7}$$

since $w - v = \dot{c}$. Further, it follows from the estimate (4.29) that

$$\|\mathscr{N}(t)\|_{E_\beta \cap W} \leqslant C(\overline{v}) \|\Psi\|_{L^\infty} \|X\|_{E_{-\beta}}, \qquad 0 \leqslant t \leqslant t_1. \tag{7.8}$$

The elimination of the 'bad' term $(w(t) - v_1)\nabla$ in the operator $A(t) - A_1$ is achieved by the following trick. We make the change of variables $(y, t) \mapsto (y_1, t) = (y + d_1(t), t)$, where

$$d_1(t) := \int_{t_1}^{t} (w(s) - v_1)\, ds, \qquad 0 \leqslant t \leqslant t_1. \tag{7.9}$$

In the new variables (y_1, t) equation (7.6) for the transversal component takes the form

$$\dot{\widetilde{X}}(t) = A_1 \widetilde{X}(t) + \widetilde{\mathscr{V}}(t) \widetilde{X}(t) + \widetilde{T}(t) + \widetilde{\mathscr{N}}(t), \qquad 0 \leqslant t \leqslant t_1, \tag{7.10}$$

where

$$\widetilde{X}(y_1, t) = (\Psi(y_1 - d_1(t), t), \Pi(y_1 - d_1(t), t)), \qquad \widetilde{\mathscr{V}}(t) = \mathscr{V}_v(y_1 - d_1) - \mathscr{V}_{v_1}(y_1) \tag{7.11}$$

and $\widetilde{T}(t)$ and $\widetilde{\mathscr{N}}(t)$ denote, respectively, the functions $T(t)$ and $\mathscr{N}(t)$ expressed in the variables (y_1, t). Recall that the matrix potential $\mathscr{V}_v$ is defined in (4.25).

Now we proceed to estimate the 'remainders' in equation (7.10). To this end we first show that the translation $d_1(t)$ is uniformly small for $0 \leqslant t \leqslant t_1$.

Lemma 7.2. *For all $t_1 < t_*$,*

$$|d_1(t)| \leqslant C_0(\overline{v}) \varepsilon^2, \qquad 0 \leqslant t \leqslant t_1, \tag{7.12}$$

where ε is defined in Definition 5.1.

Proof. In view of (6.3),

$$w(s) - v_1 = w(s) - v(s) + v(s) - v_1 = \dot{c}(s) + \int_s^{t_1} \dot{v}(\tau)\, d\tau. \tag{7.13}$$

Hence, using definitions (5.4) and (7.9), as well as the estimates (6.11), we get that

$$\begin{aligned} |d_1(t)| &= \left| \int_{t_1}^{t} (w(s) - v_1)\, ds \right| \leqslant \int_t^{t_1} \left(|\dot{c}(s)| + \int_s^{t_1} |\dot{v}(\tau)|\, d\tau \right) ds \\ &\leqslant C_0(\overline{v}) m_1^2(t_1) \int_t^{t_1} \left(\frac{1}{(1+s)^3} + \int_s^{t_1} \frac{d\tau}{(1+\tau)^3} \right) ds \leqslant C_0(\overline{v}) m_1^2(t_1) \leqslant C_0(\overline{v}) \varepsilon^2 \end{aligned} \tag{7.14}$$

for all $0 \leqslant t \leqslant t_1$. □

We shall assume henceforth that

$$\varepsilon^2 < \frac{\nu}{2C_0(\overline{v})}, \tag{7.15}$$

where ν is defined in § 5. Then, in particular,

$$|d_1(t)| < \frac{\nu}{2} < 1. \tag{7.16}$$

Let us estimate the weighted norms of the 'translated' functions $\widetilde{T}(t)$ and $\widetilde{\mathscr{N}}(t)$ in terms of the weighted norms of the functions $T(t)$ and $\mathscr{N}(t)$. From the inequality

$$(1+|y_1-d_1|)^\alpha \leqslant (1+|y_1|)^\alpha (1+|d_1|)^{|\alpha|} \leqslant C(\alpha)(1+|y_1|)^\alpha, \tag{7.17}$$

which holds for all $\alpha \in \mathbb{R}$, it follows that

$$\|\widetilde{T}(t)\|_{E_\beta} \leqslant C(\beta)\|T(t)\|_{E_\beta}, \qquad \|\widetilde{\mathscr{N}}(t)\|_{E_\beta} \leqslant C(\beta)\|\mathscr{N}(t)\|_{E_\beta}.$$

Hence, using the estimates (7.7) and (7.8) for $T(t)$ and $\mathscr{N}(t)$, we obtain similar estimates for $\widetilde{T}(t)$ and $\widetilde{\mathscr{N}}(t)$:

$$\begin{aligned} \|\widetilde{T}(t)\|_{E_\beta \cap W} &\leqslant C(\overline{v})\|X\|^2_{E_{-\beta}}, \\ \|\widetilde{\mathscr{N}}(t)\|_{E_\beta \cap W} &\leqslant C(\overline{v})\|\Psi\|_{L^\infty}\|X\|_{E_{-\beta}}, \end{aligned} \qquad 0 \leqslant t \leqslant t_1. \tag{7.18}$$

Finally, we estimate the term $\widetilde{\mathscr{V}}(t)\widetilde{X}(t)$ on the right-hand side of (7.10). We can write $\widetilde{\mathscr{V}}(t)$ in the form

$$\widetilde{\mathscr{V}}(t) = \mathscr{V}_v(y_1-d_1)-\mathscr{V}_{v_1}(y_1) = (\mathscr{V}_v(y_1-d_1)-\mathscr{V}_{v_1}(y_1-d_1))+(\mathscr{V}_{v_1}(y_1-d_1)-\mathscr{V}_{v_1}(y_1)). \tag{7.19}$$

As in the case of (7.14), one shows that $|v(t)-v_1| \leqslant C_0(\overline{v})\varepsilon^2$ for $0 \leqslant t \leqslant t_1$. Hence,

$$\begin{aligned} |V_v(y)-V_{v_1}(y)| &\leqslant |v(t)-v_1| \max_{v\in[v(t),v_1]} |\partial_v V_v(y)| \\ &\leqslant C(\overline{v})\varepsilon^2 \max_{v\in[v(t),v_1]} |F''(\psi_v(y))\partial_v\psi_v(y)|. \end{aligned} \tag{7.20}$$

Further, the inequality (7.12) implies that

$$\begin{aligned} |V_{v_1}(y)-V_{v_1}(y_1)| &\leqslant |d_1(t)| \max_{z\in[y,y_1]} |F''(\psi_{v_1}(z))\psi'_{v_1}(z)| \\ &\leqslant C(\overline{v})\varepsilon^2 \max_{z\in[y,y_1]} |F''(\psi_{v_1}(z))\psi'_{v_1}(z)|. \end{aligned} \tag{7.21}$$

Using condition **U1**, the definition (4.25), and the estimates (7.19)–(7.21), we finally obtain

$$\|\widetilde{\mathscr{V}}(t)\widetilde{X}(t)\|_{E_\beta \cap W} \leqslant C(\overline{v})\varepsilon^2\|X\|_{E_{-\beta}}, \qquad 0 \leqslant t \leqslant t_1. \tag{7.22}$$

7.2. Integral inequalities. We write equation (7.10) in the integral form

$$\widetilde{X}(t) = e^{A_1 t}\widetilde{X}(0) + \int_0^t e^{A_1(t-s)}\big[\widetilde{\mathscr{V}}(s)\widetilde{X}(s) + \widetilde{T}(s) + \widetilde{\mathscr{N}}(s)\big]\,ds, \qquad 0 \leqslant t \leqslant t_1, \tag{7.23}$$

and apply the symplectic orthogonal projection $\mathbf{P}_1^{\mathrm{c}} := \mathbf{P}_{v_1}^{\mathrm{c}}$:

$$\mathbf{P}_1^{\mathrm{c}}\widetilde{X}(t) = e^{A_1 t}\mathbf{P}_1^{\mathrm{c}}\widetilde{X}(0) + \int_0^t e^{A_1(t-s)}\mathbf{P}_1^{\mathrm{c}}\big[\widetilde{\mathscr{V}}(s)\widetilde{X}(s) + \widetilde{T}(s) + \widetilde{\mathscr{N}}(s)\big]\,ds, \quad 0 \leqslant t \leqslant t_1.$$

Here we have used the fact that the operator $\mathbf{P}_1^{\mathrm{c}}$ commutes with the group $e^{A_1 t}$. Applying the inequality (4.23), we get that

$$\|\mathbf{P}_1^{\mathrm{c}}\widetilde{X}(t)\|_{E_{-\beta}} \leqslant \frac{C\|\widetilde{X}(0)\|_{E_\beta}}{(1+t)^{3/2}} + C\int_0^t \frac{\|\widetilde{\mathscr{V}}(s)\widetilde{X}(s) + \widetilde{T}(s) + \widetilde{\mathscr{N}}(s)\|_{E_\beta}}{(1+|t-s|)^{3/2}}\,ds$$

for all $\beta > 5/2$ and $0 \leqslant t \leqslant t_1$. By (7.16), (7.18), and (7.22),

$$\begin{aligned}\|\mathbf{P}_1^{\mathrm{c}}\widetilde{X}(t)\|_{E_{-\beta}} \leqslant{}& \frac{C\|X(0)\|_{E_\beta}}{(1+t)^{3/2}}\\ &+ C\int_0^t \frac{\varepsilon^2\|X(s)\|_{E_{-\beta}} + \|X(s)\|^2_{E_{-\beta}} + \|\Psi(s)\|_{L^\infty}\|X(s)\|_{E_{-\beta}}}{(1+|t-s|)^{3/2}}\,ds\end{aligned} \tag{7.24}$$

for all $\beta > 5/2$ and $0 \leqslant t \leqslant t_1$. Similarly, by (4.27), (7.16), (7.18), and (7.22),

$$\begin{aligned}\|(\mathbf{P}_1^{\mathrm{c}}\widetilde{X}(t))_1\|_{L^\infty} \leqslant C\bigg[&\frac{\|X(0)\|_{E_\beta\cap W}}{(1+t)^{1/2}}\\ &+ \int_0^t \frac{\varepsilon^2\|X(s)\|_{E_{-\beta}} + \|X(s)\|^2_{E_{-\beta}} + \|\Psi(s)\|_{L^\infty}\|X(s)\|_{E_{-\beta}}}{(1+|t-s|)^{1/2}}\,ds\bigg]\end{aligned} \tag{7.25}$$

for all $\beta > 5/2$ and $0 \leqslant t \leqslant t_1$.

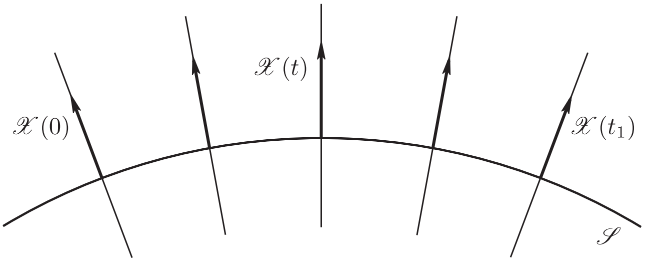

Figure 4. Symplectic orthogonality

7.3. Symplectic orthogonality. Our next objective is to replace $\mathbf{P}_1^{\mathrm{c}}\widetilde{X}(t)$ by $X(t)$ on the left-hand side of the inequalities (7.24) and (7.25). This can be done for sufficiently small ε by using the symplectic orthogonality (7.5) and the fact that the spaces $\mathscr{X}(t) := \mathbf{P}^{\rm c}_{v(t)}E$ are almost parallel for $t \in [0, t_1]$ (see Fig. 4). Consider the difference of projections $\mathbf{P}^{\rm d}_1 - \widetilde{\mathbf{P}}^{\rm d}_{v(t)}$, where

$$\widetilde{\mathbf{P}}^{\rm d}_{v(t)}X = \sum p_{jl}(v(t))\widetilde{\tau}_j(v(t))\Omega(\widetilde{\tau}_l(v(t)), X), \qquad X \in E, \tag{7.26}$$

and the projection $\mathbf{P}^{\rm d}_1 = \mathbf{P}^{\rm d}_{v(t_1)}$ is defined in (4.22). Here $\widetilde{\tau}_j(v(t))$ denotes the vectors $\tau_j(v(t))$ from (3.3) expressed in terms of y_1. We claim that this difference is uniformly small with respect to t for sufficiently small $\varepsilon > 0$. Since the τ_j' are smooth functions that decay sufficiently rapidly at infinity, it follows from the inequality (7.12) that

$$\|\widetilde{\tau}_j(v(t)) - \tau_j(v(t))\|_{E_\beta} \leqslant C(\overline{v})\varepsilon^2, \quad 0 \leqslant t \leqslant t_1, \qquad j = 1, 2. \tag{7.27}$$

Furthermore, for all $0 \leqslant t \leqslant t_1$,

$$\begin{aligned}\|\tau_j(v(t)) - \tau_j(v(t_1))\|_{E_\beta} &= \left\|\int_t^{t_1} \dot{v}(s)\partial_v\tau_j(v(s))\,ds\right\|_{E_\beta} \leqslant C\int_t^{t_1}|\dot{v}(s)|\,ds \leqslant C(\overline{v})\varepsilon^2,\\ |p_{jl}(v(t)) - p_{jl}(v(t_1))| &= \left|\int_t^{t_1} \dot{v}(s)\partial_v p_{jl}(v(s))\,ds\right| \leqslant C\int_t^{t_1}|\dot{v}(s)|\,ds \leqslant C(\overline{v})\varepsilon^2,\end{aligned} \tag{7.28}$$

since by (5.2) the quantities $|\partial_v p_{jl}(v(s))|$ are uniformly bounded. By (7.27) and (7.28),

$$\|\mathbf{P}^{\rm d}_1 - \widetilde{\mathbf{P}}^{\rm d}_{v(t)}\| < \frac{1}{2}, \qquad 0 \leqslant t \leqslant t_1, \tag{7.29}$$

for sufficiently small $\varepsilon > 0$. We have $\widetilde{\mathbf{P}}^{\rm d}_{v(t)}\widetilde{X}(t) = 0$, so from the last inequality it immediately follows that

$$\|\mathbf{P}^{\rm d}_1\widetilde{X}(t)\|_{E_{-\beta}} \leqslant \frac{1}{2}\|\widetilde{X}(t)\|_{E_{-\beta}}, \qquad 0 \leqslant t \leqslant t_1.$$

As a result, we infer from the equality $\mathbf{P}^{\rm c}_1\widetilde{X}(t) = \widetilde{X}(t) - \mathbf{P}^{\rm d}_1\widetilde{X}(t)$ and the estimate (7.16) that

$$\|X(t)\|_{E_{-\beta}} \leqslant C\|\widetilde{X}(t)\|_{E_{-\beta}} \leqslant 2C\|\mathbf{P}^{\rm c}_1\widetilde{X}(t)\|_{E_{-\beta}}, \qquad 0 \leqslant t \leqslant t_1, \tag{7.30}$$

for sufficiently small $\varepsilon > 0$, $t_* = t_*(\varepsilon)$, and all $t_1 < t_*$, where the constant C is independent of t_1. Moreover, it follows from the inequality (7.29) that for sufficiently small $\varepsilon > 0$

$$\|(\mathbf{P}^{\rm d}_1\widetilde{X}(t))_1\|_{L^\infty} \leqslant \frac{1}{2}\|\widetilde{X}(t)\|_{E_{-\beta}}, \qquad 0 \leqslant t \leqslant t_1.$$

Hence, taking into account the inequality (7.30), we get that

$$\begin{aligned}\|\Psi(t)\|_{L^\infty} = \|\widetilde{\Psi}(t)\|_{L^\infty} &\leqslant \|(\mathbf{P}^{\rm c}_1\widetilde{X}(t))_1\|_{L^\infty} + \|(\mathbf{P}^{\rm d}_1\widetilde{X}(t))_1\|_{L^\infty}\\ &\leqslant \|(\mathbf{P}^{\rm c}_1\widetilde{X}(t))_1\|_{L^\infty} + \frac{1}{2}\|\widetilde{X}(t)\|_{E_{-\beta}}\\ &\leqslant \|(\mathbf{P}^{\rm c}_1\widetilde{X}(t))_1\|_{L^\infty} + \|\mathbf{P}^{\rm c}_1\widetilde{X}(t)\|_{E_{-\beta}}, \qquad 0 \leqslant t \leqslant t_1.\end{aligned} \tag{7.31}$$

7.4. Decay of the transversal component. We can now complete the proof of Proposition 7.1. As noted above, it suffices to verify the inequality (5.6). We fix $\varepsilon > 0$ and $t_* = t_*(\varepsilon)$ to satisfy the estimates (7.15), (7.30), and (7.31). Now it is possible to replace the functions $\|\mathbf{P}_1^c\widetilde{X}(t)\|_{E_{-\beta}}$ and $\|(\mathbf{P}_1^c\widetilde{X}(t))_1\|_{L^\infty}$ on the left-hand sides of the inequalities (7.24) and (7.25) by the functions $\|X(t)\|_{E_{-\beta}}$ and $\|\Psi(t)\|_{L^\infty}$:

$$\|X(t)\|_{E_{-\beta}} \leqslant C\biggl[\frac{\|X(0)\|_{E_\beta}}{(1+t)^{3/2}} + \int_0^t \frac{\varepsilon^2\|X(s)\|_{E_{-\beta}} + \|X(s)\|^2_{E_{-\beta}} + \|\Psi(s)\|_{L^\infty}\|X(s)\|_{E_{-\beta}}}{(1+|t-s|)^{3/2}}\,ds\biggr], \qquad (7.32)$$

$$\|\Psi(t)\|_{L^\infty} \leqslant C\biggl[\frac{\|X(0)\|_{E_\beta\cap W}}{(1+t)^{1/2}} + \int_0^t \frac{\varepsilon^2\|X(s)\|_{E_{-\beta}} + \|X(s)\|^2_{E_{-\beta}} + \|\Psi(s)\|_{L^\infty}\|X(s)\|_{E_{-\beta}}}{(1+|t-s|)^{1/2}}\,ds\biggr], \qquad (7.33)$$

where $0 \leqslant t \leqslant t_1 < t_*$. We shall use these estimates to derive integral inequalities for the majorants m_1 and m_2. Multiplying both sides of (7.32) by $(1+t)^{3/2}$ and taking the supremum over $t \in [0, t_1]$, we get that

$$m_1(t_1) \leqslant C\|X(0)\|_{E_\beta} + C \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{3/2}\,ds}{(1+|t-s|)^{3/2}} \times \biggl[\frac{\varepsilon^2 m_1(s)}{(1+s)^{3/2}} + \frac{m_1^2(s)}{(1+s)^3} + \frac{m_1(s)m_2(s)}{(1+s)^2}\biggr].$$

Since $m(t)$ is a monotonically increasing function, it follows from the last inequality that

$$m_1(t_1) \leqslant Cd_0 + C[\varepsilon^2 m_1(t_1) + m_1^2(t_1) + m_1(t_1)m_2(t_1)]I_1(t_1), \qquad t_1 < t_*, \qquad (7.34)$$

where d_0 is defined in (2.8) and

$$I_1(t_1) = \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{3/2}}{(1+|t-s|)^{3/2}}\,\frac{ds}{(1+s)^{3/2}}\,.$$

Splitting the last integral into the two integrals over the intervals $0 \leqslant s \leqslant t/2$ and $t/2 \leqslant s \leqslant t$, we easily check that this integral is bounded by a constant independent of t. Therefore, by (7.34) there exists a constant C independent of t_1 such that

$$m_1(t_1) \leqslant Cd_0 + C\bigl[\varepsilon^2 m_1(t_1) + m_1^2(t_1) + m_1(t_1)m_2(t_1)\bigr], \qquad t_1 < t_*. \qquad (7.35)$$

Similarly, multiplying both sides of (7.33) by $(1+t)^{1/2}$, we obtain

$$m_2(t_1) \leqslant Cd_0 + C\bigl[\varepsilon^2 m_1(t_1) + m_1^2(t_1) + m_1(t_1)m_2(t_1)\bigr], \qquad t_1 < t_*, \qquad (7.36)$$

where the constant C is independent of t_1. Let $M(t)$ be the vector with components $m_1(t)$ and $m_2(t)$. Using the inequalities (7.35) and (7.36), we find that

$$|M(t_1)| \leqslant Cd_0 + C\bigl[\varepsilon^2|M(t_1)| + |M(t_1)|^2\bigr], \qquad t_1 < t_*.$$

Since $m_i(t_1) < \varepsilon$ by (5.5), the function $M(t_1)$ is bounded for sufficiently small d_0 and ε:

$$|M(t_1)| \leqslant C d_0, \qquad t_1 < t_*, \tag{7.37}$$

where the constant $C = C(\overline{v})$ is independent of t_*. We choose d_0 in (2.8) to be small enough that $d_0 < \varepsilon/(2C)$. Now the inequalities (5.6) for the majorants m_j are immediate from (7.37). Further, using the inequalities (6.11), (7.1), and (7.15), we get that

$$|v(t) - v(0)| \leqslant \int_0^t |\dot{v}(s)|\, ds \leqslant C_0(\overline{v})\varepsilon^2 < \frac{\nu}{2}, \qquad 0 \leqslant t < t_*,$$

that is, the first inequality in (5.6) also holds. Hence, $t_* = \infty$ and the estimate (7.37) holds for all $t_1 > 0$.

8. Soliton asymptotics

We proceed to derive the main Theorem 2.1 from the inequality (7.1). From (6.11) and (7.1) it follows that for all $t \geqslant 0$

$$|\dot{c}(t)| + |\dot{v}(t)| \leqslant C(\overline{v}, d_0)(1+t)^{-3}. \tag{8.1}$$

Hence, $c(t) = c_+ + \mathscr{O}(t^{-2})$ and $v(t) = v_+ + \mathscr{O}(t^{-2})$ as $t \to \infty$, and therefore

$$b(t) = c(t) + \int_0^t v(s)\, ds = v_+ t + q_+ + \mathscr{O}(t^{-1}), \qquad t \to \infty, \tag{8.2}$$

where c_+, v_+, and q_+ are some constants. We can write the solution $Y(x,t)$ of equation (2.1) in the form

$$Y(x,t) = Y_{v(t)}(x - b(t), t) + X(x - b(t), t). \tag{8.3}$$

Since $\|Y_{v(t)}(x - b(t), t) - Y_{v_+}(x - v_+ t - q_+, t)\|_E = \mathscr{O}(t^{-1})$, to prove the asymptotics (2.9) with remainder term (2.10) of order $t^{-1/2}$ it suffices to extract the dispersive wave $W_0(t)\Phi_+$ from the term $X(x - b(t), t)$. Substituting (8.3) into (2.1) and using (4.4), we arrive at the following inhomogeneous equation for the vector function $X(x - b(t), t) = (\Psi(x - b(t), t), \Pi(x - b(t), t))$:

$$\dot{X}(y,t) = A_v^0 X(y,t) + R(y,t), \qquad y = x - b(t), \tag{8.4}$$

where

$$A_v^0 = \begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 & v\nabla \end{pmatrix}, \qquad R(t) = \begin{pmatrix} -\dot{v}\partial_v \psi_v \\ -\dot{v}\partial_v \pi_v - V_v \Psi(t) + N(v, \Psi(t)) \end{pmatrix}.$$

Equation (8.4) in the variable $x = y + b(t)$ has the form

$$\dot{\widetilde{X}}(t) = A_0^0 \widetilde{X}(t) + \widetilde{R}(t), \qquad t \geqslant 0,$$

where $\widetilde{X}(x,t) = X(x-b(t),t)$, $\widetilde{R}(x,t) = R(x-b(t),t)$, and A_0^0 is the operator A_v^0 corresponding to $v=0$. Hence,

$$\begin{aligned}\widetilde{X}(t) &= W_0(t)\widetilde{X}(0) + \int_0^t W_0(t-s)\widetilde{R}(s)\,ds \\ &= W_0(t)\biggl(\widetilde{X}(0) + \int_0^\infty W_0(-s)\widetilde{R}(s)\,ds\biggr) - \int_t^\infty W_0(t-s)\widetilde{R}(s)\,ds \\ &= W_0(t)\Phi_+ + r_+(t),\end{aligned}$$

where $W_0(t)$ is the dynamical group of the free Klein–Gordon equation. To prove the asymptotics (2.9), it suffices to verify that

$$\begin{aligned}\Phi_+ &= \widetilde{X}(0) + \int_0^\infty W_0(-s)\widetilde{R}(s)\,ds \in E, \\ \|r_+(t)\|_E &= \biggl\|\int_t^\infty W_0(t-s)\widetilde{R}(s)\,ds\biggr\|_E = \mathscr{O}(t^{-1/2}).\end{aligned} \tag{8.5}$$

Condition (2.8) implies that $\widetilde{X}(0)\in E$. We can represent $\widetilde{R}(s)$ as the sum

$$\widetilde{R}(s) = \begin{pmatrix} -\dot v\partial_v\widetilde{\psi}_v \\ -\dot v\partial_v\widetilde{\pi}_v\end{pmatrix} + \begin{pmatrix} 0 \\ -\widetilde{V}_v\widetilde{\Psi}(s) + \widetilde{N}(v,\widetilde{\Psi}(s))\end{pmatrix} = \widetilde{R}'(s) + \widetilde{R}''(s).$$

By virtue of the inequality (8.1),

$$\|\widetilde{R}'(s)\|_E = \|R'(s)\|_E = \mathscr{O}(s^{-3}). \tag{8.6}$$

Furthermore, from the estimates (7.1)

$$\|\widetilde{V}_v\widetilde{\Psi}(s)\|_{L^2} = \|V_v\Psi(s)\|_{L^2} \leqslant C\|\Psi(s)\|_{L^2_{-\beta}} \leqslant C(\overline{v},d_0)(1+|s|)^{-3/2},$$

because the potential V_v is compactly supported. Similarly, by (7.1) and (7.8),

$$\|\widetilde{N}(v,\widetilde{\Psi}(s))\|_{L^2} = \|N(v,\Psi(s))\|_{L^2} \leqslant C(\overline{v},d_0)(1+|s|)^{-3/2}.$$

The last two inequalities imply that $\|\widetilde{R}''(s)\|_E = \mathscr{O}(s^{-3/2})$. Together with (8.6) this means that $\|\widetilde{R}(s)\|_E = \mathscr{O}(s^{-3/2})$, and hence the integrals in (8.5) converge due to the 'unitarity' of the group $W_0(t)$ in the space E. The asymptotic behaviour (8.5) for $r_+(t)$ is proved similarly.

Chapter II

Standing soliton

In this chapter we prove the asymptotic stability of kinks in the more complicated spectral situation when there is an additional discrete spectrum of the linearized dynamics. For simplicity we examine a 'standing kink' (a kink with zero velocity $v=0$) and its odd perturbations.

9. Statement of the main result

Thus, we shall be concerned only with odd solutions $Y(-x,t) = -Y(x,t)$. The space of odd states is invariant under the dynamical group of equation (2.3), because in view of condition **U1** the potential $U(\psi)$ is an even function, and hence the function $F(\psi)$ is odd. The main result here is the following theorem.

Theorem 9.1. *Assume that conditions* **U1**–**U3**, **D2**, *and* **F** *are satisfied. Let* $Y(t)$ *be the solution of the Cauchy problem* (2.3) *with odd initial data* $Y_0 \in \mathscr{E}$ *that is sufficiently close to the kink,*

$$Y_0 = (s(x), 0) + X_0, \qquad d_0 := \|X_0\|_{E_\beta \cap W} \ll 1, \tag{9.1}$$

where $\beta > 5/2$. *Then the following asymptotics holds*:

$$Y(x,t) = (s(x), 0) + W_0(t)\Phi_\pm + r_\pm(x,t), \qquad t \to \pm\infty, \tag{9.2}$$

where $\Phi_\pm \in E$ *are the asymptotic scattering states, and* $W_0(t)$ *is the dynamical group of the free Klein–Gordon equation. Moreover,*

$$\|r_\pm(t)\|_E = \mathscr{O}(|t|^{-1/3}), \qquad t \to \pm\infty. \tag{9.3}$$

10. Linearization at the soliton

Decomposing the solution of equation (2.3) into a sum $Y(t) = S + X(t)$ with $S = (s, 0)$, we obtain the equation for $X(t)$:

$$\dot{X}(t) = AX(t) + \mathscr{N}(X(t)), \qquad t \in \mathbb{R}, \tag{10.1}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\Delta + m^2 + V(x) & 0 \end{pmatrix} \quad \text{and} \quad V(x) = -F'(s(x)) - m^2 = U''(s(x)) - m^2. \tag{10.2}$$

The non-linear part $\mathscr{N}(X)$ is

$$\mathscr{N}(X) = \begin{pmatrix} 0 \\ N(\Psi) \end{pmatrix}, \qquad N(\Psi) = F(s + \Psi) - F(s) - F'(s)\Psi. \tag{10.3}$$

The continuous spectrum of A coincides with the interval $\Gamma := (-i\infty, -im] \cup [im, i\infty)$. The edge points $\pm im$ of the continuous spectrum are neither eigenvalues nor resonances of the operator A by condition **U3**. We proceed to find the discrete spectrum of A. To do so we examine the spectral equation

$$A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \Lambda\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $u = (u_1, u_2) \in L^2 \oplus L^2$. The first equation gives $u_2 = \Lambda u_1$. Substituting into the second equation, we obtain $(H + \Lambda^2)u_1 = 0$. From condition **D2** it follows that Λ^2 can assume only one value: $\Lambda^2 = -\lambda_1$. Hence, on the subspace of odd functions the operator A has two purely imaginary eigenvalues $\Lambda = \pm i\mu$, $\mu = \sqrt{\lambda_1}$. The corresponding eigenfunctions are as follows:

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \varphi_{\lambda_1} \\ i\mu\varphi_{\lambda_1} \end{pmatrix}, \qquad \overline{u} = \begin{pmatrix} \varphi_{\lambda_1} \\ -i\mu\varphi_{\lambda_1} \end{pmatrix}, \tag{10.4}$$

where φ_{λ_1} is the eigenfunction of H corresponding to the eigenvalue λ_1. Note that the function φ_{λ_1} can be assumed to be real, because the differential operator H has real coefficients.

Decay of the linearized dynamics. We consider the linearized equation

$$\dot{X}(t) = AX(t), \qquad t \in \mathbb{R}.$$

Let $\langle\,\cdot\,,\cdot\,\rangle$ denote the inner product in $L^2(\mathbb{R}, \mathbb{C}^2)$. We also define the symplectic projection P^{d} onto the eigenspace $\mathscr{X}^{\mathrm{d}}$ generated by the eigenfunctions u and $\overline{u}$:

$$P^{\mathrm{d}}X = \frac{\langle X, ju\rangle}{\langle u, ju\rangle}u + \frac{\langle X, j\overline{u}\rangle}{\langle \overline{u}, j\overline{u}\rangle}\overline{u}, \qquad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{10.5}$$

Note that if a function X is real, then its projection $P^{\mathrm{d}}X$ is also real. Let $\mathscr{X}^{\mathrm{c}}$ be the continuous-spectrum subspace of the operator A and let $P^{\mathrm{c}} = 1 - P^{\mathrm{d}}$ be the projection onto this subspace. The operator A satisfies estimates analogous to (4.23) and (4.27) in the first chapter. Namely,

$$\|e^{At}P^{\mathrm{c}}X\|_{E_{-\beta}} \leqslant C(1+t)^{-3/2}\|X\|_{E_\beta}, \qquad t \in \mathbb{R}, \tag{10.6}$$

$$\|(e^{At}P^{\mathrm{c}}X)_1\|_{L^\infty} \leqslant C(1+t)^{-1/2}(\|X\|_W + \|X\|_{E_\beta}), \qquad t \in \mathbb{R}, \tag{10.7}$$

for any $\beta > 5/2$. We shall also require the following estimate, whose proof may be found in [28]:

$$\|e^{At}(A \mp 2i\mu - 0)^{-1}P^{\mathrm{c}}X\|_{E_{-\beta}} \leqslant C(1+t)^{-3/2}\|X\|_{E_\beta}, \qquad \beta > \frac{5}{2}, \quad t \in \mathbb{R}. \tag{10.8}$$

11. Asymptotic decomposition of the dynamics

We shall seek the solution of equation (10.1) as the sum $X(t) := w(t) + f(t)$, where the function $w(t) = z(t)u + \overline{z}(t)\overline{u}$ lies in the space $\mathscr{X}^{\mathrm{d}}$, and the function $f(t)$ lies in $\mathscr{X}^{\mathrm{c}}$. Let us derive the dynamical equations for $z(t)$ and $f(t)$.

Applying the projection P^{d} to both sides of (10.1), we get that

$$\dot{z}u + \dot{\overline{z}}\,\overline{u} = Aw + P^{\mathrm{d}}\mathscr{N}. \tag{11.1}$$

Since $\langle \overline{u}, ju\rangle = 0$, $Aw = i\mu(zu - \overline{z}\,\overline{u})$, and $(P^{\mathrm{d}})^*j = jP^{\mathrm{d}}$, scalar multiplication of equation (11.1) by ju gives us

$$(\dot{z} - i\mu z)\langle u, ju\rangle = \langle \mathscr{N}, ju\rangle. \tag{11.2}$$

Applying the projection P^{c} to both sides of (10.1), we get an equation for $f(t)$:

$$\dot{f} = Af + P^{\mathrm{c}}\mathscr{N}. \tag{11.3}$$

Remark 11.1. Below we shall prove the following asymptotics for the functions $z(t)$ and $f(t)$:

$$\|f(t)\|_{E_{-\beta}} \sim t^{-1}, \quad z(t) \sim t^{-1/2}, \quad \|f_1(t)\|_{L^\infty} \sim t^{-1/2}, \qquad t \to \infty. \tag{11.4}$$

Using these asymptotics, we expand the right-hand side of equation (11.2) up to terms of order $\mathscr{O}(t^{-3/2})$ and the right-hand side of (11.3) up to terms of order $\mathscr{O}(t^{-1})$, and then we prove the asymptotics (11.4) by the method of majorants.

To begin with, we expand the non-linear term $N(x,\Psi)$ defined in (10.3) in a Taylor series,

$$N(x,\Psi) = N_2(x,\Psi) + N_R(x,\Psi), \tag{11.5}$$

where

$$N_2(x,\Psi) = \frac{F''(s(x))}{2}\Psi^2(x), \quad N_R(x,\Psi) = \frac{\Psi^3(x)}{3!}\int_0^1 (1-\rho)^2 F'''(s(x)+\rho\Psi(x))\,d\rho.$$

By condition **U2**, the function $F'''(\psi)$ vanishes in some neighbourhood of the points $\pm a$. Hence $N_R(x,\Psi)=0$ for large x, giving the following estimate:

$$\|N_R\|_{L^2_\beta \cap W^{1,1}_0} = \mathscr{R}(|z| + \|f_1\|_{L^\infty})\big(|z|^3 + |z|^2\|f\|_{E_{-\beta}} + |z|\|f_1\|_{L^\infty}\|f\|_{E_{-\beta}} + \|f_1\|^2_{L^\infty}\|f\|_{E_{-\beta}}\big). \tag{11.6}$$

We also define the symmetric bilinear form $\mathscr{N}_2[X_1,X_2] = (0, N_2[\Psi_1,\Psi_2])$ and the symmetric trilinear form $\mathscr{N}_3[X_1,X_2,X_3] = (0, N_3[\Psi_1,\Psi_2,\Psi_3])$, where

$$N_2[\Psi_1,\Psi_2] = \frac{F''(s)}{2}\Psi_1\Psi_2, \qquad N_3[\Psi_1,\Psi_2,\Psi_3] = \frac{F'''(s)}{6}\Psi_1\Psi_2\Psi_3. \tag{11.7}$$

11.1. Asymptotic decomposition of $\dot{z}$. We rewrite equation (11.2) as follows:

$$\dot{z} - i\mu z = \frac{\langle \mathscr{N}, ju\rangle}{\langle u, ju\rangle} = \frac{\langle \mathscr{N}_2[w,w] + 2\mathscr{N}_2[w,f] + \mathscr{N}_3[w,w,w], ju\rangle}{\langle u, ju\rangle} + Z_R, \tag{11.8}$$

where

$$|Z_R| = \mathscr{R}(|z| + \|f_1\|_{L^\infty})(|z|^2 + \|f\|_{E_{-\beta}})^2. \tag{11.9}$$

We have

$$\mathscr{N}_2[w,w] = (z^2+2z\overline{z}+\overline{z}^2)\mathscr{N}_2[u,u], \quad \mathscr{N}_3[w,w,w] = (z^3+3z^2\overline{z}+3z\overline{z}^2+\overline{z}^3)\mathscr{N}_3[u,u,u], \tag{11.10}$$

and thus (11.8) gives us that

$$\dot{z} = i\mu z + Z_2(z^2+2z\overline{z}+\overline{z}^2) + Z_3(z^3+3z^2\overline{z}+3z\overline{z}^2+\overline{z}^3) + (z+\overline{z})\langle f, jZ_1'\rangle + Z_R, \tag{11.11}$$

where

$$Z_2 = \frac{\langle \mathscr{N}_2[u,u], ju\rangle}{\langle u, ju\rangle}, \quad Z_3 = \frac{\langle \mathscr{N}_3[u,u,u], ju\rangle}{\langle u, ju\rangle}, \quad Z_1' = 2\frac{\mathscr{N}_2[u,u]}{\langle u, ju\rangle} \tag{11.12}$$

in view of (11.7). Note that (10.4) implies that $\langle u, ju\rangle$ is purely imaginary:

$$\langle u, ju\rangle = 2i\mu|\varphi_{\lambda_1}|^2 = i\delta, \quad \text{where } \delta = 2\mu|\varphi_{\lambda_1}|^2 > 0. \tag{11.13}$$

Hence, the coefficients Z_2, Z_3, and Z_1' are also purely imaginary.

11.2. Asymptotic decomposition of $\dot{f}$. We rewrite equation (11.3) as follows:

$$\dot{f} = Af + P^{\mathrm{c}}\mathscr{N} = Af + P^{\mathrm{c}}\mathscr{N}_2[w,w] + F_R. \tag{11.14}$$

For the remainder $F_R = F_R(x,t)$ we have

$$F_R = P^{\mathrm{c}}(\mathscr{N}_R - 2\mathscr{N}_2[f,w] - \mathscr{N}_2[f,f]) = (1-P^{\mathrm{d}})(\mathscr{N}_R - 2\mathscr{N}_2[f,w] - \mathscr{N}_2[f,f]), \tag{11.15}$$

where $\mathscr{N}_R = (0, N_R)$ and N_R is defined in (11.5). Using the estimate (11.6), we obtain

$$\|F_R\|_{E_\beta \cap W} = \mathscr{R}(|z| + \|f_1\|_{L^\infty})(|z|^3 + |z|\,\|f\|_{E_{-\beta}} + \|f_1\|_{L^\infty}\|f\|_{E_{-\beta}}). \tag{11.16}$$

12. Poincaré normal forms

In this section we shall get rid of the ‘non-resonance’ terms in (11.8) and (11.14) and obtain the so-called Poincaré ‘normal forms’ for these equations.

12.1. Normal form for $\dot{f}$. Writing equation (11.14) in more detail, we obtain

$$\dot{f} = Af + (z^2 + 2z\overline{z} + \overline{z}^2)F_2 + F_R, \qquad F_2 = P^{\mathrm{c}}\mathscr{N}_2[u,u]. \tag{12.1}$$

To single out the terms of order $z^2 \sim t^{-1}$ in (12.1), we represent f as the sum

$$f = h + k + g, \tag{12.2}$$

where

$$k = a_{20}z^2 + 2a_{11}z\overline{z} + a_{02}\overline{z}^2 \tag{12.3}$$

with coefficients $a_{ij}(x)$ such that $a_{ji}(x) = \overline{a}_{ij}(x)$, and

$$g(t) = -e^{At}k(0). \tag{12.4}$$

Note that $h(0) = f(0)$.

Lemma 12.1. *There exist functions a_{ij} in the space $E_{-\beta}$ such that the function $h = f - k - g$ obeys the equation*

$$\dot{h} = Ah + H_R, \tag{12.5}$$

where

$$H_R = F_R + H_I, \qquad H_I = \sum a_{ij}(x)\mathscr{R}(|z| + \|f_1\|_{L^\infty})|z|(|z| + \|f\|_{E_{-\beta}})^2.$$

Proof. Substituting the equalities (12.3), (12.4) into equation (12.1), we get that

$$\begin{aligned}
\dot{h} &= \dot{f} - (2a_{20}z + 2a_{11}\overline{z})\dot{z} - (2a_{11}z + 2a_{02}\overline{z})\dot{\overline{z}} - \dot{g} \\
&= Af + (z^2 + 2z\overline{z} + \overline{z}^2)F_2 + F_R \\
&\quad - (2a_{20}z + 2a_{11}\overline{z})\big(i\mu z + \mathscr{R}(|z| + \|f\|_{L^\infty})(|z| + \|f\|_{E_{-\beta}})^2\big) \\
&\quad - (2a_{11}z + 2a_{02}\overline{z})\big(-i\mu\overline{z} + \mathscr{R}(|z| + \|f_1\|_{L^\infty})(|z| + \|f\|_{E_{-\beta}})^2\big) - Ag.
\end{aligned} \tag{12.6}$$

On the other hand, (12.5) implies that

$$\dot h = A(f - a_{20}z^2 - 2a_{11}z\overline{z} - a_{02}\overline{z}^2 - g) + H_R. \tag{12.7}$$

If we compare the coefficients of z^2, $z\overline{z}$, and $\overline{z}^2$ on the right-hand sides of (12.6) and (12.7), we find that

$$F_2 - 2i\mu a_{20} = -Aa_{20}, \quad F_2 = -Aa_{11}, \quad F_2 + 2i\mu a_{02} = -Aa_{02}. \tag{12.8}$$

Since $A\colon H^{s_1+2} \oplus H^{s_2} \to H^{s_2} \oplus H^{s_1}$ is an elliptic operator with no kernel in the space of odd functions, there exists a continuous inverse operator $A^{-1}\colon H^{s_2} \oplus H^{s_1} \to H^{s_1+2} \oplus H^{s_2}$, and from the second equation in (12.8) we obtain

$$a_{11} = -A^{-1}F_2, \tag{12.9}$$

where $F_2 = P^{\mathrm{c}}\mathscr{N}_2[u,u] \in H^{s_2} \oplus H^{s_1}$ for any $s_1, s_2 > 0$. Further, the coefficients a_{20} and a_{02} are obtained from the first and third equations in (12.8):

$$a_{20} = -(A + 2i\mu - 0)^{-1}F_2, \qquad a_{02} = \overline{a}_{20} = -(A - 2i\mu - 0)^{-1}F_2. \tag{12.10}$$

As we shall show below, the estimates (10.8) are guaranteed by such a choice of inverse operators. The points $\pm 2i\mu$ lie in the continuous spectrum, and hence by the limiting absorption principle (see [1], [19]), these inverse operators do exist and act continuously from the space E_σ into the space $E_{-\sigma}$ with any $\sigma > 1/2$. □

The term H_I in the remainder H_R can be written as

$$H_I = \sum_n (A - 2i\mu n - 0)^{-1}C_n, \qquad n \in \{-1, 0, 1\}, \tag{12.11}$$

where the functions $C_n \in E^{\mathrm{c}}$ satisfy the estimate

$$\|C_n\|_{E_\beta} = \mathscr{R}(|z| + \|f\|_{E_{-\beta}})|z|(|z| + \|f\|_{E_{-\beta}})^2. \tag{12.12}$$

12.2. Normal form for $\dot z$. Substituting the decomposition (12.2) into (11.11) for z, we see that

$$\begin{aligned}\dot z = i\mu z &+ Z_2(z^2 + 2z\overline{z} + \overline{z}^2) + Z_3(z^3 + 3z^2\overline{z} + 3z\overline{z}^2 + \overline{z}^3) \\ &+ Z'_{30}z^3 + Z'_{21}z^2\overline{z} + Z'_{12}z\overline{z}^2 + Z'_{03}\overline{z}^3 + \widetilde{Z}_R,\end{aligned} \tag{12.13}$$

where

$$\begin{aligned} Z'_{30} &= \langle a_{20}, jZ'_1\rangle, \qquad & Z'_{21} &= \langle a_{11} + a_{20}, jZ'_1\rangle, \\ Z'_{03} &= \langle a_{02}, jZ'_1\rangle, \qquad & Z'_{12} &= \langle a_{02} + a_{11}, jZ'_1\rangle. \end{aligned} \tag{12.14}$$

The new remainder $\widetilde{Z}_R$ is of the form $Z_R + (z + \overline{z})\langle f - k, jZ'_1\rangle$, where Z_R satisfies the estimate (11.9). Since $f - k = g + h$, we have

$$|\langle f - k, Z'_1\rangle| \leqslant C(\|g\|_{E_{-\beta}} + \|h\|_{E_{-\beta}}).$$

This, together with (11.9), gives

$$|\widetilde{Z}_R| = \mathscr{R}_1(|z| + \|f\|_{L^\infty})\big[(|z|^2 + \|f\|_{E_{-\beta}})^2 + |z|\,\|g\|_{E_{-\beta}} + |z|\,\|h\|_{E_{-\beta}}\big]. \tag{12.15}$$

It is worth pointing out that the resonance terms involving $z^2\overline{z} = |z|^2 z$ are of special importance in equation (12.13). Namely, Poincaré's method of normal forms lets us eliminate all the polynomial terms on the right-hand side of (12.13) except for the first term and the resonance terms.

We claim that the real part of the coefficient Z'_{21} is strictly negative if the non-degeneracy condition **F** is fulfilled. By (11.12), (12.9), and (12.10) it follows that

$$\begin{aligned} Z'_{21} = &- \left\langle A^{-1}P^{\rm c}\mathscr{N}_2[u,u], 2j\,\frac{\mathscr{N}_2[u,u]}{\langle u, ju\rangle} \right\rangle \\ &- \left\langle (A-2i\mu-0)^{-1}P^{\rm c}\mathscr{N}_2[u,u], 2j\,\frac{\mathscr{N}_2[u,u]}{\langle u, ju\rangle} \right\rangle. \end{aligned} \tag{12.16}$$

The operator $A^{-1}P^{\rm c}j$ is self-adjoint, and hence $\langle A^{-1}P^{\rm c}j\mathscr{N}_2[u,u], 2\,\mathscr{N}_2[u,u]\rangle$ is a real number. Consequently, the first term on the right-hand side of (12.16) is purely imaginary. Therefore,

$$\begin{aligned} \operatorname{Re} Z'_{21} &= 2\operatorname{Re}\frac{\langle (A-2i\mu-0)^{-1}P^{\rm c}\mathscr{N}_2[u,u], j\mathscr{N}_2[u,u]\rangle}{i\delta} \\ &= \frac{2}{\delta}\operatorname{Im}\langle R(2i\mu+0)P^{\rm c}\mathscr{N}_2[u,u], j\mathscr{N}_2[u,u]\rangle, \end{aligned}$$

where $R(\lambda) = (A-\lambda)^{-1}$, $\operatorname{Re}\lambda > 0$, is the resolvent of the operator A. Since the projection $P^{\rm c}$ commutes with the resolvent, we have

$$R(2i\mu+0)P^{\rm c} = P^{\rm c}R(2i\mu+0)P^{\rm c}.$$

Furthermore, $(P^{\rm c})^* j = jP^{\rm c}$, and therefore

$$\operatorname{Re} Z'_{21} = \frac{2}{\delta}\operatorname{Im}\langle R(2i\mu+0)\alpha, j\alpha\rangle, \qquad \alpha = P^{\rm c}\mathscr{N}_2[u,u]. \tag{12.17}$$

Now we employ the following spectral representation (see [7], formula (2.1.9)):

$$\begin{aligned} &\langle R(2i\mu+0)\alpha, j\alpha\rangle \\ &\quad = \frac{1}{i}\int_m^\infty \theta(\lambda)\,d\lambda\left(\frac{\langle\alpha, ju(i\lambda)\rangle\langle u(i\lambda), j\alpha\rangle}{i\lambda-2i\mu-0} + \frac{\langle\alpha, j\overline{u}(i\lambda)\rangle\langle\overline{u}(i\lambda), j\alpha\rangle}{-i\lambda-2i\mu-0}\right) \\ &\quad = \int_m^\infty \theta(\lambda)\,d\lambda\left(\frac{\langle u(i\lambda), j\alpha\rangle\overline{\langle u(i\lambda), j\alpha\rangle}}{\lambda-2\mu+i0} + \frac{\langle\overline{u}(i\lambda), j\alpha\rangle\overline{\langle\overline{u}(i\lambda), j\alpha\rangle}}{\lambda+2\mu-i0}\right), \end{aligned}$$

where $\theta(\lambda) = 1/(2\pi N^2(\lambda)\sqrt{\lambda-m}\,)$ and $N(\lambda)$ is a certain real normalizing factor. Taking into account the equality $1/(\nu+i0) = \text{p.v.}(1/\nu) - i\pi\delta(\nu)$, where p.v. denotes the Cauchy principal value, we have

$$\begin{aligned} \langle R(2i\mu+0)\alpha, j\alpha\rangle = &\int_m^\infty \theta(\lambda)\,d\lambda\left(\frac{|\langle u(i\lambda), j\alpha\rangle|^2}{\lambda-2\mu} + \frac{|\langle\overline{u}(i\lambda), j\alpha\rangle|^2}{\lambda+2\mu}\right) \\ &- i\pi\theta(2\mu)|\langle u(2i\mu), j\alpha\rangle|^2. \end{aligned}$$

The integrand is real, hence

$$\operatorname{Im}\langle R(2i\mu+0)\alpha, j\alpha\rangle = -\pi\theta(2\mu)|\langle u(2i\mu), j\alpha\rangle|^2 < 0, \tag{12.18}$$

because $\theta(2\mu) > 0$ and

$$\begin{aligned}\langle u(2i\mu), j\alpha\rangle &= \langle u(2i\mu), jP^{\mathrm{c}}\mathscr{N}_2[u,u]\rangle = \langle u(2i\mu), j\mathscr{N}_2[u,u]\rangle \\ &= -\int u_1(2i\mu)(x) N_2[u,u](x)\,dx \\ &= -\frac{1}{2}\int \varphi_{4\lambda_1}(x)F''(s(x))\varphi_{\lambda_1}^2(x)\,dx \neq 0\end{aligned}$$

in view of condition **F**. As a result it follows from (12.17) and (12.18) that

$$\operatorname{Re} Z'_{21} < 0. \tag{12.19}$$

Now we apply Poincaré's method of normal forms to equation (12.13).

Lemma 12.2. *There exist coefficients such that the new function*

$$z_1 = z + c_{20}z^2 + c_{11}z\overline{z} + c_{02}\overline{z}^2 + c_{30}z^3 + c_{12}z\overline{z}^2 + c_{03}\overline{z}^3 \tag{12.20}$$

satisfies the equation

$$\dot{z}_1 = i\mu z_1 + iK|z_1|^2 z_1 + \widehat{Z}_R, \tag{12.21}$$

where

$$\operatorname{Re}(iK) = \operatorname{Re} Z'_{21} < 0. \tag{12.22}$$

The same estimate as for $\widetilde{Z}_R$ holds for the remainder $\widehat{Z}_R$:

$$|\widehat{Z}_R| = \mathscr{R}_1(|z| + \|f\|_{L^\infty})\big[(|z|^2 + \|f\|_{E_{-\beta}})^2 + |z|\,\|g\|_{E_{-\beta}} + |z|\,\|h\|_{E_{-\beta}}\big]. \tag{12.23}$$

Proof. Substituting the expression (12.20) for the function z_1 into equation (12.13) gives

$$\begin{aligned}\dot{z}_1 &= (1 + 2c_{20}z + c_{11}\overline{z} + 3c_{30}z^2 + c_{12}\overline{z}^2)\dot{z} + (c_{11}z + 2c_{02}\overline{z} + 2c_{12}z\overline{z} + 3c_{03}\overline{z}^2)\dot{\overline{z}} \\ &= i\mu z + Z_2(z^2 + 2z\overline{z} + \overline{z}^2) + Z_3(z^3 + 3z^2\overline{z} + 3z\overline{z}^2 + \overline{z}^3) \\ &\qquad + Z'_{30}z^3 + Z'_{21}z^2\overline{z} + Z'_{12}z\overline{z}^2 + Z'_{03}\overline{z}^3 + \widetilde{Z}_R \\ &\qquad + (2c_{20}z + c_{11}\overline{z})\big(i\mu z + Z_2(z^2 + 2z\overline{z} + \overline{z}^2) + \mathscr{O}(|z|^3) + \widetilde{Z}_R\big) \\ &\qquad + (3c_{30}z^2 + c_{12}\overline{z}^2)\big(i\mu z + \mathscr{O}(|z|^2) + \widetilde{Z}_R\big) \\ &\qquad + (c_{11}z + 2c_{02}\overline{z})\big(-i\mu\overline{z} + \overline{Z}_2(\overline{z}^2 + 2z\overline{z} + z^2) + \mathscr{O}(|z|^3) + \overline{\widetilde{Z}}_R\big) \\ &\qquad + (2c_{12}z\overline{z} + 3c_{03}\overline{z}^2)\big(-i\mu\overline{z} + \mathscr{O}(|z|^2) + \overline{\widetilde{Z}}_R\big).\end{aligned} \tag{12.24}$$

On the other hand, putting (12.20) into (12.21) gives

$$\dot{z}_1 = i\mu(z + c_{20}z^2 + c_{11}z\overline{z} + c_{02}\overline{z}^2 + c_{30}z^3 + c_{12}z\overline{z}^2 + c_{03}\overline{z}^3) + iKz^2\overline{z} + \mathscr{O}(|z|^4) + \widehat{Z}_R. \tag{12.25}$$

Comparing the coefficients of z^2, $z\overline{z}$, and $\overline{z}^2$ on the right-hand sides of (12.24) and (12.25), we find that

$$c_{20} = \frac{iZ_2}{\mu}\,, \qquad c_{11} = -\frac{2iZ_2}{\mu}\,, \qquad c_{02} = -\frac{3iZ_2}{\mu}\,. \tag{12.26}$$

Further, comparing the coefficients of $z^2\overline{z}$, we get that

$$iK = 3Z_3 + Z'_{21} + (4c_{20} - c_{11} - 2c_{20})Z_2. \tag{12.27}$$

Since the coefficients Z_2 and Z_3 defined in (11.12) are purely imaginary, the inequality (12.22) follows from the last equality. The estimate (12.23) for the remainder $\widehat{Z}_R$ is easily verified. □

Multiplying (12.21) by $\overline{z}_1$ and taking the real part, we obtain for $y = |z_1|^2$ the equation

$$\dot{y} = 2\,\mathrm{Re}(iK)\,y^2 + Y_R, \tag{12.28}$$

where

$$|Y_R| = \mathscr{R}_1(|z| + \|f_1\|_{L^\infty})|z|\big[(|z|^2 + \|f\|_{E_{-\beta}})^2 + |z|\,\|g\|_{E_{-\beta}} + |z|\,\|h\|_{E_{-\beta}}\big]. \tag{12.29}$$

13. Majorants

In this section we define majorants and obtain uniform estimates for them.

13.1. Initial conditions and estimate for g. To start with we formulate our assumptions about the smallness of the initial data for the functions z, f, and h. According to condition (9.1) the initial conditions can be assumed to satisfy the following inequalities:

$$|z(0)| \leqslant \varepsilon^{1/2}, \tag{13.1}$$

$$\|f(0)\|_{E_\beta} = \|h(0)\|_{E_\beta} \leqslant \varepsilon^{3/2}h_0, \tag{13.2}$$

$$\|f(0)\|_{E_\beta \cap W} \leqslant \varepsilon^{1/2}f_0, \tag{13.3}$$

where h_0 and f_0 are fixed constants and $\varepsilon > 0$ is a small number. We have $|z_1|^2 \leqslant |z|^2 + \mathscr{R}(|z|)|z|^3$ by (12.20). Hence,

$$y_0 = y(0) = |z_1(0)|^2 \leqslant \varepsilon + C(|z(0)|)\varepsilon^{3/2}. \tag{13.4}$$

Let us also estimate $g(t) = -e^{At}k(0)$, where $k(0) = a_{20}z^2(0) + a_{11}z(0)\overline{z}(0) + a_{02}\overline{z}^2(0)$ and the coefficients $a_{ij} \in E_{-\beta}$ are defined in (12.9) and (12.10). Since the coefficients a_{ij} also lie in E^{c}, it follows from (10.6) and (13.1) that

$$\|g(t)\|_{E_{-\beta}} \leqslant \frac{C|z(0)|^2}{(1+t)^{3/2}} \leqslant \frac{C\varepsilon}{(1+t)^{3/2}}\,, \qquad \beta > \frac{5}{2}\,. \tag{13.5}$$

13.2. System of majorants. We define the following functions of $T \geqslant 0$:

$$\mathscr{M}_1(T) = \max_{0\leqslant t\leqslant T} |z(t)| \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-1/2}, \tag{13.6}$$

$$\mathscr{M}_2(T) = \max_{0\leqslant t\leqslant T} \|f_1(t)\|_{L^\infty} \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-1/2} \log^{-1}(2+\varepsilon t), \tag{13.7}$$

$$\mathscr{M}_3(T) = \max_{0\leqslant t\leqslant T} \|h(t)\|_{E_{-\beta}} \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-3/2} \log^{-1}(2+\varepsilon t). \tag{13.8}$$

We also set $\mathscr{M} = (\mathscr{M}_1, \mathscr{M}_2, \mathscr{M}_3)$. The main purpose of this subsection is to prove the uniform boundedness of $\mathscr{M}(T)$ for sufficiently small $\varepsilon > 0$.

13.3. Estimates of the remainders. We first estimate the remainders in terms of the corresponding majorants.

I. Consider the remainder Y_R defined in (12.28). Using the equality $f = k+g+h$ and the estimate (12.29), we get that

$$\begin{aligned}
|Y_R| &= \mathscr{R}_2(|z| + \|f_1\|_{L^\infty})|z|\bigl[(|z|^2 + \|g\|_{E_{-\beta}} + \|h\|_{E_{-\beta}})^2 + |z|(\|g\|_{E_{-\beta}} + \|h\|_{E_{-\beta}})\bigr] \\
&= \mathscr{R}(\mathscr{M})\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{1/2} \mathscr{M}_1 \Biggl[\biggl(\frac{\varepsilon}{1+\varepsilon t}\mathscr{M}_1^2 + \frac{\varepsilon}{(1+t)^{3/2}} \\
&\qquad + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2} \log(2+\varepsilon t)\mathscr{M}_3\biggr)^2 \\
&\qquad + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{1/2} \mathscr{M}_1 \left(\frac{\varepsilon}{(1+t)^{3/2}} + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2} \log(2+\varepsilon t)\mathscr{M}_3\right)\Biggr] \\
&= \mathscr{R}(\mathscr{M})\, \frac{\varepsilon^{5/2}}{(1+\varepsilon t)^2\sqrt{\varepsilon+\varepsilon t}} \log(2+\varepsilon t)(1+|\mathscr{M}|)^5.
\end{aligned}$$

Hence,

$$|Y_R| = \mathscr{R}(\mathscr{M})\, \frac{\varepsilon^{5/2}}{(1+\varepsilon t)^2\sqrt{\varepsilon t}} \log(2+\varepsilon t)(1+|\mathscr{M}|)^5. \tag{13.9}$$

II. To estimate the remainder F_R we employ (11.14). From (12.2) and (11.16),

$$\begin{aligned}
\|F_R\|_{E_\beta\cap W} &= \mathscr{R}(|z| + \|f_1\|_{L^\infty})\bigl[|z|^3 + \bigl(|z| + \|f_1\|_{L^\infty}\bigr)\bigl(|z|^2 + \|g\|_{E_{-\beta}} + \|h\|_{E_{-\beta}}\bigr)\bigr] \\
&= \mathscr{R}(\mathscr{M})\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2} \Bigl[\mathscr{M}_1^3 + \bigl(\mathscr{M}_1 + \log(2+\varepsilon t)\mathscr{M}_2\bigr) \\
&\qquad \times \left(\mathscr{M}_1^2 + \frac{1}{(1+t)^{1/2}} + \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{1/2} \log(2+\varepsilon t)\mathscr{M}_3\right)\Bigr], \quad \beta > \frac{5}{2}.
\end{aligned}$$

Hence, the remainder F_R satisfies the estimate

$$\|F_R\|_{E_\beta\cap W} = \mathscr{R}(\mathscr{M})\left(\frac{\varepsilon}{1+\varepsilon t}\right)^{3/2} \log(2+\varepsilon t)\bigl((\mathscr{M}_1^2+1)(\mathscr{M}_1+\mathscr{M}_2)+\varepsilon^{1/2}(1+|\mathscr{M}|)^2\bigr). \tag{13.10}$$

III. Next, we estimate the remainder $\widetilde{F}_R = P^{\mathrm{c}}\mathscr{N}_2[w,w] + F_R$. By (11.10),

$$\|P^{\mathrm{c}}\mathscr{N}_2[w,w]\|_{E_\beta\cap W} = \mathscr{R}(\mathscr{M})\,\frac{\varepsilon}{1+\varepsilon t}\,\mathscr{M}_1^2.$$

This, together with the estimate (13.10), implies that

$$\|\widetilde{F}_R\|_{E_\beta\cap W} = \mathscr{R}(\mathscr{M})\,\frac{\varepsilon}{1+\varepsilon t}\,\bigl(\mathscr{M}_1^2 + \varepsilon^{1/2}(1+|\mathscr{M}|)^3\bigr). \tag{13.11}$$

IV. Finally, we examine the remainder $H_R = F_R + H_I$, where H_I is defined in (12.11) and the coefficients C_n satisfy the estimate (12.12). We estimate C_n in terms of the majorants. From (12.12),

$$\begin{aligned}\|C_n\|_{E_\beta} &= \mathscr{R}(|z| + \|f\|_{E_{-\beta}})|z|(|z| + \|g\|_{E_{-\beta}} + \|h\|_{E_{-\beta}})^2\\ &= \mathscr{R}(\mathscr{M})\Bigl(\frac{\varepsilon}{1+\varepsilon t}\Bigr)^{1/2}\mathscr{M}_1\biggl[\Bigl(\frac{\varepsilon}{1+\varepsilon t}\Bigr)^{1/2}\mathscr{M}_1 + \frac{\varepsilon}{(1+t)^{3/2}}\\ &\qquad + \Bigl(\frac{\varepsilon}{1+\varepsilon t}\Bigr)^{3/2}\log(2+\varepsilon t)\mathscr{M}_2\biggr]^2.\end{aligned}$$

Hence,

$$\|C_n\|_{E_\beta} = \mathscr{R}(\mathscr{M})\Bigl(\frac{\varepsilon}{1+\varepsilon t}\Bigr)^{3/2}\bigl(\mathscr{M}_1^3 + \varepsilon^{1/2}(1+|\mathscr{M}|)^3\bigr), \qquad n = 0, \pm 1. \tag{13.12}$$

13.4. Estimates via the majorants. Here we shall use the majorants to estimate solutions of dynamical equations, thereby obtaining relations between the majorants themselves.

I. We first estimate the solution $y(t)$ of equation (12.28), which is the Riccati equation. As in Proposition 5.6 of [7], the solution of this equation with initial function satisfying the inequality (13.4) and the remainder satisfying the estimate (13.9) has the estimate

$$\Bigl|y - \frac{y_0}{1 + 2y_0 \operatorname{Im} Kt}\Bigr| \leqslant \mathscr{R}(\mathscr{M})\,\frac{\varepsilon^{5/2}}{(1+\varepsilon t)^2\sqrt{\varepsilon t}}\log(2+\varepsilon t)(1+|\mathscr{M}|)^5. \tag{13.13}$$

Furthermore, by the estimates (13.4) and (13.13),

$$y \leqslant \mathscr{R}(\mathscr{M})\biggl[\frac{\varepsilon}{1+\varepsilon t} + \Bigl(\frac{\varepsilon}{1+\varepsilon t}\Bigr)^{3/2}\log(2+\varepsilon t)(1+|\mathscr{M}|)^5\biggr].$$

We have $|z|^2 \leqslant y + \mathscr{R}(|z|)|z|^3$, hence

$$|z|^2 \leqslant \mathscr{R}(\mathscr{M})\biggl[\frac{\varepsilon}{1+\varepsilon t} + \Bigl(\frac{\varepsilon}{1+\varepsilon t}\Bigr)^{3/2}\log(2+\varepsilon t)(1+|\mathscr{M}|)^5 + \Bigl(\frac{\varepsilon}{1+\varepsilon t}\Bigr)^{3/2}\mathscr{M}_1^3\biggr].$$

Taking into account the definition (13.6) of the first majorant $\mathscr{M}_1$, we have

$$\mathscr{M}_1^2 = \mathscr{R}(\mathscr{M})\bigl(1 + \varepsilon^{1/2}(1+|\mathscr{M}|)^5\bigr). \tag{13.14}$$

II. Further, let us examine equation (11.14) for f. The solution of this equation can be represented as

$$f(t)=e^{At}f(0)+\int_0^t e^{A(t-\tau)}\widetilde{F}_R(\tau)\,d\tau.$$

Using the estimates (10.7), (13.3), and (13.11), we find that

$$\begin{aligned}\|f_1\|_{L^\infty}&\leqslant\frac{C}{(1+t)^{1/2}}\|f(0)\|_{E_\beta\cap W}+\int_0^t\frac{C}{(1+(t-\tau))^{1/2}}\|\widetilde{F}_R(\tau)\|_{E_\beta\cap W}\,d\tau\\&\leqslant C\biggl[f_0\Bigl(\frac{\varepsilon}{1+t}\Bigr)^{1/2}+\mathscr{R}(\mathscr{M})\bigl(\mathscr{M}_1^2+\varepsilon^{1/2}(1+|\mathscr{M}|)^3\bigr)\int_0^t\frac{d\tau}{(t-\tau)^{1/2}}\,\frac{\varepsilon}{1+\varepsilon\tau}\biggr]\\&\leqslant C\Bigl(\frac{\varepsilon}{1+\varepsilon t}\Bigr)^{1/2}\log(2+\varepsilon t)\bigl[f_0+\mathscr{R}(\mathscr{M})\bigl(\mathscr{M}_1^2+\varepsilon^{1/2}(1+|\mathscr{M}|)^3\bigr)\bigr].\end{aligned}$$

This, together with the definition (13.7) of the second majorant $\mathscr{M}_2$, implies that

$$\mathscr{M}_2=\mathscr{R}(\mathscr{M})\bigl(\mathscr{M}_1^2+\varepsilon^{1/2}(1+|\mathscr{M}|)^3\bigr).\tag{13.15}$$

III. Finally, let us examine equation (12.5) for h. The solution $h(t)$ of this equation is given by

$$h(t)=e^{At}h(0)+\int_0^t e^{A(t-\tau)}H_R(\tau)\,d\tau.$$

Using the estimates (10.6), (10.8), (13.2), (13.10), and (13.12), we see that

$$\begin{aligned}\|h\|_{E_{-\beta}}&\leqslant\frac{C}{(1+t)^{3/2}}\|h(0)\|_{E_\beta}+\int_0^t\frac{C}{(1+(t-\tau))^{3/2}}\\&\qquad\times\Bigl(\|F_R(\tau)\|_{E_\beta}+\sum_m\|C_m(\tau)\|_{E_\beta}\Bigr)\,d\tau\\&\leqslant C\biggl[h_0\Bigl(\frac{\varepsilon}{1+t}\Bigr)^{3/2}+\mathscr{R}(\mathscr{M})\bigl((\mathscr{M}_1^2+1)(\mathscr{M}_1+\mathscr{M}_2)+\varepsilon^{1/2}(1+|\mathscr{M}|)^2\bigr)\\&\qquad\times\int_0^t\frac{\log(2+\varepsilon\tau)\,d\tau}{(1+(t-\tau))^{3/2}}\Bigl(\frac{\varepsilon}{1+\varepsilon\tau}\Bigr)^{3/2}\\&\qquad+\sum_m\mathscr{R}(\mathscr{M})\bigl(\mathscr{M}_1^3+\varepsilon^{1/2}(1+|\mathscr{M}|)^3\bigr)\int_0^t\frac{d\tau}{(1+(t-\tau))^{3/2}}\Bigl(\frac{\varepsilon}{1+\varepsilon\tau}\Bigr)^{3/2}\biggr].\end{aligned}$$

Therefore,

$$\begin{aligned}\|h\|_{E_{-\beta}}\leqslant C\Bigl(\frac{\varepsilon}{1+\varepsilon t}\Bigr)^{3/2}\log(2+\varepsilon t)\bigl[h_0+\mathscr{R}(\mathscr{M})\bigl((\mathscr{M}_1^2+1)(\mathscr{M}_1+\mathscr{M}_2)\\+\varepsilon^{1/2}(1+|\mathscr{M}|)^3\bigr)\bigr].\end{aligned}\tag{13.16}$$

Consequently, using definition (13.8), we get that

$$\mathscr{M}_3=\mathscr{R}(\mathscr{M})\bigl[(\mathscr{M}_1^2+1)(\mathscr{M}_1+\mathscr{M}_2)+\varepsilon^{1/2}(1+|\mathscr{M}|)^3\bigr].\tag{13.17}$$

13.5. Uniform estimates of the majorants. We claim that the majorants $\mathscr{M}_i$ are uniformly bounded with respect to T and ε for sufficiently small ε.

Putting together the estimates (13.14), (13.15), and (13.17) for the majorants, we obtain the following inequality:

$$\mathscr{M}^2 = \mathscr{R}(\mathscr{M})\big[1 + (\mathscr{M}_1^4 + 1)(\mathscr{M}_1^2 + \mathscr{M}_2^2) + \varepsilon^{1/2}(1 + |\mathscr{M}|)^6\big].$$

Replacing $\mathscr{M}_1^2$ and $\mathscr{M}_2$ on the right-hand side by their estimates (13.14) and (13.15), we see that

$$\mathscr{M}^2 = \mathscr{R}(\mathscr{M})(1 + \varepsilon^{1/2}F(\mathscr{M}))$$

for some continuous function $F(\mathscr{M})$. This proves the uniform boundedness of the function $\mathscr{M}(T)$, since $\mathscr{M}(0)$ is small and the function $\mathscr{M}(T)$ is continuous. Thus, we have shown that for sufficiently small ε there exists a constant M independent of T and ε such that $|\mathscr{M}(T)| \leqslant M$. The following estimates are direct consequences of this inequality and the definitions (13.6)–(13.8) of the majorants $\mathscr{M}_i$:

$$|z(t)| \leqslant M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}, \qquad \|f_1(t)\|_{L^\infty} \leqslant M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\log(1+\varepsilon t),$$
$$\|f(t)\|_{E_{-\beta}} \leqslant M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big), \qquad \|h(t)\|_{E_{-\beta}} \leqslant M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(1+\varepsilon t), \tag{13.18}$$

where $\beta > 5/2$. We have proved the following result.

Theorem 13.1. *Assume that the hypotheses of Theorem 9.1 are satisfied. Then for a sufficiently small $\varepsilon > 0$ there exist functions $z(t) \in C^1(\mathbb{R})$ and $f(x,t) \in C(\mathbb{R}, E)$ and a constant $M > 0$ such that for all $t \geqslant 0$ the solution of equation (2.3) can be written in the form*

$$Y(x,t) = S(x) + (z(t) + \overline{z}(t))u + f(x,t). \tag{13.19}$$

Moreover, the functions $z(t)$ and $f(x,t)$ satisfy the estimates

$$|z(t)| \leqslant M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}, \quad \|f(t)\|_{E_{-\beta}} \leqslant M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big), \qquad \beta > \frac{5}{2}, \quad t \geqslant 0. \tag{13.20}$$

14. Long-time asymptotic behaviour

14.1. Long-time behaviour of $z(t)$. Note that the estimate (12.23) for $\widehat{Z}_R$ differs from the estimate for Y_R only by the factor $|z|$, and so

$$|\widehat{Z}_R(t)| \leqslant C\varepsilon^2(1+\varepsilon t)^{-3/2}(\varepsilon t)^{-1/2}\log(2+\varepsilon t), \tag{14.1}$$

by (13.9) and the boundedness of the majorants.

On the other hand, since $|z_1|^2 = y$ it follows from the inequality (13.13) that

$$\bigg|\,|z_1(t)|^2 - \frac{y_0}{1 + 2\operatorname{Im} K y_0 t}\bigg| \leqslant \frac{C\varepsilon^{5/2}\log(2+\varepsilon t)}{(1+\varepsilon t)^2\sqrt{\varepsilon t}}\,.$$

Hence we may rewrite equation (12.21) as

$$\dot{z}_1 = i\mu z_1 + iK\frac{y_0}{1+2\,\mathrm{Im}\,Ky_0 t}\,z_1 + Z_1, \tag{14.2}$$

where the remainder Z_1 satisfies the estimate

$$|Z_1(t)| \leqslant C\varepsilon^2(1+\varepsilon t)^{-3/2}(\varepsilon t)^{-1/2}\log(2+\varepsilon t) \tag{14.3}$$

in view of (13.18) and (14.1). We have $y_0 = \varepsilon + \mathscr{O}(\varepsilon^{3/2})$ by (13.4), and hence we can replace the constant $2\,\mathrm{Im}\,Ky_0$ in (14.2) by the constant $k\varepsilon$. Next, let $\rho = \mathrm{Re}\,K/\,\mathrm{Im}\,K$. Then the solution z_1 of equation (14.2) can be written as

$$\begin{aligned} z_1(t) &= \frac{e^{i\mu t}}{(1+k\varepsilon t)^{1/2-i\rho}}\left[z_1(0) + \int_0^t e^{-i\mu s}(1+k\varepsilon s)^{1/2-i\rho}Z_1(s)\,ds\right] \\ &= z_\infty\frac{e^{i\mu t}}{(1+k\varepsilon t)^{1/2-i\rho}} + z_R(t), \end{aligned}$$

where

$$\begin{aligned} z_\infty &= z_1(0) + \int_0^\infty e^{-i\mu s}(1+k\varepsilon s)^{1/2-i\rho}Z_1(s)\,ds, \\ z_R(t) &= -\int_t^\infty e^{i\mu t}\left(\frac{1+k\varepsilon s}{1+k\varepsilon t}\right)^{1/2-i\rho} Z_1(s)\,ds. \end{aligned}$$

Further, $|z_R(t)| \leqslant C\varepsilon(1+\varepsilon t)^{-1}\log(2+\varepsilon t)$ in view of the estimate (14.3) for the remainder Z_1. Thus, the function $z_1(t)$ has the following asymptotic behaviour:

$$z_1(t) = z_\infty\frac{e^{i\mu t}}{(1+k\varepsilon t)^{1/2-i\rho}} + \mathscr{O}(t^{-1}\log t), \qquad t\to+\infty. \tag{14.4}$$

Since $z = z_1 + \mathscr{O}(t^{-1})$ by (12.20) and (13.18), it is immediately seen from (14.4) that $z(t)$ has the asymptotic behaviour

$$z(t) = z_\infty\frac{e^{i\mu t}}{(1+k\varepsilon t)^{1/2-i\rho}} + \mathscr{O}(t^{-1}\log t), \qquad t\to+\infty. \tag{14.5}$$

14.2. Soliton asymptotics. We proceed to prove the main Theorem 9.1. According to Theorem 13.1, the solution $Y(x,y)$ of equation (2.3) can be written as

$$Y(t) = S + w(t) + f(t), \quad \text{where } w(t) = z(t)u + \overline{z}(t)\overline{u}, \tag{14.6}$$

and the functions $z(t)$ and $f(t)$ satisfy the estimates (13.20). We can incorporate the function $w(t)$ into the remainder $r_+(t)$ of the asymptotics (9.2), because $z(t)\sim t^{-1/2}$ by (14.5). Therefore, to find the asymptotic behaviour it suffices to single out the dispersive wave $W_0(t)\Phi_+$ from the function $f(t)$. We write equation (11.14) in the form

$$\dot{f} = A_0 f + \mathscr{V}f + P^{\mathrm{c}}\mathscr{N}, \tag{14.7}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ -\Delta+m^2 & 0\end{pmatrix}, \qquad \mathscr{V} = \begin{pmatrix} 0 & 0 \\ V & 0\end{pmatrix}$$

and $\mathscr{N}$ is given in (10.3). For the solution $f(t)$ of equation (14.7), we use the integral representation:

$$\begin{aligned} f(t) &= W_0(t)f(0) + \int_0^t W_0(t-\tau)Q(\tau)\,d\tau \\ &= W_0(t)\Big(f(0) + \int_0^\infty W_0(-\tau)Q(\tau)\,d\tau\Big) - \int_t^\infty W_0(t-\tau)Q(\tau)\,d\tau \\ &= W_0(t)\phi_+ + r_+(t), \end{aligned} \tag{14.8}$$

where $Q(t)$ is the vector function with coordinates

$$Q_1 = (P^{\mathrm{c}}\mathscr{N})_1 = -(P^{\mathrm{d}}\mathscr{N})_1, \quad Q_2 = (P^{\mathrm{c}}\mathscr{N})_2 - Vf_1 = (P^{\mathrm{c}}\mathscr{N}_2[w,w])_2 + (F_R)_2 - Vf_1. \tag{14.9}$$

In order to derive the asymptotic behaviour (9.2), it suffices to check that all the integrals in (14.8) converge in the norm of the space E, and also that

$$\|r_+(t)\|_E = \mathscr{O}(t^{-1/3}), \qquad t \to \infty. \tag{14.10}$$

Since the function φ_{λ_1} is real, it follows from the definition (10.5) and the equality (11.13) that

$$Q_1 = -(P^{\mathrm{d}}\mathscr{N})_1 = \frac{i}{\delta}\varphi_{\lambda_1}\big(\langle \mathscr{N}, ju\rangle - \langle \mathscr{N}, j\overline{u}\rangle\big) = 0.$$

Consider the function $Q_2(t)$. Using (10.5), (11.10), and (11.12), we get that

$$(P^{\mathrm{c}}\mathscr{N}_2[w,w])_2 = N_2[w,w] - (P^{\mathrm{d}}\mathscr{N}_2[w,w])_2 = (z^2 + 2z\overline{z} + \overline{z}^2)(N_2[u,u] - 2i\mu u_1 Z_2).$$

Hence, taking into account the equalities (12.2), (12.3), we represent the function $Q_2(t)$ as

$$Q_2 = q_{20}z^2 + 2q_{11}z\overline{z} + q_{02}\overline{z}^2 + Q_R, \tag{14.11}$$

where $q_{ij} = N_2[u_1,u_1] - 2iZ_2\mu u_1 - V(a_{ij})_1$, $Q_R = (F_R)_2 - V(f_1 - k_1)$, and $(a_{ij})_1$ and k_1 are the first components of the vector functions a_{ij} and k, respectively. In view of (11.15) and (13.18),

$$\|(F_R)_2\|_{L^2} = \mathscr{O}(t^{-3/2}\log t), \qquad t \to \infty.$$

Using the equality (12.2) and the estimates (13.5) and (13.18), we find that

$$\|V(f_1 - k_1)\|_{L^2} = \|V(g_1 + h_1)\|_{L^2} = \mathscr{O}(t^{-3/2}\log t), \qquad t \to \infty.$$

Hence the remainder Q_R in (14.11) has the asymptotic behaviour

$$\|Q_R\|_{L^2} = \mathscr{O}(t^{-3/2}\log t), \qquad t \to \infty. \tag{14.12}$$

Therefore, the integrals in (14.8) with the function Q_R converge in E, and the contribution of this function to the remainder $r_+(t)$ is of order $\mathscr{O}(t^{-1/2}\log t)$.

It remains to evaluate the contributions to $r_+(t)$ from the quadratic terms on the right-hand side of (14.11). Clearly, the functions $q_{ij}(x)$ lie in L^2, but the functions $z^2(t)$, $\overline{z}^2(t)$, and $z(t)\overline{z}(t)$ decrease slowly, like $\mathscr{O}(t^{-1})$. Thus, we cannot assert that the integrals converge absolutely. Nevertheless, we can define these integrals as limits. For example,

$$\int_t^\infty W_0(t-\tau)\begin{pmatrix}0\\ q_{11}(\tau)\end{pmatrix} z\overline{z}\,d\tau := \lim_{T\to\infty}\int_t^T W_0(t-\tau)\begin{pmatrix}0\\ q_{11}(\tau)\end{pmatrix} z\overline{z}\,d\tau. \qquad (14.13)$$

We prove that these limits exist in the space E.

Let us first examine the integral (14.13). From the asymptotics (14.5) it follows that $z\overline{z}\sim(1+k\varepsilon t)^{-1}$. We show that the contribution of the integral (14.13) to $r_+(t)$ is of order $\mathscr{O}(t^{-1})$.

Lemma 14.1. *Assume that $q(x)\in L^2(\mathbb{R})$. Then*

$$I(t) := \left\|\int_t^\infty W_0(-\tau)\begin{pmatrix}0\\ q\end{pmatrix}\frac{d\tau}{1+\tau}\right\|_E = \mathscr{O}(t^{-1}), \qquad t\to\infty. \qquad (14.14)$$

Proof. We set $\omega=\omega(\xi)=\sqrt{\xi^2+m^2}$. Then

$$I(t)\sim\left\|\int_t^\infty\begin{pmatrix}-\sin(\omega\tau)\,\widehat{q}(\xi)\\ -\cos(\omega\tau)\,\widehat{q}(\xi)\end{pmatrix}\frac{d\tau}{1+\tau}\right\|_{L^2\oplus L^2}\leqslant\frac{C}{1+t}\left\|\frac{\widehat{q}(\xi)}{\omega(\xi)}\right\|_{L^2}\leqslant\frac{C_1}{1+t}, \qquad (14.15)$$

because the formula for integration by parts gives us that

$$\left|\int_t^\infty\frac{e^{i\omega\tau}}{1+\tau}\,d\tau\right| = \left|\int_t^\infty\frac{de^{i\omega\tau}}{i\omega(1+\tau)}\,d\tau\right|\leqslant\left|\frac{e^{i\omega\tau}}{\omega(1+t)}\right|+\left|\int_t^\infty\frac{e^{i\omega\tau}}{\omega(1+\tau)^2}\,d\tau\right|$$
$$\leqslant\frac{C}{\omega(1+t)}\,. \qquad (14.16)$$

The proof is complete.

Now let us examine the integrals (14.13) with the functions $q_{20}(x)z^2$ and $q_{02}(x)\overline{z}^2$ instead of the function $q_{11}(x)z\overline{z}$ and show that the contribution of these integrals to the remainder $r_+(t)$ is of order $\mathscr{O}(t^{-1/3})$. From the asymptotics (14.5),

$$z^2\sim\frac{e^{2i\mu\tau}}{(1+k\varepsilon t)^{1-2i\rho}}\,, \qquad \overline{z}^2\sim\frac{e^{-2i\mu\tau}}{(1+k\varepsilon t)^{1+2i\rho}}\,.$$

In addition, it is readily checked that $q_{02},q_{20}\in L^1(\mathbb{R})$. Therefore, it remains to prove the following result.

Lemma 14.2. *Let $q(x)\in L^2(\mathbb{R})\cap L^1(\mathbb{R})$. Then*

$$I_\pm(t) := \left\|\int_t^\infty W_0(-\tau)\begin{pmatrix}0\\ q\end{pmatrix}\frac{e^{\pm 2i\mu\tau}\,d\tau}{(1+\tau)^{1\mp 2i\rho}}\right\|_E = \mathscr{O}(t^{-1/3}), \qquad t\to\infty. \qquad (14.17)$$

Proof. We consider only the integral $I_-(t)$ involving the exponential $e^{-2i\mu\tau}$ (the integral with $e^{+2i\mu\tau}$ is dealt with similarly). For simplicity we shall drop the factor $(1+t)^{2i\rho}$. As in the case of (14.15), we have

$$I_-(t)\sim\left\|\int_t^\infty\begin{pmatrix}-\sin(\omega\tau)\,\widehat{q}(\xi)\\ -\cos(\omega\tau)\,\widehat{q}(\xi)\end{pmatrix}\frac{e^{-2i\mu\tau}\,d\tau}{1+\tau}\right\|_{L^2\oplus L^2}. \qquad (14.18)$$

Writing $\sin\omega\tau$ and $\cos\omega\tau$ as linear combinations of the exponentials $e^{i\omega\tau}$ and $e^{-i\omega\tau}$, we obtain two integrals involving the exponentials $e^{-i(\omega+2\mu)\tau}$ and $e^{i(\omega-2\mu)\tau}$, respectively. The integral involving the 'non-resonance' factor $e^{-i(\omega+2\mu)\tau}$ is estimated as in (14.15)–(14.16), and its contribution to $r_+(t)$ is of order $\mathscr{O}(t^{-1})$. It remains to show that

$$J(t) = \bigg\| \int_t^\infty \frac{e^{i(\omega-2\mu)\tau}\widehat{q}(\xi)\,d\tau}{1+\tau} \bigg\|_{L^2} = \mathscr{O}(t^{-1/3}). \tag{14.19}$$

Given a fixed $0<\alpha<1$, we define the function

$$\chi_\tau(\xi) = \begin{cases} 1, & |\omega(\xi)-2\mu| \leqslant (1+\tau)^{-\alpha}, \\ 0, & |\omega(\xi)-2\mu| > (1+\tau)^{-\alpha}. \end{cases}$$

Then

$$\begin{aligned} J(t) &\leqslant \bigg\| \int_t^\infty \frac{e^{i(2\omega-\mu)\tau}\chi_\tau(\xi)\widehat{q}(\xi)\,d\tau}{1+\tau} \bigg\|_{L^2} + \bigg\| \int_t^\infty \frac{e^{i(2\omega-\mu)\tau}(1-\chi_\tau(\xi))\widehat{q}(\xi)\,d\tau}{1+\tau} \bigg\|_{L^2} \\ &= J_1(t) + J_2(t). \end{aligned}$$

Since the function $\widehat{q}(\xi)$ is bounded and since $\|\chi_\tau\|_{L^2} \leqslant (1+\tau)^{-\alpha/2}$, we have $J_1(t) \leqslant \|\widehat{q}\|_{L^\infty}(1+t)^{-\alpha/2}$. On the other hand, integrating by parts, we get that

$$\begin{aligned} J_2(t) &= \bigg\| \int_t^\infty \frac{(1-\chi_\tau(\xi))\widehat{q}(\xi)\,de^{i(2\omega-\mu)\tau}}{(2\omega-\mu)(1+\tau)} \bigg\|_{L^2} \\ &\leqslant \frac{C(1+t)^\alpha}{1+t}\|\widehat{q}\|_{L^2} + C\int_t^\infty \frac{(1+\tau)^\alpha\,d\tau}{(1+\tau)^2}\|\widehat{q}\|_{L^2} \leqslant \frac{C\|\widehat{q}\|_{L^2}}{(1+t)^{1-\alpha}}. \end{aligned}$$

Equating degrees, $\alpha/2 = 1-\alpha$, we have $\alpha = 2/3$. □

Chapter III

Examples of non-linear potentials

In this chapter we shall construct examples of non-linear potentials satisfying the spectral conditions of the first and second chapters. We first consider piecewise parabolic potentials glued at two points, and then approximate them by smooth functions.

15. Piecewise parabolic potentials

Consider equation (1.1)

$$\ddot{\psi}(x,t) = \psi''(x,t) - U_0'(\psi(x,t)), \qquad x\in\mathbb{R}, \tag{15.1}$$

with a piecewise parabolic potential

$$U_0(\psi) = \begin{cases} (1-b\psi^2)/2, & |\psi| \leqslant \gamma, \\ d(\psi\mp 1)^2/2, & \pm\psi \geqslant \gamma, \end{cases} \tag{15.2}$$

where $b, d > 0$ and $0 < \gamma < 1$ are constants. We want to determine conditions on b and d under which $U_0(\psi) \in C^1(\mathbb{R})$. Equating the values of the function U_0 and its derivative at γ gives

$$b = \frac{1}{\gamma}, \quad d = \frac{1}{1-\gamma}, \qquad 0 < \gamma < 1. \tag{15.3}$$

We note that the second derivative $U_0''(\psi)$ is a piecewise constant function with discontinuities at the points $\psi = \pm\gamma$. Consider the stationary equation

$$s_0''(x) = U_0'(s_0(x)) = \begin{cases} -bs_0(x), & 0 < s_0(x) \leqslant \gamma, \\ d(s_0(x) - 1), & s_0(x) > \gamma. \end{cases} \tag{15.4}$$

We find a non-zero odd solution (kink) of this equation:

$$s_0(x) = \begin{cases} C \sin \sqrt{b}\, x, & 0 < x \leqslant q, \\ Ae^{-\sqrt{d}\, x} + 1, & x > q, \end{cases} \tag{15.5}$$

where $C > \gamma$, $A < 0$, and $q = (1/\sqrt{b}\,) \arcsin(\gamma/C)$. Equating the values of the function $s_0(x)$ and its derivative at $x = q$ and using (15.3) for b and d, we see that

$$C = \sqrt{\gamma}\,, \quad A = (\gamma - 1)e^{\sqrt{\gamma/(1-\gamma)}\ \arcsin \sqrt{\gamma}}, \quad q = \sqrt{\gamma}\ \arcsin \sqrt{\gamma}\,. \tag{15.6}$$

15.1. The linearized equation. We linearize equation (15.1) near the kink $s_0(x)$ by representing the solution $\psi(t)$ of this equation as the sum $\psi(t) = s_0 + \phi(t)$. Substituting this into (15.1), we get that

$$\ddot{\phi}(x,t) = \phi''(x,t) - U_0'(s_0(x) + \phi(x,t)) + U_0'(s_0(x)).$$

Taking into account (15.2), we write the last equation as

$$\ddot{\phi}(t) = -H_0\phi(t) + \mathscr{N}(\phi(t)), \qquad t \in \mathbb{R},$$

where

$$H_0 = -\frac{d^2}{dx^2} + W_0(x), \qquad W_0(x) = U_0''(s_0(x)) = \begin{cases} -b, & |x| \leqslant q, \\ d, & |x| > q. \end{cases} \tag{15.7}$$

The continuous spectrum of the operator H_0 coincides with $[d, \infty)$, and the discrete spectrum lies in the interval $[0, d]$. The eigenfunction $\varphi(x)$ corresponding to the eigenvalue λ satisfies the equation

$$\begin{cases} -\varphi''(x) - b\varphi(x) = \lambda\varphi(x), & |x| \leqslant q, \\ -\varphi''(x) + d\varphi(x) = \lambda\varphi(x), & |x| > q. \end{cases} \tag{15.8}$$

The eigenvalue $\lambda_0 = 0$ is the ground state, and the positive even function $\varphi_0(x) = s_0'(x)$ is the corresponding eigenfunction. Hence, the eigenfunction $\varphi_1(x)$ corresponding to the next eigenvalue $\lambda_1 > 0$ (if it exists) must be odd.

15.2. Odd eigenfunctions. From equation (15.8) it is seen that the odd eigenfunctions have the form

$$\varphi(x) = \begin{cases} B\sin\beta x, & |x| \leqslant q, \\ A(\operatorname{sgn} x)e^{-\alpha|x|}, & |x| > q, \end{cases} \tag{15.9}$$

where $\alpha = \sqrt{d-\lambda} > 0$ and $\beta = \sqrt{b+\lambda} > 0$. Equating the values of the function $\varphi(x)$ and its left and right derivatives at the point $x = q$, we have

$$Ae^{-\alpha q} = B\sin\beta q, \qquad -A\alpha e^{-\alpha q} = B\beta\cos\beta q, \tag{15.10}$$

where

$$\alpha^2 + \beta^2 = b + d. \tag{15.11}$$

The system (15.10) has a non-zero solution only if its determinant is zero, that is,

$$-\alpha = \beta\cot\beta q. \tag{15.12}$$

We multiply both sides of equation (15.12) by q and let $\xi = \beta q$, $\eta = \alpha q$. Taking into account (15.11), we obtain the system of equations

$$-\eta = \xi\cot\xi, \qquad \xi^2 + \eta^2 = R^2, \tag{15.13}$$

where $R = q\sqrt{b+d}$ denotes the radius of the circle on the (ξ,η)-plane. Substituting the expressions for b, d, and q in terms of the parameter γ from (15.3) and (15.6), we see that

$$R = \frac{q}{\sqrt{\gamma(1-\gamma)}} = \frac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}}\,. \tag{15.14}$$

The solution of the system (15.13) can be found graphically (see Fig. 5). Considering that $\eta > 0$, we get the following result:

$$\begin{array}{rl} R \in (0, \pi/2]: & \text{the system (15.13) has no solutions,} \\ R \in (\pi/2, 3\pi/2]: & \text{the system (15.13) has a unique solution,} \\ R \in (3\pi/2, 5\pi/2]: & \text{the system (15.13) has two solutions,} \\ \multicolumn{2}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \end{array} \tag{15.15}$$

Let γ_k, $k \in \mathbb{N}$, denote the solution of the equation

$$\frac{\arcsin\sqrt{\gamma_k}}{\sqrt{1-\gamma_k}} = \frac{k\pi}{2}\,, \qquad k \in \mathbb{N}. \tag{15.16}$$

Solving this equation numerically, we find that

$$\gamma_1 \sim 0.64643,\ \gamma_2 \sim 0.8579,\ \gamma_3 \sim 0.92472,\ \gamma_4 \sim 0.95359,\ \gamma_5 \sim 0.96856,\ \dots\,. \tag{15.17}$$

By (15.15),

$$\begin{array}{rl} \gamma \in (0, \gamma_1]: & \text{there are no non-zero odd eigenfunctions,} \\ \gamma \in (\gamma_1, \gamma_3]: & \text{there exists one linearly independent odd eigenfunction,} \\ \gamma \in (\gamma_3, \gamma_5]: & \text{there are two linearly independent odd eigenfunctions,} \\ \multicolumn{2}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \end{array}$$

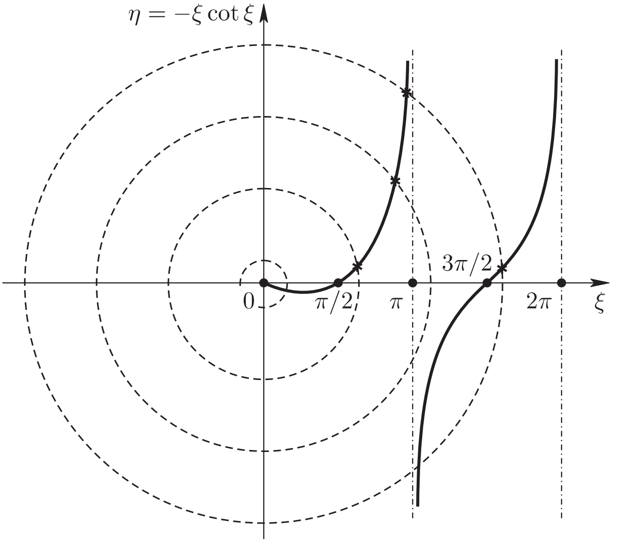

Figure 5. Graphical solution of the system (15.13)

In particular, for $\gamma \in (\gamma_1, \gamma_3]$ we have one odd eigenfunction with the corresponding eigenvalue

$$\lambda_1 = \lambda_1(\gamma) = \beta^2 - b = \frac{\xi^2}{q^2} - b = \frac{1}{\gamma}\left(\frac{\xi^2}{\arcsin^2 \sqrt{\gamma}} - 1\right) = \frac{1}{\gamma}\left(\frac{\sin^2 \xi}{1-\gamma} - 1\right), \quad (15.18)$$

where ξ is the solution of the equation

$$\frac{\xi^2}{\sin^2 \xi} = \frac{\arcsin^2 \sqrt{\gamma}}{1-\gamma}\,. \quad (15.19)$$

15.3. Even eigenfunctions. From equation (15.8) it follows that even eigenfunctions have the form

$$\varphi(x) = \begin{cases} B\cos\beta x, & |x| \leqslant q, \\ Ae^{-\alpha|x|}, & |x| > q. \end{cases} \quad (15.20)$$

Proceeding as in the case of (15.13), we obtain the following equations for the parameters $\xi = \beta q$ and $\eta = \alpha q$:

$$\eta = \xi \tan \xi, \qquad \xi^2 + \eta^2 = R^2. \quad (15.21)$$

The solution of this system can also be found graphically (see Fig. 6). The result is:

$$\begin{array}{rl} R \in (0, \pi]: & \text{the system (15.21) has one solution,} \\ R \in (\pi, 2\pi]: & \text{the system (15.21) has two solutions,} \\ \multicolumn{2}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{array} \quad (15.22)$$

We note that, for any $\gamma \in (0, 1)$, the function $\xi = \arcsin \sqrt{\gamma} \in (0, \pi/2)$ is a solution of equation (15.21). This solution corresponds to the eigenvalue $\lambda = 0$ and the first

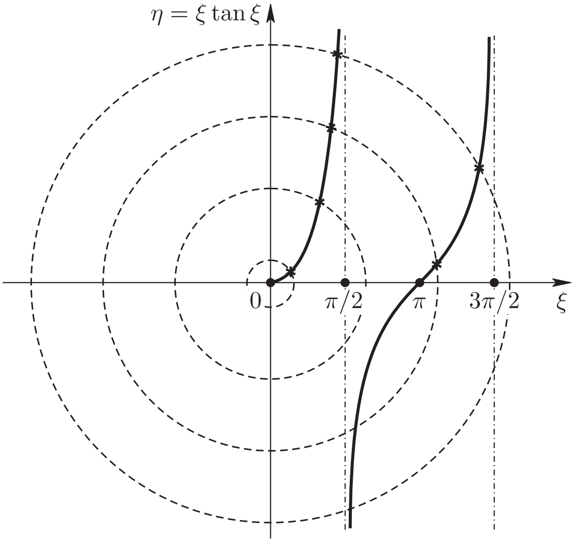

Figure 6. Graphical solution of the system (15.21)

even eigenfunction. Thus:

for $\gamma \in (0, \gamma_2]$:	there exists one linearly independent even eigenfunction,
for $\gamma \in (\gamma_2, \gamma_4]$:	there exist two linearly independent even eigenfunctions,
. . . .	. . . .

Conclusions: for $\gamma \in (0, \gamma_1]$ there exists one eigenvalue $\lambda_0 = 0$; for $\gamma \in (\gamma_1, \gamma_2]$ there exist two eigenvalues $\lambda_0 = 0$ and $0 < \lambda_1 < d$, and so on (see Fig. 7).

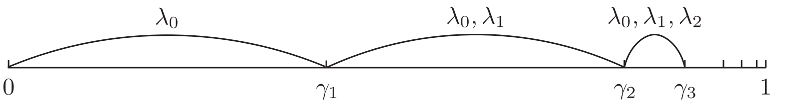

Figure 7. Discrete spectrum

15.4. Spectral conditions. It is readily checked that conditions **U1** and **U2** (except for the smoothness at $\psi = \pm\gamma$) hold for the potential U_0. From (15.9) and (15.20) it follows that resonance can then occur only for $\alpha = 0$, hence for $\gamma = \gamma_k$, $k \in \mathbb{N}$. Thus, condition **U3** is satisfied for $\gamma \in (0, 1) \setminus \{\gamma_k\}$. Also, condition **D1** holds for $\gamma \in (0, \gamma_1]$. Therefore, **for $\gamma \in (0, \gamma_1)$ all the spectral conditions of the first chapter are satisfied except for the smoothness condition at the points $\pm\gamma$.**

In order to satisfy the conditions of the second chapter, it suffices to find the values $\gamma \in (\gamma_1, \gamma_2)$ satisfying conditions (1.7) and **F**, because we have already proved that the operator H_0 has exactly two eigenvalues ($\lambda_0 = 0$ and $\lambda_1 \in (0, d)$) for these γ.

Lemma 15.1. *Condition* (1.7) *holds for all* $\gamma \in (\gamma_1, \gamma_2)$. *Condition* **F** *holds for all* $\gamma \in (\gamma_1, \gamma_2)$ *except for a unique point* γ_*.

Proof. 1) In view of (15.18) and (15.19) the inequality (1.7) with $m^2 = d$ is equivalent to the inequality

$$\frac{4}{\gamma}\Big(\frac{\sin^2\xi(\gamma)}{1-\gamma} - 1\Big) > \frac{1}{1-\gamma}\,,$$

where $\xi(\gamma) \in (\pi/2, \pi)$ is the solution of equation (15.19). After some simple transformations, we get that $4\cos^2\xi(\gamma) < 3\gamma$. Hence, it suffices to verify that

$$\frac{\pi}{2} < \xi(\gamma) < \pi - \arccos\frac{\sqrt{3\gamma}}{2} \tag{15.23}$$

for all $\gamma \in (\gamma_1, \gamma_2)$. The function $\xi/\sin\xi$ is monotonically increasing on the interval $(\pi/2, \pi)$, and thus the inequalities (15.23) are equivalent to

$$\frac{\pi}{2} < \frac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}} < \frac{2(\pi - \arccos(\sqrt{3\gamma}/2))}{\sqrt{4-3\gamma}}\,.$$

As a result, it follows that (1.7) is satisfied for $\gamma_1 < \gamma < \alpha$, where α is the solution of the equation

$$\frac{\arcsin\sqrt{\alpha}}{\sqrt{1-\alpha}} = \frac{2(\pi - \arccos(\sqrt{3\alpha}/2))}{\sqrt{4-3\alpha}}\,.$$

Approximate calculations show that $\alpha \sim 0.921485 > \gamma_2$, and hence the inequality (1.7) holds for all $\gamma \in (\gamma_1, \gamma_2)$.

2) We write condition **F** as

$$\int U_0'''(s_0(x))\varphi_{4\lambda_1}(x)\varphi^2_{\lambda_1}(x)\,dx = \int \frac{d}{dx}U_0''(s_0(x))\frac{\varphi_{4\lambda_1}(x)\varphi^2_{\lambda_1}(x)}{s_0'(x)}\,dx \neq 0. \tag{15.24}$$

By (15.7)

$$\frac{d}{dx}U_0''(s_0(x)) = (b+d)\delta(x-q) - (b+d)\delta(x+q),$$

so (15.24) means that $\varphi_{4\lambda_1}(q)\varphi^2_{\lambda_1}(q) \neq 0$. From formula (15.9) it follows that $\varphi_{\lambda_1}(q) = Ae^{-\alpha q} \neq 0$. We show that the equality $\varphi_{4\lambda_1}(q) = 0$ holds for only one value $\gamma \in (\gamma_1, \gamma_2)$. The function $\varphi_{4\lambda_1}$ is an odd solution of the equation

$$\begin{cases} -\varphi''_{4\lambda_1}(x) - b\varphi_{4\lambda_1}(x) = 4\lambda_1\varphi_{4\lambda_1}(x), & |x| \leqslant q,\\ -\varphi''_{4\lambda_1}(x) + d\varphi_{4\lambda_1}(x) = 4\lambda_1\varphi_{4\lambda_1}(x), & |x| > q. \end{cases}$$

Hence, $\varphi_{4\lambda_1}(q) = \sin\beta q$, where $\beta = \sqrt{b+4\lambda_1} > 0$. As a result, the equality $\varphi_{4\lambda_1}(q) = 0$ is satisfied only if $\beta q = k\pi$, $k \in \mathbb{N}$, which is equivalent to the condition

$$\sqrt{1+4\gamma\lambda_1(\gamma)}\arcsin\sqrt{\gamma} = k\pi, \qquad k \in \mathbb{N}.$$

Substituting here the expression for $\lambda_1(\gamma)$ from (15.18)–(15.19), we see that

$$\frac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}}\sqrt{4\sin^2\xi-3(1-\gamma)}=k\pi,\qquad \frac{\xi^2}{\sin^2\xi}=\frac{\arcsin^2\sqrt{\gamma}}{1-\gamma}\,. \tag{15.25}$$

Since

$$0<\frac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}}\sqrt{4\sin^2\xi-3(1-\gamma)}<2\pi\quad\text{for }\gamma\in(\gamma_1,\gamma_2),$$

it follows that $k=1$. Let $\theta=\arcsin\sqrt{\gamma}$. Then the system (15.25) is equivalent to the system

$$4\xi^2-3\theta^2=\pi^2,\qquad \frac{\sin\xi}{\xi}=\frac{\cos\theta}{\theta}\,. \tag{15.26}$$

We find the solution graphically. Let us express θ in terms of ξ from the first and second equation and consider the corresponding functions $\theta_1(\xi)$ and $\theta_2(\xi)$. The function $\theta_1(\xi):=\sqrt{(4\xi^2-\pi^2)/3}$ is increasing on the interval $(\xi(\gamma_1),\xi(\gamma_2))$. In addition,

$$\theta_1'(\xi)=\frac{1}{\sqrt{3}}\,\frac{4\xi}{\sqrt{4\xi^2-\pi^2}}>\frac{1}{\sqrt{3}}\,\frac{4(\pi/2)}{\sqrt{4(3\pi/4)^2-\pi^2}}=\frac{4}{\sqrt{15}}>1,\quad \xi(\gamma_1)<\xi<\xi(\gamma_2), \tag{15.27}$$

because $\xi(\gamma_1)=\pi/2$ and $\xi(\gamma_2)\sim 2.3137<3\pi/4$. The second function $\theta_2(\xi)$ is implicitly given by the equation $(\sin\xi)/\xi=(\cos\theta)/\theta$. Its derivative is

$$\theta_2'(\xi)=\frac{\sin\xi-\xi\cos\xi}{\xi^2}\,\frac{\theta^2}{\cos\theta+\theta\sin\theta}>0,\qquad \frac{\pi}{2}<\xi<\xi(\gamma_2). \tag{15.28}$$

Moreover, from (15.26),

$$\theta_2'(\xi)=\frac{\theta}{\xi}\,\frac{(\sin\xi)/\xi-\cos\xi}{(\cos\theta)/\theta+\sin\theta}<\frac{(\sin\xi)/\xi-\cos\xi}{(\sin\xi)/\xi+\sin\theta}<1,\qquad \frac{\pi}{2}<\xi<\xi(\gamma_2), \tag{15.29}$$

since $|\cos\xi|<|\cos\xi(\gamma_2)|<\sqrt{2}/2$ and $\sin\theta=\sqrt{\gamma}>\sqrt{\gamma_1}>\sqrt{2}/2$ by (15.17). Furthermore,

$$\theta_2\Big(\frac{\pi}{2}\Big)>\theta_1\Big(\frac{\pi}{2}\Big)=0,\qquad \theta_2(\xi(\gamma_2))\sim 1.1843<\theta_1(\xi(\gamma_2))\sim 1.9616. \tag{15.30}$$

From (15.27)–(15.30) it follows that $\theta_1(\xi)=\theta_2(\xi)$ for only one value $\xi(\gamma_*)\in(\pi/2,\xi(\gamma_2))$ (see Fig. 8). Solving the system (15.26) numerically, we get $\gamma_*\sim 0.7925$. Thus, condition **F** holds for all $\gamma\in(\gamma_1,\gamma_2)$ except for the one point γ_*. □

Thus: **for $\gamma\in(\gamma_1,\gamma_*)\cup(\gamma_*,\gamma_2)$ all the conditions of the second chapter are satisfied except for the smoothness condition for the potential at the points $\pm\gamma$.**

16. Smooth approximations

Let us approximate the potential U_0 by smooth potentials. We consider an even function $h(\psi)\in C_0^\infty(\mathbb{R})$ such that $h(\psi)\geqslant 0$, $\operatorname{supp}h\subset[-1,1]$, and $\int h(\psi)\,d\psi=1$. Given any $\varepsilon\in(0,\gamma)$, we set

$$\widetilde{U}_\varepsilon(\psi):=\frac{1}{\varepsilon}\int h\Big(\frac{\psi-\psi'}{\varepsilon}\Big)U_0(\psi')\,d\psi'. \tag{16.1}$$

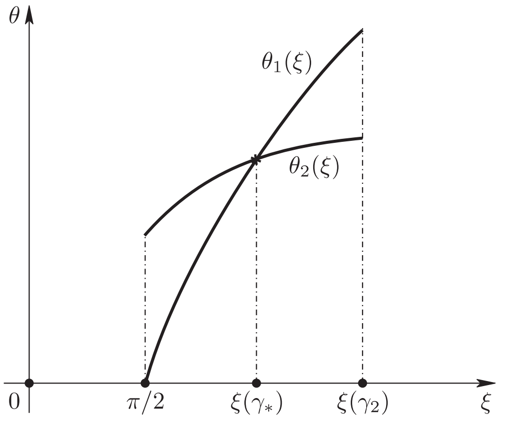

Figure 8. The functions θ_1 and θ_2

The function $\widetilde{U}_\varepsilon(\psi) \geqslant 0$ is a smooth even function symmetric with respect to the points $\psi = \pm 1$ in some neighbourhood of these points. In addition,

$$\widetilde{U}_\varepsilon(\psi) - U_0(\psi) = \begin{cases} \mu_\varepsilon > 0, & |\psi| \geqslant \gamma + \varepsilon, \\ -\nu_\varepsilon < 0, & |\psi| \leqslant \gamma - \varepsilon, \end{cases}$$

where $\mu_\varepsilon, \nu_\varepsilon = \mathscr{O}(\varepsilon^2)$. Setting $U_\varepsilon(\psi) = \widetilde{U}_\varepsilon(\psi) - \mu_\varepsilon$, we have

$$U_\varepsilon(\psi) = \begin{cases} U_0(\psi), & |\psi| \geqslant \gamma + \varepsilon, \\ U_0(\psi) - \mu_\varepsilon - \nu_\varepsilon, & |\psi| \leqslant \gamma - \varepsilon. \end{cases} \tag{16.2}$$

Clearly, $\sup |U_\varepsilon(\psi) - U_0(\psi)| \leqslant C\varepsilon$ with some constant $C > 0$. Also, $U_\varepsilon'''(\psi) \leqslant 0$ for $\psi \leqslant 0$. The corresponding kink is an odd solution of the equation $s_\varepsilon''(x) - U_\varepsilon'(s_\varepsilon(x)) = 0$. Integrating, we get that

$$\int_0^{s_\varepsilon(x)} \frac{ds}{\sqrt{2U_\varepsilon(s_\varepsilon)}} = x, \qquad x \in \mathbb{R}. \tag{16.3}$$

Hence, $s_\varepsilon(x)$ is a monotonically increasing function, and $s_\varepsilon(x) \to \pm 1$ as $x \to \pm\infty$. We have $|s_\varepsilon(x) - s_0(x)| \leqslant C_1\varepsilon$ by (16.2). Therefore, $|\,|s_\varepsilon(x)| - \gamma| \geqslant \varepsilon$ for $|\,|x| - q| \geqslant \delta$, where

$$\delta \to 0, \qquad \varepsilon \to 0. \tag{16.4}$$

Consequently, $W_\varepsilon(x) := U_\varepsilon''(s_\varepsilon(x)) = W_0(x)$ for $||x| - q| \geqslant \delta$, and

$$|W_\varepsilon(x) - W_0(x)| \leqslant b + d, \qquad x \in \mathbb{R}. \tag{16.5}$$

Let $w_\varepsilon(x) = W_\varepsilon(x) - W_0(x)$. Using (16.4) and (16.5), we find that

$$\|w_\varepsilon\|_{L^2(\mathbb{R})} \to 0, \qquad \varepsilon \to 0. \tag{16.6}$$

It is readily seen that conditions **U1** and **U2** hold for the potentials U_ε. We claim that condition **U3** is fulfilled. According to Theorem 7.2 of [31], the no-resonance condition at an edge point of the continuous spectrum is equivalent to the requirement that the resolvent be bounded at this point. The resolvents $R_0(\omega)$ and $R_\varepsilon(\omega)$ of the operators H_0 and $H_\varepsilon := -d^2/dx^2 + W_\varepsilon(x)$ are related by the equation

$$R_\varepsilon(\omega) = R_0(\omega)(1 + w_\varepsilon R_0(\omega))^{-1}, \tag{16.7}$$

from which it follows by (16.6) that the potentials $U_\varepsilon(\psi)$ satisfy condition **U3** for sufficiently small $\varepsilon > 0$ and all $\gamma \in (0,1) \setminus \{\gamma_k\}$.

From (16.6) and (16.7) it also follows that the eigenvalues of the operator H_ε converge to eigenvalues of the operator H_0 as $\varepsilon \to 0$. As a result:

i) for all $\gamma \in (0,\gamma_1)$ and sufficiently small ε, the operator H_ε has only the one eigenvalue zero, that is, the potentials U_ε satisfy the spectral condition **D1**;

ii) for $\gamma \in (\gamma_1,\gamma_2)$ and sufficiently small ε, the operator H_ε has two eigenvalues $\lambda_0 = 0$ and $0 < \lambda_1(\varepsilon) < d$, with $4\lambda_1(\varepsilon) > d$, that is, the potentials U_ε satisfy the spectral condition **D2**.

It is easily checked that condition **F** is also satisfied for small ε and all $\gamma \in (\gamma_1,\gamma_2) \setminus \{\gamma_*\}$.

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