

Weighted Energy Decay for 1D Dirac Equation

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Abstract

We obtain a dispersive long-time decay in weighted energy norms for solutions of the 1D Dirac equation with generic potential. The decay extends the results obtained by Jensen, Kato and Murata for the Schrödinger equations.

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1 Introduction

In this paper, we establish a dispersive long time decay for the solutions to 1D Dirac equation

$$i\dot{\psi}(x, t) = \mathcal{H}\psi(x, t) := i\alpha\psi'(x, t) + m\beta\psi(x, t) + \mathcal{V}(x)\psi(x, t), \quad x \in \mathbb{R}, \quad m > 0 \quad (1.1)$$

in weighted energy norms. Here $\psi(x, t) \in \mathbb{C}^2$ for $(x, t) \in \mathbb{R}^2$,

$$\alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1.2)$$

and $\mathcal{V}(x)$ a given Hermitian matrix potential:

$$\mathcal{V}(x) = \begin{pmatrix} \mathcal{V}_{11}(x) & \mathcal{V}_{12}(x) \\ \mathcal{V}_{21}(x) & \mathcal{V}_{22}(x) \end{pmatrix}, \quad \mathcal{V}_{11}(x), \mathcal{V}_{22}(x) \in \mathbb{R}, \quad \mathcal{V}_{21}(x) = \overline{\mathcal{V}_{12}(x)}, \quad x \in \mathbb{R}. \quad (1.3)$$

Hence, the Dirac operator \mathcal{H} is Hermitian. The matrices α and β satisfy the relations

$$\alpha^2 = \beta^2 = I, \quad \alpha\beta + \beta\alpha = 0 \quad (1.4)$$

For $s, \sigma \in \mathbb{R}$, let us denote by $H_\sigma^s = H_\sigma^s(\mathbb{R}^3)$ the weighted Sobolev spaces [1], with the finite norms

$$\|u\|_{H_\sigma^s} = \|\langle x \rangle^\sigma \langle \nabla \rangle^s u\|_{L^2(\mathbb{R})} < \infty, \quad \langle x \rangle = (1 + |x|^2)^{1/2}$$

We assume that

$$|\mathcal{V}_{11}(x)| + |\mathcal{V}_{22}(x)| + |\mathcal{V}_{12}(x)| + |\mathcal{V}'_{12}(x)| \leq C\langle x \rangle^{-\beta}, \quad x \in \mathbb{R} \quad (1.5)$$

for some $\beta > 5$. Then the multiplication by \mathcal{V}_{12} is bounded operators $H_s^1 \rightarrow H_{s+\beta}^1$ for any $s \in \mathbb{R}$.

We restrict ourselves to “nonsingular case” when the truncated resolvent of the operator \mathcal{H} is bounded at the edge points $\lambda = \pm m$ of the continuous spectrum.

Our main result is the following long time decay of the solutions to (1.1): in the nonsingular case

$$\|\mathcal{P}_c \psi(t)\|_{L_{-\sigma}^2} = \mathcal{O}(|t|^{-3/2}), \quad t \rightarrow \pm\infty \quad (1.6)$$

for initial data $\psi_0 = \psi(0) \in L_\sigma^2 := H_\sigma^0 \otimes \mathbb{C}^2$ with $\sigma > 5/2$ where \mathcal{P}_c is a Riesz projector onto the continuous spectrum of the operator \mathcal{H} . The decay is desirable for the study of asymptotic stability and scattering for solutions to nonlinear Dirac equations.

Let us comment on previous results in this direction. The decay of type (1.6) in weighted norms has been established first by Jensen and Kato [7] for the Schrödinger equation in the dimension $n = 3$. The result has been extended to all other dimensions by Jensen and Nenciu [5, 6, 8], and to more general PDEs of the Schrödinger type by Murata [13]. In [9]-[11] the decay of type (1.6) in weighted energy norms has been proved for the Klein-Gordon equations. For Dirac equations the Strichartz estimates were established in [12]. The decay $\sim t^{-1}$ in L^∞ norm were established in [2] for Dirac equations with small potential. The decay of type (1.9) in weighted norms for and Dirac equation without any smallness conditions on the potential was not obtained before.

Let us comment on our techniques. We extend our approach [9] to the Dirac equation. It is well known that the decay (1.6) violates for the free 1D Klein-Gordon and Dirac equation corresponding to $\mathcal{V}(x) = 0$ when the solutions slow decay, like $\sim t^{-1/2}$. Hence, the decay (1.6) cannot be deduced by perturbation arguments from the corresponding estimate for the free equation. The slow decay is caused by the “zero resonance function” $\psi(x) = \text{const}$ corresponding to the end point $\lambda = 0$ of the continuous spectrum of the operator d^2/dx^2 .

Main idea of our approach is a spectral analysis of the “bad” term, with the slow decay $\sim t^{-1/2}$. Namely, we show that the bad term does not contribute to the high energy component of solution to the free equation, and the high energy component decays like $t^{-3/2}$. Then we prove the decay $\sim t^{-3/2}$ for the high energy component of solution to perturbed equation (1.1) using finite Born series and convolutions. For the proof we apply a gauge transformation to obtain a suitable expression for the resolvent of the operator \mathcal{H} via resolvent of the corresponding “squared Dirac operator” which is a matrix Schrödinger operator with a perturbation. The perturbation does not contain differential operators which allows us to apply the estimates (3.15) obtained in [7] for Schrödinger operator.

For the low energy component of solution to perturbed equation, the decay $\sim t^{-3/2}$ follows in the “nonsingular case” by methods [7, 13].

Our paper is organized as follows. In Section 2 we obtain the time decay for the solution to the free Dirac equation and state the spectral properties of the free resolvent which follow from the corresponding known properties of the free Schrödinger resolvent. In Section 3 we obtain spectral properties of the perturbed resolvent and prove the decay (1.6).

2 Free Dirac equation

First, we consider the free Dirac equation:

$$i\psi_t(x, t) = \mathcal{H}_0\psi(x, t) := i\alpha\psi'(x, t) + m\beta\psi(x, t) \quad (2.1)$$

Denote by $\mathcal{U}(t) : \psi(\cdot, 0) \rightarrow \psi(\cdot, t)$ the dynamical group of the equation (2.1). It is strongly continuous group in $L^2 := L^2(\mathbb{R}) \otimes \mathbb{C}^2$. The group is unitary that follows from the charge conservation.

2.1 Spectral properties

We state spectral properties of the free dynamical group $\mathcal{U}(t)$ applying known results of [1, 13] which concern the corresponding spectral properties of the free Schrödinger dynamical group. For $t > 0$ and $\psi_0 = \psi(0) \in L^2$, the solution $\psi(t)$ to the free equation (2.1) admits the spectral Fourier-Laplace representation

$$\theta(t)\psi(t) = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-i(\omega+i\varepsilon)t} \mathcal{R}_0(\omega + i\varepsilon)\psi_0 d\omega, \quad t \in \mathbb{R} \quad (2.2)$$

with any $\varepsilon > 0$ where $\theta(t)$ is the Heaviside function, $\mathcal{R}_0(\omega) = (\mathcal{H}_0 - \omega)^{-1}$ is the resolvent of the operator \mathcal{H}_0 . The representation follows from the stationary equation $\omega\tilde{\psi}^+(\omega) = \mathcal{H}_0\tilde{\psi}^+(\omega) + i\psi_0$ for the Fourier-Laplace transform $\tilde{\psi}^+(\omega) := \int_{\mathbb{R}} \theta(t)e^{i\omega t}\psi(t)dt$, $\omega \in \mathbb{C}^+ := \{\text{Im}\omega > 0\}$. The

solution $\psi(t)$ is continuous bounded function of $t \in \mathbb{R}$ with the values in L^2 by the charge conservation for the free equation (2.1). Hence, $\tilde{\psi}^+(\omega) = -i\mathcal{R}_0(\omega)\psi_0$ is analytic function of $\omega \in \mathbb{C}^+$ with the values in L^2 , bounded for $\omega \in \mathbb{R} + i\varepsilon$. Therefore, the integral (2.2) converges in the sense of distributions of $t \in \mathbb{R}$ with the values in L^2 . Similarly to (2.2),

$$\theta(-t)\psi(t) = -\frac{1}{2\pi i} \int_{\mathbb{R}} e^{-i(\omega-i\varepsilon)t} \mathcal{R}_0(\omega-i\varepsilon)\psi_0 d\omega, \quad t \in \mathbb{R} \quad (2.3)$$

The resolvent $\mathcal{R}_0(\omega)$ can be expressed in terms of the resolvent $R_0(\zeta) = (-\partial_x^2 - \zeta)^{-1}$ of the free Schrödinger operator. Indeed, (1.4) implies

$$(\mathcal{H}_0 - \omega)(\mathcal{H}_0 + \omega) = (i\alpha\partial_x + m\beta - \omega)(i\alpha\partial_x + m\beta + \omega) = (-\partial_x^2 + m^2 - \omega^2) \quad (2.4)$$

Therefore,

$$\mathcal{R}_0(\omega) = (i\alpha\partial_x + m\beta + \omega)R_0(\omega^2 - m^2) \quad (2.5)$$

where $R_0(\zeta)$ is the operator with the integral kernel

$$R_0(\zeta, x-y) = -\frac{\exp(i\sqrt{\zeta}|x-y|)}{2i\sqrt{\zeta}}, \quad \zeta \in \mathbb{C} \setminus [0, \infty), \quad \text{Im} \zeta^{1/2} > 0 \quad (2.6)$$

Denote by $\mathcal{L}(B_1, B_2)$ the Banach space of bounded linear operators from a Banach space B_1 to a Banach space B_2 . Explicit formula (2.6) obviously implies the properties of $R_0(\zeta)$ (cf. [1, 13]):

- i) $R_0(\zeta)$ is analytic function of $\zeta \in \mathbb{C} \setminus [0, \infty)$ with the values in $\mathcal{L}(H^0, H^2)$;
- ii) For $\zeta > 0$, the convergence holds

$$R_0(\zeta \pm i\varepsilon) \rightarrow R_0(\zeta \pm i0), \quad \varepsilon \rightarrow 0+ \quad (2.7)$$

in $\mathcal{L}(H_\sigma^0, H_{-\sigma}^2)$ with $\sigma > 1/2$, uniformly in $\zeta \geq r$ for any $r > 0$.

iii) In $\mathcal{L}(H_\sigma^0; H_{-\sigma}^2)$ with $\sigma > 5/2$ the asymptotics holds

$$R_0(\zeta) = A_0\zeta^{-1/2} + A_1 + \mathcal{O}(\zeta^{1/2}), \quad R_0'(\zeta) = -\frac{1}{2}A_0\zeta^{-3/2} + \mathcal{O}(\zeta^{-1/2}), \quad R_0''(\zeta) = \mathcal{O}(\zeta^{-5/2}) \quad (2.8)$$

as $\zeta \rightarrow 0$, $\zeta \in \mathbb{C} \setminus [0, \infty)$. Here

$$A_0 = \text{Op} \left[\frac{i}{2} \right] \in \mathcal{L}(H_\sigma^0; H_{-\sigma}^2), \quad \sigma > 1/2 \quad (2.9)$$

$$A_1 = \text{Op} \left[-\frac{1}{2}|x-y| \right] \in \mathcal{L}(H_\sigma^0; H_{-\sigma}^2), \quad \sigma > 3/2$$

iv) For $l = 0, 1$, $k = 0, 1, 2, \dots$ and $\sigma > 1/2 + k$ the asymptotics hold

$$\|R_0^{(k)}(\zeta)\|_{\mathcal{L}(H_\sigma^0, H_{-\sigma}^l)} = \mathcal{O}(|\zeta|^{-\frac{1-l+k}{2}}), \quad \zeta \rightarrow \infty, \quad \zeta \in \mathbb{C} \setminus (0, \infty) \quad (2.10)$$

Let us denote $\Gamma := (-\infty, -m) \cup (m, \infty)$, The properties i) – iv) and formula (2.5) imply

Lemma 2.1. *i) The resolvent $\mathcal{R}_0(\omega)$ is analytic function of $\omega \in \mathbb{C} \setminus \bar{\Gamma}$ with the values in $\mathcal{L}(L^2, L^2)$.*

ii) For $\omega \in \Gamma$, the convergence holds

$$\mathcal{R}_0(\omega \pm i\varepsilon) \rightarrow \mathcal{R}_0(\omega \pm i0), \quad \varepsilon \rightarrow 0+ \quad (2.11)$$

in $\mathcal{L}(L_\sigma^2, L_{-\sigma}^2)$ with $\sigma > 1/2$, uniformly in $|\omega| \geq m + r$ for any $r > 0$.

iii) In $\mathcal{L}(L_\sigma^2; L_{-\sigma}^2)$ with $\sigma > 5/2$ the asymptotics hold

$$\left. \begin{aligned} \mathcal{R}_0(\omega) &= \mathcal{A}_0^\pm(\omega \mp m)^{-1/2} + \mathcal{A}_1^\pm + \mathcal{O}((\omega \mp m)^{1/2}) \\ \mathcal{R}'_0(\omega) &= -\frac{1}{2}\mathcal{A}_0^\pm(\omega \mp m)^{-3/2} + \mathcal{O}((\omega \mp m)^{-1/2}) \\ \mathcal{R}''_0(\omega) &= \mathcal{O}((\omega \mp m)^{-3/2}) \end{aligned} \right| \omega \rightarrow \pm m, \quad \omega \in \mathbb{C} \setminus \bar{\Gamma} \quad (2.12)$$

Here

$$\mathcal{A}_0^\pm = \text{Op} \left[\frac{i\sqrt{m}}{\sqrt{\pm 2}} \begin{pmatrix} \pm 1 & 1 \\ 1 & \pm 1 \end{pmatrix} \right] \in \mathcal{L}(L_\sigma^2; L_{-\sigma}^2), \quad \sigma > 1/2 \quad (2.13)$$

$$\mathcal{A}_1^\pm = \text{Op} \left[\frac{-m|x-y|}{2} \begin{pmatrix} \pm 1 & 1 \\ 1 & \pm 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \text{sgn}(x-y) & 0 \\ 0 & \text{sgn}(x-y) \end{pmatrix} \right] \in \mathcal{L}(L_\sigma^2; L_{-\sigma}^2), \quad \sigma > 3/2$$

iv) For $k = 0, 1, 2, \dots$ and $\sigma > 1/2 + k$ the asymptotics hold

$$\|\mathcal{R}_0^{(k)}(\omega)\|_{\mathcal{L}(L_\sigma^2, L_{-\sigma}^2)} = \mathcal{O}(1) \quad \omega \rightarrow \infty, \quad \omega \in \mathbb{C} \setminus \Gamma \quad (2.14)$$

Corollary 2.2. *For $t \in \mathbb{R}$ and $\psi_0 \in L_\sigma^2$ with $\sigma > 1/2$, the group $\mathcal{U}(t)$ admits the integral representation*

$$\mathcal{U}(t)\psi_0 = \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} \left[\mathcal{R}_0(\omega + i0) - \mathcal{R}_0(\omega - i0) \right] \psi_0 \, d\omega \quad (2.15)$$

where the integral converges in the sense of distributions of $t \in \mathbb{R}$ with the values in $L_{-\sigma}^2$.

Proof. Summing up the representations (2.2) and (2.3), and sending $\varepsilon \rightarrow 0+$, we obtain (2.15) by the Cauchy theorem and Lemma 2.1. \square

2.2 Time decay

Here we prove the time decay (1.6) for the free Dirac equation (2.1). Let $G(t) = \text{Op}[G(x-y, t)]$, where $G(x, t)$ is the fundamental solution to the Klein-Gordon operator $\partial_t^2 - \partial_x^2 + m^2$:

$$G(x, t) = \frac{1}{2} \theta(t - |x|) J_0(m\sqrt{t^2 - x^2})$$

where J_0 is the Bessel function. Since

$$(\partial_t + \alpha \partial_x - im\beta)(\partial_t - \alpha \partial_x + im\beta) = \partial_t^2 - \partial_x^2 + m^2$$

then for the integral kernel $\mathcal{U}(x - y, t)$ of the operator $\mathcal{U}(t)$ we have

$$\mathcal{U}(x, t) = (\partial_t + \alpha \partial_x - im\beta)G(x, t) \quad (2.16)$$

The asymptotics of the Bessel functions [14] imply

$$G(x, t) \sim t^{-1/2}, \quad \partial_t G(x, t) \sim t^{-1/2}, \quad \partial_x G(x, t) \sim t^{-1/2}, \quad x \in \mathbb{R}, \quad t \rightarrow \infty$$

Hence (2.16) implies

$$\mathcal{U}(x, t) \sim t^{-1/2}, \quad x \in \mathbb{R}, \quad t \rightarrow \infty$$

and then the free group $\mathcal{U}(t)$ decays like $t^{-1/2}$. The slow decay is caused by the presence of resonance at the edge points $\zeta = \pm m$ of the continuous spectrum.

In [9] we obtain similarly (2.15) the integral representation for the operator $G(t)$:

$$G(t)\psi_0 = \frac{1}{2\pi} \int_{\Gamma} e^{-i\omega t} \left[R_0((\omega + i0)^2 - m^2) - R_0((\omega - i0)^2 - m^2) \right] \psi_0 \, d\omega \quad (2.17)$$

for $t \in \mathbb{R}$ and $\psi_0 \in L^2_{\sigma}$ with $\sigma > 1/2$. Let us introduce the *low energy* and *high energy* components of $G(t)$ and $\mathcal{U}(t)$:

$$G_l(t) = \frac{1}{2\pi} \int_{\Gamma} e^{-i\omega t} l(\omega) \left[R_0((\omega + i0)^2 - m^2) - R_0((\omega - i0)^2 - m^2) \right] \, d\omega \quad (2.18)$$

$$G_h(t) = \frac{1}{2\pi} \int_{\Gamma} e^{-i\omega t} h(\omega) \left[R_0((\omega + i0)^2 - m^2) - R_0((\omega - i0)^2 - m^2) \right] \, d\omega \quad (2.19)$$

$$\mathcal{U}_l(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} l(\omega) \left[\mathcal{R}_0(\omega + i0) - \mathcal{R}_0(\omega - i0) \right] \, d\omega \quad (2.20)$$

$$\mathcal{U}_h(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} h(\omega) \left[\mathcal{R}_0(\omega + i0) - \mathcal{R}_0(\omega - i0) \right] \, d\omega \quad (2.21)$$

where $l(\omega) \in C_0^{\infty}(\mathbb{R})$ is an even function, $\text{supp } l \in [-m - 2\varepsilon, m + 2\varepsilon]$, $l(\omega) = 1$ if $|\omega| \leq m + \varepsilon$, and $h(\omega) = 1 - l(\omega)$. In [9] we have proved that $G_h(t)$ decays like $t^{-3/2}$. Here we will prove that $\mathcal{U}_h(t)$ also decays like $t^{-3/2}$.

Theorem 2.3. *Let $\sigma > 5/2$. Then the decay holds*

$$\|\mathcal{U}_h(t)\|_{\mathcal{L}(L^2_{\sigma}; L^2_{-\sigma})} \leq C(1 + |t|)^{-3/2}, \quad t \in \mathbb{R} \quad (2.22)$$

Proof. First, taking into account (2.16), (2.19), (2.21) we obtain that

$$\mathcal{U}_h(t) = (\partial_t + \alpha \partial_x - im\beta)G_h(t) \quad (2.23)$$

Further, [9, Theorem 2.7] implies that for $\sigma > 5/2$

$$\|G_h(t)\|_{\mathcal{L}(H^2_{\sigma}; H^1_{-\sigma})} + \|\partial_t G_h(t)\|_{\mathcal{L}(L^2_{\sigma}; L^2_{-\sigma})} \leq C(1 + |t|)^{-3/2}, \quad t \in \mathbb{R} \quad (2.24)$$

Then (2.23) and (2.24) imply (2.22). \square

3 Perturbed equation

To prove the long time decay for the perturbed equation, we first establish the spectral properties of its generator \mathcal{H} .

3.1 Spectral properties

Similarly to [13, formula (3.1)], let us introduce a generalized eigenspaces \mathbf{M}^\pm of the operator \mathcal{H} :

$$\mathbf{M}^\pm = \{\psi \in H_{-1/2-0}^1 : (1 + \mathcal{A}_1^\pm \mathcal{V})\psi \in \mathfrak{R}(\mathcal{A}_0^\pm), \quad \mathcal{A}_0^\pm \mathcal{V}\psi = 0\}$$

Where \mathcal{A}_0^\pm and \mathcal{A}_1^\pm are defined in (2.14), $\mathfrak{R}(\mathcal{A}_0^\pm)$ is the range of \mathcal{A}_0^\pm . Below we assume that

$$\mathbf{M}^\pm = 0 \tag{3.1}$$

Denote by $\mathcal{R}(\omega) = (\mathcal{H} - \omega)^{-1}$, $\omega \in \mathbb{C} \setminus \Gamma$, the resolvents of the operators \mathcal{H} . Next Lemma is the vector version of [13, Theorem 7.2].

Lemma 3.1. *Let the conditions (1.5) and (3.1) hold. Then the families $\{\mathcal{R}(\pm m + \varepsilon) : \pm m + \varepsilon \in \mathbb{C} \setminus \bar{\Gamma}, |\varepsilon| < \delta\}$ are bounded in the operator norm of $\mathcal{L}(L_\sigma^2, L_{-\sigma}^2)$ for any $\sigma > 3/2$ and sufficiently small δ .*

Asymptotics (2.12) and lemma 3.1 imply

Proposition 3.2. *Let the conditions (1.5) and (3.1) hold. Then the asymptotics hold*

$$\left. \begin{aligned} \mathcal{R}(\omega) - \mathcal{R}(\pm m) &= \mathcal{O}(|\omega \mp m|^{1/2}) \\ \mathcal{R}'(\omega) &= \mathcal{O}(|\omega \mp m|^{-1/2}) \\ \mathcal{R}''(\omega) &= \mathcal{O}(|\omega \mp m|^{-3/2}) \end{aligned} \right| \omega \rightarrow \pm m, \quad \omega \in \mathbb{C} \setminus \bar{\Gamma} \tag{3.2}$$

in $\mathcal{L}(L_\sigma^2, L_{-\sigma}^2)$ with $\sigma > 5/2$.

Proof. Lemma 3.1 implies that for any $\sigma > 3/2$ the operators $(1 + \mathcal{R}_0(\omega)\mathcal{V})^{-1} = 1 - \mathcal{R}(\omega)\mathcal{V}$ and $(1 + \mathcal{V}\mathcal{R}_0(\omega))^{-1} = 1 - \mathcal{V}\mathcal{R}(\omega)$ are bounded in $\mathcal{L}(L_{-\sigma}^2, L_{-\sigma}^2)$ and in $\mathcal{L}(L_\sigma^2, L_\sigma^2)$ respectively for $|\omega \mp m| < \delta$, $\omega \in \mathbb{C} \setminus \bar{\Gamma}$ with δ sufficiently small. Asymptotics (2.12) imply

$$\left. \begin{aligned} \mathcal{R}(\omega) &= (1 + \mathcal{R}_0(\omega)\mathcal{V})^{-1}\mathcal{R}_0(\omega) = (1 + \mathcal{R}_0(\omega)\mathcal{V})^{-1} \left(\mathcal{A}_0^\pm \frac{1}{\sqrt{\omega \mp m}} + \mathcal{O}(1) \right) \\ \mathcal{R}(\omega) &= \mathcal{R}_0(\omega)(1 + \mathcal{V}\mathcal{R}_0(\omega))^{-1} = \left(\mathcal{A}_0^\pm \frac{1}{\sqrt{\omega \mp m}} + \mathcal{O}(1) \right) (1 + \mathcal{V}\mathcal{R}_0(\omega))^{-1} \end{aligned} \right| \omega \rightarrow \pm m, \quad \omega \in \mathbb{C} \setminus \bar{\Gamma}$$

in $\mathcal{L}(L_\sigma^2, L_{-\sigma}^2)$ with $\sigma > 3/2$. Hence, the boundedness $\mathcal{R}(\omega)$, $(1 + \mathcal{R}_0(\omega)\mathcal{V})^{-1}$ and $(1 + \mathcal{V}\mathcal{R}_0(\omega))^{-1}$ at the points $\omega = \pm m$ in corresponding norms imply that

$$(1 + \mathcal{R}_0(\omega)\mathcal{V})^{-1}\mathcal{A}_0^\pm = \mathcal{O}(\sqrt{\omega \mp m}), \quad \mathcal{A}_0^\pm(1 + \mathcal{V}\mathcal{R}_0(\omega))^{-1} = \mathcal{O}(\sqrt{\omega \mp m}), \quad \omega \rightarrow 0, \quad \omega \in \mathbb{C} \setminus \bar{\Gamma} \tag{3.3}$$

in $\mathcal{L}(L_\sigma^2, L_{-\sigma}^2)$ with $\sigma > 3/2$. Therefore,

$$\|(1 + \mathcal{R}_0(\omega)\mathcal{V})^{-1}[1]\|_{L_{-\sigma}^2} = \mathcal{O}(\sqrt{\omega \mp m}), \quad \omega \rightarrow \pm m, \quad \omega \in \mathbb{C} \setminus \bar{\Gamma}, \quad \sigma > 3/2 \tag{3.4}$$

and for any $f \in L^2_\sigma$ with $\sigma > 3/2$

$$\int [(1 + \mathcal{V}\mathcal{R}_0(\omega))^{-1}f](x)dx = \mathcal{O}(\sqrt{\omega \mp m}), \quad \omega \rightarrow \pm m, \quad \omega \in \mathbb{C} \setminus \bar{\Gamma}, \quad \sigma > 3/2 \quad (3.5)$$

Taking into account the identities

$$\mathcal{R}' = (1 + \mathcal{R}_0\mathcal{V})^{-1}\mathcal{R}'_0(1 + \mathcal{V}\mathcal{R}_0)^{-1}, \quad \mathcal{R}'' = \left[(1 + \mathcal{R}_0\mathcal{V})^{-1}\mathcal{R}''_0 - 2\mathcal{R}'\mathcal{V}\mathcal{R}'_0 \right] (1 + \mathcal{V}\mathcal{R}_0)^{-1}$$

we obtain from (3.4)-(3.5) the asymptotics (3.2) for $\mathcal{R}'(\omega)$ and $\mathcal{R}''(\omega)$. Finally, the asymptotics (3.2) for $\mathcal{R}(\omega)$ follow by integration the asymptotics (3.2) for $\mathcal{R}'(\omega)$. \square

To obtain other properties of $\mathcal{R}(\omega)$ we express $\mathcal{R}(\omega)$ in the resolvent of a matrix Schrödinger operator. First, we introduce the operator $\tilde{\mathcal{H}}$ similar to \mathcal{H} with matrix potential $\tilde{\mathcal{V}}(x)$ satisfying $\tilde{\mathcal{V}}_{11}(x) = \tilde{\mathcal{V}}_{22}(x) = 0$. Namely, let us introduce the matrix of gauge transformation

$$C = \begin{pmatrix} \exp\left(-i \int_{-\infty}^x \mathcal{V}_{11}(y)dy\right) & 0 \\ 0 & \exp\left(i \int_{-\infty}^x \mathcal{V}_{22}(y)dy\right) \end{pmatrix} \quad (3.6)$$

This matrix function is bounded by the conditions (1.3), and

$$C^{-1}\mathcal{H}C = \tilde{\mathcal{H}} \quad (3.7)$$

where

$$\tilde{\mathcal{H}} := i\alpha\partial_x + \tilde{\mathcal{V}}, \quad \tilde{\mathcal{V}} = \begin{pmatrix} 0 & \tilde{\mathcal{V}}_{12} \\ \tilde{\mathcal{V}}_{21} & 0 \end{pmatrix} \quad (3.8)$$

with

$$\tilde{\mathcal{V}}_{12}(x) = \exp\left(i \int_{-\infty}^x (\mathcal{V}_{11}(y) + \mathcal{V}_{22}(y))dy\right)(\mathcal{V}_{12}(x) + m), \quad \tilde{\mathcal{V}}_{21}(x) = \overline{\tilde{\mathcal{V}}_{12}(x)} \quad (3.9)$$

where we have used the conditions (1.3). By (3.7) the spectral properties of \mathcal{H} and $\tilde{\mathcal{H}}$ are identical. Further, we have

$$\begin{aligned} (\tilde{\mathcal{H}} - \omega)(\tilde{\mathcal{H}} + \omega) &= (i\alpha\partial_x + \tilde{\mathcal{V}}(x) - \omega)(i\alpha\partial_x + \tilde{\mathcal{V}}(x) + \omega) \\ &= \begin{pmatrix} -\partial_x^2 + m^2 - \omega^2 + 2m\operatorname{Re}\mathcal{V}_{12} + |\mathcal{V}_{12}|^2 & -i\tilde{\mathcal{V}}'_{12} \\ i\tilde{\mathcal{V}}'_{21} & -\partial_x^2 + m^2 - \omega^2 + 2m\operatorname{Re}\mathcal{V}_{12} + |\mathcal{V}_{12}|^2 \end{pmatrix} = \mathbf{H} - (\omega^2 - m^2) \end{aligned} \quad (3.10)$$

Here $\mathbf{H} = \mathbf{H}_0 + V$ is the matrix Schrödinger operator with

$$\mathbf{H}_0 = \begin{pmatrix} -\partial_x^2 & 0 \\ 0 & -\partial_x^2 \end{pmatrix}, \quad V = \begin{pmatrix} 2m\operatorname{Re}\mathcal{V}_{12} + |\mathcal{V}_{12}|^2 & -i\tilde{\mathcal{V}}'_{12} \\ i\tilde{\mathcal{V}}'_{21} & 2m\operatorname{Re}\mathcal{V}_{12} + |\mathcal{V}_{12}|^2 \end{pmatrix} \quad (3.11)$$

Remark 3.3. Due (1.3), the perturbation V is Hermitian and does not contain differential operators. This fact is essential for obtaining high energy decay (3.31), (3.33), and (3.35).

Formula (3.10) implies the following representation for the resolvent $\tilde{\mathcal{R}}(\omega) = (\tilde{\mathcal{H}} - \omega)^{-1}$:

$$\tilde{\mathcal{R}}(\omega) = (\tilde{\mathcal{H}} + \omega)R(\omega^2 - m^2) = (i\alpha\partial_x + \tilde{\mathcal{V}}(x) + \omega)R(\omega^2 - m^2) \quad (3.12)$$

where $R(\zeta) = (\mathbf{H} - \zeta)^{-1}$, $\zeta \in \mathbb{C} \setminus [0, \infty)$, the resolvent of the operators \mathbf{H} .

Due to (3.7) we have $\tilde{\mathcal{R}}(\omega) = C\mathcal{R}(\omega)C^{-1}$. The resolvent $\tilde{\mathcal{R}}(\omega)$ admits the low energy asymptotics of type (3.2) since matrix function (3.6) is bounded. Namely

$$\left. \begin{aligned} \tilde{\mathcal{R}}(\omega) - \tilde{\mathcal{R}}(\pm m) &= \mathcal{O}(|\omega \mp m|^{1/2}) \\ \tilde{\mathcal{R}}'(\omega) &= \mathcal{O}(|\omega \mp m|^{-1/2}) \\ \tilde{\mathcal{R}}''(\omega) &= \mathcal{O}(|\omega \mp m|^{-3/2}) \end{aligned} \right| \omega \rightarrow \pm m, \quad \omega \in \mathbb{C} \setminus \bar{\Gamma} \quad (3.13)$$

in $\mathcal{L}(L_\sigma^2, L_{-\sigma}^2)$ with $\sigma > 5/2$.

Below we need the limiting absorption principle and high energy decay for the resolvent $\tilde{\mathcal{R}}(\omega)$. First, we obtain these properties for the resolvent $R(\zeta)$ of the matrix Schrödinger operator \mathbf{H} .

Lemma 3.4. *Let the conditions (1.5) and (3.1) hold. Then*

- i) $R(\zeta)$ is meromorphic function of $\zeta \in \mathbb{C} \setminus [0, \infty)$.
- ii) For $\zeta > 0$, the convergence (limiting absorption principle) holds

$$R(\zeta \pm i\varepsilon) \rightarrow R(\zeta \pm i0), \quad \varepsilon \rightarrow 0+ \quad (3.14)$$

in $\mathcal{L}(H_\sigma^0, H_{-\sigma}^2)$ with $\sigma > 1/2$.

iii) For $l = 0, 1$, the asymptotics hold

$$\|R_j^{(k)}(\zeta)\|_{\mathcal{L}(H_\sigma^0, H_{-\sigma}^l)} = \mathcal{O}(|\zeta|^{-\frac{1-l+k}{2}}), \quad \zeta \rightarrow \infty, \quad \zeta \in \mathbb{C} \setminus [0, \infty) \quad (3.15)$$

with $\sigma > 1/2 + k$ for $k = 0, 1, 2$.

Proof. *Step i)* The statement i) follows from Lemma 2.1-i), the Born splitting

$$R(\zeta) = R_0(\zeta)(1 + VR_0(\zeta))^{-1} \quad (3.16)$$

and the Gohberg-Bleher theorem [3, 4] since the operators $VR_0(\zeta)$ is a compact operator in L^2 for $\zeta \in \mathbb{C} \setminus [0, \infty)$.

Step ii) The convergence (3.14) follows from the vector version of Agmon's theorem [1, Theorem 3.3 and Lemma 4.2] taking into account the absence of embedded eigenvalues follows from the theory of ordinary differential equations since $V \in L^1$.

Step iii) The asymptotics (3.15) follows from (2.10) by the vector version of [7, Theorem 9.2]. \square

Lemma 3.4 and formula (3.12) imply

Lemma 3.5. *Let the conditions (1.5) and (3.1) hold. Then*

- i) $\tilde{\mathcal{R}}(\omega)$ is meromorphic function of $\omega \in \mathbb{C} \setminus \bar{\Gamma}$ with the values in $\mathcal{L}(L^2, L^2)$;
- ii) For $\omega \in \Gamma$, the convergence (limiting absorption principle) holds

$$\tilde{\mathcal{R}}(\omega \pm i\varepsilon) \rightarrow \tilde{\mathcal{R}}(\omega \pm i0), \quad \varepsilon \rightarrow 0+ \quad (3.17)$$

in $\mathcal{L}(L_\sigma^2, L_{-\sigma}^2)$ with $\sigma > 1/2$;

iii) For $k = 0, 1, 2$ and $\sigma > 1/2 + k$ the asymptotics hold

$$\|\tilde{\mathcal{R}}^{(k)}(\omega)\|_{\mathcal{L}(L_\sigma^2, L_{-\sigma}^2)} = \mathcal{O}(1), \quad |\omega| \rightarrow \infty, \quad \omega \in \mathbb{C} \setminus \Gamma \quad (3.18)$$

Remark 3.6. *The reduction to the Schrödinger operator \mathbf{H} allow us to deduce Lemma 3.5 from Lemma 3.4 applying known results of Agmon [1] and Jensen-Kato [7].*

3.2 Time decay

In this section we combine the spectral properties of the perturbed resolvent $\mathcal{R}(\omega)$ and time decay for the unperturbed dynamics using the (finite) Born perturbation series. Denote by Σ the set of eigenvalues of the operators \mathcal{H} . Lemma 3.1 imply that the eigenvalues cannot accumulate to the points $\pm m$ and then the set Σ is finite. Our main result is the following.

Theorem 3.7. *Let conditions (1.5) and (3.1) hold. Then*

$$\|e^{-it\mathcal{H}} - \sum_{\omega_j \in \Sigma} e^{-i\omega_j t} P_j\|_{\mathcal{L}(L^2_\sigma, L^2_{-\sigma})} = \mathcal{O}(|t|^{-3/2}), \quad t \rightarrow \pm\infty \quad (3.19)$$

with $\sigma > \frac{5}{2}$, where P_j are the Riesz projectors onto the corresponding eigenspaces of operator \mathcal{H} .

Proof. By (3.7) it suffices to prove that for $\sigma > 5/2$

$$\|e^{-it\tilde{\mathcal{H}}} - \sum_{\omega_j \in \Sigma} e^{-i\omega_j t} \tilde{P}_j\|_{\mathcal{L}(L^2_\sigma, L^2_{-\sigma})} = \mathcal{O}(|t|^{-3/2}), \quad t \rightarrow \pm\infty \quad (3.20)$$

where $\tilde{P}_j = C^{-1}P_jC$ are the Riesz projectors onto the corresponding eigenspaces of operator $\tilde{\mathcal{H}}$. Lemma 3.5 and asymptotics (3.13) imply similarly to (2.15), that

$$e^{-it\tilde{\mathcal{H}}}\psi_0 - \sum_{\omega_j \in \Sigma} e^{-i\omega_j t} \tilde{P}_j\psi_0 = \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} [\tilde{\mathcal{R}}(\omega + i0) - \tilde{\mathcal{R}}(\omega - i0)] \psi_0 d\omega = \psi_l(t) + \psi_h(t) \quad (3.21)$$

where

$$\tilde{P}_j\psi_0 := -\frac{1}{2\pi i} \int_{|\omega - \omega_j| = \delta} \tilde{\mathcal{R}}(\omega)\psi_0 d\omega$$

with a small $\delta > 0$, and

$$\psi_l(t) = \frac{1}{2\pi i} \int_{\Gamma} l(\omega) e^{-i\omega t} [\tilde{\mathcal{R}}(\omega + i0) - \tilde{\mathcal{R}}(\omega - i0)] \psi_0 d\omega \quad (3.22)$$

$$\psi_h(t) = \frac{1}{2\pi i} \int_{\Gamma} h(\omega) e^{-i\omega t} [\tilde{\mathcal{R}}(\omega + i0) - \tilde{\mathcal{R}}(\omega - i0)] \psi_0 d\omega \quad (3.23)$$

where $l(\omega)$ and $h(\omega)$ are defined in Section 2.2. Further we analyze $\psi_l(t)$ and $\psi_h(t)$ separately.

3.2.1 Low energy decay

We consider only the integral over $(m, m + 2\varepsilon)$. The integral over $(-m - 2\varepsilon, -m)$ is dealt with in the same way. We prove the desired decay of $\psi_l(t)$ using a special case of Lemma 10.2 from [7]. Denote by \mathbf{B} a Banach space with the norm $\|\cdot\|$.

Lemma 3.8. *Let $F \in C([m, a], \mathbf{B})$, satisfy*

$$F(m) = F(a) = 0, \quad \|F''(\omega)\| = \mathcal{O}(|\omega - m|^{-3/2}), \quad \omega \rightarrow m \quad (3.24)$$

Then

$$\int_m^a e^{-it\omega} F(\omega) d\omega = \mathcal{O}(t^{-3/2}), \quad t \rightarrow \infty \quad (3.25)$$

Due to (3.13), we can apply Lemma 3.8 with $F = l(\omega)(\tilde{\mathcal{R}}(\omega + i0) - \tilde{\mathcal{R}}(\omega - i0))$, $\mathbf{B} = \mathcal{L}(L_\sigma^2, L_{-\sigma}^2)$, $a = m + 2\varepsilon$ with a small $\varepsilon > 0$ and $\sigma > 5/2$, to get

$$\|\psi_l(t)\|_{L_{-\sigma}^2} \leq C(1 + |t|)^{-3/2} \|\psi_0\|_{L_\sigma^2} \quad t \in \mathbb{R}, \quad \sigma > 5/2 \quad (3.26)$$

3.2.2 High energy decay

The resolvents $R(\zeta)$, $R_0(\zeta)$ are related by the Born perturbation series

$$R(\zeta) = R_0(\zeta)I - R_0(\zeta)VR_0(\zeta) + R_0(\zeta)VR_0(\zeta)VR(\zeta), \quad \zeta \in \mathbb{C} \setminus [0, \infty) \quad (3.27)$$

which follows by iteration of $R(\zeta) = R_0(\zeta)I - R_0(\zeta)VR(\zeta)$. Then by (3.12) we have

$$\begin{aligned} \tilde{\mathcal{R}}(\omega) &= (i\alpha\partial_x + \tilde{\mathcal{V}}(x) + \omega) \left[R_0(\omega^2 - m^2)I - R_0(\omega^2 - m^2)VR_0(\omega^2 - m^2) \right. \\ &\quad \left. + R_0(\omega^2 - m^2)VR_0(\omega^2 - m^2)VR(\omega^2 - m^2) \right] \end{aligned} \quad (3.28)$$

Let us substitute the series (3.28) into the spectral representation (3.23) for $\psi_h(t)$:

$$\begin{aligned} \psi_h(t) &= \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} h(\omega) (i\alpha\partial_x + \tilde{\mathcal{V}}(x) + \omega) \left[R_0(\zeta_+) - R_0(\zeta_-) \right] \psi_0 d\omega \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} h(\omega) (i\alpha\partial_x + \tilde{\mathcal{V}}(x) + \omega) \left[R_0(\zeta_+)VR_0(\zeta_+) - R_0(\zeta_-)VR_0(\zeta_-) \right] \psi_0 d\omega \\ &\quad + \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} h(\omega) (i\alpha\partial_x + \tilde{\mathcal{V}}(x) + \omega) \left[R_0(\zeta_+)VR_0(\zeta_+)VR(\zeta_+) - R_0(\zeta_-)VR_0(\zeta_-)VR(\zeta_-) \right] \psi_0 d\omega \\ &= \psi_{h1}(t) + \psi_{h2}(t) + \psi_{h3}(t), \quad t \in \mathbb{R} \end{aligned}$$

where $\zeta_+ = (\omega + i0)^2 - m^2$, $\zeta_- = (\omega - i0)^2 - m^2$. We analyze each term ψ_{hk} , $k = 1, 2, 3$ separately.

Step i) By (2.5), (2.19) and (2.21) for the first term $\psi_{h1}(t)$ we have

$$\psi_{h1}(t) = -i(\tilde{\mathcal{V}}(x) - m\beta)\mathcal{G}_h(t)\psi_0 + \mathcal{U}_h(t)\psi_0$$

Hence, (2.22) and (2.24) imply that

$$\|\psi_{h1}(t)\|_{L_{-\sigma}^2} \leq C(1 + |t|)^{-3/2} \|\psi_0\|_{L_\sigma^2}, \quad t \in \mathbb{R}, \quad \sigma > 5/2 \quad (3.29)$$

Step ii) Let us represent the second term $\psi_{h2}(t)$ as

$$\psi_{h2}(t) = \varphi_1(t) + \varphi_2(t) \quad (3.30)$$

where

$$\varphi_1(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} h(\omega) (\tilde{\mathcal{V}} - m\beta) \left[R_0(\zeta_+)VR_0(\zeta_+) - R_0(\zeta_-)VR_0(\zeta_-) \right] \psi_0 d\omega$$

$$\begin{aligned}\varphi_2(t) &= \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} h(\omega) (i\alpha\partial_x + m\beta + \omega) \left[R_0(\zeta_+) V R_0(\zeta_+) - R_0(\zeta_-) V R_0(\zeta_-) \right] \psi_0 d\omega \\ &= \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} h(\omega) \left[\mathcal{R}_0(\omega + i0) V R_0(\zeta_+) - \mathcal{R}_0(\omega - i0) V R_0(\zeta_-) \right] \psi_0 d\omega\end{aligned}$$

Let us consider the first term $\varphi_1(t)$. Denote

$$P(\omega) = h(\omega) (\tilde{V} - m\beta) \left[R_0(\zeta_+) V R_0(\zeta_+) - R_0(\zeta_-) V R_0(\zeta_-) \right] \psi_0$$

We have

$$\text{supp } P(\omega) \in \Gamma_\varepsilon := (-\infty, -m - \varepsilon) \cup (m + \varepsilon, \infty)$$

and $P'' \in L^1(\Gamma_\varepsilon; \mathcal{L}(L_\sigma^2, L_{-\sigma}^2))$ with $\sigma > 5/2$ by (2.10) with $l = 0$ and $k = 2$ since V does not contain differential operators, see Remark 3.3. Then, two times partial integration implies that

$$\|\varphi_1(t)\|_{L_{-\sigma}^2} \leq C(1 + |t|)^{-2} \|\psi_0\|_{L_\sigma^2}, \quad t \in \mathbb{R}, \quad \sigma > 5/2 \quad (3.31)$$

Now we consider the second term $\varphi_2(t)$. Denote $h_1(\omega) = \sqrt{h(\omega)}$ (we can assume that $h(\omega) \geq 0$ and $h_1 \in \mathbb{C}_0^\infty(\mathbb{R})$). We set

$$\phi_{h_1} = \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} h_1(\omega) \left[R_0(\zeta_+) - R_0(\zeta_-) \right] \psi_0 d\omega = -i\mathcal{G}_{h_1}(t)\psi_0$$

It is obvious that for ϕ_{h_1} the inequality (3.29) also holds. Namely,

$$\|\phi_{h_1}(t)\|_{L_{-\sigma}^2} \leq C(1 + |t|)^{-3/2} \|\psi_0\|_{L_\sigma^2}, \quad t \in \mathbb{R}, \quad \sigma > 5/2$$

Now the term $\varphi_2(t)$ can be rewritten as a convolution.

Lemma 3.9. (cf. [9, Lemma 3.11]) *The convolution representation holds*

$$\varphi_2(t) = i \int_0^t \mathcal{U}_{h_1}(t - \tau) V \phi_{h_1}(\tau) d\tau, \quad t \in \mathbb{R} \quad (3.32)$$

where the integral converges in $L_{-\sigma}^2$ with $\sigma > 5/2$.

Applying Theorem 2.3 with h_1 instead of h to the integrand in (3.32), we obtain that

$$\|\mathcal{U}_{h_1}(t - \tau) V \phi_{h_1}(\tau)\|_{L_{-\sigma}^2} \leq \frac{C \|V \phi_{h_1}(\tau)\|_{L_{\sigma'}^2}}{(1 + |t - \tau|)^{3/2}} \leq \frac{C_1 \|\phi_{h_1}(\tau)\|_{L_{\sigma' - \beta}^2}}{(1 + |t - \tau|)^{3/2}} \leq \frac{C_2 \|\psi_0\|_{L_\sigma^2}}{(1 + |t - \tau|)^{3/2} (1 + |\tau|)^{3/2}} \quad (3.33)$$

where $\sigma' \in (5/2, \beta - 5/2)$. Therefore, integrating here in τ , we obtain by (3.32) that

$$\|\varphi_2(t)\|_{L_{-\sigma}^2} \leq C(1 + |t|)^{-3/2} \|\psi_0\|_{L_\sigma^2}, \quad t \in \mathbb{R}, \quad \sigma > 5/2 \quad (3.34)$$

Step iii) Denote by $Q(\omega)$ the integrand in ψ_{h_3} . Since $Q'' \in L^1(\Gamma_\varepsilon; \mathcal{L}(L_\sigma^2, L_{-\sigma}^2))$ with $\sigma > 5/2$ by (2.10) and (3.15) then, two times partial integration implies that

$$\|\psi_{h_3}(t)\|_{L_{-\sigma}^2} \leq C(1 + |t|)^{-2} \|\psi_0\|_{L_\sigma^2}, \quad t \in \mathbb{R}, \quad \sigma > 5/2 \quad (3.35)$$

Finally, the bounds (3.29), (3.31), (3.34) and 3.35) imply

$$\|\psi_h(t)\|_{L^2_{-\sigma}} \leq C(1 + |t|)^{-3/2} \|\psi_0\|_{L^2_{\sigma}}, \quad t \in \mathbb{R}, \quad \sigma > 5/2$$

Theorem 3.7 is proved. \square

Corollary 3.10. *The asymptotics (3.20) imply (1.6) with the projector*

$$\mathcal{P}_c := 1 - \mathcal{P}_d, \quad \mathcal{P}_d = \sum_{\omega_j \in \Sigma} P_j \quad (3.36)$$

4 Application to the asymptotic completeness

We apply the obtained results to prove the asymptotic completeness by standard Cook's argument.

Theorem 4.1. *Let conditions (1.5) and (3.1) hold. Then*

i) For solution to (1.1) with any initial function $\psi(0) \in L^2$, the long time asymptotics hold

$$\psi(t) = \sum_{\omega_j \in \Sigma} e^{-i\omega_j t} \psi_j + \mathcal{U}(t)\phi_{\pm} + r_{\pm}(t) \quad (4.1)$$

where ψ_j are the corresponding eigenfunctions, $\phi_{\pm} \in L^2$ are the scattering states, and

$$\|r_{\pm}(t)\|_{L^2} \rightarrow 0, \quad t \rightarrow \pm\infty \quad (4.2)$$

ii) Furthermore,

$$\|r_{\pm}(t)\|_{L^2} = \mathcal{O}(|t|^{-1/2}) \quad (4.3)$$

if $\psi(0) \in L^2_{\sigma}$ with $\sigma > 5/2$.

Proof. Denote $\mathcal{X}_d := \mathcal{P}_d L^2$, $\mathcal{X}_c := \mathcal{P}_c L^2$. For $\psi(0) \in \mathcal{X}_d$ the asymptotics (4.1) obviously hold with $\phi_{\pm} = 0$ and $r_{\pm}(t) = 0$. Hence, it remains to prove for $\psi(0) \in \mathcal{X}_c$ the asymptotics

$$\psi(t) = \mathcal{U}(t)\phi_{\pm} + r_{\pm}(t) \quad (4.4)$$

with the remainder satisfying (4.2). Moreover, it suffices to prove the asymptotics (4.4), (4.3) for $\psi(0) \in \mathcal{X}_c \cap L^2_{\sigma}$ with $\sigma > 5/2$ since the space L^2_{σ} is dense in L^2 , while the group $\mathcal{U}(t)$ is unitary in L^2 . In this case Theorem 3.7 implies the decay

$$\|\psi(t)\|_{L^2_{-\sigma}} \leq C(1 + |t|)^{-3/2} \|\psi(0)\|_{L^2_{\sigma}}, \quad t \rightarrow \pm\infty \quad (4.5)$$

The function $\psi(t)$ satisfies the equation (1.1),

$$i\dot{\psi}(t) = (\mathcal{H}_0 + \mathcal{V})\psi(t)$$

Hence, the corresponding Duhamel equation reads

$$\psi(t) = \mathcal{U}(t)\psi(0) + \int_0^t \mathcal{U}(t - \tau)\mathcal{V}\psi(\tau)d\tau, \quad t \in \mathbb{R} \quad (4.6)$$

Let us rewrite (4.6) as

$$\psi(t) = \mathcal{U}(t) \left[\psi(0) + \int_0^{\pm\infty} \mathcal{U}(-\tau) \mathcal{V} \psi(\tau) d\tau \right] - \int_t^{\pm\infty} \mathcal{U}(t-\tau) \mathcal{V} \psi(\tau) d\tau = \mathcal{U}(t) \phi_{\pm} + r_{\pm}(t) \quad (4.7)$$

It remains to prove that $\phi_{\pm} \in L^2$ and (4.3) holds. Let us consider the sign “+” for the concreteness. The unitarity of $\mathcal{U}(t)$ in L^2 , the condition (1.5) and the decay (4.5) imply that for $\sigma' \in (5/2, \min\{\sigma, \beta\})$

$$\int_0^{\infty} \|\mathcal{U}(-\tau) \mathcal{V} \psi(\tau)\|_{L^2} d\tau \leq C \int_0^{\infty} \|\mathcal{V} \psi(\tau)\|_{L^2} d\tau \leq C_1 \int_0^{\infty} \|\psi(\tau)\|_{L^2_{-\sigma'}} d\tau \leq C_2 \int_0^{\infty} \frac{\|\psi(0)\|_{L^2_{\sigma}} d\tau}{(1+\tau)^{-3/2}} < \infty$$

since $|\mathcal{V}(x)| \leq C \langle x \rangle^{-\beta} \leq C' \langle x \rangle^{-\sigma'}$. Hence, $\phi_+ \in L^2$. The estimate (4.3) follows similarly. \square

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