On Dispersive Estimates for Discrete Schrödinger and Klein-Gordon Equations

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Abstract

We derive the long-time asymptotics for solutions of the discrete 3D Schrödinger and Klein-Gordon equations.

Keywords: discrete Schrödinger and Klein-Gordon equations, lattice, Cauchy problem, long-time asymptotics, weighed norms.

2000 Mathematics Subject Classification: 39A11, 35L10.

1 Introduction

We consider the 3D discrete version of the Schrödinger equation,

$$\left\{\begin{array}{l} i\dot\psi(x,t)=H\psi(x,t):=(-\Delta+V(x))\,\psi(x,t) \\[2ex] \psi\big|_{t=0}=\psi_0 \end{array}\right. \quad \Bigg| \quad x\in\mathbb{Z}^3,\quad t\in\mathbb{R}, \tag{1.1}$$

Schr

where Δ stands for the difference Laplacian in $\mathbb{Z}^3$, defined by

$$\Delta\psi(x)=\sum_{|y-x|=1}\psi(y)-6\psi(x),\quad x\in\mathbb{Z}^3,\quad \psi:\mathbb{Z}^3\to\mathbb{C}$$

Denote by $\mathcal{V}$ the set of real valued functions on the lattice $\mathbb{Z}^3$ with finite supports. For the potential V, we assume that $V\in\mathcal{V}$. Then for $\psi_0\in l^2=l^2(\mathbb{Z}^3)$ there exists the unique solution $\psi(x,t)\in C(\mathbb{R},l^2)$ to Cauchy problem (1.1), and the charge $\|\psi(\cdot,t)\|_{l^2}=\text{const}$ is conserved.

It is well known that with the help of the Fourier-Laplace transform in respect to variable t one can deduce the properties of nonstationary equation from the properties of resolvent $R(\omega)=(H-\omega)^{-1}$ of the Schrödinger operator H.

[1] Supported partly by FWF, DFG and RFBR grants

We are going to use the weighted Hilbert spaces $l^2_\sigma = l^2_\sigma(\mathbb{Z}^3)$ with the norms

$$\|u\|_{l^2_\sigma} = \left\|(1+x^2)^{\sigma/2}u\right\|_{l^2}, \quad \sigma \in \mathbb{R}.$$

Let us denote

$$B(\sigma,\,\sigma') = \mathcal{L}(l^2_\sigma,\, l^2_{\sigma'})$$

the space of bounded linear operators from l^2_σ to $l^2_{\sigma'}$.

The spectrum of the operator H consists of the continuous spectrum and of the real eigenvalues μ_j, $j = 1, ..., n$. Note that $n \le N$, where N is the number of points in the support of V (see. [3], Theorem 13^{bis}, Chapter I).

Note that the continuous spectrum of the operator H coincides with the interval $[0, 12]$, which is the range of the symbol $\phi(\theta) = 4(\sin^2\frac{\theta_1}{2} + \sin^2\frac{\theta_2}{2} + \sin^2\frac{\theta_3}{2})$ of the difference Laplace operator $H_0 = -\Delta$.

We give special attention to the points $\omega_k = 4k \in [0, 12]$, $k = 0, 1, 2, 3$, which are critical values of the symbol, i.e. the values of the symbol in the critical points.

Our main results are as follows. For “a generic potential” $V \in \mathcal{V}$ (see Definition 3.3), we obtain
a) the existence of the limits $R(\omega \pm i0)$ (“limiting absorption principle”) on the continuous spectrum in the norm of $B(\sigma; -\sigma)$ with $\sigma > 3/2$;
b) the Puiseux expansion for the resolvent at the singular spectral points ω_k:

$$R(\omega_k + \omega) = D_k + \mathcal{O}(\sqrt{\omega}), \quad \omega \to 0, \tag{1.2}$$

in the norm of $B(\sigma; -\sigma)$ with $\sigma > 7/2$.

Then for initial data $\psi_0 \in l^2_\sigma$ with $\sigma > 11/2$ we obtain the following long-time asymptotics:

$$\|e^{-itH} - \sum_{j=1}^{n} e^{-it\mu_j} P_j\|_{\mathcal{B}(\sigma,-\sigma)} = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \tag{1.3}$$

Here P_j are the orthogonal projections in l^2 onto the eigenspaces of H, corresponding to the discrete eigenvalues μ_j.

We also obtain similar results for the discrete Klein-Gordon equation:

$$\left\{ \begin{array}{l} \ddot{\psi}(x,t) = (\Delta - m^2 - V(x))\,\psi(x,t) \\ \\ \psi\big|_{t=0} = \psi_0,\ \dot{\psi}\big|_{t=0} = \pi_0 \end{array} \right| \quad x \in \mathbb{Z}^3, \quad t \in \mathbb{R}. \tag{1.4}$$

Let us comment on previous results in this direction. For the first time the difference Schrödinger equation was considered by Eskina [2]. She proved the limiting absorption principle for matrix elements of the resolvent. The asymptotic expansion of the matrix element of the resolvent $R(\omega)$ at the critical points ω_k was obtained by Islami and Vainberg [4] in 2D case. They used this expansion to prove the long time asymptotics for the solutions of the Cauchy problem for the difference wave equation. The main feature which differs the present paper from [4] is that here all asymptotic expansions hold in the weighted functional spaces l^2_σ, not on compacts. Such expansions are desirable for the study of nonlinear evolutionary equations.

The asymptotic expansion of the resolvent and the long time asymptotics (1.3) for hyperbolic PDEs in $\mathbb{R}^n$ (continuous case) were obtained earlier in [9], [13], [14], and for the

Schrödinger equation in [5], [6], [10]; also see [11] for an up-to-date review. We use the main ideas of the papers.

The results of present paper extend the results of [7] and [8] from difference 1D and 2D equations to difference 3D equations. In 3D case the analytical problems is more difficult because of several type of critical points.

The paper is organized as follows. In §2 we prove the limiting absorption principle and derive the Puiseux asymptotic of the free resolvent. In §3 we extend the results to perturbed resolvent. In §4 we prove the long-time asymptotics (1.3). In §5 we consider the discrete Klein-Gordon equation. In Appendix C we apply the obtained results to construct asymptotic scattering states.

2 Free resolvent

We start with an investigation of the unperturbed problem for the equation (1.1) with $V(x)=0$. The discrete Fourier transform of $u(x)\in l^2(\mathbb{Z}^3)$ is defined by the formula

$$\widehat{u}(\theta)=\sum_{x\in\mathbb{Z}^3}u(x)e^{i\theta x},\ \theta\in T^3:=\mathbb{R}^3/2\pi\mathbb{Z}^3,$$

After taking the Fourier transform, the operator $H_0=-\Delta$ becomes the operator of multiplication by $\phi(\theta):=6-2\sum\limits_{j=1}^{3}\cos\theta_j=4\sum\limits_{j=1}^{3}\sin^2\frac{\theta_j}{2}$:

$$-\widehat{\Delta u}(\theta)=\phi(\theta)\widehat{u}(\theta),\quad \theta\in T^3. \tag{2.1}$$

Thus, the spectrum of the operator H_0 coincides with the range of the function ϕ, that is $\mathrm{Spec}H_0=\Sigma:=[0,12]$. Denote by $R_0(\omega)=(H_0-\omega)^{-1}$ the resolvent of the difference Laplacian. Then the kernel of the resolvent $R_0(\omega)$ reads

$$R_0(\omega,x-y)=\frac{1}{8\pi^3}\int\limits_{T^3}\frac{e^{-i\theta(x-y)}}{\phi(\theta)-\omega}\,d\theta,\ \omega\in\mathbb{C}\setminus\Sigma. \tag{2.2}$$

Lemma 2.1. *The free resolvent $R_0(\omega)$ is an analytic function of $\omega\in\mathbb{C}\setminus\Sigma$ with the values in $\mathcal{B}(\sigma,\sigma')$ for any $\sigma,\sigma'\in\mathbb{R}$.*

Proof. For a fixed $\omega\in\mathbb{C}\setminus\Sigma$, we have $\phi(\theta)-\omega\neq 0$ for $\theta\in T^3$. Therefore, $\phi(\theta+i\xi)-\omega\neq 0$ for $\theta\in T^3$, $\xi\in\mathbb{R}^3$, if $\xi\neq 0$ is sufficiently small. Hence, the function $1/(\phi(\theta)-\omega)$ admits analytic continuation into a complex neighbourhood of the torus of type $\{\theta+i\xi:\theta\in T^3,\ \xi\in\mathbb{R}^3:|\xi|<\delta(\omega)\}$ with an $\delta(\omega)>0$. Therefore the Paley-Wiener arguments imply that

$$|R_0(\omega,x-y)|\le C(\delta)e^{-\delta|x-y|}$$

for any $\delta<\delta(\omega)$. Hence, $R_0(\omega)\in\mathcal{B}(\sigma,\sigma')$ by the Schur lemma. $\square$

2.1 Limiting absorption principle

Now we are interested in the traces of the analytic function $R_0(\omega)$ at the cut Σ. Consider

$$R_0(\omega\pm i\varepsilon,z)=\frac{1}{8\pi^3}\int\limits_{T^3}\frac{e^{-i\theta z}}{\phi(\theta)-\omega\mp i\varepsilon}\,d\theta,\quad z\in\mathbb{Z}^3,\quad \omega\in\Sigma,\quad \varepsilon>0. \tag{2.3}$$

Note that the limiting distribution $\dfrac{1}{\phi(\theta)-\omega\mp i0}$ is well defined if ω is not a critical value of the function $\phi(\theta)$, i.e. $\omega\neq 0,4,8,12$. The following limiting absorption principle holds:

LAP **Proposition 2.2.** *For $\sigma>3/2+k$ the following limits exist as $\varepsilon\to 0+$:*

$$\partial_\omega^k R_0(\omega\pm i\varepsilon)\ \xrightarrow{\mathcal{B}(\sigma,-\sigma)}\ \partial_\omega^k R_0(\omega\pm i0),\quad \omega\in\Sigma\setminus\{0,4,8,12\}. \tag{2.4}$$ lim

Proof. i) Let $k=0$. First we prove the convergence (2.4) for any fixed z, follow [2]. Let $\chi_j(\theta)$, $j=1,...,l$ are the sufficient small partition of unity on the torus T^3, which will be specified below. Then

$$R_0(\omega\pm i\varepsilon,z)=\sum_{j=1}^{l}\frac{1}{8\pi^3}\int\limits_{D_j}\frac{\chi_j(\theta)e^{-i\theta z}}{\phi(\theta)-\omega\mp i\varepsilon}\,d\theta=\sum_{j=1}^{l}P_j(\omega\pm i\varepsilon,z), \tag{2.5}$$ esk1

where D_j is the support of the function χ_j. If $\{\phi(\theta)=\omega\}\cap D_j=\emptyset$, then the function $P_j(\omega\pm i\varepsilon,z)$ is continuous for $\varepsilon\geq 0$ and

$$|P_j(\omega\pm i\varepsilon,z)|<C_j<\infty,\quad z\in Z^3,\quad \varepsilon\geq 0. \tag{2.6}$$ R0j

Now let $S_j=\{\phi(\theta)=\omega\}\cap D_j$. Then any $\theta\in D_j$ can be uniquely represented as $\theta=s+tn(s)$ where $s\in S_j$, and $n(s)$ is the external normal vector to S_j at the point s of unit length. Let us introduce the new variables (s,t). Then

$$P_j(\omega\pm i\varepsilon,z)=\frac{1}{8\pi^3}\int\limits_{S_j}e^{-isz}ds\int\limits_{-a(s)}^{b(s)}\frac{\chi_j(s+tn(s))e^{-itn(s)z}J(s,t)}{t\psi(s+tn(s))\mp i\varepsilon}\,dt. \tag{2.7}$$ esk2

where $J(s,t)$ is the Jacobian, ψ is the smooth function, and $a(s)$, $b(s)>0$. Note that $J(s,t)|_{t=0}=1$, and $\psi(s+tn(s))|_{t=0}=|\nabla\phi(s)|\neq 0$, since $\omega\in\Sigma\setminus\{0,4,8,12\}$ is not a critical value of $\phi(\theta)$. We will prove the following lemma:

SP **Lemma 2.3.** *Let $\varphi(t,z)$ be the smooth function satisfies*

$$|\varphi(t,z)|\leq C,\quad |\partial_t\varphi(t,z)|\leq C|z|,\quad t\in[-\delta,\delta],\ z\in\mathbb{Z}^3, \tag{2.8}$$ prp

and let $\psi(t)$ be the smooth function such that $\psi(t)\neq 0$ if $t\in[-\delta,\delta]$. Consider

$$F(\pm\varepsilon,z)=\int\limits_{-\delta}^{\delta}\frac{\varphi(t,z)}{t\psi(t)\mp i\varepsilon}dt.$$

Then $F(\pm\varepsilon,z)\to F(\pm 0,z)$ as $\varepsilon\to 0+$, $\forall z\in\mathbb{Z}^3$, and

$$\sup_{\varepsilon\in(0,1]}|F(\pm\varepsilon,z)|\leq C\Big(\ln(1+|z|)+1\Big),\quad z\in\mathbb{Z}^3.$$

Proof. Let us rewrite $F(\pm\varepsilon, z)$ as

$$\begin{aligned} F(\pm\varepsilon, z) &= \varphi(0,z)\int\limits_{-\delta}^{\delta}\frac{dt}{t\psi(0)\mp i\varepsilon} - \varphi(0,z)\int\limits_{-\delta}^{\delta}\frac{(\psi(t)-\psi(0))tdt}{(t\psi(t)\mp i\varepsilon)(t\psi(0)\mp i\varepsilon)} \\ &+ \int\limits_{-\delta}^{\delta}\frac{\varphi(t,z)-\varphi(0,z)}{t\psi(t)\mp i\varepsilon}dt. \end{aligned}$$

Then,

$$\begin{aligned} F(\pm\varepsilon, z) \to F(\pm 0, z) = \pm i\pi\frac{\varphi(0,z)}{\psi(0)} - \frac{\varphi(0,z)}{\psi(0)}\int\limits_{-\delta}^{\delta}\frac{\psi(t)-\psi(0)}{t\psi(t)}dt \\ + \int\limits_{-\delta}^{\delta}\frac{\varphi(t,z)-\varphi(0,z)}{t\psi(t)}dt \end{aligned} \tag{2.9}$$

as $\varepsilon \to 0+$. By (2.8) the first and the second summand in RHS of (2.9) can be estimated by the constant which does not depend on $|z| \in \mathbb{Z}^3$. Let us estimate the third summand in RHS of (2.9). For $|z| \le 1/\delta$ this summand also can be estimated by the constant. For $|z| > 1/\delta$ we obtain by (2.8) that

$$\begin{aligned} \int\limits_{-\delta}^{\delta}\Big|\frac{\varphi(t,z)-\varphi(0,z)}{t\psi(t)}\Big|dt &= \int\limits_{|t|<1/|z|}\ldots + \int\limits_{1/|z|<|t|<\delta}\ldots \\ &\le \frac{1}{|z|}C|z| + C\ln|z| \le C\ln|z|. \end{aligned}$$

Lemma is proved. $\square$

Lemma 2.3 implies that $P_j(\omega \pm i\varepsilon, z) \to P_j(\omega \pm i0, z)$ as $\varepsilon \to 0+$, $\forall z \in \mathbb{Z}^3$ and

$$\sup_{\varepsilon\in(0,1]}|P_j(\omega\pm i\varepsilon, z)| \le C_j\Big(\ln(1+|z|)+1\Big).$$

Evidently, that the whole resolvent R_0 satisfies the similar properties. Hence by the Lebesgue dominated convergence theorem

$$\sum_{x,y\in\mathbb{Z}^3}(1+|x|^2)^{-\sigma}|R_0(\omega\pm i\varepsilon, x-y) - R_0(\omega\pm i0, x-y)|^2(1+|y|^2)^{-\sigma} \to 0,\ \varepsilon\to 0+$$

with $\sigma > 3/2$. Then the Hilbert-Schmidt norm of the difference $R_0(\omega\pm i\varepsilon) - R_0(\omega\pm i0)$ converges to zero. Proposition 2.2 in the case $k = 0$ is proved.

ii) In the case $k \neq 0$ we use integration by parts. For instance, let us consider $k = 1$. Since

$\nabla\phi(\theta)\neq 0$ for $\theta\in D_j$ then there exists $i\in\{1,2,3\}$ such that $\partial_i\phi(\theta)\neq 0$ for $\theta\in D_j$. Hence,

$$\begin{aligned}
P_j'(\omega\pm i\varepsilon,z)&=\frac{1}{8\pi^3}\int\limits_{D_j}\frac{\chi_j(\theta)e^{-i\theta z}}{(\phi(\theta)-\omega\mp i\varepsilon)^2}\,d\theta\\
&=-\frac{1}{8\pi^3}\int\limits_{D_j}\partial_i\Big(\frac{1}{\phi(\theta)-\omega\mp i\varepsilon}\Big)\frac{\chi_j(\theta)e^{-i\theta z}\,d\theta}{\partial_i\phi(\theta)}\\
&=\frac{1}{8\pi^3}\int\limits_{D_j}\frac{1}{\phi(\theta)-\omega\mp i\varepsilon}\partial_i\Big(\frac{\chi_j(\theta)e^{-i\theta z}}{\partial_i\phi(\theta)}\Big)\,d\theta.
\end{aligned}$$

The further proof is similar to the case $k=0$. Differentiating the exponent implies additional factor z_i and then the value of σ increase on one unit. $\square$

2.2 Asymptotics near critical points

Further we need the information on behavior of the resolvent $R_0(\omega)$ near the critical points ω_k. We consider "elliptic" points $\omega_1=0$, $\omega_4=12$ and "hyperbolic" points $\omega_2=4$, $\omega_3=8$ separately.

2.2.1 Elliptic points

Here we construct the Puiseux expansion of the free resolvent $R_0(\omega)$ near the point $\omega_1=0$ (the expansion near the point $\omega_4=12$ can be construct similarly).

Proposition 2.4. *Let $N=0,1,2...$ and $\sigma>N+3/2$. Then the following expansion holds in $\mathcal{B}(\sigma,-\sigma)$:*

$$R_0(\omega)=\sum_{k=0}^{N}A_k\omega^{k/2}+\mathcal{O}(\omega^{(N+1)/2}),\ |\omega|\to 0,\ \arg\omega\in(0,2\pi). \tag{2.10}$$

Here $A_k\in\mathcal{B}(\sigma,-\sigma)$ with $\sigma>k+1/2$.

Proof. The resolvent $R_0(\omega\pm i0)$ is represented by the integral (2.2). Fix $0<\delta<1$ and consider $0<|\omega|<\delta^2/2$. We identify T^3 with the cube $[-\pi,\pi]^3$ and represent $R_0(\omega,z)$, $z=x-y$, as the sum

$$R_0(\omega,z)=\frac{1}{8\pi^3}\int\limits_{B_\delta}\frac{e^{-i\theta z}}{\phi(\theta)-\omega}\,d\theta+\frac{1}{8\pi^3}\int\limits_{T^3\setminus B_\delta}\frac{e^{-i\theta z}}{\phi(\theta)-\omega}\,d\theta=R_{01}(\omega,z)+R_{02}(\omega,z),$$

where B_δ is the ball of radius δ. Since $\phi(\theta)=|\theta|^2+\mathcal{O}(|\theta|^4)$, then $R_{02}(\omega,z)$ is analytic function of ω in $|\omega|\le\delta^2/2$, and

$$|\partial_\omega^j R_{02}(\omega,z)|\le\frac{C_{j,N}}{(|z|+1)},\quad |\omega|\le\delta^2/2,\quad z\in\mathbb{Z}^3.$$

Hence it suffices to prove the asymptotics of type (2.10) for R_{01}. For simplicity we suppose that $\phi(\theta)=|\theta|^2$ (In the case $\phi(\theta)=|\theta|^2+\mathcal{O}(|\theta|^4)$ the scheme of proving is similar and differ only the technical details). Let us choose the system of coordinate in which the direction of the axe θ_3 coincides with the direction of vector z and rewrite $R_{01}(\omega, z)$ as

$$R_{01}(\omega, z) = \frac{1}{8\pi^3} \int\limits_{|n|=1} \Big(\int\limits_0^{\delta} \frac{e^{-ir|z|n_3} r^2 dr}{r^2 - \omega} \Big) dS(n).$$

Here $r = |\theta|$, $\theta = rn$. Then

$$\begin{aligned} R_{01}(\omega, z) &= \frac{1}{8\pi^3} \int\limits_{S_+} \Bigg(\int\limits_0^{\delta} \frac{e^{-ir|z|n_3} r^2 dr}{r^2 - \omega} + \int\limits_0^{\delta} \frac{e^{ir|z|n_3} r^2 dr}{r^2 - \omega} \Bigg) dS(n) \\ &= \frac{1}{8\pi^3} \int\limits_{S_+} \Bigg(\int\limits_{-\delta}^{\delta} \frac{e^{-ir|z|n_3} r^2 dr}{r^2 - \omega} \Bigg) dS(n), \end{aligned} \tag{2.11}$$

R2

where $S_+ = \{|n| = 1,\ n_3 > 0\}$. The integrand in the RHS of (2.11) has one simple pole at the lower half-plane. Let us apply the Cauchy residue theorem:

$$\begin{aligned} R_{01}(\omega, z) = &-\frac{i\sqrt{\omega}}{32\pi^4} \int\limits_{S_1^+} e^{i\sqrt{\omega}|z|n_3} dS(n) + \frac{1}{8\pi^3} \int\limits_{S_+} \Bigg(\int\limits_{\Gamma_\delta} \frac{e^{-ir|z|n_3} r^2\ dr}{r^2 - \omega} \Bigg) dS(n) \\ &= R_{01}^1(\omega, z) + R_{01}^2(\omega, z). \end{aligned} \tag{2.12}$$

R3

Here $\Gamma_\delta = \{|r| = \delta, \operatorname{Im} r < 0\}$. For the first summand the asymptotics of type (2.10) are evident. Let us consider the second summand in the RHS of (2.12):

$$\begin{aligned} R_{01}^2(\omega, z) &= \frac{1}{8\pi^3} \int\limits_0^{2\pi} d\beta \int\limits_0^{\pi/2} d\alpha \int\limits_{\Gamma_\delta} \frac{e^{-ir|z|\cos\alpha} r^2 \sin\alpha\ dr}{r^2 - \omega} \\ &= \frac{i}{4\pi^2|z|} \int\limits_{\Gamma_\delta} \frac{r dr}{r^2 - \omega} \int\limits_0^{\pi/2} de^{-ir|z|\cos\alpha} = \frac{i}{4\pi^2|z|} \int\limits_{\Gamma_\delta} \frac{r(1 - e^{-ir|z|}) dr}{r^2 - \omega}, \quad |z| \neq 0, \end{aligned}$$

where α is the angle between z and θ. Since $r^2 = \delta^2$ on Γ_δ, then the operator value function $R_{01}^2(\omega)$ is analytic in $|\omega| < \delta^2/2$. Moreover, the function $R_{01}^2(\omega)$ and all its derivatives in respect to ω are bounded in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 1/2$. Hence R_{01}^2 admits an expansion of type (2.10). ☐

dif1 **Remark 2.5.** *The expansion (2.10) can been differentiated $N + 1$ times in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > N + 3/2$:*

$$\partial_\omega^r R_0(\omega) = \partial_\omega^r \Big(\sum_{k=0}^{N} A_k \omega^{k/2} \Big) + \mathcal{O}(\omega^{(N+1)/2 - r}), \quad 1 \leq r \leq N + 1.$$

Proof. For the proof let us note, that each differentiation of the resolvent in respect to ω increase the power of pole of the integrand in the RHS of (2.11) on one unit. Therefore, the calculation of the corresponding residue leads to differentiation of the exponent and then to appear extra factors $|z|$. Hence the value of σ increase on one unit. ☐

2.2.2 Hyperbolic points

as-hyp

Here we construct the Puiseux expansion of the free resolvent $R_0(\omega)$ near the "hyperbolic" point $\omega_2 = 4$ (the expansion near the point $\omega_3 = 8$ can be construct similarly). The main contribution into (2.2) is given by the corresponding critical points $(0,0,\pi)$, $(0,\pi,0)$ and $(\pi,0,0)$ of hyperbolic type.

R0-exphyp **Proposition 2.6.** *Let $N = -1, 0, 1, ...$ and $\sigma > 2N + 7/2$. Then in $\mathcal{B}(\sigma, -\sigma)$ the expansion holds:*

$$R_0(4+\omega) = \sum_{k=0}^{N} E_k\omega^k + \sqrt{\omega}\sum_{k=0}^{N} B_k\omega^k + \mathcal{O}(\omega^{N+1}),\ |\omega| \to 0,\ \operatorname{Im}\omega > 0. \tag{2.13}$$

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Here the operators $E_k, B_k \in \mathcal{B}(\sigma, -\sigma)$ with $\sigma > 2k + 3/2$. In the case $\operatorname{Im}\omega < 0$ *the similar expansion holds.*

Proof. For $\omega = \omega_2 = 4$ the denominator of the integral (2.2) vanishes along the curve $\phi(\theta) = 4$. We will study main contribution of points $(0,0,\pi)$, $(0,\pi,0)$ and $(\pi,0,0)$ of the curve which are critical points of $\phi(\theta)$. The contribution of other points of the curve can be proved by methods of Section 2.1. For concreteness, let us consider the integral over a neighborhood of the point $(\pi,0,0)$. Let $\zeta(\theta)$ be a smooth cutoff function, equal 1 in a neighborhood of the point $(\pi,0,0)$ (the other properties of $\zeta(\theta)$ we specified below). For $\operatorname{Im}\omega > 0$ denote

$$\begin{aligned} Q(\omega, z) &= \frac{1}{8\pi^3}\int \frac{e^{-iz\theta}\ \zeta(\theta)\ d\theta}{\phi(\theta) - 4 - \omega} \\ &= \frac{e^{-iz_3\pi}}{8\pi^3}\int \frac{e^{-iz\theta'}\zeta_1(\theta')\ d\theta'}{4\sin^2\frac{\theta_1}{2} + 4\sin^2\frac{\theta_2}{2} - 4\sin^2\frac{\theta_3'}{2} - \omega}, \end{aligned}$$

where $\theta_3' = \theta_3 - \pi$, $\theta' = (\theta_1, \theta_2, \theta_3')$, $\zeta_1(\theta') = \zeta(\theta)$. We suppose that $\zeta_1(\theta')$ is symmetric in θ_1, θ_2 and θ_3'. Then the exponent in the numerator can be substituted by its even part, so we have

$$\begin{aligned} Q(\omega, z) &= \frac{e^{-iz_3\pi}}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\cos(z_1\theta_1)\ \cos(z_2\theta_2)\ \cos(z_3\theta_3)\ \zeta_1(\theta)d\theta}{4\sin^2\frac{\theta_1}{2} + 4\sin^2\frac{\theta_2}{2} - 4\sin^2\frac{\theta_3}{2} - \omega} \\ &= \frac{e^{-iz_3\pi}}{\pi^3}Q_1(\omega, z). \end{aligned}$$

Let us obtain the expansion of type (2.13) for Q_1. We change the variables: $s_i = 2\sin\frac{\theta_i}{2}$, and choose the cutoff function ζ such that $\zeta_1(\theta) = \zeta_2(|s|^2)$, with smooth function ζ_2. Then

$$\begin{aligned} Q_1(\omega, z) &= \int_0^\infty\int_0^\infty\int_0^\infty \frac{F(z, s_1^2, s_2^2, s_3^2)\zeta_2(|s|^2)\ ds}{s_1^2 + s_2^2 - s_3^2 - \omega}, \\ F(z, s_1^2, s_2^2, s_3^2) &= \prod_i \frac{2\cos(2z_i \arcsin s_i/2)}{\sqrt{4 - s_i^2}}. \end{aligned}$$

Now we use the cylindrical variables: $s_1 = \tau\cos\varphi, s_2 = \tau\sin\varphi, s_3 = s_3$. Then

$$
\begin{aligned}
Q_1(\omega, z) &= \int_0^\infty\int_0^\infty \frac{F_1(z,\tau^2,s_3^2)\zeta_2(\tau^2+s_3^2)\tau d\tau ds_3}{\tau^2 - s_3^2 - \omega},\\
F_1(z,\tau^2,s_3^2) &= \int_0^{\pi/2} F(z,\tau^2\cos^2\varphi, \tau^2\sin^2\varphi, s_3^2)d\varphi.
\end{aligned} \tag{2.14} \qquad \boxed{\texttt{Q1}}
$$

We change the variables once more:

$$
\rho_1 = \tau^2 - s_3^2 = R^2\cos 2\psi, \quad \rho_2 = 2\tau s_3 = R^2\sin 2\psi,
$$

where R, ψ are the polar coordinates on the plane (τ, s_3). Then $|\rho|^2 = \rho_1^2 + \rho_2^2 = R^4$, hence, $|\rho| = R^2$, $\tau^2 = (|\rho| + \rho_1)/2$, $s_3^2 = (|\rho| - \rho_1)/2$, $d\rho_1 d\rho_2 = 4|\rho| d\tau ds_3$ and

$$
Q_1(\omega, z) = \int_0^\infty \Big(\int_{\mathbb{R}} \frac{h(|\rho|,\rho_1,z)}{(\rho_1-\omega)|\rho|}\sqrt{|\rho|+\rho_1}d\rho_1\Big)d\rho_2, \tag{2.15} \qquad \boxed{\texttt{h-int}}
$$

where $h(|\rho|,\rho_1,z) = F_1(z,\frac{|\rho|+\rho_1}{2},\frac{|\rho|-\rho_1}{2})\zeta_2(|\rho|)/4\sqrt{2}$. Now we can specify all needed properties of cutoff function:

$$
\operatorname{supp}\zeta_2(|\rho|)\cap\{\rho\in\mathbb{R}^2 : \rho_2 \ge 0\} \subset \Pi = \{(\rho_1,\rho_2) : -\delta \le \rho_1 \le \delta,\ 0 \le \rho_2 \le \delta\}
$$

with some $0 < \delta < 1$. We consider $0 < |\omega| \le \delta/2$, $\operatorname{Im}\omega > 0$. Denote $r = |\rho|$. The function $h(r,\rho_1,z)$ can be expanded into the following finite Taylor series with respect to ρ_1:

$$
h(r,\rho_1,z) = h_0(r,z) + h_1(r,z)\rho_1 + ... + h_N(r,z)\rho_1^N + H_N(r,\rho_1,z)\rho_1^N, \tag{2.16} \qquad \boxed{\texttt{hk}}
$$

where $h_k(r,z)$ are polynomial in z of order $2k$, and

$$
\begin{aligned}
&|H_N(r,\rho_1,z)| \le C|z|^{2N},\\
&|\partial_{\rho_1}H_N(r,\rho_1,z)| \le C|z|^{2N+2}, \quad (\rho_1,\rho_2)\in[-\delta,\delta]\times[0,\delta].
\end{aligned} \tag{2.17} \qquad \boxed{\texttt{HN-est}}
$$

Let us substitute (hk 2.16) into (h-int 2.15). Then

$$
Q_1(\omega,z) = \sum_{k=0}^N J_k(\omega,z) + \tilde{J}_N(\omega,z), \tag{2.18} \qquad \boxed{\texttt{JJ}}
$$

where

$$
\begin{aligned}
J_k(\omega,z) &= \int_\Pi \frac{h_k(r,z)\rho_1^k\sqrt{r+\rho_1}}{(\rho_1-\omega)r}d\rho_1 d\rho_2,\\
\tilde{J}_N(\omega,z) &= \int_\Pi \frac{H_N(r,\rho_1,z)\rho_1^N d\rho_1 d\rho_2}{(\rho_1-\omega)r}\sqrt{r+\rho_1}.
\end{aligned}
$$

Step i). First we consider the summands $J_k(\omega,z)$, $k=0,1,...,N$:

$$
\begin{aligned}
J_k(\omega,z) &= \int\limits_{\Pi}\frac{h_k(r,z)\sqrt{r+\rho_1}}{r}\Big(\rho_1^{k-1}+\omega\rho_1^{k-2}+...+\omega^{k-1}+\frac{\omega^k}{\rho_1-\omega}\Big)d\rho_1 d\rho_2\\
&=\sum_{k=0}^{k-1}a_{k,j}(z)\omega^j+\omega^k\int\limits_{\Pi}\frac{h_k(r,z)\sqrt{r+\rho_1}}{(\rho_1-\omega)r}d\rho_1 d\rho_2 \qquad (2.19)\\
&=\sum_{j=0}^{k-1}a_{k,j}(z)\omega^j+\omega^k\int\limits_0^{\delta}\frac{h_k(r,z)}{\sqrt{r}}dr\int\limits_0^{\pi}\frac{\sqrt{1+\cos\psi}\;d\psi}{\cos\psi-\omega/r},
\end{aligned}
$$

where $a_{k,j}(z)$ are polynomial of order $2k$. Let us calculate the integral

$$
\begin{aligned}
\int\limits_0^{\pi}\frac{\sqrt{1+\cos\psi}\;d\psi}{\cos\psi-\omega/r} &= \int\limits_0^{\pi}\frac{2\sqrt{2}d(\sin\frac{\psi}{2})}{1-2\sin^2\frac{\psi}{2}-\frac{\omega}{r}}\\
&=\int\limits_0^1\frac{-\sqrt{2}dt}{t^2-\frac{r-\omega}{2r}}=\frac{-\sqrt{r}}{\sqrt{r-\omega}}\log\frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}+\frac{\pi i\sqrt{r}}{\sqrt{r-\omega}}. \qquad (2.20)
\end{aligned}
$$

Here $\sqrt{r}\geq 0$, function $z=\sqrt{r-\omega}$ is analytic in $\operatorname{Im}\omega>0$ with the values in $\operatorname{Im}z<0$, $\operatorname{Re}z>0$, and function $\zeta=\log w$ is analytic in $|w|<1$, $\operatorname{Im}w>0$, where $\log(-1)=\pi i$.

Substitute (2.20) into (2.19), we get

$$
J_k(\omega,z)=\sum_{j=0}^{k-1}a_{k,j}(z)\omega^j+\omega^k\int\limits_0^{\delta}\left(\pi i-\log\frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right)\frac{h_k(r,z)dr}{\sqrt{r-\omega}} \qquad (2.21)
$$

Let us expand $h_k(r,z)$ into the following finite Taylor series with respect to r:

$$
h_k(r,z)=h_{k,0}(z)+h_{k,1}(z)r+...+h_{k,N-k}(z)r^{N-k}+H_{k,N-k}(r,z)r^{N-k}, \qquad (2.22)
$$

where $h_{k,j}(z)$ are polynomial of order $2(k+j)$, and $|H_{k,N-k}(r,z)|\leq C|z|^{2N}$, $0\leq r\leq\delta$. The following lemma is true

Lemma 2.7. *Let* $0<|\omega|<\delta/2$, $\operatorname{Im}\omega>0$. *Then*

$$
I_l=\int_0^{\delta}\left(\pi i-\log\frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right)\frac{r^l dr}{\sqrt{r-\omega}}=s_l(\omega)+C_l\omega^l\sqrt{\omega}, \qquad (2.23)
$$

where s_l *are analytic in* $0<|\omega|<\delta/2$, $\operatorname{Im}\omega>0$, $C_l\in\mathbb{R}$.

We shall prove this lemma in Appendix A. Now (2.21)-(2.23) imply that for $0<|\omega|<\delta/2$, $\operatorname{Im}\omega>0$

$$
\begin{aligned}
J_k(\omega,z) &= \sum_{j=0}^{N}b_{k,j}(z)\omega^j+\omega^k\sqrt{\omega}\sum_{j=0}^{N-k}c_{k,j}(z)\omega^j+\tilde{a}_{N,k}(\omega,z)\omega^{N+1}\\
&+\omega^k\int\limits_0^{\delta}\left(\pi i-\log\frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right)\frac{H_{k,N-k}(r,z)r^{N-k}dr}{\sqrt{r-\omega}}, \qquad (2.24)
\end{aligned}
$$

where $|b_{k,j}(z)| \le C|z|^{2N}$, $|c_{k,j}(z)| \le C|z|^{2(k+j)}$, and $|\tilde a_{N,k}(\omega,z)| \le C|z|^{2N}$. Further,

$$\int_0^\delta \left(\pi i - \log \frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right) \frac{H_{k,N-k}(r,z) r^{N-k} dr}{\sqrt{r-\omega}} = \int_0^{2|\omega|} + \int_{2|\omega|}^{\delta} = \mathcal{I}_1 + \mathcal{I}_2. \tag{2.25}$$

HNk

In $\mathcal{I}_1$ we change the variable: $r = |\omega|\tau$. Then

$$\begin{aligned} |\mathcal{I}_1| &= |\omega|^{N-k}\sqrt{|\omega|} \left| \int_0^2 \left(\pi i - \log \frac{1-\sqrt{\frac{\tau-\omega/|\omega|}{2\tau}}}{1+\sqrt{\frac{\tau-\omega/|\omega|}{2\tau}}}\right) \frac{H_{k,N-k}(|\omega|\tau,z)\tau^{N-k} d\tau}{\sqrt{\tau-\omega/|\omega|}} \right| \\ &\le C|z|^{2N}|\omega|^{N-k}\sqrt{|\omega|}. \end{aligned} \tag{2.26}$$

I1

Let us expend $\sqrt{r-\omega}$ and the function in brackets into the finite Taylor series with respect to ω/r. Then

$$\begin{aligned} \mathcal{I}_2 &= \int_{2|\omega|}^{\delta} H_{k,N-k}(r,z) r^{N-k-1/2} \\ &\quad \times \Big(d_0 + d_1\frac{\omega}{r} + \cdots + d_{N-k}\frac{\omega^{N-k}}{r^{N-k}} + \hat d_{N-k}(\omega/r)\frac{\omega^{N-k}}{r^{N-k}}\Big) dr \\ &= \int_{2|\omega|}^{\delta} H_{k,N-k}(r,z) \\ &\quad \times \Big(d_0 r^{N-k-1/2} + d_1\omega r^{N-k-3/2} + \cdots + d_{N-k}\omega^{N-k} r^{-1/2}\Big) dr + \tilde u_{N-k}(\omega,z) \\ &= \int_0^\delta H_{k,N-k}(r,z) \\ &\quad \times \Big(d_0 r^{N-k-1/2} + d_1\omega r^{N-k-3/2} + \cdots + d_{N-k}\omega^{N-k} r^{-1/2}\Big) dr + \hat u_{N-k}(\omega,z) \\ &= \sum_{j=0}^{N-k} u_j(z)\omega^j + \hat u_{N-k}(\omega,z), \end{aligned} \tag{2.27}$$

I2

where $|\hat d_{N-k}(\omega/r)| \le C$, $|u_j(z)| \le C|z|^{2N}$, $|\tilde u_{N-k}(\omega,z)|$, $|\hat u_{N-k}(\omega,z)| \le C|z|^{2N}|\omega|^{N-k}$. Now (2.24)-(2.27) imply that

hk-int12

$$J_k(\omega,z) = \sum_{j=0}^{N} d_{k,j}(z)\omega^j + \omega^k\sqrt{\omega}\sum_{j=0}^{N-k} c_{k,j}(z)\omega^j + \tilde d_{N,k}(\omega,z), \tag{2.28}$$

Jk

where $|d_{k,j}(z)| \le C|z|^{2N}$, $|c_{k,j}(z)| \le C|z|^{2(k+j)}$ and $|\tilde d_{N,k}(\omega,z)| \le C|z|^{2N}|\omega|^N$.

Step ii). It remains to consider the summand $\tilde J_N(\omega,z)$ in the RHS of (2.18):

$$\begin{aligned}\tilde J_N(\omega,z)&=\int\limits_{\Pi}\frac{H_N(r,\rho_1,z)\sqrt{r+\rho_1}}{r}\\&\quad\times\Big(\rho_1^{N-1}+\omega\rho_1^{N-2}+\dots+\omega^{N-1}+\frac{\omega^N}{\rho_1-\omega}\Big)d\rho_1d\rho_2\\&=\sum_{j=0}^{N-1}w_j(z)\omega^j+\omega^N\int\limits_{\Pi}\frac{H_N(r,\rho_1,z)\sqrt{r+\rho_1}\;d\rho_1d\rho_2}{(\rho_1-\omega)r},\end{aligned}\tag{2.29}$$ JN

where $|w_j(z)|\le C|z|^{2N}$. The following estimate is true

rem-est1 **Lemma 2.8.** *Let $0<|\omega|<\delta/2$, $\operatorname{Im}\omega>0$. Then*

$$\Big|\int\limits_{\Pi}\frac{H_N(r,\rho_1,z)\sqrt{r+\rho_1}\;d\rho_1d\rho_2}{(\rho_1-\omega)r}\Big|\le C|z|^{2N}\ln^2|z|,\ |z|>1.\tag{2.30}$$ Rem-est1

We shall prove the lemma in Appendix B.

Step iii). Finally, (2.18), (2.28)-(2.30) imply that

$$Q_1(\omega,z)=\sum_{k=0}^{N}q_k(z)\omega^k+\sqrt{\omega}\sum_{k=0}^{N}p_k(z)\omega^k+\widehat q_N(\omega,z),\ |\omega|\to0,$$

where $|\widehat q_N(\omega,z)|\le C|z|^{2N}\ln^2|z||\omega|^N$. Further, $p_k(z)=\mathcal O(|z|^{2k})$, and $q_k(z)=\mathcal O(|z|^{2N})$ for $0\le k\le N$. Therefore, $q_k(z)=\mathcal O(|z|^{2k})$, since $q_k(z)$ do not depend on N. $\square$

32 **Corollary 2.9.** *Let $\sigma>3/2$. Then in $\mathcal B(\sigma,-\sigma)$ the expansion holds:*

$$R_0(4+\omega)=\mathcal O(1),\ |\omega|\to0,\ \operatorname{Im}\omega>0.\tag{2.31}$$ exp-h

diff2 **Corollary 2.10.** *The expansion (2.13) can be differentiated. More precisely,*

$$\partial_\omega R_0(4+\omega)=\frac{B_0}{2\sqrt\omega}+\mathcal O(1),\quad|\omega|\to0,\quad\operatorname{Im}\omega>0,\tag{2.32}$$ pa1

in $\mathcal B(\sigma,-\sigma)$ with $\sigma>7/2$,

$$\partial^2_\omega R_0(4+\omega)=-\frac{B_0}{4\omega\sqrt\omega}+\mathcal O(\omega^{-1/2}),\quad|\omega|\to0,\quad\operatorname{Im}\omega>0,\tag{2.33}$$ pa2

in $\mathcal B(\sigma,-\sigma)$ with $\sigma>11/2$.

Proof. It is sufficient to obtain the asymptotics of type (2.32) and (2.33) for Q_1 defined in (2.14). Formula (2.14) implies

$$\begin{aligned}\partial_\omega Q_1(\omega,z)&=\int\limits_0^\infty\int\limits_0^\infty\frac{F_1(z,\tau^2,s^2)\zeta_2(\tau^2+s^2)\tau d\tau ds}{(\tau^2-s^2-\omega)^2}\\&=\int\limits_0^\infty ds\int\limits_0^\infty F_1(z,\tau^2,s^2)\zeta_2(\tau^2+s^2)\partial_\tau\frac{-1/2}{\tau^2-s^2-\omega}d\tau\\&=-\frac12\int\limits_0^\infty\frac{F_1(z,0,s^2)\zeta_2(s^2)}{s^2+\omega}\;ds+\int\limits_0^\infty\int\limits_0^\infty\frac{\partial^2_\tau\big(F_1(z,\tau^2,s^2)\zeta_2(\tau^2+s^2)\big)\tau}{\tau^2-s^2-\omega}\;d\tau ds\\&=S_1(\omega,z)+S_2(\omega,z).\end{aligned}\tag{2.34}$$ fint

The asymptotics of $S_2(\omega, z)$ are similar to (2.31):

$$S_2(\omega) = \mathcal{O}(1), \quad |\omega| \to 0, \quad \operatorname{Im}\omega > 0, \tag{2.35}$$

in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 7/2$. Let us note, that the differentiation in respect to τ^2 implies extra factors $|z|^2$ and then the value of σ increase on two units by comparison with (2.31).

Consider $S_1(\omega, z)$:

$$S_1(\omega, z) = \int\limits_0^\delta \frac{(F_2(z, s^2) - F_2(z, |\omega|))ds}{s^2 + \omega} + \int\limits_0^\delta \frac{F_2(z, |\omega|)ds}{s^2 + \omega} \tag{2.36}$$

For the function $F_2(z, s^2) = -F_1(z, 0, s^2)\zeta_2(s^2)/2$ the bounds hold:

$$|F_2(z, s^2)| \le C, \quad |\partial_{s^2} F_2(z, s^2)| \le C|z|^2, \quad |\partial_{s^2}^2 F_2(z, s^2)| \le C|z|^4. \tag{2.37}$$

Let us estimate the first integral in the RHS of (2.36) using the second bound (2.37):

$$\Big|\int\limits_0^\delta \frac{(F_2(z, s^2) - F_2(z, |\omega|))ds}{s^2 + \omega}\Big| \le C|z|^2 \int\limits_0^\delta \frac{|s^2 - |\omega||}{|s^2 + \omega|} ds \le C|z|^2$$

Let us calculate the second integral in the RHS of (2.36):

$$\int\limits_0^\delta \frac{F_2(z, |\omega|)ds}{s^2 + \omega} = F_2(z, |\omega|)\frac{1}{2\sqrt{-\omega}}\big(\ln\frac{\delta - \sqrt{-\omega}}{\delta + \sqrt{-\omega}} - \pi i\big) = p(z)\frac{1}{\sqrt{\omega}} + q(\omega, z), \tag{2.38}$$

where

$$|p(z)| + |q(\omega, z)| \le C, \quad 0 < |\omega| < \delta/2, \quad \operatorname{Im}\omega > 0.$$

Therefore,

$$S_1(\omega) = \frac{P_1}{\sqrt{\omega}} + \mathcal{O}(1),\ |\omega| \to 0,\ \operatorname{Im}\omega > 0 \tag{2.39}$$

in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 7/2$. From (2.35) (2.39) the asymptotics for the first derivative follow. Further, let us consider the second derivative:

$$\begin{aligned}
\partial_\omega^2 Q_1(\omega, z) &= \int\limits_0^\infty\int\limits_0^\infty \frac{2F_1(z, \tau^2, s^2)\zeta_2(\tau^2 + s^2)\tau d\tau ds}{(\tau^2 - s^2 - \omega)^3} \\
&= \int\limits_0^\infty ds \int\limits_0^\infty F_1(z, \tau^2, s^2)\zeta_2(\tau^2 + s^2)\partial_\tau \frac{-1/2}{(\tau^2 - s^2 - \omega)^2} d\tau \\
&= \frac{1}{2}\int\limits_0^\infty \frac{F_1(z, 0, s^2)\zeta_2(s^2)}{(s^2 + \omega)^2}\, ds + \int\limits_0^\infty\int\limits_0^\infty \frac{\partial_{\tau^2}\big(F_1(z, \tau^2, s^2)\zeta_2(\tau^2 + s^2)\big)\tau}{(\tau^2 - s^2 - \omega)^2}\, d\tau ds. \\
&= U_1(\omega, z) + U_2(\omega, z).
\end{aligned} \tag{2.40}$$

The proof of asymptotics for $U_2(\omega, z)$ is similar to the proof of asymptotics for the first derivative. We obtain

$$U_2(\omega) = \frac{P_2}{\sqrt{\omega}} + \mathcal{O}(1), \quad |\omega| \to 0, \quad \operatorname{Im}\omega > 0,$$

in $\mathcal{B}(\sigma,-\sigma)$ with $\sigma>11/2$. Let us consider $U_1(\omega,z)$:

$$U_1(\omega,z)=-\int\limits_0^\delta\frac{F_2(z,s^2)}{(s^2+\omega)^2}\,ds$$
$$=-\int\limits_0^\delta\frac{F_2(z,s^2)-F_2(z,|\omega|)-F_2'(z,|\omega|)(s^2-|\omega|)}{(s^2+\omega)^2}\,ds+\int\limits_0^\delta\frac{F_2(z,|\omega|))}{(s^2+\omega)^2}\,ds$$
$$+\int\limits_0^\delta\frac{F_2'(z,|\omega|))}{s^2+\omega}\,ds=U_{11}(\omega,z)+U_{12}(\omega,z)+U_{13}(\omega,z).$$

The last bound (2.37) implies

$$U_{11}(\omega,z)|\le C|z|^4. \tag{2.41}$$

Further, we obtain similar to (2.38)

$$U_{13}(\omega,z)=\frac{p_1(z)}{\sqrt{\omega}}+q_1(\omega,z),\quad |p_1(z)|+|q_1(\omega,z)|\le C|z|^2,\quad 0<|\omega|<\frac{\delta^2}{2},\ \ \mathrm{Im}\,\omega>0. \tag{2.42}$$

Finally,

$$\begin{aligned}U_{12}(\omega,z)&=F_2(z,|\omega|)\Big(\frac{1}{4\omega\sqrt{-\omega}}\big(\pi i-\ln\frac{1+\sqrt{-\omega}/\delta}{1-\sqrt{-\omega}/\delta}\big)+\frac{\delta}{2\omega(\delta^2+\omega)}\Big)\\&=F_2(z,|\omega|)\Big(\frac{1}{4\omega\sqrt{-\omega}}\big(\pi i-\frac{2\sqrt{-\omega}}{\delta}+\frac{2\omega\sqrt{-\omega}}{3\delta^3}+\dots\big)+\frac{1}{2\omega\delta}\Big(1-\frac{\omega}{\delta^2}+\dots\Big)\Big)\\&=\frac{s_2(z)}{\omega\sqrt{\omega}}+\frac{p_2(z)}{\sqrt{\omega}}+q_2(\omega,z),\quad |s_2(z)|+|p_2(z)|+|q_2(\omega,z)|\le C,\\&\quad 0<|\omega|<\frac{\delta^2}{2},\quad \mathrm{Im}\,\omega>0.\end{aligned} \tag{2.43}$$

From (2.41)–(2.43) the asymptotics for the second derivative follow. □

3 Perturbed resolvent

3.1 Limiting absorption principle

For the perturbed resolvent the limiting absorption principle holds.

Proposition 3.1. *Let $V\in\mathcal{V}$, $\sigma>3/2$. Then the limits exist as $\varepsilon\to 0+$*

$$R(\omega\pm i\varepsilon)\ \xrightarrow{B(\sigma,-\sigma)}\ R(\omega\pm i0),\quad \omega\in\Sigma\setminus\{0,4,8,12\}. \tag{3.1}$$

Proof. Let $\omega\in\Sigma\setminus\{0,4,8,12\}$ and $\sigma>3/2$. Since the potential V has a finite support, then Proposition 2.2 implies

$$I+VR_0(\omega\pm i\varepsilon)\ \xrightarrow{\mathcal{B}(\sigma,\sigma)}\ I+VR_0(\omega\pm i0),\quad \varepsilon\to 0+\,.$$

The operator $I+VR_0(\omega\pm i0)$ has only a trivial kernel (see [12, Theorem 10]), and the operator $VR_0(\omega\pm i0)$ is finite dimensional. Hence, the operator $I+VR_0(\omega\pm i0)$ is invertible, and moreover

$$\big(I+VR_0(\omega\pm i\varepsilon)\big)^{-1} \xrightarrow{\mathcal{B}(\sigma,\sigma)} \big(I+VR_0(\omega\pm i0)\big)^{-1}, \quad \varepsilon\to 0+\,. \tag{3.2}$$

Further, (2.4), (3.2) and identity $R=R_0(I+VR_0)^{-1}$ imply the existence of the limits (3.1). $\square$

Remark 3.2. *For $\omega\in\Sigma\setminus\{0,4,8,12\}$ the derivatives $\partial_\omega^k R(\omega\pm i0)$ belong to $\mathcal{B}(\sigma,-\sigma)$ with $\sigma>3/2+k$.*

Proof. The statement follows from Proposition 2.2 and the identity (see [5, Theorem 9.2])

$$R^{(k)}=\Big[(1-RV)R_0^{(k)}-\left(\begin{array}{c}k\\1\end{array}\right)R'VR_0^{k-1}-...\Big](1-VR),\quad k=1,2,...$$

$\square$

3.2 Asymptotics near critical points

In this sections we are going to obtain an asymptotic expansion for the perturbed resolvent $R(\omega)$ near the critical points ω_k, $k=1,2,3,4$.

Definition 3.3. *i) A set $\mathcal{W}\subset\mathcal{V}$ is called generic, if for each $V\in\mathcal{V}$ we have $\alpha V\in\mathcal{W}$, with the possible exception of a discrete set of $\alpha\in\mathbb{R}$.*
ii) We say that a property holds for a "generic" V, if it holds for all V from a generic subset $\mathcal{W}\subset\mathcal{V}$.

Theorem 3.4. *Let $\sigma>3/2$. Then for "generic" $V\in\mathcal{V}$ the following expansion holds:*

$$R(\omega)=D_1+\mathcal{O}(\sqrt{\omega}),\qquad |\omega|\to 0,\ \arg\omega\in(0,2\pi) \tag{3.3}$$

in the norm of $\mathcal{B}(\sigma,-\sigma)$.

Proof. We use the relation

$$R(\omega)=T^{-1}(\omega)R_0(\omega),\quad \text{where}\quad T(\omega):=I+R_0(\omega)V. \tag{3.4}$$

According to (2.10),

$$T(\omega)=I+A_0V+O(\sqrt{\omega}),\quad |\omega|\to 0,\quad \arg\omega\in(0,2\pi). \tag{3.5}$$

Let us prove that for "generic" $V\in\mathcal{V}$ the operator $T(\omega)$ is invertible in $l^2_{-\sigma}$ for sufficient small $|\omega|>0$. It is suffices to prove that the operator $T(0)=I+A_0V$ is invertible in $l^2_{-\sigma}$, or the operator with the kernel

$$(1+x^2)^{-\sigma/2}(\delta(x-y)+A_0V(y))(1+y^2)^{\sigma/2}$$

is invertible in l^2. Let us consider the operator

$$\mathcal{A}(\alpha)=\mathrm{Op}[(1+x^2)^{-\sigma/2}\big(\delta(x-y)+\alpha A_0V(y)\big)(1+y^2)^{\sigma/2}]=1+\alpha\mathcal{K},\qquad \alpha\in\mathbb{C}.$$

For $\sigma > 3/2$

$$K(x,y) = (1+x^2)^{-\sigma/2} A_0 V(y)(1+y^2)^{\sigma/2} \in l^2(\mathbb{Z}^2 \times \mathbb{Z}^2).$$

Hence, $K(x,y)$ is a Hilbert-Schmidt kernel, and accordingly the operator $\mathcal{K} = \mathrm{Op}(K(x,y))$: $l^2 \to l^2$ is compact. Further, $\mathcal{A}(\alpha)$ is analytic in $\alpha \in \mathbb{C}$, and $\mathcal{A}(0)$ is invertible. It follows that $\mathcal{A}(\alpha)$ is invertible for all $\alpha \in \mathbb{C}$ outside a discrete set; see [1]. Thus we could replace the original potential V by αV with α arbitrarily close to 1, if necessary, to have $T(0)$ invertible.

Now (3.4) and (3.5) imply that for sufficiently small $|\omega| > 0$

$$R(\omega) = (I + T(0) + \mathcal{O}(\sqrt{\omega}))^{-1}(A_0 + \mathcal{O}(\sqrt{\omega})) = T(0)^{-1}A_0 + O(\sqrt{\omega}). \tag{3.6}$$

□

Remark 3.5. *i) The expansion of the resolvent near the second elliptic point $\omega_4 = 12$ is similar to the expansion (3.3).*
ii) The expansion of type (3.3) near the hyperbolic points $\omega_2 = 4$ and $\omega_3 = 8$ require larger value of σ. Namely, for "generic" $V \in \mathcal{V}$

$$R(\omega_k + \omega) = D_k + \mathcal{O}(\sqrt{\omega}), \quad |\omega| \to 0, \quad \mathrm{Im}\,\omega > 0, \quad k = 1, 2$$

in the norm of $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 7/2$.
iii) These expansion can be differentiated two times in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 5/2$ for elliptic points and with $\sigma > 11/2$ for hyperbolic points. In these cases $\partial_\omega^2 R(\omega_k + \omega) = \mathcal{O}(\omega^{-3/2})$, $k = 1, 2, 3, 4$.

4 Long-time asymptotics

Now we apply Lemma 4.2 below, which is a version of Lemma 10.2 from [5] to prove the following theorem

Theorem 4.1. *Let $\sigma > 11/2$. Then for "generic" $V \in \mathcal{V}$ the following asymptotics hold*

$$\left\| e^{-itH} - \sum_{j=1}^{n} e^{-it\mu_j} P_j \right\|_{B(\sigma,-\sigma)} = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \tag{4.1}$$

Here P_j denote the projections on the eigenspaces corresponding to the eigenvalues $\mu_j \in \mathbb{R} \setminus [0, 12]$, $j = 1, \ldots, n$.

Proof. The estimate (4.1) is based on the formula

$$e^{-itH} = -\frac{1}{2\pi i} \oint\limits_{|\omega|=C} e^{-it\omega} R(\omega) d\omega, \; C > \max\{12; |\mu_j|, \, j = 1, ..., n\}. \tag{4.2}$$

The integral above is equal to the sum of residues at the poles of $R(\omega)$ and the integral over the contour around the segment [0,12], i.e.

$$\begin{aligned} e^{-itH} - \sum_{j=1}^{n} e^{-it\mu_j} P_j &= \frac{1}{2\pi i} \int\limits_{[0,12]} e^{-it\omega} (R(\omega + i0) - R(\omega - i0))\, d\omega \\ &= \int\limits_{[0,12]} e^{-it\omega} P(\omega) d\omega. \end{aligned}$$

The main contribution into the long-time asympotics gives the integrals over the neighbourhoods of the critical points. For example, let us consider the integral over the neighbourhood of the point $\omega_1 = 0$. Expansion (3.3) and Remark 3.5 imply

$$\partial^k P(\omega) = \mathcal{O}(\partial^k \sqrt{\omega}),\ \omega \to +0,\ \omega \in \mathbb{R},\ k = 0, 1, 2 \tag{4.3}$$

in $B(\sigma, -\sigma)$ with $\sigma > 11/2$. The following result is a special case of [5, Lemma 10.2].

Lemma 4.2. *Assume $\mathcal{B}$ is a Banach space, $a > 0$, and $F \in C(0, a; \mathcal{B})$ satisfies $F(0) = F(a) = 0$, $F' \in L^1(0, a; \mathcal{B})$, as well as $F''(\omega) = \mathcal{O}(\omega^{-3/2})$ as $\omega \to 0$. Then*

$$\int\limits_{[0,a]} e^{-it\omega} F(\omega) d\omega = \mathcal{O}(t^{-3/2}), \quad t \to \infty.$$

Set $F(\omega) = \zeta(\omega)P(\omega)$, where ζ is the smooth function, $\zeta(\omega) = 1$ for $\omega \in [-1/2, 1/2]$, $\operatorname{supp}\zeta \in (-1, 1)$; $a = 1$, $\mathcal{B} = B(\sigma, -\sigma)$ with $\sigma > 11/2$. Then due to (4.3) we can apply Lemma 4.2 to get

$$\int\limits_{[0,1]} e^{-it\omega} \zeta(\omega) P(\omega) = \mathcal{O}(t^{-3/2}), \quad t \to \infty,$$

in $B(\sigma, -\sigma)$ with $\sigma > 11/2$.

The integrals over the neighborhoods of the other critical points can be estimated similarly. $\square$

5 Klein-Gordon equation

Now we extend the results of Sections 3-4 to the case of the Klein-Gordon equation (1.4). Denote $\mathbf{\Psi}(t) \equiv \big(\psi(\cdot, t), \dot\psi(\cdot, t)\big)$, $\mathbf{\Psi}_0 \equiv \big(\psi_0, \pi_0\big)$. Then (1.4) becomes

$$i\dot{\mathbf{\Psi}}(t) = \mathbf{H}\mathbf{\Psi}(t) = \begin{pmatrix} 0 & i \\ i(\Delta - m^2 - V) & 0 \end{pmatrix} \mathbf{\Psi}(t), \quad t \in \mathbb{R}; \quad \mathbf{\Psi}(0) = \mathbf{\Psi}_0,$$

The resolvent $\mathbf{R}(\omega)$ can be expressed in term of the resolvent $R(\omega)$ as

$$\mathbf{R}(\omega) = \begin{pmatrix} \omega R(\omega^2 - m^2) & iR(\omega^2 - m^2) \\ -i(1 + \omega^2 R(\omega^2 - m^2)) & \omega R(\omega^2 - m^2) \end{pmatrix}, \quad \omega^2 - m^2 \in \mathbb{C} \setminus [0, 12]. \tag{5.1}$$

Representation (5.1) and the properties of $R(\omega)$ imply the following long time asymptotics:

Let $\sigma > 11/2$ and $\mathbf{\Psi}_0 \in l^2_\sigma \oplus l^2_\sigma$. Then for "generic" $V \in \mathcal{V}$

$$\Big\| e^{-it\mathbf{H}}\mathbf{\Psi}_0 - \sum_{\pm} \sum_{j=1}^{n} e^{-it\nu_j^\pm} \mathbf{P}_j^\pm \mathbf{\Psi}_0 \Big\|_{l^2_{-\sigma} \oplus l^2_{-\sigma}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty.$$

Here $\mathbf{P}_j^\pm$ are the projections onto the eigenspaces corresponding to the eigenvalues $\nu_j^\pm = \pm\sqrt{m^2 + \mu_j}$, $j = 1, \dots, n$.

6 Appendix A

Let us prove Lemma 2.7 by induction. For $l=0$ we get

$$
\begin{aligned}
I_0 &= \int\limits_0^\delta \left(\pi i - \log \frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right) \frac{dr}{\sqrt{r-\omega}} \\
&= 2\sqrt{r-\omega}\left(\pi i - \log \frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right)\Bigg|_0^\delta + 2\sqrt{2}\omega \int\limits_0^\delta \frac{dr}{\sqrt{r}(r+\omega)} \\
&= 2\sqrt{\delta-\omega}\left(\pi i - \log \frac{1-\sqrt{\frac{\delta-\omega}{2\delta}}}{1+\sqrt{\frac{\delta-\omega}{2\delta}}}\right) - i2\sqrt{2\omega}\log\frac{\sqrt{r}-i\sqrt{\omega}}{\sqrt{r}+i\sqrt{\omega}}\Bigg|_0^\delta \\
&= \tilde{s}_0(\omega) - i2\sqrt{2\omega}\log\frac{1-i\sqrt{\frac{\omega}{\delta}}}{1+i\sqrt{\frac{\omega}{\delta}}} - \pi\sqrt{2\omega} = s_0(\omega) + C_0\sqrt{\omega},
\end{aligned}
$$

where $\tilde{s}_0$, s_0 are the analytic functions of ω, $C_0 = -\pi\sqrt{2}$.

Further, for $l \geq 1$ we get

$$
\begin{aligned}
I_l &= \int_0^\delta \left(\pi i - \log \frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right) \frac{r^l dr}{\sqrt{r-\omega}} = 2\delta^l\sqrt{\delta-\omega}\left(\pi i - \log \frac{1-\sqrt{\frac{\delta-\omega}{2\delta}}}{1+\sqrt{\frac{\delta-\omega}{2\delta}}}\right) \\
&- 2l\int\limits_0^\delta \frac{r^{l-1}(r-\omega)}{\sqrt{r-\omega}}\left(\pi i - \log \frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right) dr + 2\sqrt{2}\omega\int\limits_0^\delta \frac{r^l dr}{\sqrt{r}(r+\omega)} \\
&= \tilde{s}_l(\omega) - 2lI_l + 2l\omega I_{l-1} \\
&+ 2\sqrt{2}\omega\int\limits_0^\delta \frac{dr}{\sqrt{r}}\left(r^{l-1} - \omega r^{l-2} + \cdots + (-\omega)^{l-1} + \frac{(-\omega)^l}{r+\omega}\right) \\
&= \tilde{s}_l(\omega) - 2lI_l + 2l\omega I_{l-1} + \tilde{\tilde{s}}_l(\omega) \\
&- i2\sqrt{2\omega}(-\omega)^l\log\frac{1-i\sqrt{\frac{\omega}{\delta}}}{1+i\sqrt{\frac{\omega}{\delta}}} - 2\pi\sqrt{2\omega}(-\omega)^l,
\end{aligned}
$$

where $\tilde{s}_l$, $\tilde{\tilde{s}}_l$ are the analytic functions of ω. Hence, $I_l = s_l(\omega) + C_l\omega^l\sqrt{\omega}$, where s_l are the analytic functions of ω and $C_l \in \mathbb{R}$.

7 Appendix B

Here we prove Lemma 2.8. We estimate only the integral over $\Pi_+ = \{0 \leq \rho_1, \rho_2 \leq \delta\}$. The integral over $\Pi \setminus \Pi_+$ can be estimated similarly. Let us split the integral over Π_+ into two integrals:

$$\int\limits_{\Pi_+} \frac{H_N(r,\rho_1,z)\sqrt{r+\rho_1}\ d\rho_1 d\rho_2}{(\rho_1-\omega)r}$$

$$= \int\limits_{\Pi_+} \frac{(H_N(r,\rho_1,z)-H_N(r,|\omega|,z))\sqrt{r+\rho_1}d\rho_1 d\rho_2}{(\rho_1-\omega)r}$$

$$+ \int\limits_{\Pi_+} \frac{H_N(r,|\omega|,z)\sqrt{r+\rho_1}d\rho_1 d\rho_2}{(\rho_1-\omega)r} = J_1 + J_2.$$

Similar to (2.20) we obtain

$$J_2 = \int\limits_0^\delta \left(\pi i - \log \frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right) \frac{H_N(r,|\omega|,z)dr}{\sqrt{2(r-\omega)}}.$$

Note that

$$\Big|\log\Big|\frac{\sqrt{2r}-\sqrt{r-\omega}}{\sqrt{2r}+\sqrt{r-\omega}}\Big|\Big| = \Big|\log\Big|\frac{r+\omega}{(\sqrt{2r}+\sqrt{r-\omega})^2}\Big|\Big|$$

$$\le |\log|r+\omega|| + 2|\log|\sqrt{2r}+\sqrt{r-\omega}|| \le 2|\log|r-|\omega|||.$$

Then (2.8) implies

$$|J_2| \le C|z|^{2N} \int\limits_0^\delta \frac{1+|\log|r-|\omega|||}{|\sqrt{r-|\omega|}|} dr \le C|z|^{2N}.$$

Further, for $|z|>1$ let us split J_1 as

$$J_1 = J_{11} + J_{12} + J_{13},$$

where J_{11} is integral over

$$\Pi_1 = \{(\rho_1,\rho_2) \in \Pi_+ : |r| < 1/|z|^{4/3}\},$$

J_{12} is integral over

$$\Pi_2 = \{(\rho_1,\rho_2) \in \Pi_+ \setminus \Pi_1 : |\rho_1 - |\omega|| < 1/|z|^{8/3}\},$$

and J_{13} is integral over

$$\Pi_3 = \Pi_+ \setminus (\Pi_1 \cup \Pi_2)$$

(see Picture 1).

By (2.17) and the inequality $|\rho_1-\omega| \ge |\rho_1 - |\omega||$ we get

$$|J_{11}| \le C|z|^{2N+2} \int\limits_{\Pi_1} \frac{|\rho_1-|\omega||\sqrt{r+\rho_1}d\rho_1 d\rho_2}{|\rho_1-\omega|r} \le C|z|^{2N+2} \int\limits_{\Pi_1} \frac{\sqrt{r+\rho_1}d\rho_1 d\rho_2}{r}$$

$$\le C|z|^{2N+2} \int\limits_0^{\pi/2} d\psi \int\limits_0^{1/|z|^{4/3}} \sqrt{r+r\cos\psi}\ dr$$

$$\le C|z|^{2N+2} \int\limits_0^{\pi/2} \cos\frac{\psi}{2} d\psi \int\limits_0^{1/|z|^{4/3}} \sqrt{r}dr \le C|z|^{2N}.$$

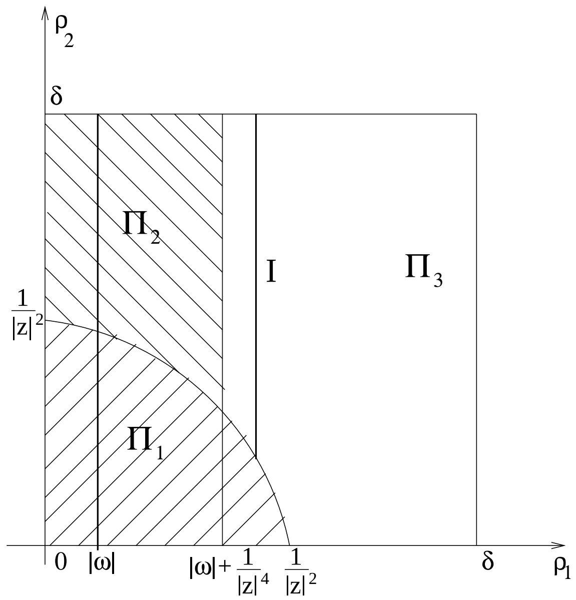

Figure 1: Case $|\omega| - 1/|z|^{8/3} < 0$. Picture1

$$\leq C|z|^{2N+2} \int_0^{\pi/2} d\psi \int_0^{1/|z|^{4/3}} \sqrt{r + r\cos\psi}\; dr \leq C|z|^{2N+2} \int_0^{\pi/2} \cos\frac{\psi}{2} d\psi \int_0^{1/|z|^{4/3}} \sqrt{r} dr \leq C|z|^{2N}.$$

For the second integral we obtain similarly

$$\begin{aligned} |J_{12}| &\leq C|z|^{2N+2} \int_{\Pi_2} \frac{\sqrt{r+\rho_1} d\rho_1 d\rho_2}{r} \\ &\leq C|z|^{2N+2} \int_{\Pi_2} \frac{d\rho_1 d\rho_2}{\sqrt{r}} \leq C|z|^{2N+2} |z|^{2/3} \frac{\delta}{|z|^{8/3}} \leq C|z|^{2N}, \end{aligned}$$

since $1/\sqrt{r} \leq |z|^{2/3}$ for $(\rho_1, \rho_2) \in \Pi_2$, and $|\Pi_2| \leq 2\delta/|z|^{8/3}$. Finally, HN-est (2.17) implies

$$|J_{13}| \leq C|z|^{2N} \int_{\Pi_3} \frac{d\rho_1 d\rho_2}{|\rho_1 - |\omega|| \sqrt{\rho_1^2 + \rho_2^2}} \leq C|z|^{2N} \ln^2 |z|,$$

since for any vertical interval $I \in \Pi_3$ we get

$$\int_I \frac{d\rho_2}{\sqrt{\rho_1^2 + \rho_2^2}} = \ln(\rho_2 + \sqrt{\rho_1^2 + \rho_2^2}) \leq C \ln |z|.$$

8 Appendix C. Asymptotic completeness

cat-sect

We apply the obtained results to construct the asymptotic scattering states. Let u_k be the eigenfunctions of operator H, corresponding eigenvalues μ_k, and $U_0(t)$ be the dynamical group of free Schrödinger equation.

Theorem 8.1. *i) Let $\sigma > 11/2$ and $\psi_0 \in l^2_\sigma$. Then for "generic" $V \in \mathcal{V}$ for solution to (1.1) the following long time asymptotics hold*

$$\psi(\cdot, t) = \sum_{k=1}^{n} C_k e^{-it\mu_k} u_k + U_0(t)\phi_\pm + r_\pm(t), \quad t \to \pm\infty, \tag{8.1}$$

where $\phi_\pm \in l^2$ are the corresponding scattering states, and

$$\|r_\pm(t)\|_{l^2} = \mathcal{O}(|t|^{-1/2})$$

Proof. For concreteness we consider the case $t \to +\infty$. Let us apply the projector P^c onto the continuous spectrum of the operator H to both sides of (1.1):

$$iP^c\dot{\psi} = P^c H\psi = H_0 P^c\psi + VP^c\psi \tag{8.2}$$

since P^c and H commute. Applying the Duhamel representation to equation (8.2) we obtain

$$P^c\psi(t) = U_0(t)P^c\psi(0) + \int_0^t U_0(t-\tau)VP^c\psi(\tau)d\tau, \quad t \in \mathbb{R}. \tag{8.3}$$

We can rewrite (8.3) as

$$P^c\psi(t) = U_0(t)\Big(P^c\psi(0) + \int_0^\infty U_0(-\tau)VP^c\psi(\tau)d\tau\Big)$$
$$- \int_t^\infty U_0(t-\tau)VP^c\psi(\tau)d\tau = U_0(t)\phi + r_+(t).$$

Let us show that the integrals converge, and the function

$$\phi_+ = P^c\psi(0) + \int_0^\infty U_0(-\tau)VP^c\psi(\tau)d\tau$$

belongs to l^2. Indeed, (4.1) implies

$$\begin{aligned}\int_0^\infty \|U_0(-\tau)VP^c\psi(\tau)\|_{l^2}d\tau &= \int_0^\infty \|VP^c\psi(\tau)\|_{l^2}d\tau \le C\int_0^\infty \|P^c\psi(\tau)\|_{l^2_{-\sigma}}d\tau \\ &\le C\int_0^\infty t^{-3/2}\|\psi(0)\|_{l^2_\sigma}d\tau \le C\end{aligned}$$

Here we used the unitarity of $U_0(t)$ in l^2 and the identity $P^c = I - P^d$, where P^d is the projector onto the discrete spectrum, which consists of the exponentially decreasing functions. The estimate for $r_+(t)$ follows similarly. □

For the Klein-Gordon equation the asymptotics of type (8.1) also hold.

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