On convergence to equilibrium distribution for Dirac equation

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Abstract

We consider the Dirac equation in $\mathbb{R}^3$ with a potential, and study the distribution μ_t of the random solution at time $t \in \mathbb{R}$. The initial measure μ_0 has zero mean, a translation-invariant covariance, and a finite mean charge density. We also assume that μ_0 satisfies a mixing condition of Rosenblatt- or Ibragimov-Linnik-type. The main result is the long time convergence of projection of μ_t onto the continuous spectral space. The limiting measure is Gaussian.

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1 Introduction

This paper can be considered as a continuation of our papers [5]-[8], [11] which concern the long time convergence to equilibrium distribution for the linear wave, Klein-Gordon and Schrödinger equations.

The convergence should clarify the distinguished role of the canonical Maxwell-Boltzmann-Gibbs equilibrium distribution in statistical physics. One of fundamental examples is the Kirchhoff-Planck black body radiation law which specify the equilibrium distribution for the Maxwell equations, and served as a basis for creation of quantum mechanics. The law likely should be correlation function of limiting equilibrium measure for coupled Maxwell-Schrödinger or Maxwell-Dirac equations.

Our ultimate goal would be the proof of the convergence for nonlinear hyperbolic PDEs. At the moment, a unique result in nonlinear case has been proved by Jaksic and Pillet for wave equation coupled to a nonlinear finite dimensional Hamiltonian system [12].

The main peculiarity of the problem is the time-reversibility of dynamical equations. For infinite particle systems this difficulty was discussed in Boltzmann-Zermelo debates (1896-1897). Many attempts were made to deduce the convergence from an ergodicity for such systems by H. Poincaré, G. Birkhoff, A. Hinchin, and many others. However, the ergodicity is not proved until now.

In 1980 R. Dobrushin and Yu. Suhov have introduced a totally new idea for obtaining the convergence to equilibrium measures imposing a mixing condition on initial distributions [4] in the context of infinite particle systems.

We develop this approach for hyperbolic PDEs. In [5]-[8], [10]-[11] the convergence to equilibrium distributions has been proved for the linear wave, Klein-Gordon and Schrödinger equations with potentials, for the harmonic crystal, and for the free Dirac equation. The initial distribution are translation invariant and satisfy the mixing condition of Rosenblatt or Ibragimov-Linnik type.

Here we consider the linear Dirac equation with the Maxwell potentials in $\mathbb{R}^3$:

$$\left\{ \begin{array}{l} i\dot{\psi}(x,t) = H\psi(x,t) := [-i\alpha\cdot\nabla + \beta m + V(x)]\psi(x,t) \\ \psi(x,0) = \psi_0(x) \end{array} \right| \quad x \in \mathbb{R}^3 \tag{1.1}$$

where $\psi(x,t) \in \mathbb{C}^4$, $m > 0$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. The hermitian matrices $\beta = \alpha_0$ and α_k satisfy the following relations:

$$\left\{ \begin{array}{l} \alpha_k^* = \alpha_k, \\ \alpha_k\alpha_l + \alpha_l\alpha_k = 2\delta_{kl}I \end{array} \right| \quad k,l = 0,1,2,3,4.$$

The standard form of the Dirac matrices α_k and β (in 2×2 blocks) is

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad (k = 1,2,3), \tag{1.2}$$

where I denotes the unit matrix, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

We assume the following conditions:

E1. The potential $V \in C^\infty(\mathbb{R}^3)$ is a hermitian 4×4 matrix function such that

$$|\partial^\alpha V(x)| \le C(\alpha)\langle x\rangle^{-\rho-|\alpha|}, \quad \langle x\rangle^\sigma = (1+|x|^2)^{\sigma/2} \tag{1.4}$$

with some $\rho > 5$.

E2. The operator H presents neither resonance nor eigenvalue at thresholds.

Under the condition **E2** the operator H has a finite set of eigenvalues $\omega_j \in (-m, m)$, $j = 1, ..., N$ with the corresponding eigenfunctions $\zeta_j^1, ..., \zeta_j^{k_j}$, where k_j is the multiplicity of ω_j. Denote by P_j the Riesz projection onto the corresponding eigenspaces and by

$$P_c := 1 - P_d, \quad P_d = \sum_j P_j \tag{1.5}$$

the projections onto the continuous and discrete spectral spaces of H.

We fix an arbitrary $\delta > 0$ such that $5 + \delta < \rho$ and consider the solutions $\psi(x,t) \in \mathbb{C}^4$ with initial data $\psi_0(x)$ which are supposed to be a random element of the weighted Sobolev space $\mathcal{H} = L^2_{-5/2-\delta}$, see Definition 2.1 below. The distribution of ψ_0 is a Borel probability measure μ_0 on $\mathcal{H}$ with zero mean satisfying some additional assumptions, see Conditions **S1-S3** below. Denote by μ_t, $t \in \mathbb{R}$, the measure on $\mathcal{H}$, giving the distribution of the random solution $\psi(t)$ to problem (1.1). We identify the complex and real spaces $\mathbb{C}^4 \equiv \mathbb{R}^8$, and $\otimes$ stands for the tensor product of real vectors. The correlation functions of the initial measure are supposed to be translation-invariant:

$$Q_0(x,y) := E\Big(\psi_0(x) \otimes \psi_0(y)\Big) = q_0(x-y), \;\; x, y \in \mathbb{R}^3. \tag{1.6}$$

We also assume that the initial mean charge density is finite:

$$e_0 := E|\psi_0(x)|^2 = \operatorname{tr} q_0(0) < \infty, \quad x \in \mathbb{R}^3. \tag{1.7}$$

Finally, we assume that the measure μ_0 satisfies a mixing condition of a Rosenblatt- or Ibragimov-Linnik type, which means that

$$\psi_0(x) \;\; \textit{and} \;\; \psi_0(y) \;\; \textit{are asymptotically independent as} \;\; |x-y| \to \infty. \tag{1.8}$$

Let $P_c^*\mu_t$ denote the projection of μ_t onto the space $\mathcal{H}_c := P_c\mathcal{H}$. Our main result is the (weak) convergence of $P_c^*\mu_t$ to a limiting measure ν_∞,

$$P_c^*\mu_t \rightharpoonup \nu_\infty, \;\; t \to \infty, \tag{1.9}$$

which is an equilibrium Gaussian measure on $\mathcal{H}_c$. A similar convergence holds for $t \to -\infty$ since our system is time-reversible.

The convergence (1.9) for the free Dirac equation with $V(x) \equiv 0$ has been proved in [8]. The case of the perturbed Dirac equation with $V \ne 0$ requires new constructions due to the absence an explicit formula for the solution. To reduce the case of perturbed equation to the case of free equation we formally need a scattering theory for the solutions of infinite global charge. We manage a dual scattering theory for finite charge solutions to avoid the infinite charge scattering theory:

$$P_c U'(t)\phi = U_0'(t)W\phi + r(t)\phi, \quad t \geq 0. \tag{1.10}$$

Here $U_0'(t)$ and $U'(t)$ are a 'formal adjoint' to the dynamical groups $U_0(t)$ and $U(t)$ of the free equation with $V \equiv 0$ and equation (1.1) with $V \neq 0$ respectively. The remainder $r(t)$ is small in the mean:

$$E|\langle \psi_0, r(t)\phi\rangle|^2 \to 0, \ \ t \to \infty. \tag{1.11}$$

where $\langle \cdot, \cdot \rangle$ is defined in (2.20). This version of scattering theory is based on the weighted energy decay established in [2].

2 Main results

2.1 Well posedness

Definition 2.1. *For $s, \sigma \in \mathbb{R}$, let us denote by $H^s_\sigma = H^s_\sigma(\mathbb{R}^3, \mathbb{C}^4)$ the weighted Sobolev spaces with the finite norms*

$$\|\psi\|_{H^s_\sigma} = \|\langle x\rangle^\sigma \langle \nabla \rangle^s \psi\|_{L^2} < \infty.$$

We set $L^2_\sigma = H^0_\sigma$. Note, that the multiplication by $V(x)$ is bounded operator $L^2_\sigma \to L^2_{\sigma+\rho}$. The finite speed of propagation for equation (1.1) implies

Proposition 2.2. *i) For any $\psi_0 \in L^2_{-\sigma}$ with $0 \leq \sigma \leq \rho$ there exists a unique solution $\psi(\cdot, t) \in C(\mathbb{R}, L^2_{-\sigma})$ to the Cauchy problem (1.1).*
ii) For any $t \in \mathbb{R}$, the operator $U(t) : \psi_0 \mapsto \psi(\cdot, t)$ is continuous in $L^2_{-\sigma}$.

Proof. Fist, consider the free Dirac equation:

$$\begin{cases} \dot\chi(x,t) = H_0\chi(x,t) = (-\alpha \cdot \nabla - i\beta m)\chi(x,t) \ \ x \in \mathbb{R}^3, \\ \chi(x,0) = \psi_0(x). \end{cases} \tag{2.12}$$

Let $s \in \mathbb{R}$ and $\psi_0 \in L^2_s$. In the Fourier space the solution to (2.12) reads

$$\hat\chi(k,t) = e^{i(\alpha\cdot k - \beta m)t}\hat\psi_0(k).$$

Since $\hat\psi_0 \in H^s$ then $\hat\chi(\cdot, t) \in H^s$ and the bounds hold

$$\|\chi(\cdot,t)\|_{L^2_s} = C\|\hat\chi(\cdot,t)\|_{H^s} \leq C_s(t)\|\hat\psi_0\|_{H^s} \leq C_s'(t)\|\psi_0\|_{L^2_s}\,. \tag{2.13}$$

Now consider perturbed equation (1.1). Let $0 \leq \sigma \leq \rho$ and $\psi_0 \in L^2_{-\sigma}$. We seek the solution to (1.1) in the form

$$\psi(x,t) = \chi(x,t) + \phi(x,t), \tag{2.14}$$

where $\chi(t) = U_0(t)\psi_0 \in L^2_{-\sigma}$ is the solution to free equation (2.12), and

$$\dot\phi(x,t) = H\phi(x,t) + V\chi(x,t), \quad \phi(x,0) = 0. \tag{2.15}$$

Since $\phi(0) = 0$ and $V\chi \in L^2$ then there exists the unique solution $\phi(t) \in L^2$ to (2.15) which is given by Duhamel representation:

$$\phi(t) = \int_0^t U(t-\tau)V\chi(\tau)d\tau.$$

Finally, by charge conservation for the Dirac equation we obtain

$$\|U(t-\tau)V\chi(\tau)\|_{L^2_{-\sigma}} \le \|U(t-\tau)V\chi(\tau)\|_{L^2} = \|V\chi(\tau)\|_{L^2} \le C\|\chi(\tau)\|_{L^2_{-\rho}} \le C\|\chi(\tau)\|_{L^2_{-\sigma}} < \infty\,.$$

□

2.2 Random solution. Convergence to equilibrium

Let (Ω, Σ, P) be a probability space with expectation E and $\mathcal{B}(\mathcal{H})$ denote the Borel σ-algebra in $\mathcal{H}$. We assume that $\psi_0 = \psi_0(\omega, \cdot)$ in (1.1) is a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. In other words, $(\omega, x) \mapsto \psi_0(\omega, x)$ is a measurable map $\Omega \times \mathbb{R}^3 \to \mathbb{C}^4$ with respect to the (completed) σ-algebras $\Sigma \times \mathcal{B}(\mathbb{R}^3)$ and $\mathcal{B}(\mathbb{C}^4)$. Then, owing to Proposition 2.2, $\psi(t) = U(t)\psi_0$ is again a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. We denote by $\mu_0(d\psi_0)$ a Borel probability measure in $\mathcal{H}$ giving the distribution of the random function ψ_0. Without loss of generality, we assume $(\Omega, \Sigma, P) = (\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu_\prime)$ and $\psi_0(\omega, x) = \omega(x)$ for $\mu_0(d\omega) \times dx$-almost all $(\omega, x) \in \mathcal{H} \times \mathbb{R}^3$.

Definition 2.3. *μ_t is a probability measure on $\mathcal{H}$ which gives the distribution of $\psi(t)$:*

$$\mu_t(B) = \mu_0(U(-t)B), \quad \forall B \in \mathcal{B}(\mathcal{H}), \;\; t \ge 0. \tag{2.16}$$

Denote by $P_c^*\mu_t$ the projection of measure μ_t onto $\mathcal{H}_c = P_c\mathcal{H}$:

$$P_c^*\mu_t(B) = \mu_t(P_c^{-1}B), \quad \forall B \in \mathcal{B}(\mathcal{H}_c), \;\; t \ge 0. \tag{2.17}$$

Our main goal is to derive the weak convergence of $P_c^*\mu_t$ in the Hilbert space $P_cH^{-\varepsilon}_{-\sigma}$ for any $\varepsilon > 0$, and $\sigma > 5/2 + \delta$:

$$P_c^*\mu_t \;\; \overset{P_cH^{-\varepsilon}_{-\sigma}}{\longrightarrow} \;\; \nu_\infty \;\text{ as }\; t \to \infty, \tag{2.18}$$

where ν_∞ is a Borel probability measure on $P_cH^{-\varepsilon}_{-\sigma}$. By definition, this means the convergence

$$\int f(\psi)P_c^*\mu_t(d\psi) \to \int f(\psi)\nu_\infty(d\psi) \;\text{ as }\; t \to \infty. \tag{2.19}$$

for any bounded and continuous functional $f(\psi)$ in $P_cH^{-\varepsilon}_{-\sigma}$.

Set $\mathcal{R}\psi \equiv (\operatorname{Re}\psi, \operatorname{Im}\psi) = \{\operatorname{Re}\psi_1, \dots, \operatorname{Re}\psi_4, \operatorname{Im}\psi_1, \dots, \operatorname{Im}\psi_4\}$ for $\psi = (\psi_1, \dots \psi_4) \in \mathbb{C}^4$ and denote by $\mathcal{R}^j\psi$ the j-th component of the vector $\mathcal{R}\psi$, $j = 1, ..., 8$. The brackets $(\cdot, \cdot)$ mean the inner product in the real Hilbert spaces $L^2 \equiv L^2(\mathbb{R}^3)$, in $L^2 \otimes \mathbb{R}^N$, or in some their different extensions. For $\psi(x), \phi(x) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$, write

$$\langle \psi, \phi \rangle := (\mathcal{R}\psi, \mathcal{R}\phi) = \sum_{j=1}^{8} (\mathcal{R}^j\psi, \mathcal{R}^j\phi). \tag{2.20}$$

Definition 2.4. *The correlation functions of the measure* μ_0 *are defined by*

$$Q_0^{ij}(x,y) \equiv E\Big(\mathcal{R}^i\psi_0(x)\mathcal{R}^j\psi_0(y)\Big) \quad \textit{for almost all} \;\; x,y\in\mathbb{R}^3, \quad i,j=1,...,8, \tag{2.21}$$

provided that the expectations in the right-hand side are finite.

Denote by D the space of complex- valued functions in $C_0^\infty(\mathbb{R}^3)$ and write $\mathcal{D} := [D]^4$. For a Borel probability measure μ denote by $\hat{\mu}$ the characteristic functional (the Fourier transform)

$$\hat{\mu}(\phi) \equiv \int \exp(i\langle\psi,\phi\rangle)\,\mu(d\psi), \;\; \phi\in\mathcal{D}.$$

A measure μ is said to be Gaussian (with zero expectation) if its characteristic functional is of the form

$$\hat{\mu}(\phi) = \exp\{-\frac{1}{2}\mathcal{Q}(\phi,\phi)\}, \;\; \phi\in\mathcal{D},$$

where $\mathcal{Q}$ is a real nonnegative quadratic form on $\mathcal{D}$. A measure μ on $\mathcal{H}$ is said to be translation-invariant if

$$\mu(T_hB) = \mu(B), \quad B\in\mathcal{B}(\mathcal{H}), \quad h\in\mathbb{R}^3,$$

where $T_h\psi(x) = \psi(x-h)$, $x\in\mathbb{R}^3$.

2.3 Mixing condition

Let $O(r)$ be the set of all pairs of open bounded subsets $\mathcal{A},\mathcal{B}\subset\mathbb{R}^3$ at the distance not less than r, $\operatorname{dist}(\mathcal{A},\,\mathcal{B})\geq r$, and let $\sigma(\mathcal{A})$ be the σ-algebra in $\mathcal{H}$ generated by the linear functionals $\psi\mapsto\langle\psi,\phi\rangle$, where $\phi\in\mathcal{D}$ with $\operatorname{supp}\phi\subset\mathcal{A}$. Define the Ibragimov-Linnik mixing coefficient of a probability measure μ_0 on $\mathcal{H}$ by the rule (cf. [9, Def. 17.2.2])

$$\varphi(r) \equiv \sup_{(\mathcal{A},\mathcal{B})\in O(r)} \; \sup_{\substack{A\in\sigma(\mathcal{A}),\,B\in\sigma(\mathcal{B})\\ \mu_0(B)>0}} \frac{|\mu_0(A\cap B)-\mu_0(A)\mu_0(B)|}{\mu_0(B)}. \tag{2.22}$$

Definition 2.5. *We say that the measure* μ_0 *satisfies the strong uniform Ibragimov-Linnik mixing condition if*

$$\varphi(r)\to 0 \quad \text{as} \quad r\to\infty. \tag{2.23}$$

We specify the rate of decay of φ below (see Condition **S3**).

2.4 Main assumptions and results

We assume that the measure μ_0 has the following properties **S0–S3**:

S0 μ_0 has zero expectation value,

$$E\psi_0(x)\equiv 0, \;\; x\in\mathbb{R}^3.$$

S1 μ_0 has translation invariant correlation functions,

$$Q_0^{ij}(x,y) \equiv E\Big(\mathcal{R}^i\psi_0(x)\mathcal{R}^j\psi_0(y)\Big) = q_0^{ij}(x-y), \quad i,j=1,...,8 \tag{2.24}$$

for almost all $x,y\in\mathbb{R}^3$.
S2 μ_0 has finite mean charge density, i.e. Eqn (1.7) holds.
S3 μ_0 satisfies the strong uniform Ibragimov-Linnik mixing condition, with

$$\int_0^\infty r^2\varphi^{1/2}(r)dr < \infty. \tag{2.25}$$

Remark 2.6. *The examples of measures on $L^2_{loc}(\mathbb{R}^3)$ satisfying properties* **S0**-**S3** *have been constructed in [5] (see §§2.6.1-2.6.2). The measures on $L^2_{-\sigma}$ with any $\sigma > 3/2$ can be construct similarly.*

Introduce the following 8×8 real valued matrices (in 4×4 blocks)

$$\Lambda_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & i\alpha_2 \\ -i\alpha_2 & 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} \alpha_3 & 0 \\ 0 & \alpha_3 \end{pmatrix}, \quad \Lambda_0 = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}. \tag{2.26}$$

Note that $\Lambda_k^T = \Lambda_k$, $k=1,2,3$, $\Lambda_0^T = -\Lambda_0$. Write

$$\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3), \quad P = \Lambda\cdot\nabla + m\Lambda_0. \tag{2.27}$$

For almost all $x,y\in\mathbb{R}^3$, introduce the matrix-valued function

$$Q_\infty(x,y) \equiv \Big(Q_\infty^{ij}(x,y)\Big)_{i,j=1,...,8} = \Big(q_\infty^{ij}(x-y)\Big)_{i,j=1,...,8}. \tag{2.28}$$

Here

$$\hat q_\infty(k) = \frac{1}{2}\hat q_0(k) - \frac{1}{2}\hat{\mathcal{P}}(k)\hat P(k)\hat q_0(k)\hat P(k), \tag{2.29}$$

$\hat P(k) = -i\Lambda\cdot k + m\Lambda_0$, $\hat{\mathcal{P}}(k) = 1/(k^2+m^2)$, and $\hat q_0(k)$ is the Fourier transform of the correlation matrix of the measure μ_0 (see 2.24). We formally have

$$q_\infty(z) = \frac{1}{2}q_0(z) + \frac{1}{2}\mathcal{P} * Pq_0(z)P \tag{2.30}$$

where $\mathcal{P}(z) = e^{-m|z|}/(4\pi|z|)$ is the fundamental solution for the operator $-\Delta + m^2$, and $*$ stands for the convolution of distributions.

Lemma 2.7. *Let conditions* **S0**, **S2** *and* **S3** *hold. Then*

$$q_0 \in L^p(\mathbb{R}^3), \quad p \ge 1. \tag{2.31}$$

Proof. Conditions **S0**, **S2** and **S3** imply (cf. [9, Lemma 17.2.3]) that

$$|q_0^{ij}(z)| \le Ce_0\varphi^{1/2}(|z|), \quad z\in\mathbb{R}^3, \quad i,j=1,...,8.$$

The mixing coefficient φ is bounded, hence

$$\int |q_0^{ij}(z)|^p dz \le C\int \varphi^{p/2}(|z|)dz \le C_1\int_0^\infty r^2\varphi^{1/2}(r)dr < \infty$$

by (2.25). □

Lemma 2.7 with $p = 2$ imply that $\hat{q}_0 \in L^2$. Hence, $\hat{q}_\infty \in L^2$ by (2.29), and q_∞ also belongs to L^2 by (2.30).

Denote by $\mathcal{Q}_\infty$ a real quadratic form on L^2 defined by

$$\mathcal{Q}_\infty(\phi,\phi) \equiv (Q_\infty(x,y), \mathcal{R}\phi(x) \otimes \mathcal{R}\phi(y)) = \sum_{i,j=1}^{8} \int_{\mathbb{R}^3\times\mathbb{R}^3} Q_\infty^{ij}(x,y)\mathcal{R}^i\phi(x)\mathcal{R}^j\phi(y)dxdy$$

Corollary 2.8. *The form $\mathcal{Q}_\infty$ is continuous on L^2 because $\hat{q}_0(k)$ and then $\hat{q}_\infty(k)$ are bounded by Lemma 2.7 and formula (2.29).*

Our main result is the following:

Theorem 2.9. *Let $m > 0$, and let conditions* **E1–E2**, **S0–S3** *hold. Then*
i) the convergence in (2.18) holds for any $\varepsilon > 0$ and $\sigma > 5/2 + \delta$.
ii) the limiting measure μ_∞ is a Gaussian equilibrium measure on $\mathcal{H}_c$.
iii) the characteristic functional of ν_∞ is of the form

$$\hat{\nu}_\infty(\phi) = \exp\{-\frac{1}{2}\mathcal{Q}_\infty(W\phi, W\phi)\}, \ \ \phi \in \mathcal{D},$$

where $W : \mathcal{D} \to L^2$ is a linear continuous operator.

2.5 Remark on various mixing conditions for initial measure

We use the *strong uniform* Ibragimov-Linnik mixing condition for the simplicity of our presentation. The *uniform* Rosenblatt mixing condition [13] with a higher degree > 2 in the bound (1.7) is also sufficient. In this case we assume that there exists an ϵ, $\epsilon > 0$, such that

$$\sup_{x\in\mathbb{R}^3} E|\psi_0(x)|^{2+\epsilon} < \infty.$$

Then condition (2.25) requires the following modification:

$$\int_0^\infty r\alpha^p(r)dr < \infty, \quad p = \min(\epsilon/(2+\epsilon), 1/2),$$

where $\alpha(r)$ is the Rosenblatt mixing coefficient defined as in (2.22), but without the denominator $\mu_0(B)$. The statements of Theorem 2.9 and their proofs remain essentially unchanged.

3 Free Dirac equation

Here we consider the free Dirac equation (2.12) We have

$$(\partial_t - \alpha\cdot\nabla - i\beta m)(\partial_t + \alpha\cdot\nabla + i\beta m) = \partial_t^2 - \Delta + m^2$$

Then the fundamental solution $G(x,t)$ of the free Dirac operator reads

$$G_t(x) = (\partial_t - \alpha\cdot\nabla - i\beta m)\mathcal{E}_t(x) \tag{3.1}$$

where $\mathcal{E}_t(x)$ is the fundamental solution of the Klein-Gordon operator $\partial_t^2 - \Delta + m^2$:

$$\mathcal{E}_t(x) = F^{-1}_{k\to x}\frac{\sin\omega t}{\omega}, \quad \omega = \omega(k) = \sqrt{|k|^2 + m^2}. \tag{3.2}$$

Using the notations (2.26) and (2.27), we obtain in real form

$$\mathcal{R}\chi(t) = \mathcal{G}_t * \mathcal{R}\psi_0, \quad \mathcal{G}_t = (\partial_t - P)\mathcal{E}_t. \tag{3.3}$$

The convolution exists since the distribution $\mathcal{E}_t(x)$ is supported by the ball $|x| \le t$. Now we derive an explicit formula for the correlation function

$$Q_t(x,y) = q_t(x-y) = E\Big(\mathcal{R}\chi(x,t)\otimes\mathcal{R}\chi(y,t)\Big) \tag{3.4}$$

Lemma 3.1. *(cf. [8, Formula (4.6)]) The correlation function $Q_t(x,y)$ reads*

$$\begin{aligned} Q_t(x,y) &= q_t(x-y) = F^{-1}_{k\to x-y}\Big[\frac{1+\cos 2\omega t}{2}\hat q_0(k) - \frac{\sin 2\omega t}{2\omega}(\hat q_0(k)P(k) - P(k)\hat q_0(k)) \\ &- \frac{1-\cos 2\omega t}{2\omega^2}P(k)\hat q_0(k)P(k)\Big] \end{aligned} \tag{3.5}$$

Proof. Applying the Fourier transform to (3.3) we obtain

$$\widehat{\mathcal{R}\chi}(k,t) = \hat{\mathcal{G}}_t(k)\widehat{\mathcal{R}\psi}_0(k) = \Big(\cos\omega t - \hat P(k)\frac{\sin\omega t}{\omega}\Big)\hat\psi_0(k) \tag{3.6}$$

By translation invariance condition (2.24) we have

$$E(\widehat{\mathcal{R}\psi}_0(k)\otimes\widehat{\mathcal{R}\psi}_0(k')) = F_{x\to k, y\to k'}q_0(x-y) = (2\pi)^3\delta(k-k')\hat q_0(k)$$

Then (3.6) implies that

$$E(\widehat{\mathcal{R}\chi}(k,t)\otimes\widehat{\mathcal{R}\chi}(k',t)) = (2\pi)^3\delta(k-k')\hat{\mathcal{G}}_t(k)\hat q_0(k)\hat{\mathcal{G}}_t^*(k)$$

Therefore,

$$\hat q_t(k) = \hat{\mathcal{G}}_t(k)\hat q_0(k)\hat{\mathcal{G}}_t^*(k) = \Big(\cos\omega t - \hat P(k)\frac{\sin\omega t}{\omega}\Big)\hat q_0(k)\Big(\cos\omega t + \hat P(k)\frac{\sin\omega t}{\omega}\Big)$$

since $\hat P^*(k) = -\hat P(k)$. Hence (3.5) follows. □

Corollary 3.2. *For any $z\in\mathbb{R}^3$ the convergence holds*

$$q_t(z) \to q_\infty(z), \quad t\to\infty$$

where $q_\infty(z)$ is defined in (2.30).

Proof. The convergence follows from (3.5) since the integrals with the oscillatory functions converge to zero. □

Below we will need the following lemma:

Lemma 3.3. *Let Conditions* **S0–S3** *hold. Then for any $\sigma > 3/2$ the bound holds*

$$\sup_{t\geq 0} E\|\chi(\cdot,t)\|^2_{L^2_{-\sigma}} < \infty \tag{3.7}$$

Proof. Denote

$$e_t(x) := E|\chi(x,t)|^2, \quad x\in\mathbb{R}^3.$$

The mathematical expectation is finite for almost all $x\in\mathbb{R}^3$ by (2.13) with $s=-\sigma$ and the Fubini theorem. Moreover, $e_t(x)=e_t$ for almost all $x\in\mathbb{R}^3$ by **S1**. Formula (3.5) implies

$$\begin{aligned} q_t(0) &= \frac{1}{(2\pi)^3}\int\Big[\cos^2(\omega t)\hat q_0(k) - \frac{\sin 2\omega t}{2\omega}(\hat q_0(k)P(k) - P(k)\hat q_0(k)) \\ &- \frac{\sin^2\omega t}{\omega^2}P(k)\hat q_0(k)P(k)\Big]dk, \end{aligned} \tag{3.8}$$

Then $e_t = \operatorname{tr} q_t(0) \leq Ce_0$. Hence for $\sigma > 3/2$ we obtain

$$E\|\chi(\cdot,t)\|^2_{L^2_{-\sigma}} = e_t\int(1+|x|^2)^{-\sigma}dx \leq C(\nu)e_0$$

and then (3.7) follows. □

We will use also the following result:

Proposition 3.4. *(see [8, Proposition 2.8], [5, Proposition 3.3]). Let Conditions* **S0–S3** *hold. Then for any $\phi\in\mathcal{D}$,*

$$E\exp\{i\langle U_0(t)\psi_0,\phi\rangle\} \to \exp\{-\frac{1}{2}\mathcal{Q}_\infty(\phi,\phi)\}, \ \ t\to\infty. \tag{3.9}$$

Remark 3.5. *In [8] the phase space $L^2_{loc}(\mathbb{R}^3)\otimes\mathbb{C}^4$ has been considered. Nevertheless, all the steps of proving the convergence (3.9) in [8] remain true if we change $L^2_{loc}(\mathbb{R}^3)\otimes\mathbb{C}^4$ by $L^2_{-\sigma}$ with $\sigma > 3/2$.*

4 Perturbed Dirac equation.

4.1 Scattering Theory

To deduce Theorem 2.9 we construct the dual scattering theory (1.10) for finite energy solutions using the Boussaid results, [2].

Lemma 4.1. *(see [2, Theorem 1.1]) Let conditions* **E1-E2** *hold and $\sigma > 5/2$. Then the bound holds*

$$\|P_cU(t)\psi\|_{L^2_{-\sigma}} \leq C(1+|t|)^{-3/2}\|\psi\|_{L^2_\sigma}, \quad t\in\mathbb{R}. \tag{4.1}$$

Note that for $\psi_0\in L^2$ the solutions $U_0(t)\psi_0$ and $U(t)\psi_0$ to problems (2.12) and (1.1), respectively, also belong to L^2 and the charge conservation holds:

$$\|U(t)\psi_0\| = \|\psi_0\|, \quad \|U_0(t)\psi_0\| = \|\psi_0\|. \tag{4.2}$$

Here and below $\|\cdot\|$ is the norm in L^2.

For $t \in \mathbb{R}$, introduce the operators $U_0'(t)$ and $U'(t)$ which are conjugate to the operators $U_0(t)$ and $U(t)$ on L^2:

$$(\psi, U_0'(t)\phi) = (U_0(t)\psi, \phi), \quad (\psi, U'(t)\phi) = (U(t)\psi, \phi), \quad \psi, \phi \in L^2. \tag{4.3}$$

Here $(\cdot, \cdot)$ stands for the hermitian scalar product in $L^2(\mathbb{R}^3, \mathbb{C}^4)$. The adjoint groups admit a convenient description:

Lemma 4.2. *For $\phi \in L^2$ the function $U_0'(t)\phi_0 = \phi(\cdot, t)$ is the solution to*

$$\dot{\phi}(x,t) = [\alpha \cdot \nabla + i\beta m]\phi(x,t), \quad \phi(x,0) = \phi_0(x). \tag{4.4}$$

Proof. Differentiating the first equation of (4.3) with $\psi, \phi \in \mathcal{D}$, we obtain

$$(\psi, \dot{U}_0'(t)\phi) = (\dot{U}_0(t)\psi, \phi). \tag{4.5}$$

The group $U_0(t)$ has the generator

$$\mathcal{A}_0 = -\alpha \cdot \nabla - i\beta m. \tag{4.6}$$

Therefore, the generator of $U_0'(t)$ is the conjugate operator

$$\mathcal{A}_0' = \alpha \cdot \nabla + i\beta m. \tag{4.7}$$

Hence, (4.4) holds, where $\dot{\phi}(t) = \mathcal{A}_0'\phi(t)$. □

Similarly, we obtain

Lemma 4.3. *For $\phi \in L^2$ the function $U'(t)\phi = \phi(x,t)$ is the solution to*

$$\dot{\phi}(x,t) = [\alpha \cdot \nabla + i\beta m + iV]\phi(x,t), \quad \phi(x,0) = \phi(x). \tag{4.8}$$

Corollary 4.4. *i) $U_0'(t) = U_0(-t)$, $U'(t) = U(-t)$.*
ii) For any $\phi \in L^2$ the uniform bounds hold:

$$\|U_0'(t)\phi\| = \|\phi\|, \quad \|U'(t)\phi\| = \|\phi\|, \quad t \geq 0. \tag{4.9}$$

iii) Under assumptions **E1**-**E2** *for $U'(t)$ a bound of type (4.1) also holds:*

$$\|P_c U'(t)\psi\|_{L^2_{-\sigma}} \leq C(1+|t|)^{-3/2}\|\psi\|_{L^2_\sigma}, \quad t \in \mathbb{R} \tag{4.10}$$

with $\sigma > 5/2$.

Now we formulate the scattering theory in the dual representation.

Theorem 4.5. *Let conditions* **E1–E2** *and* **S0–S3** *hold and $\sigma > 5/2$. Then there exist linear operators $W, r(t) : L^2_\sigma \to L^2$ such that for $\phi \in L^2_\sigma$*

$$P_c U'(t)\phi = U_0'(t)W\phi + r(t)\phi, \;\; t \geq 0. \tag{4.11}$$

and the bounds hold

$$\begin{aligned} \|r(t)\phi\| &\leq C(1+t)^{-1/2}\|\phi\|_{L^2_\sigma}, && (4.12)\\ E|\langle\psi_0, r(t)\phi\rangle|^2 &\leq C(1+t)^{-1}\|\phi\|^2_{L^2_\sigma}, \quad t > 0. && (4.13) \end{aligned}$$

Proof. We apply the Cook method, [14, Theorem XI.4]. Fix $\phi \in L^2_\sigma$ and define $W\phi$, formally, as

$$W\phi = \lim_{t\to+\infty} U_0'(-t)P_cU'(t)\phi = \phi + \int_0^{+\infty} \frac{d}{d\tau}U_0'(-\tau)P_cU'(\tau)\phi\, d\tau. \tag{4.14}$$

We have to prove the convergence of the last integral in the norm of L^2. First, observe that

$$\frac{d}{d\tau}U_0'(\tau)\phi = \mathcal{A}_0'U_0'(\tau)\phi, \quad \frac{d}{d\tau}U'(\tau)\phi = \mathcal{A}'U'(\tau)\phi,\ \tau \geq 0$$

where $\mathcal{A}_0'$ and $\mathcal{A}'$ are the generators to the groups $U_0'(\tau)$, $U'(\tau)$, respectively. Therefore,

$$\frac{d}{d\tau}U_0'(-\tau)P_cU'(\tau)\phi = U_0'(-\tau)(\mathcal{A}' - \mathcal{A}_0')P_cU'(\tau)\phi. \tag{4.15}$$

We have $\mathcal{A}' - \mathcal{A}_0' = iV$. Furthermore, **E2**, (4.9), (4.10) imply that

$$\begin{aligned} \|U_0'(-\tau)(\mathcal{A} - \mathcal{A}_0)P_cU'(\tau)\phi\| &\leq C\ \|(\mathcal{A} - \mathcal{A}_0)P_cU'(\tau)\phi\| = C\ \|VU'(\tau)\phi\| \\ &\leq C_1\ \|U'(\tau)\phi\|_{L^2_{-\rho}} \leq C_2(1+\tau)^{-3/2}\|\phi\|_{L^2_\sigma}, \quad \tau \geq 0. \end{aligned} \tag{4.16}$$

Hence, the convergence of the integral in the right hand side of (4.14) follows.
Further, (4.11) and (4.14) imply

$$r(t)\phi = P_cU'(t)\phi - U_0'(t)W\phi = -U_0'(t)\int_t^\infty \frac{d}{d\tau}U_0'(-\tau)P_cU'(\tau)\phi\, d\tau.$$

Hence (4.12) follows by (4.9), (4.15) and (4.16).

It remains to prove (4.13). Applying the Shur lemma we obtain

$$\begin{aligned} E|\langle\psi_0, r(t)\phi\rangle|^2 &= \langle q_0(x-y), r(t)\phi(x)\otimes r(t)\phi(y)\rangle \\ &\leq \|q_0\|_{L^1}\ \|r(t)\phi\|^2. \end{aligned} \tag{4.17}$$

Hence, (4.13) follows by (2.31) with $p = 1$ and (4.12). □

4.2 Convergence to equilibrium distribution

Theorem 2.9 can be derived from Propositions 4.6-4.7 below by using the same arguments as in [15, Theorem XII.5.2].

Proposition 4.6. *The family of the measures $\{P_c^*\mu_t, t \in \mathbb{R}\}$ is weakly compact in $P_cH^{-\varepsilon}_{-\sigma}$ for any $\varepsilon > 0$ and $\sigma > 5/2 + \delta$.*

Proposition 4.7. *For any $\phi \in \mathcal{D}$*

$$\widehat{P_c^*\mu_t}(\phi) \equiv \int \exp(i\langle\psi,\phi\rangle)P_c^*\mu_t(d\psi) \to \exp\{-\frac{1}{2}\mathcal{Q}_\infty(W\phi, W\phi)\},\ \ t\to\infty. \tag{4.18}$$

Proposition 4.6 provides the existence of the limiting measures of the family $P_c^*\mu_t$, and Proposition 4.7 provides the uniqueness of the limiting measure, and hence the convergence (2.19). We deduce these propositions with the help of Theorem 4.5.

Proof of Proposition 4.6. First, we prove the bound

$$\sup_{t\geq 0} E\|P_c U(t)\psi_0\|_{\mathcal{H}} < \infty, \tag{4.19}$$

Representation (2.14) implies

$$P_c U(t)\psi_0 = P_c\chi(x,t) + P_c\phi(x,t), \tag{4.20}$$

where $\chi(x,t) = U_0(t)\psi_0$, and $\phi(x,t)$ is the solution to (2.15). Therefore,

$$E\|P_c U(t)\psi_0\|_{\mathcal{H}} \leq E\|P_c\chi(t)\|_{\mathcal{H}} + E\|P_c\phi(t)\|_{\mathcal{H}}. \tag{4.21}$$

Bound (3.7) implies

$$\sup_{t\geq 0} E\|\chi(t)\|_{\mathcal{H}} < \infty. \tag{4.22}$$

Further, we have by the Cauchy-Schwartz inequality

$$E\|(\chi(t),\zeta_j)\zeta_j\|_{L^2_{-\sigma}} \leq C\|\zeta_j\|_{L^2_{-\sigma}}\|\zeta_j\|_{L^2_\sigma} E\|\chi(t)\|_{L^2_{-\sigma}} \leq C_j E\|\chi(t)\|_{L^2_{-\sigma}}, \quad \sigma = 5/2+\delta$$

since the eigenfunctions $\zeta_j \in L^2_s$ with any s, see Appendix. Therefore

$$\sup_{t\geq 0} E\|P_c\chi(t)\|_{\mathcal{H}} < \infty$$

since $P_c\chi(x,t) = \chi(x,t) - P_d\chi(x,t)$ by (1.5).

It remains to estimate the second term in the RHS of (4.21). Choose a $\delta_1 > 0$ such that $\delta_1 < \rho - 5 - \delta$. It is possible due to **E1**. Then the Duhamel representation (2.15) and bounds (4.1) and (4.22) imply

$$\begin{aligned} E\|P_c\phi(t)\|_{\mathcal{H}} &\leq \int_0^t E\|P_c U(t-s)V\chi(s)\|_{L^2_{-5/2-\delta}}\,ds \leq C\int_0^t (1+t-s)^{-3/2} E\|V\chi(t)\|_{L^2_{5/2+\delta_1}}\,ds \\ &\leq C_1\int_0^t (1+t-s)^{-3/2} E\|\chi(t)\|_{L^2_{5/2+\delta_1-\rho}}\,ds \leq C_2, \quad t>0 \end{aligned} \tag{4.23}$$

since $5/2+\delta_1-\rho < -5/2-\delta$. Now (4.21)– (4.23) imply (4.19).

Now Proposition 4.6 follows from (4.19) by Prokhorov theorem [15, Lemma II.3.1] as in the proof of [15, Theorem XII.5.2]. □

Proof of Proposition 4.7 We have

$$\int \exp(i\langle\psi,\phi\rangle)P_c^*\mu_t(d\psi) = \int \exp(i\langle P_c\psi,\phi\rangle)\mu_t(d\psi) = E\exp i\langle P_c U(t)\psi_0,\phi\rangle$$

Bound (4.13) and Cauchy-Schwartz inequality imply

$$\begin{aligned} |E\exp i\langle P_c U(t)\psi_0,\phi\rangle - E\exp i\langle U_0(t)\psi_0, W\phi\rangle| &= |E\exp i\langle\psi_0, P_c U'(t)\phi\rangle - E\exp i\langle\psi_0, U_0'(t)W\phi\rangle| \\ &\leq E|\langle\psi_0, r(t)\phi\rangle| \leq (E\langle\psi_0, r(t)\phi\rangle^2)^{1/2} \to 0 \end{aligned}$$

as $t \to \infty$. It remains to prove that

$$E \exp i\langle \psi_0, U_0'(t) W\phi\rangle \to \exp\{-\frac{1}{2}\mathcal{Q}_\infty(W\phi, W\phi)\}, \quad t \to \infty. \tag{4.24}$$

The convergence does not follow directly from Proposition 3.4 since $W\phi \notin \mathcal{D}$. We can approximate $W\phi \in L^2$ by functions from $\mathcal{D}$ since $\mathcal{D}$ is dense in L^2. Hence, for any $\varepsilon > 0$ there exists $\phi_\varepsilon \in \mathcal{D}$ such that

$$\|W\phi - \phi_\varepsilon\| \le \varepsilon. \tag{4.25}$$

By the triangle inequality

$$\begin{aligned}
& |E \exp i\langle \psi_0, U_0'(t) W\phi\rangle - \exp\{-\frac{1}{2}\mathcal{Q}_\infty(W\phi, W\phi)\}| \\
\le \; & |E \exp i\langle \psi_0, U_0'(t) W\phi\rangle - E \exp i\langle \psi_0, U_0'(t)\phi_\varepsilon\rangle| \\
& + E|\exp i\langle U_0(t)\psi_0, \phi_\varepsilon\rangle - \exp\{-\frac{1}{2}\mathcal{Q}_\infty(\phi_\varepsilon, \phi_\varepsilon)\}| \\
& + |\exp\{-\frac{1}{2}\mathcal{Q}_\infty(\phi_\varepsilon, \phi_\varepsilon)\} - \exp\{-\frac{1}{2}\mathcal{Q}_\infty(W\phi, W\phi)\}|.
\end{aligned} \tag{4.26}$$

Let us estimate each term in the RHS of (4.26). Theorem 4.5 implies that uniformly in $t > 0$

$$\begin{aligned}
E|\langle \psi_0, U_0'(t)(W\phi - \phi_\varepsilon)\rangle| & \le (E|\langle \psi_0, U_0'(t)(W\phi - \phi_\varepsilon)\rangle|^2)^{1/2} \le \|q_0\|_{L^1}^{1/2} \|U_0'(t)(W\phi - \phi_\varepsilon)\| \\
& \le C\|W\phi - \phi_\varepsilon\| \le C\varepsilon.
\end{aligned}$$

Then the first term is $\mathcal{O}(\varepsilon)$ uniformly in $t > 0$. The second term converges to zero as $t \to \infty$ by Proposition 3.4 since $\phi_\varepsilon \in \mathcal{D}$. At last, the third term is $\mathcal{O}(\varepsilon)$ by (4.25) and the continuity of the quadratic form $\mathcal{Q}_\infty(\phi, \phi)$ in $L^2 \otimes \mathbb{C}^4$. The continuity follows from Corollary 2.8. Now convergence (4.24) follows since $\varepsilon > 0$ is arbitrary. □

5 Appendix: Decay of eigenfunctions

Here we prove the spatial decay of eigenfunctions.

Lemma 5.1. *Let V satisfy* **E1**, *and $\psi(x) \in L^2(\mathbb{R}^3)$ be an eigenfunction of the Dirac operator corresponding to a eigenvalue $\lambda \in (-m, m)$, i.e.*

$$H\psi(x) = \lambda\psi(x), \quad x \in \mathbb{R}^3.$$

Then $\psi \in L^2_s$ for all $s \in \mathbb{R}$.

Proof. Denote by $R_0(\lambda) = (H_0 - \lambda)^{-1}$ the resolvent of the free Dirac equation. The equation $(H_0 + V - \lambda)\psi = 0$ implies

$$\psi = R_0(\lambda) f, \quad \text{where} \quad f = -V\psi \in L^2_{2+\rho} \tag{5.1}$$

From the identity

$$(-i\alpha \cdot \nabla + \beta m - \lambda)(i\alpha \cdot \nabla - \beta m - \lambda) = \Delta - m^2 + \lambda^2$$

it follows that

$$R_0(\lambda) = \frac{i\alpha \cdot \nabla - \beta m - \lambda}{\Delta - m^2 + \lambda^2} \tag{5.2}$$

Hence, in the Fourier transform, the first equation of (5.1) reads

$$\hat{\psi}(k) = \frac{(-\alpha \cdot k + \beta m + \lambda)\hat{f}(k)}{k^2 + m^2 - \lambda^2}$$

Since $|\lambda| < m$, we have

$$\|\psi\|_{L^2_{2+\rho}} = C\|\hat{\psi}\|_{H^{2+\rho}} \leq C_1\|\hat{f}\|_{H^{2+\rho}} = C_2\|f\|_{L^2_{2+\rho}} \leq C_3\|\psi\|_{L^2_2}$$

Hence, $\psi \in L^2_s$ with any $s \in \mathbb{R}$ by induction. □

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