On Global attraction to solitary waves for Klein-Gordon equation with concentrated nonlinearity

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Abstract

The global attraction is proved for the nonlinear 3D Klein-Gordon equation with a nonlinearity concentrated at one point. Our main result is the convergence of each "finite energy solution" to the manifold of all solitary waves as $t \to \pm\infty$. This global attraction is caused by the nonlinear energy transfer from lower harmonics to the continuous spectrum and subsequent dispersion radiation.

We justify this mechanism by the following strategy based on *inflation of spectrum by the nonlinearity*. We show that any *omega-limit trajectory* has the time-spectrum in the spectral gap $[-m,m]$ and satisfies the original equation. Then the application of the Titchmarsh Convolution Theorem reduces the spectrum of each omega-limit trajectory to a single frequency $\omega \in [-m,m]$.

1 Introduction

The paper concerns a nonlinear interaction of the Klein-Gordon field with a point oscillator. The point interaction are widely used in physical works. One of the well-known application in dimension one is the Kronig-Penney model [25]. I n 3D case a rigorous mathematical definition of point interactions was given by Berezin and Faddeev [6]. For the numerous literature concerning the models with a point interactions we refer to [5].

In the case of the Schrödinger equations the nonlinear point interaction was justified in [9, 10] as a scaling limit of a regularized nonlinear Schrödinger dynamics. We suppose that for the Klein-Gordon equations a justification can be done by suitable modification of methods [9, 10], but it still remains an open question.

We consider the system governed by the following equations

$$\left\{ \begin{array}{l} \ddot\psi(x,t) = (\Delta - m^2)\psi(x,t) + \zeta(t)\delta(x) \\ \\ \lim\limits_{x\to 0}(\psi(x,t) - \zeta(t)G(x)) = F(\zeta(t)) \end{array} \right| \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \quad m > 0, \tag{1.1}$$

where $G(x)$ is the Green's function of operator $-\Delta + m^2$ in $\mathbb{R}^3$, i.e.

$$G(x) = \frac{e^{-m|x|}}{4\pi|x|}. \tag{1.2}$$

The nonlinearity $F(\zeta)$ admits a real-valued potential

$$F(\zeta) = \partial_{\overline{\zeta}} U(\zeta), \quad \zeta \in \mathbb{C}, \quad U \in C^2(\mathbb{C}), \tag{1.3}$$

where $\partial_{\overline{\zeta}} := \frac{1}{2}(\partial_1 + i\partial_2)$ with $\zeta_1 := \operatorname{Re}\zeta$ and $\zeta_2 := \operatorname{Im}\zeta$. We assume that the potential $U(\zeta)$ is U(1)-invariant, where U(1) stands for the unitary group $e^{i\theta}$, $\theta \in \mathbb{R} \bmod 2\pi$. Namely, we assume that there exists $u \in C^2(\mathbb{R})$ such that

$$U(\zeta) = u(|\zeta|^2), \quad \zeta \in \mathbb{C}. \tag{1.4}$$

Conditions (1.3) and (1.4) imply that

$$F(\zeta) = b(|\zeta|^2)\zeta, \quad \zeta \in \mathbb{C}. \tag{1.5}$$

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where $b(\cdot) = u'(\cdot) \in C^1(\mathbb{R})$ is real valued. Therefore

$$F(e^{i\theta}\zeta) = e^{i\theta}F(\zeta), \quad \theta \in \mathbb{R}, \quad \zeta \in \mathbb{C}. \tag{1.6}$$

This symmetry implies that $e^{i\theta}\psi(x,t)$ is a solution to the system (1.1) if $\psi(x,t)$ is. The system (1.1) admits soliton solutions $\psi_\omega(x)e^{-i\omega t}$ with some $\omega \in (-m,m)$ and $\psi_\omega \in L^2(\mathbb{R}^3)$. Our main goal is the global attraction

$$\psi(x,t) \sim \psi_{\omega_\pm}(x)e^{-i\omega_\pm t}, \quad t \to \pm\infty,$$

for all solutions from the Hilbert space $\mathscr{D}_F$ (see Definition 2.1), where the asymptotics hold in local L^2-seminorms.

Similar global attraction was established for the first time i) in [13]–[17] for 1D wave and 1D Klein-Gordon equations coupled to a nonlinear oscillator, ii) in [18, 19] for t nD Klein-Gordon and Dirac equations with mean field interaction, and iii) in [8] for discrete in space and in time nD Klein-Gordon equation equations interacting with a nonlinear oscillator.

In the context of the Schrödinger and wave equations the point interaction of type (1.1) was introduced in [1, 2, 5, 26, 27, 28], where the well-posedness of the Cauchy problem and the blow up solutions were studied. In our recent paper [23] we proved the well-posedness for system (1.1).

The asymptotic stability of solitary waves have been obtained in [7, 21, 20] for 1D Schrödinger equation coupled to nonlinear oscillator, in [22] for 1D discrete Klein-Gordon equation coupled to nonlinear oscillator, and in [3, 4] for 3D Schrödinger equation with concentrated nonlinearity.

Global attraction to stationary state for 3D wave equation with the point interaction has been proved for the first time in our recent paper [24]. However, the global attraction for 3D Klein-Gordon equation equations with the point interaction has not been studied up to now.

Let us comment on our methods. First, we represent the solution as a sum of dispersive and singular components. The dispersive component is a solution to the free Klein-Gordon equation, and the singular component is a solution to the coupled system of the Klein-Gordon equation with delta-like sources and of the first-order nonlinear integro-differential equation which control the dynamics of the coefficients $\zeta(t)$ (see equation (3.3)). The right hand side of this equation is the value of the dispersive component at the singular point $x = 0$.

The dispersive component vanishes asymptotically in the local seminorms and one remains with the contribution of the singular part only. We show that the singular component converges in the chosen topology to a solitary wave which is a standing wave with a single frequency.

Further, we extract the omega-limit trajectories of the singular component via the compactness argument. Here the key role is played by the absolute continuity of the spectral density $\tilde{\zeta}(\omega)$ outside the spectral gap. The absolute continuity is a nonlinear version of Kato's theorem on the absence of the embedded eigenvalues and provides the dispersion decay for the high energy component. Any omega-limit trajectory is the solution to (1.1) with a function $\eta(t)$ instead of $\zeta(t)$, which is a solution to a homogeneous nonlinear integro-differential equation (5.2). The Fourier transform of $\eta(t)$ is a quasimeasure. The theory of quasimeasures helps to prove the spectral inclusion (6.5).

Finally, we apply the Titchmarsh convolution theorem (see [11, Theorem 4.3.3]) to conclude that the support of the distributional Fourier transform of each omega-limit trajectory is a singleton, i.e. the spectrum of each omega-limit trajectory has a single frequency. The Titchmarsh theorem controls the inflation of spectrum by the nonlinearity. Physically, these arguments justify the following binary mechanism of the energy radiation, which is responsible for the attraction to the solitary waves: (i) the nonlinear energy transfer from the lower to higher harmonics, and (ii) the subsequent dispersion decay caused by the energy radiation to infinity.

The general scheme of the proof bring to mind the approach of [16, 17]. Nevertheless the Klein-Gordon equation with the point interaction requires new ideas due to a more singular character. As a consequence, the formulation of the problem and the techniques used are not a straightforward generalization of the one-dimensional result [16] and the result [17] for 3D equation with mean field interaction.

Our paper is organized as follows. In Section 2 we formulate the main theorem. In Section 3 we separate the first dispersive component and study its decay properties. In Section 4 we construct spectral representation for the remaining singular component, and prove absolute continuity of its spectrum outside the spectral gap. In section 5 we establish compactness for the singular component. In Section 6 we study omega-limit trajectories of the solution. In Section 7 we prove the main theorem and in Appendix we calculate some Fourier transforms.

2 Main results

Model

We fix a nonlinear function $F:\mathbb{C}\to\mathbb{C}$ and define the domain

$$D_F=\{\psi\in L^2(\mathbb{R}^3):\psi(x)=\psi_{reg}(x)+\zeta G(x),\ \ \psi_{reg}\in H^2(\mathbb{R}^3),\ \ \zeta\in\mathbb{C},\ \ \psi_{reg}(0)=F(\zeta)\}\tag{2.1}$$

which generally is not a linear space. Let H_F be a nonlinear operator on the domain D_F defined by

$$H_F\psi=(\Delta-m^2)\psi_{reg},\quad \psi\in D_F.\tag{2.2}$$

The system (1.1) for $\psi(t)\in D_F$ reads

$$\ddot\psi(x,t)=H_F\psi(x,t),\quad x\in\mathbb{R}^3,\quad t\in\mathbb{R}.\tag{2.3}$$

Let us introduce the phase space for equation (2.3). Denote the space

$$\dot D=\{\pi\in L^2(\mathbb{R}^3):\pi(x)=\pi_{reg}(x)+\eta G(x),\ \ \pi_{reg}\in H^1(\mathbb{R}^3),\ \ \eta\in\mathbb{C}\}\tag{2.4}$$

Obviously, $D_F\subset\dot D$.

Definition 2.1. (i) $\mathscr{D}_F$ is the space of the states $\Psi=(\psi(x),\pi(x))\in D_F\oplus\dot D$ equipped with the finite norm

$$\|\Psi\|^2_{\mathscr{D}_F}:=\|\psi_{reg}\|^2_{H^2(\mathbb{R}^3)}+\|\pi_{reg}\|^2_{H^1(\mathbb{R}^3)}+|\zeta|^2+|\eta|^2.\tag{2.5}$$

(ii) $\mathscr{X}$ is the Hilbert space of the states $\Psi=(\psi(x),\pi(x))\in H^2(\mathbb{R}^3)\oplus H^1(\mathbb{R}^3)$ equipped with the finite norm

$$\|\Psi\|^2_{\mathscr{X}}:=\|\psi\|^2_{H^2(\mathbb{R}^3)}+\|\pi\|^2_{H^1(\mathbb{R}^3)}.\tag{2.6}$$

Definition 2.2. $H^s_{loc}=H^s_{loc}(\mathbb{R}^3)$, $s=0,1,2,...$, denotes the Fréchet space with finite seminorms

$$\|\psi\|_{H^s_R}:=\|\psi\|_{H^s(B_R)},\quad R>0,\tag{2.7}$$

where B_R is the ball of radius R.

Denote $L^2_{loc}=H^0_{loc}$, $\mathscr{L}^2_{loc}=L^2_{loc}\oplus L^2_{loc}$ and $\mathscr{X}_{loc}=H^2_{loc}\oplus H^1_{loc}$. We set for $\Psi=(\psi,\pi)$

$$\|\Psi\|^2_{\mathscr{L}^2_R}=\|\psi\|^2_{L^2_R}+\|\pi\|^2_{L^2_R},\qquad \|\Psi\|^2_{\mathscr{X}_R}=\|\psi\|^2_{H^2_R}+\|\pi\|^2_{H^1_R},\quad R>0.$$

Remark 2.3. The spaces $\mathscr{L}^2_{loc}$ are metrisable. The metrics can be defined by

$$\mathrm{dist}_{\mathscr{L}^2_{loc}}(\Psi_1,\Psi_2)=\sum_{R=1}^{\infty}2^{-R}\frac{\|\Psi_1-\Psi_2\|_{\mathscr{L}^2_R}}{1+\|\Psi_1-\Psi_2\|_{\mathscr{L}^2_R}}.\tag{2.8}$$

Global well-posedness

For the global well-posedness, we assume that

$$U(\zeta)\to\infty,\quad |\zeta|\to\infty.\tag{2.9}$$

Denote $\|\cdot\|=\|\cdot\|_{L^2(\mathbb{R}^3)}$. The next theorem is proved in [23].

Theorem 2.4. *Let conditions (1.3), (1.4) and (2.9) hold. Then*

(i) *For every initial data $\Psi(0)=\Psi_0=(\psi_0,\pi_0)\in\mathscr{D}$ the Cauchy problem for (2.3) has a unique solution $\psi(t)$ such that*

$$\Psi(t)=(\psi(t),\dot\psi(t))\in C(\mathbb{R},\mathscr{D}_F).$$

(ii) *The energy is conserved:*

$$\mathscr{H}(\Psi(t)):=\frac{1}{2}\Big(\|\dot\psi(t)\|^2+\|\nabla\psi_{reg}(t)\|^2+m^2\|\psi_{reg}(t)\|^2\Big)+U(\zeta(t))=\mathrm{const},\quad t\in\mathbb{R}.\tag{2.10}$$

(iii) *The following a priori bound holds*

$$|\zeta(t)|\le C(\Psi_0)\quad t\in\mathbb{R}.\tag{2.11}$$

Solitary waves and the main theorem

Definition 2.5. (i) The solitary waves of equation (2.3) are solutions of the form

$$\psi(x,t)=e^{-i\omega t}\psi_\omega(x),\quad \omega\in\mathbb{R},\quad \psi_\omega\in L^2(\mathbb{R}^3). \tag{2.12}$$

(ii) The solitary manifold is the set $\mathbf{S}=\{\Psi_\omega=(\psi_\omega,-i\omega\psi_\omega)\colon \omega\in\mathbb{R}\}$, where ψ_ω are the amplitudes of solitary waves.

The identity (1.6) implies that the set $\mathbf{S}$ is invariant under multiplication by $e^{i\theta}$, $\theta\in\mathbb{R}$. Let us note that since $F(0)=0$ by (1.5), then for any $\omega\in\mathbb{R}$ there is a zero solitary wave with $\psi_\omega(x)\equiv 0$.

Lemma 2.6. *(Existence of solitary waves). Assume that $F(\zeta)$ satisfies (1.5). Then nonzero solitary waves may exist only for $\omega\in(-m,m)$. The amplitudes of solitary waves are given by*

$$\psi_\omega(x)=q_\omega\frac{e^{-\sqrt{m^2-\omega^2}|x|}}{4\pi|x|}\in L^2(\mathbb{R}^3),\quad \omega\in(-m,m), \tag{2.13}$$

where q_ω is the solution to

$$m-\sqrt{m^2-\omega^2}=4\pi b(|q_\omega|^2). \tag{2.14}$$

Proof. We can split $\psi(x,t)=\psi_{reg}(x,t)+\zeta(t)G(x)$, where

$$\psi_{reg}(x,t)=q_\omega e^{-i\omega t}\,\frac{e^{-\sqrt{m^2-\omega^2}|x|}-e^{-m|x|}}{4\pi|x|},\qquad \zeta(t)=q_\omega e^{-i\omega t}.$$

Evidently, $\psi_{reg}(\cdot,t)\in H^2(\mathbb{R}^3)$, $\zeta(\cdot)\in C_b(\mathbb{R})$. Finally, the second equation of (1.1) together with (1.5) give

$$q_\omega e^{-i\omega t}\frac{m-\sqrt{m^2-\omega^2}}{4\pi}=q_\omega e^{-i\omega t}b(|q_\omega|^2).$$

□

At last, we assume that the nonlinearity is polynomial. This assumption is crucial in our argument since it will allow us to apply the Titchmarsh convolution theorem. Now all our assumptions on F can be summarized as follows.

Assumption A $$F(\zeta)=\partial_{\overline{\zeta}}U(\zeta),\qquad U(\zeta)=\sum_{n=0}^{N}u_n|\zeta|^{2n},\quad u_n\in\mathbb{R},\quad u_N>0,\quad N\geq 2. \tag{2.15}$$

In particular, this assumption guarantees that the nonlinearity F satisfies the bound (2.9) from Theorem 2.4. Our main result is the following theorem.

Theorem 2.7 (Main Theorem). *Let Assumption (2.15) be satisfied. Then for any $(\psi_0,\pi_0)\in\mathscr{D}_F$ the solution $\Psi(t)=(\psi(t),\dot\psi(t))$ to* (2.3) *with $(\psi,\dot\psi)|_{t=0}=(\psi_0,\pi_0)$ converges to solitary manifold $\mathbf{S}$ in the space $\mathscr{L}^2_{loc}$:*

$$\lim_{t\to\pm\infty}\operatorname{dist}_{\mathscr{L}^2_{loc}}(\Psi(t),\mathbf{S})=0, \tag{2.16}$$

where $\operatorname{dist}_{\mathscr{L}^2_{loc}}(\cdot,\cdot)$ *defined in (2.8).*

It suffices to prove Theorem 2.7 for $t\to+\infty$. We will only consider the solution $\psi(x,t)$ restricted to $t\geq 0$.

3 Dispersive component

Let J_1 be the Bessel function of order 1, and θ be the Heaviside function. In [23] we proved that the solution $\psi(x,t)$ to (2.3) with initial data $\psi_0=\psi_{0,reg}+\zeta_0G\in D_F$, $\pi_0=\pi_{0,reg}+\dot\zeta_0G\in D$ is given by

$$\psi(x,t):=\psi_f(x,t)+\frac{\theta(t-|x|)}{4\pi|x|}\zeta(t-|x|)-\frac{m}{4\pi}\int_0^t\frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}}\zeta(t-s)ds,\quad t\geq 0. \tag{3.1}$$

Here $\psi_f(x,t) \in C([0,\infty), L^2(\mathbb{R}^3))$ is a unique solution to the Cauchy problem for the free Klein-Gordon equation.

$$\ddot{\psi}_f(x,t) = (\Delta - m^2)\psi_f(x,t), \quad \psi_f(x,0) = \psi_0(x), \quad \dot{\psi}_f(x,0) = \pi_0(x), \tag{3.2}$$

and $\zeta(t) \in C^1([0,\infty))$ is a unique solution to the Cauchy problem for the following first-order nonlinear integro-differential equation with delay

$$\frac{\dot{\zeta}(t)}{4\pi} - \frac{m}{4\pi}\zeta(t) + \frac{m}{4\pi}\int_0^t \frac{J_1(ms)}{s}\zeta(t-s)ds + F(\zeta(t)) = \lambda(t), \quad t \geq 0, \quad \zeta(0) = \zeta_0, \tag{3.3}$$

where $\lambda(t) := \lim\limits_{x\to 0} \psi_f(x,t) \in C([0,\infty))$. Note that the limit is well defined, $\lambda(t)$ is continuous for $t > 0$, and it admits a limit as $t \to +0$ (see [23]). The integral in (3.3) is bounded for all $t \geq 0$ due to well known properties of the Bessel function J_1: $J_1(r) \sim r^{-1/2}$ for $r \to \infty$, and $J_1(r) \sim r$ as $r \to 0$ (see for example [29]). Now we study the decay properties of the dispersive component $\psi_f(x,t)$ for $t \to \infty$.

Proposition 3.1. *$\psi_f(x,t)$ decays in $\mathscr{X}_{loc}$ seminorms. That is, $\forall R > 0$*

$$\left\|(\psi_f(t), \dot{\psi}_f(t)\right\|_{\mathscr{X}_R} \to 0, \qquad t \to \infty. \tag{3.4}$$

Proof. We split $\psi_f(x,t)$ as

$$\psi_f(x,t) = \psi_{f,reg}(x,t) + \psi_{f,G}(x,t), \quad t \geq 0,$$

where $\psi_{f,reg}$ and $\psi_{f,G}$ are defined as solutions to the following Cauchy problems:

$$\ddot{\psi}_{f,reg}(x,t) = (\Delta - m^2)\psi_{f,reg}(x,t), \qquad (\psi_{f,reg}, \dot{\psi}_{f,reg})|_{t=0} = (\psi_{0,reg}, \pi_{0,reg}), \tag{3.5}$$

$$\ddot{\psi}_{f,G}(x,t) = (\Delta - m^2)\psi_{f,G}(x,t), \qquad (\psi_{f,G}, \dot{\psi}_{f,G})|_{t=0} = (\zeta_0 G, \dot{\zeta}_0 G), \tag{3.6}$$

Since $(\psi_{0,reg}, \pi_{0,reg}) \in \mathscr{X}$, then evidently,

$$(\psi_{f,reg}, \dot{\psi}_{f,reg}) \in C_b([0,\infty), \mathscr{X}). \tag{3.7}$$

The following lemma states well known decay in local seminorms for the free Klein-Gordon equation.

Lemma 3.2. *cf. [16, Lemma 3.1]) Let $(u_0, v_0) \in \mathscr{X}$. Then $\forall R > 0$*

$$\|\mathscr{U}(t)(u_0, v_0)\|_{\mathscr{X}_R} \to 0, \qquad t \to \infty, \tag{3.8}$$

where $\mathscr{U}(t)$ is the dynamical group of the free Klein-Gordon equation.

Therefore, the first dispersive component $\psi_{f,reg}(x,t)$ decays in $\mathscr{X}_{loc}$ seminorms. That is, $\forall R > 0$

$$\left\|(\psi_{f,reg}(\cdot,t), \dot{\psi}_{f,reg}(\cdot,t))\right\|_{\mathscr{X}_R} \to 0, \qquad t \to \infty. \tag{3.9}$$

Now we consider the second dispersive component $\psi_{f,G}$.

Lemma 3.3. *$\psi_{f,G}(x,t)$ decays in $\mathscr{X}_{loc}$ seminorms. That is, $\forall R > 0$*

$$\left\|(\psi_{f,G}(t), \dot{\psi}_{f,G}(t)\right\|_{\mathscr{X}_R} \to 0, \qquad t \to \infty. \tag{3.10}$$

Proof. Let $\eta(x)$ be a smooth function with a support in B_1, such that $\eta(x) = 1$ for $x \in B_{1/2}$. We split G as

$$G = \eta G + (1-\eta)G.$$

Lemma 3.2 implies that

$$\|\mathscr{U}(t)\big(\zeta_0(1-\eta)G, \dot{\zeta}_0(1-\eta)G\big)\|_{\mathscr{X}_R} \to 0, \qquad t \to \infty, \quad \forall R > 0,$$

since $\big(\zeta_0(1-\eta)G, \dot{\zeta}_0(1-\eta)G\big) \in \mathscr{X}$. Hence it suffices to prove that

$$\|(u(t), \dot{u}(t))\|_{\mathscr{X}_R} \to 0, \qquad t \to \infty, \tag{3.11}$$

where $(u(t),\dot{u}(t)) := \mathscr{U}(t)\big(\zeta_0\eta G,\, \dot{\zeta}_0\eta G\big)$. The matrix kernel $\mathscr{U}(x-y,t)$ of the dynamical group $\mathscr{U}(t)$ can be written as

$$\mathscr{U}(x-y,t) = \begin{pmatrix} \dot{U}(x-y,t) & U(x-y,t) \\ \ddot{U}(x-y,t) & \dot{U}(x-y,t) \end{pmatrix}, \quad x,y\in\mathbb{R}^3, \quad t>0, \tag{3.12}$$

where

$$U(z,t) = \frac{\delta(t-|z|)}{4\pi t} - \frac{m}{4\pi}\frac{\theta(t-|z|)J_1(m\sqrt{t^2-|z|^2})}{\sqrt{t^2-|z|^2}}, \quad z\in\mathbb{R}, \quad t>0. \tag{3.13}$$

Well known asymptotics of the Bessel function imply that

$$\Big|\partial_t^k\partial_z^\beta \frac{J_1(m\sqrt{t^2-|z|^2})}{\sqrt{t^2-|z|^2}}\Big| \le C(1+t)^{-3/2}, \quad t\ge 2|z|, \quad k=0,1,2, \quad |\beta|\le 2. \tag{3.14}$$

Hence, for $|x|\le R$ and $t>2(R+1)$, we obtain

$$|u(x,t)| + |\Delta u(x,t)| + |\dot{u}(x,t)| + |\nabla\dot{u}(x,t)| \le Ct^{-3/2}.$$

Then (3.11) follows. □

Finally, (3.8) and (3.10) imply (3.4). □

Corollary 3.4. *From (3.4) immediately follows that*

$$\lambda(t) = \psi_f(0,t) \to 0, \quad t\to\infty. \tag{3.15}$$

In conclusion, let us show that

$$\psi_{f,G}(t) \in C_b([0,\infty), L^2(\mathbb{R}^3)). \tag{3.16}$$

Indeed, the energy conservation for the free Klein-Gordon equation implies that

$$\mathscr{U}(t)(0,G) = (U(t)G, \dot{U}(t)G) \in C_b([0,\infty), H^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)).$$

Then

$$\psi_{f,G}(t) = \zeta_0\dot{U}(t)G + \dot{\zeta}_0 U(t)G \in C_b([0,\infty), L^2(\mathbb{R}^3)).$$

4 Singular component

Complex Fourier-Laplace transform

In notation (3.1) define the functions

$$\psi_S(x,t) := \frac{\theta(t-|x|)}{4\pi|x|}\zeta(t-|x|) - \frac{m}{4\pi}\int_0^t \frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}}\zeta(t-s)ds \in C([0,\infty), L^2(\mathbb{R}^3)), \quad t\ge 0. \tag{4.1}$$

It is easy to verify that $\psi_S(x,t)$ is the solution to the Cauchy problem

$$\ddot{\psi}_S(x,t) = (\Delta - m^2)\psi_S(x,t) + \zeta(t)\delta(x), \quad \psi_S(x,0)=0, \quad \dot{\psi}_S(x,0)=0. \tag{4.2}$$

The energy conservation (2.10) and a priory bound (2.11) imply that $\psi(t)\in C_b([0,\infty),L^2(\mathbb{R}^3))$. Hence (3.1), (3.7) and (3.16) give that

$$\psi_S(t) \in C_b([0,\infty), L^2(\mathbb{R}^3)). \tag{4.3}$$

Let us analyze the Fourier-Laplace transform of $\psi_S(x,t)$:

$$\tilde{\psi}_S(x,\omega) = \mathscr{F}_{t\to\omega}[\theta(t)\psi_S(x,t)] := \int_0^\infty e^{i\omega t}\psi_S(x,t)\,dt, \quad \omega\in\mathbb{C}^+, \quad x\in\mathbb{R}^3, \tag{4.4}$$

where $\mathbb{C}^+ := \{z\in\mathbb{C}:\ \mathrm{Im}\, z>0\}$. Note that $\tilde{\psi}_S(\cdot,\omega)$ is an L^2-valued analytic function of $\omega\in\mathbb{C}^+$ due to (4.3). Equation (4.2) implies that

$$-\omega^2\tilde{\psi}_S(x,\omega) = (\Delta - m^2)\tilde{\psi}_S(x,\omega) + \tilde{\zeta}(\omega)\delta(x), \quad \omega\in\mathbb{C}^+, \quad x\in\mathbb{R}^3, \tag{4.5}$$

where $\tilde{\zeta}(\omega)$ is the Fourier-Laplace transform of $\zeta(t)$:

$$\tilde{\zeta}(\omega) = \mathscr{F}_{t\to\omega}[\theta(t)\zeta(t)] = \int_0^\infty e^{i\omega t}\zeta(t)\,dt. \tag{4.6}$$

Applying the Fourier transform to (4.5), we get

$$\hat{\tilde{\psi}}_S(\xi,\omega) = \frac{\tilde{\zeta}(\omega)}{\xi^2+m^2-\omega^2}, \qquad \xi\in\mathbb{R}^3, \qquad \omega\in\mathbb{C}^+. \tag{4.7}$$

Denote

$$\varkappa(\omega) = \sqrt{\omega^2-m^2}, \qquad \operatorname{Im}\varkappa(\omega)>0, \qquad \omega\in\mathbb{C}^+. \tag{4.8}$$

Then $\varkappa(\omega)$ is the analytic function on $\mathbb{C}^+$, and $\tilde{\psi}_S(x,\omega)$ is given by

$$\tilde{\psi}_S(x,\omega) = \tilde{\zeta}(\omega)V(x,\omega), \quad V(x,\omega) = \frac{e^{i\varkappa(\omega)|x|}}{4\pi|x|}, \qquad \omega\in\mathbb{C}^+. \tag{4.9}$$

We then have, formally, for any $\varepsilon>0$:

$$\psi_S(x,t) = \frac{1}{2\pi}\int_{\operatorname{Im}\omega=\varepsilon} e^{-i\omega t}\tilde{\zeta}(\omega)V(x,\omega)\,d\omega = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\tilde{\zeta}(\omega+i0)V(x,\omega+i0)\,d\omega = \mathscr{F}^{-1}_{\omega\to t}\big[\tilde{\zeta}(\omega)V(x,\omega)\big]. \tag{4.10}$$

Traces on the real line

By (4.3) the Fourier transform $\tilde{\psi}_S(\cdot,\omega) = \mathscr{F}_{t\to\omega}[\theta(t)\psi_S(\cdot,t)]$ is a tempered L^2-valued distribution of $\omega\in\mathbb{R}$. It is the boundary value of the analytic function (4.4) in the following sense:

$$\tilde{\psi}_S(\cdot,\omega) = \lim_{\varepsilon\to0+}\tilde{\psi}_S(\cdot,\omega+i\varepsilon), \qquad \omega\in\mathbb{R}, \tag{4.11}$$

where the convergence holds in $\mathscr{S}'(\mathbb{R},L^2(\mathbb{R}^3))$. Indeed,

$$\tilde{\psi}_S(\cdot,\omega+i\varepsilon) = \mathscr{F}_{t\to\omega}[\theta(t)\psi_S(\cdot,t)e^{-\varepsilon t}],$$

while $\theta(t)\psi_S(\cdot,t)e^{-\varepsilon t} \underset{\varepsilon\to0+}{\longrightarrow} \theta(t)\psi_S(\cdot,t)$ in $\mathscr{S}'(\mathbb{R},L^2(\mathbb{R}^3))$. Therefore, (4.11) holds by the continuity of the Fourier transform $\mathscr{F}_{t\to\omega}$ in $\mathscr{S}'(\mathbb{R})$.

Similarly to (4.11), the distribution $\tilde{\zeta}(\omega)$, $\omega\in\mathbb{R}$, is the boundary values of the analytic in $\mathbb{C}^+$ function $\tilde{\zeta}(\omega)$, $\omega\in\mathbb{C}^+$:

$$\tilde{\zeta}(\omega) = \lim_{\varepsilon\to0+}\tilde{\zeta}(\omega+i\varepsilon), \quad \omega\in\mathbb{R}, \tag{4.12}$$

since the function $\theta(t)\zeta(t)$ is bounded. The convergence holds in the space of tempered distributions $\mathscr{S}'(\mathbb{R})$.

Let us justify that the representation (4.9) for $\tilde{\psi}_S(x,\omega)$ is also valid when $\omega\in\mathbb{R}\setminus\{-m;m\}$, if the multiplication in (4.9) is understood in the sense of distribution. Namely,

Lemma 4.1. *$V(x,\omega)$ is a smooth function of $\omega\in\mathbb{R}\setminus\{-m;m\}$ for any fixed $x\in\mathbb{R}^3\setminus\{0\}$, and the identity*

$$\tilde{\psi}_S(x,\omega) = \tilde{\zeta}(\omega)V(x,\omega), \quad \omega\in\mathbb{R}\setminus\{-m;m\} \tag{4.13}$$

holds in the sense of distributions.

Proof. This lemma follows from (4.11) and (4.12) by the smoothness of $V(x,\omega)$ for $\omega\neq\pm m$. □

Absolutely continuous spectrum

Note that $\mathbb{R}\setminus(-m,m)$ coincides with the continuous spectrum of the free Klein-Gordon equation, and the function $\omega\varkappa(\omega)$ is positive for $\omega\in\mathbb{R}\setminus[-m,m]$.

Proposition 4.2. *(cf. [18, Proposition 2.3] The distribution $\tilde{\zeta}(\omega+i0)$ is absolutely continuous for $|\omega|>m$ and satisfies*

$$\int_{|\omega|>m}|\tilde{\zeta}(\omega)|^2\mathscr{M}(\omega)\,d\omega<\infty, \quad \text{where} \quad \mathscr{M}(\omega) = \frac{\varkappa(\omega)}{\omega}. \tag{4.14}$$

Remark 4.3. Recall that $\tilde{\zeta}(\omega)$, $\omega\in\mathbb{R}$, is defined by (4.12) as the trace distribution: $\tilde{\zeta}(\omega)=\tilde{\zeta}(\omega+i0)$.

Proof. For any $\delta>0$ denote $I_\delta=(-\infty,-m-\delta]\cup[m+\delta,\infty)$. It suffices to prove that

$$\int_{I_\delta}|\tilde{\zeta}(\omega)|^2\mathscr{M}(\omega)\,d\omega\le C, \tag{4.15}$$

with some constant $C>0$ which does not depend on δ. First, the Parseval identity applied to

$$\tilde{\psi}_S(x,\omega+i\varepsilon)=\int_0^\infty\psi_S(x,t)e^{i\omega t-\varepsilon t}\,dt,\quad \varepsilon>0,$$

gives

$$\int_{\mathbb{R}}\|\tilde{\psi}_S(\cdot,\omega+i\varepsilon)\|^2_{L^2(\mathbb{R}^3)}\,d\omega=2\pi\int_0^\infty\|\psi_S(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\,e^{-2\varepsilon t}\,dt.$$

Since $\sup_{t\ge0}\|\psi_S(\cdot,t)\|_{L^2(\mathbb{R})}<\infty$ by (4.3), we may bound the right-hand side by C_1/ε, with some $C_1>0$. Taking into account (4.9), we arrive at the key inequality

$$\int_{\mathbb{R}}|\tilde{\zeta}(\omega+i\varepsilon)|^2\|V(\cdot,\omega+i\varepsilon)\|^2_{L^2(\mathbb{R}^3)}\,d\omega\le\frac{C_1}{\varepsilon}. \tag{4.16}$$

Lemma 4.4. *There exists $n\in\mathbb{N}$ such that for any $\delta>0$ and $0<\varepsilon\le\delta/n$*

$$\|V(\cdot,\omega+i\varepsilon)\|^2_{L^2(\mathbb{R}^3)}\ge\frac{\mathscr{M}(\omega)}{16\pi\varepsilon},\qquad \omega\in I_\delta. \tag{4.17}$$

Proof. The explicit formula (4.9) for $V(x,\omega+i\varepsilon)$ implies

$$\|V(\cdot,\omega+i\varepsilon)\|^2_{L^2(\mathbb{R}^3)}=\frac{1}{4\pi}\int_0^\infty|e^{i\varkappa(\omega+i\varepsilon)r}|^2dr=\frac{1}{8\pi\,\mathrm{Im}\,\varkappa(\omega+i\varepsilon)}. \tag{4.18}$$

Further, for $\omega\in I_\delta$ and $0<\varepsilon\le\delta/n$ with sufficiently large $n\in\mathbb{N}$, we have

$$\mathrm{Im}\,\varkappa(\omega+i\varepsilon)=\mathrm{Im}\sqrt{(\omega+i\varepsilon)^2-m^2}=\varkappa(\omega)\mathrm{Im}\sqrt{1+(2i\varepsilon\omega-\varepsilon^2)/\varkappa^2(\omega)}\le\frac{2\varepsilon\omega}{\varkappa(\omega)}. \tag{4.19}$$

Finally, (4.18) and (4.19) imply (4.17). □

Substituting (4.17) into (4.16), we obtain the bound

$$\int_{I_\delta}|\tilde{\zeta}(\omega+i\varepsilon)|^2\mathscr{M}(\omega)\,d\omega\le16\pi C_1,\qquad 0<\varepsilon\le\delta/n. \tag{4.20}$$

We conclude that the set of functions $g_{\delta,\varepsilon}(\omega)=\tilde{\zeta}(\omega+i\varepsilon)\sqrt{\mathscr{M}(\omega)}$, $0<\varepsilon\le\varepsilon(\delta)$ defined for $\omega\in I_\delta$, is bounded in the Hilbert space $L^2(I_\delta)$, and, by the Banach Theorem, is weakly compact. The convergence of the distributions (4.12) implies the weak convergence $g_{\delta,\varepsilon}\underset{\varepsilon\to0+}{\rightharpoonup}g_\delta$ in the Hilbert space $L^2(I_\delta)$. The limit function $g_\delta(\omega)$ coincides with the distribution $\tilde{\zeta}(\omega)\sqrt{\mathscr{M}(\omega)}$ restricted onto I_δ. This proves the bound (4.15) and finishes the proof of the proposition. □

5 Compactness

We are going to prove compactness of the set of translations of $\{\psi_S(x,t+s):\ s\ge0\}$. We start from the following lemma

Lemma 5.1. *For any sequence $s_j\to\infty$ there exists an infinite subsequence (which we also denote by s_j) such that*

$$\zeta(t+s_j)\to\eta(t),\quad j\to\infty,\quad t\in\mathbb{R} \tag{5.1}$$

for some $\eta\in C_b(\mathbb{R})$. The convergence is uniform on $[-T,T]$ for any $T>0$. Moreover, $\eta(t)$ is the solution to

$$\frac{1}{4\pi}\dot{\eta}(t)-\frac{m}{4\pi}\eta(t)+\frac{m}{4\pi}\int_0^\infty\frac{J_1(ms)}{s}\eta(t-s)ds+F(\eta(t))=0,\quad t\in\mathbb{R}. \tag{5.2}$$

Proof. Theorem 2.4-iv), Corollary 3.4 and equation (3.3) imply that $\zeta \in C_b^1(\mathbb{R})$. Then (5.1) follows from the Arzelá-Ascoli theorem. Further, for any $t \in \mathbb{R}$ we get

$$\int_0^{t+s_j} \frac{J_1(ms)}{s} \zeta(t+s_j-s)ds \to \int_0^{\infty} \frac{J_1(ms)}{s} \eta(t-s)ds, \quad j \to \infty \tag{5.3}$$

by the Lebesgue dominated convergence theorem. Then equation (3.3) for $\zeta(t)$ together with (3.15) and (5.3) imply (5.2). □

Lemma 5.1 imply

Lemma 5.2. *The convergences hold*

$$\psi_S(\cdot, t+s_j) \to \beta_S(\cdot, t) := \frac{\eta(t-|x|)}{4\pi|x|} - \frac{m}{4\pi} \int_0^{\infty} \frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}} \eta(t-s)ds, \quad j \to \infty, \quad t \in \mathbb{R}, \tag{5.4}$$

$$\dot\psi_S(\cdot, t+s_j) \to \dot\beta_S(\cdot, t) = \frac{\dot\eta(t-|x|)}{4\pi|x|} - \frac{m}{4\pi} \int_0^{\infty} \frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}} \dot\eta(t-s)ds, \quad j \to \infty, \quad t \in \mathbb{R}, \tag{5.5}$$

in the topology of $C_b([-T,T], L^2_{loc})$ *for any* $T > 0$.

Proof. The convergence (5.4) follows immediately from (4.1), (5.1) and the Lebesgue dominated convergence theorem. Let us prove the convergence of $\dot\psi_S(\cdot, t+s_j)$. Equations (3.3) and (5.2) imply that

$$\dot\zeta(t+s_j) \to \dot\eta(t), \quad j \to \infty, \tag{5.6}$$

uniformly on $[-T,T]$ for any $T > 0$. Further, differentiating (4.1) for $t > |x|$ gives

$$\dot\psi_S(x,t) = \frac{\dot\zeta(t-|x|)}{4\pi|x|} - \frac{m}{4\pi} \frac{J_1(m\sqrt{t^2-|x|^2})}{\sqrt{t^2-|x|^2}} \zeta(0) - \frac{m}{4\pi} \int_0^t \frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}} \dot\zeta(t-s)ds,$$

which imply (5.7) by (5.6). □

Remark 5.3. From (4.3) it follows that

$$\beta_S(\cdot, t) \in L^{\infty}(\mathbb{R}, L^2(\mathbb{R}^3)). \tag{5.7}$$

6 Nonlinear spectral analysis

We call *an omega-limit trajectory* any function $\beta_S(x,t)$ that can appear as a limit in (5.4). Proposition 3.1 demonstrates that the long-time asymptotics of the solution $\psi(x,t)$ in L^2_{loc} depends only on the singular component $\psi_S(x,t)$. Namely, the convergences (5.4), and system (1.1) together with (3.1), (3.4) and (3.15) imply that any $\beta_S(x,t)$ is a solution to (1.1) with $\eta(t)$ instead $\zeta(t)$:

$$\left\{ \begin{array}{l} \ddot\beta_S(x,t) = (\Delta - m^2)\beta_S(x,t) + \eta(t)\delta(x) \\ \\ \lim\limits_{x\to 0}(\beta_S(x,t) - \eta(t)G(x)) = F(\eta(t)) \end{array} \right| \quad t \in \mathbb{R}.$$

In this section we prove the following proposition.

Proposition 6.1. *Every omega-limit trajectory is a solitary wave, that is,*

$$\beta_S(x,t) = \psi_{\omega_+}(x)e^{-i\omega_+ t}, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \tag{6.1}$$

with some $\omega_+ \in \mathbb{R}$.

6.1 Reduction of spectrum

Lemma 6.2. $\operatorname{supp}\tilde{\eta} \subset [-m,m]$.

Proof. Due to (5.1) and the continuity of the Fourier transform in $\mathscr{S}'(\mathbb{R})$, we have

$$\alpha(\omega)\tilde{\zeta}(\omega)e^{-i\omega s_l} \xrightarrow{\mathscr{S}'} \alpha(\omega)\tilde{\eta}(\omega), \quad j \to \infty.$$

for any $\alpha \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\alpha \cap [-m,m] = \emptyset$. The products $\alpha(\omega)\tilde{\zeta}(\omega)$ are absolutely continuous measures since $\tilde{\zeta}(\omega)$ is locally L^2 for $\omega \in \mathbb{R} \setminus [-m,m]$ by Proposition 4.2. Then $\tilde{\eta}(\omega) = 0$ for $\omega \notin [-m,m]$ by the Riemann-Lebesgue Theorem. □

Using (4.13) and taking into account that $V(x,\omega)$ is smooth for $\omega \neq \pm m$ and $x \neq 0$, we obtain the following relation, which holds in the sense of distributions:

$$\tilde{\beta}_S(x,\omega) = \tilde{\eta}(\omega)V(x,\omega), \qquad \omega \in \mathbb{R} \setminus \{\pm m\}. \tag{6.2}$$

Since $V(x,\omega) \neq 0$ for $\omega \in \mathbb{R}$ it follows from Lemma 6.2 that

$$\operatorname{supp}\tilde{\beta}_S(x,\cdot) \subset [-m,m]. \tag{6.3}$$

6.2 Spectral inclusion and the Titchmarsh theorem

We will derive (6.1) from the following identity

$$\eta(t) = Ce^{-i\omega_+ t}, \quad t \in R, \quad \omega_+ \in [-m,m], \tag{6.4}$$

which will be proven in three steps. We start with an investigation of $\operatorname{supp}\tilde{\eta}$.

Lemma 6.3. *The following spectral inclusion holds:*

$$\operatorname{supp}\widetilde{F(\eta)} \subset \operatorname{supp}\tilde{\eta}. \tag{6.5}$$

Proof. Applying the Fourier transform to (5.2), we get by the theory of quasimeasures (see [16]) that

$$\widetilde{F(\eta)}(\omega) = \frac{1}{4\pi}(i\omega + m - m\tilde{K}(\omega))\tilde{\eta}(\omega) = \frac{1}{4\pi}(m - \sqrt{m^2-\omega^2})\tilde{\eta}(\omega), \quad |\omega| \leq m, \tag{6.6}$$

where $\tilde{K}(\omega) = \frac{1}{m}\left(\sqrt{m^2-\omega^2} + i\omega\right)$ is the Fourier transform of the function $K(t) = \theta(t)J_1(mt)/t \in L^1(\mathbb{R})$ (see Appendix), and $\tilde{\eta}(\omega)$ is a quasimeasure. Then (6.5) follows. □

The second step is the following lemma

Lemma 6.4. *For any omega-limit trajectory*

$$|\eta(t)| = \text{const}, \quad t \in \mathbb{R}. \tag{6.7}$$

Proof. Our main assumption (2.15) implies that the function $F(\eta(t))$ admits the representation

$$F(\eta(t)) = a_\eta(t)\eta(t), \tag{6.8}$$

where, according to (2.15)

$$a_\eta(t) = \sum_{n=1}^{N} 2nu_n|\eta(t)|^{2n-2}. \tag{6.9}$$

Both function $\eta(t)$ and $a_\eta(t)$ are bounded continuous functions in $\mathbb{R}$ by Lemma 5.1. Hence, $\eta(t)$ and $a_\eta(t)$ are tempered distributions. According to (6.3) $\operatorname{supp}\tilde{\eta} \subset [-m,m]$, $\operatorname{supp}\tilde{\overline{\eta}} \subset [-m,m]$, and then $\tilde{a}_\eta$ also has a bounded support. Denote $\mathbf{F} = \operatorname{supp}\tilde{F}(\eta)$, $\mathbf{A} = \operatorname{supp}\tilde{a}_\eta$, $\mathbf{Z} = \operatorname{supp}\tilde{\eta}$. Then the spectral inclusion (6.5) gives

$$\mathbf{F} \subset \mathbf{Z}. \tag{6.10}$$

On the other hand, applying the Titchmarsh convolution theorem (see [11, Theorem 4.3.3]) to (6.8), we obtain

$$\inf \mathbf{F} = \inf \mathbf{A} + \inf \mathbf{Z}, \quad \sup \mathbf{F} = \sup \mathbf{A} + \sup \mathbf{Z}. \tag{6.11}$$

From (6.10) and (6.11) it follows that $\inf \mathbf{A} = \sup \mathbf{A} = 0$, and hence $\mathbf{A} \subset \{0\}$. Thus, we conclude that $\operatorname{supp} \tilde{a}_\eta = \mathbf{A} \subset \{0\}$, and therefore the distribution $\tilde{a}_\eta(\omega)$ is a finite linear combination of $\delta(\omega)$ and its derivatives. Then $a_\eta(t)$ is a polynomial in t; since $a_\eta(t)$ is bounded by Lemma 5.1, we conclude that $a_\eta(t) = \text{const}$. Finally, (6.7) follows since $a_\eta(t)$ is a polynomial in $|\eta(t)|$, and its degree $2N-2 \geq 2$ by (2.15) and (6.9). □

Now (6.7) means that $\eta(t)\overline{\eta}(t) \equiv C = \text{const}$, and then $\tilde{\eta} * \tilde{\overline{\eta}} = 2\pi C \delta(\omega)$. Hence, if η is not identically zero, the Titchmarsh theorem implies that $\mathbf{Z} = \omega_+ \in [-m, m]$. Indeed,

$$0 = \sup \mathbf{Z} + \sup(-\mathbf{Z}) = \sup \mathbf{Z} - \inf \mathbf{Z},$$

and hence $\inf \mathbf{Z} = \sup \mathbf{Z}$. Therefore, $\tilde{\eta}$ is a finite linear combination of $\delta(\omega - \omega_+)$ and its derivatives. But the derivatives could not be present because of the boundedness of $\eta(t)$. Thus $\tilde{\eta} \sim \delta(\omega - \omega_+)$, which implies (6.4).

Proof of Proposition 6.1. Substituting (6.4) in the RHS of (5.4), we obtain

$$\begin{aligned} \beta_S(x,t) &= \frac{Ce^{-i\omega_+(t-|x|)}}{4\pi|x|} - \frac{mC}{4\pi}\int_0^\infty \frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}} e^{-i\omega_+(t-s)}ds \\ &= \frac{Ce^{-i\omega_+ t}}{4\pi}\Big(\frac{e^{i\omega_+|x|}}{|x|} - m\tilde{L}(x,\omega_+)\Big) = \frac{Ce^{-\sqrt{m^2-\omega_+^2}|x|}}{4\pi|x|}e^{-i\omega_+ t}. \end{aligned} \tag{6.12}$$

Here $\tilde{L}(x,\omega_+) = \frac{1}{|x|m}\big(e^{i|x|\omega_+} - e^{i|x|\sqrt{\omega_+^2-m^2}}\big)$ is the Fourier transform of the function $L(x,t) = \frac{\theta(t-|x|)J_1(m\sqrt{t^2-|x|^2})}{\sqrt{t^2-|x|^2}}$ (see Appendix). Hence, (6.1) holds and $\beta_S(x,t)$ is a solitary wave. □

Remark 6.5. If (6.4) holds with some $|\omega_+| < m$, then (6.1) follows immediately from (6.2).

7 Proof of Theorem 2.7

Due to Proposition 3.1 it suffices to prove that

$$\lim_{t\to\infty} \operatorname{dist}_{\mathscr{L}^2_{loc}}(\Psi_S(t), \mathbf{S}) = 0, \tag{7.1}$$

where $\Psi_S(t) = (\psi_S(t), \dot{\psi}_S(t))$. Assume by contradiction that there exists a sequence $s_j \to \infty$ such that

$$\operatorname{dist}_{\mathscr{L}^2_{loc}}(\Psi_S(s_j), \mathbf{S}) \geq \delta, \quad \forall j \tag{7.2}$$

for some $\delta > 0$. According to Lemmas 5.1 and 5.2, and formula (6.1) there exist a subsequence s_{j_k} of the sequence s_j and an amplitude ψ_{ω_+} such that the following convergences hold

$$\Psi_S(t + s_{j_k}) \to (\psi_{\omega_+}e^{-i\omega_+ t}, -i\omega_+\psi_{\omega_+}e^{-i\omega_+ t}), \quad j_k \to \infty, \quad t \in \mathbb{R}.$$

This implies that $\Psi_S(s_{j_k}) \to (\psi_{\omega_+}, -i\omega\psi_{\omega_+})$, which contradict (7.2). This completes the proof of Theorem 2.7. □

A Appendix. Fourier transforms

Here we calculate the Fourier transforms of $L(x,t) = \theta(t-|x|)J_1(m\sqrt{t^2-|x|^2})/\sqrt{t^2-|x|^2}$ and $K(t) = \theta(t)J_1(mt)/t = L(0,t)$, which we have used in (6.6) and (6.12). Recall, that the function

$$U(x,t) = \frac{\delta(t-|x|)}{4\pi|x|} - \frac{m}{4\pi}L(x,t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R} \tag{A.1}$$

is the fundamental solution to the Klein-Gordon equation:

$$\ddot{U}(x,t) = (\Delta - m^2)U(x,t) + \delta(x)\delta(t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}.$$

Applying the Fourier transforms in t, we obtain

$$(\Delta - m^2 + \omega^2)\tilde{U}(x,\omega) = -\delta(x), \quad x \in \mathbb{R}^3, \quad \omega \in \mathbb{R}.$$

Note that $\tilde{U}(\cdot,\omega)$ is an analytic and bounded function of $\omega \in \mathbb{C}^+$ with values in tempered distributions on $\mathbb{R}^3$. Moreover, $\tilde{U}(\cdot,\omega)$ is a radial distribution, and hence, it coincides with $V(\cdot,\omega)$ by (4.8)–(4.9):

$$\tilde{U}(x,\omega) = V(x,\omega) = \frac{e^{i\varkappa(\omega)|x|}}{4\pi|x|}, \quad \omega \in \overline{\mathbb{C}}^+.$$

Therefore,

$$\tilde{L}(x,\omega) = \frac{4\pi}{m}\Big(\frac{e^{i\omega|x|}}{4\pi|x|} - V(x,\omega)\Big) = \frac{1}{m|x|}\Big(e^{i\omega|x|} - e^{-\sqrt{m^2-\omega^2}|x|}\Big), \quad |\omega| \le m. \tag{A.2}$$

Passing to the limit, we obtain

$$\tilde{K}(\omega) = \lim_{x\to 0}\tilde{L}(x,m) = \frac{1}{m}\Big(\sqrt{m^2-\omega^2} + i\omega\Big), \quad |\omega| \le m, \tag{A.3}$$

Remark A.1. Formula (A.3) agrees with [12, Sections 1.12(4) and 2.12 (5)], which are cosine and sine transforms of $K(t)$.

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