

ON DISPERSION DECAY FOR 3D KLEIN-GORDON EQUATION

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ABSTRACT. We improve previous results on dispersion decay for 3D Klein-Gordon equation with generic potential. We develop a novel approach, which allows us to establish the decay in more strong norms and to weaken assumptions on the potential.

1. Introduction. We are concerned with three-dimensional Klein-Gordon equation

$$\ddot{\psi}(x, t) = \Delta\psi(x, t) - m^2\psi(x, t) + V(x)\psi(x, t), \quad x \in \mathbb{R}^3, \quad m > 0. \quad (1)$$

In vector form

$$i\dot{\Psi}(t) = \mathcal{K}\Psi(t), \quad (2)$$

where

$$\Psi(t) = \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} 0 & i \\ i(\Delta - m^2 + V(x)) & 0 \end{pmatrix}.$$

Our goal is the improvement of previous results on dispersion decay of solutions. We suggest a novel approach which allows us to establish the decay in more strong norms and to weaken assumptions on the potential. We assume that $V(x)$ is a continuous real function, and

$$|V(x)| \leq C\langle x \rangle^{-\beta}, \quad x \in \mathbb{R}^3, \quad \langle x \rangle = (1 + |x|^2)^{1/2}, \quad (3)$$

for some $\beta > 3$. We restrict ourselves to the “regular case” when the point 0 is neither eigenvalue nor resonance for the Schrödinger operator $\mathcal{H} = -\Delta - V(x)$. Equivalently, the truncated resolvent of the operator \mathcal{H} is bounded at the point 0.

To formulate our result, we introduce weighted Sobolev spaces $H_\sigma^s = H_\sigma^s(\mathbb{R}^3)$ with $s, \sigma \in \mathbb{R}$, associated with the norms

$$\|\psi\|_{H_\sigma^s} = \|\langle x \rangle^\sigma \langle \nabla \rangle^s \psi\|_{L^2} < \infty.$$

We denote $L_\sigma^2 = H_\sigma^0$.

Definition 1.1. E_σ is the complex Hilbert space $H_\sigma^1 \oplus L_\sigma^2$ of vector-functions $\Psi = (\psi, \pi)$ with the norm

$$\|\Psi\|_{E_\sigma} = \|\psi\|_{H_\sigma^1} + \|\pi\|_{L_\sigma^2} < \infty.$$

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Our main result is the following

Theorem 1.2. *Let (??) hold and $\sigma > 3/2$. Then in the regular case,*

$$\|U(t)P_c(\mathcal{K})\|_{E_\sigma \rightarrow E_{-\sigma}} \leq C(1 + |t|)^{-3/2}, \quad t \in \mathbb{R}, \tag{4}$$

where $U(t)$ is the dynamical group of the equation (??), and $P_c(\mathcal{K})$ is the Riesz projection onto the continuous spectrum of \mathcal{K} .

The Strichartz estimates for the 3D Klein-Gordon equations were obtained in [?, ?, ?]; the long-time decay $\sim t^{-3/2}$ in local energy norms was established first by Vainberg [?]. In [?, ?] we obtained the weighted energy decay (??) for $\sigma > 5/2$ under an additional assumption on the decay of $\nabla V(x)$. Now we improve this result to $\sigma > 3/2$ assuming (??) only.

For the 3D Schrödinger equation the dispersion decay in weighted norms has been established first by Jensen and Kato [?]: for $\sigma > 5/2$

$$\|e^{i\mathcal{H}t}P_c(\mathcal{H})\|_{L^2_\sigma \rightarrow L^2_{-\sigma}} \leq C(1 + |t|)^{-3/2}, \quad t \in \mathbb{R} \tag{5}$$

in the regular case. Later, Goldberg and Schlag [?] proved the dispersion decay in $L^1 \rightarrow L^\infty$ norm:

$$\|e^{i\mathcal{H}t}P_c(\mathcal{H})\|_{L^1 \rightarrow L^\infty} = \mathcal{O}(|t|^{-3/2}), \quad t \rightarrow \pm\infty, \tag{6}$$

which implies (??) with $\sigma > 3/2$.

The approach [?, ?] to the Schrödinger equation relies on the spectral Fourier-Laplace representation

$$e^{i\mathcal{H}t}P_c(\mathcal{H}) = \frac{1}{2\pi i} \int_0^\infty e^{-i\omega t} [R(\omega + i0) - R(\omega - i0)] d\omega, \tag{7}$$

where $R(\omega) = (\mathcal{H} - \omega)^{-1}$ is the resolvent of the Schrödinger operator \mathcal{H} . Low energy and high energy parts are considered separately. The required dispersion decay follows from the regularity of resolvent for small ω and its decay in weighted norms for large ω (see [?, ?]).

The methods [?] cannot be directly applied to the Klein-Gordon equation, since the results of [?, ?] imply the boundedness of the corresponding resolvent $\mathcal{R}(\omega) = (\mathcal{K} - \omega)^{-1}$ only (see the discussion in Introduction of [?]). Similarly, the methods [?] are also inapplicable since the $L^1 \rightarrow L^\infty$ bound of type (??) fails for the Klein-Gordon equation (see below).

Our approach relies on the Born expansion for the dynamical group $\mathcal{U}(t)$,

$$\begin{aligned} \mathcal{U}(t) &= \mathcal{U}_0(t) - i \int_0^t \mathcal{U}_0(t-s) \mathcal{V} \mathcal{U}_0(s) ds \\ &+ (-i)^k \int_0^t \mathcal{U}_0(t-s_1) \int_0^{s_1} \mathcal{V} \mathcal{U}_0(s_1-s_2) \dots \int_0^{s_{k-1}} \mathcal{V} \mathcal{U}_0(s_{k-1}-s_k) ds_k \dots ds_1 + \mathcal{U}_k(t), \end{aligned}$$

where $\mathcal{U}_0(t)$ is the dynamical group of the free equation and \mathcal{V} is the matrix (??).

The decay of type (??) for the first term $\mathcal{U}_0(t)$ in weighted energy norms was established in [?] using an analog of the strong Huygens principle. Then the decay for the next terms follows by estimates for convolutions and by the condition (??) on the potential.

The main difficulty is to prove the decay of type (??) for the remainder $\mathcal{U}_k(t)$,

$$\mathcal{U}_k(t) = \frac{(-1)^{k+1}}{2\pi i} \int_{\Gamma} e^{-i\omega t} ([(\mathcal{R}_0 \mathcal{V})^k \mathcal{V} \mathcal{R} \mathcal{V} \mathcal{R}_0](\omega + i0) - [(\mathcal{R}_0 \mathcal{V})^k \mathcal{V} \mathcal{R} \mathcal{V} \mathcal{R}_0](\omega - i0)) d\omega,$$

where $\Gamma = (-\infty, -m] \cup [m, \infty)$, and $\mathcal{R}_0(\omega)$ stands for the free resolvent corresponding to $V = 0$. We derive this decay proving the time decay of $\mathcal{U}_k(t)$ and of $\nabla \mathcal{U}_k(t)$ in $L^1 \oplus L^1 \rightarrow L^\infty \oplus L^\infty$ norm. For the proof we develop a streamlined version of the approach [?].

In conclusion, we note that for the free Schrödinger operator the decay of type (??) in $L^1 \rightarrow L^\infty$ norm follows from the boundedness of its integral kernel for $|t| \geq 1$ and from the uniform decay of the kernel. Namely,

$$\|e^{-i\Delta t}\|_{L^1 \rightarrow L^\infty} = \sup_{x, y \in \mathbb{R}^3} \left| \frac{e^{i|x-y|^2/4t}}{(4\pi it)^{3/2}} \right| \simeq |t|^{-3/2}. \quad (8)$$

On the other hand, for $\mathcal{U}_0(t)$ the decay in $L^1 \oplus L^1 \rightarrow L^\infty \oplus L^\infty$ norm does not hold since its kernel is unbounded,

$$\mathcal{U}_0^{12}(t, x, y) = \frac{\delta(t - |x - y|)}{4\pi t} - \frac{m}{4\pi} \frac{\theta(t - |x - y|) J_1(m\sqrt{t^2 - |x - y|^2})}{\sqrt{t^2 - |x - y|^2}}. \quad (9)$$

This difference reflects the distinct character of the wave propagation for the relativistic and nonrelativistic equations. Namely, the singularities of solutions to the Schrödinger equation are concentrated at $t = 0$ and disappear at infinity for $t \neq 0$ due to infinite speed of propagation. On the other hand, in the case of Klein-Gordon equation, the singularities move with bounded velocities, thus they are present forever in the space.

Our paper is organised as follows. In Section ?? we recall the known spectral properties of the Schrödinger resolvent. In Sections ?? and ?? we derive some properties of the finite Born series. In Section ?? we prove our main result. In Section ?? we apply our approach to the Schrödinger equation.

Let us note that the dispersion decay in weighted norms plays an important role in proving asymptotic stability of solitons in associated nonlinear equations [?, ?, ?, ?, ?].

2. The Schrödinger resolvent. Here we collect the properties of the resolvent $R(\omega) = (\mathcal{H} - \omega)^{-1}$ obtained in [?, ?] (see also [?] where the full proofs of these properties can be found). We suppose that the condition (??) holds with some $\beta > 1$. Then

R1. $R(\omega) : L^2 \rightarrow L^2$ is a meromorphic function of $\omega \in \mathbb{C} \setminus [0, \infty)$; the poles of $R(\omega)$ are located at a finite set of eigenvalues $\omega_j < 0$.

R2. For $\omega > 0$ and $\sigma > 1/2$

$$\|R(\omega \pm i\varepsilon) - R(\omega \pm i0)\|_{L^2_\sigma \rightarrow L^2_{-\sigma}} \rightarrow 0, \quad \varepsilon \rightarrow 0+.$$

R3. Let $\beta > 3$. Then for $\sigma > 1/2 + k$

$$\|R^{(k)}(\omega)\|_{L^2_\sigma \rightarrow L^2_{-\sigma}} = \mathcal{O}(|\omega|^{-\frac{1+k}{2}}), \quad |\omega| \rightarrow \infty, \quad \omega \in \mathbb{C} \setminus [0, \infty), \quad k = 0, 1, 2. \quad (10)$$

R4. Let $\beta > 2$. Then in the regular case, $R(\omega)$ is a continuous operator function at $\omega = 0$ with the values in $B(L^2_\sigma, L^2_{-\sigma})$ for any $\sigma_1, \sigma_2 > 1/2$ with $\sigma_1 + \sigma_2 > 2$.

R5. Let $\beta > 3$. Then in the regular case, for $\omega \in \mathbb{C} \setminus [0, \infty)$,

$$\|R(\omega)\|_{L^2_{\sigma_1} \rightarrow L^2_{-\sigma_2}} = \mathcal{O}(1), \quad \omega \rightarrow 0, \quad \sigma_1, \sigma_2 > 1/2, \quad \sigma_1 + \sigma_2 > 2, \quad (11)$$

$$\|R^{(k)}(\omega)\|_{L^2_{\sigma} \rightarrow L^2_{-\sigma}} = \mathcal{O}(|\omega|^{\frac{1}{2}-k}), \quad \omega \rightarrow 0, \quad \sigma > 1/2 + k, \quad k = 1, 2. \quad (12)$$

In particular, all these properties hold for the free resolvent $R_0(\omega) = (-\Delta - \omega)^{-1}$.

Denote $R^{\pm}(\lambda^2) = R(\lambda^2 \pm i0)$, $R_0^{\pm}(\lambda^2) = R_0(\lambda^2 \pm i0)$, $\lambda \geq 0$. The integral kernel of $R_0^{\pm}(\lambda^2)$ has an explicit representation

$$R_0^{\pm}(\lambda^2) = \frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}, \quad \lambda \geq 0. \quad (13)$$

The asymptotics (??)–(??) imply

Lemma 2.1. *Let (??) hold with $\beta > 3$. Then in the regular case*

$$\|\partial_{\lambda}^k R^{\pm}(\lambda^2)\|_{L^2_{\sigma_1} \rightarrow L^2_{-\sigma_2}} \leq C(1 + \lambda)^{-1}, \quad \lambda \geq 0, \quad (14)$$

where $\sigma_1, \sigma_2 > 1/2$, $\sigma_1 + \sigma_2 > 2$ for $k = 0$, and $\sigma_1, \sigma_2 > 1/2 + k$ for $k = 1, 2$.

Proof. First, we have

$$\|\partial_{\lambda}^k R^{\pm}(\lambda^2)\|_{L^2_{\sigma_1} \rightarrow L^2_{-\sigma_2}} = \mathcal{O}(1), \quad \lambda \rightarrow 0, \quad (15)$$

where $\sigma_1, \sigma_2 > 1/2$, $\sigma_1 + \sigma_2 > 2$ for $k = 0$, and $\sigma_1, \sigma_2 > 1/2 + k$ for $k = 1, 2$. Asymptotics (??) with $k = 0$ follow from (??). For $k = 1, 2$ we apply the formulas (see for example [?, Formulas (17.9), (17.11)])

$$R' = R'_0 + RV R'_0 + R'_0 V R + RV R'_0 V R,$$

$$R'' = R''_0 + RV R''_0 + R''_0 V R + RV R''_0 V R + 2R' V R'_0 + 2R' V R'_0 V R.$$

Similarly, asymptotics (??) implies

$$\|\frac{\partial^k}{\partial \lambda^k} R^{\pm}(\lambda^2)\|_{L^2_{\sigma_1} \rightarrow L^2_{-\sigma_2}} = \mathcal{O}(\lambda^{-1}), \quad \lambda \rightarrow \infty, \quad \sigma_1, \sigma_2 > 1/2 + k, \quad k = 0, 1, 2.$$

□

3. The Born series. Iteration the Schrödinger resolvent identity yields the finite Born series

$$R^{\pm} = \sum_{k=0}^N (R_0^{\pm} V)^k R_0^{\pm} + (R_0^{\pm} V)^N R^{\pm} V R_0^{\pm}, \quad N \geq 0. \quad (16)$$

For any $N \geq 1$ and $\lambda \geq 0$ denote

$$\begin{aligned} \Pi_N^{\pm}(\lambda) &= (R_0^{\pm}(\lambda^2) V)^{N-1} R^{\pm}(\lambda^2), \\ \Lambda_N^{\pm}(\lambda) &= R_0^{\pm}(\lambda^2) V \Pi_N^{\pm}(\lambda) V R_0^{\pm}(\lambda^2) = (R_0^{\pm}(\lambda^2) V)^N R^{\pm}(\lambda^2) V R_0^{\pm}(\lambda^2). \end{aligned} \quad (17)$$

Here we prove some properties of Π_N^{\pm} and Λ_N^{\pm} which we will need below. We denote by $a+$ any number $a + \varepsilon$ with an arbitrary small, but fixed $\varepsilon > 0$.

Lemma 3.1. *Let (??) hold with some $\beta > 2$. Then in the regular case,*

$$\|\Pi_N^+(\lambda) - \Pi_N^-(\lambda)\|_{L^2_{1+} \rightarrow L^2_{-1-}} \rightarrow 0, \quad \lambda \rightarrow 0, \quad N \geq 1. \quad (18)$$

Proof. We have

$$\begin{aligned} \|R_0^+(\lambda^2) - R_0^-(\lambda^2)\|_{L_{1+}^2 \rightarrow L_{-1-}^2} &\rightarrow 0, \\ \|R^+(\lambda^2) - R^-(\lambda^2)\|_{L_{1+}^2 \rightarrow L_{-1-}^2} &\rightarrow 0, \quad \lambda \rightarrow 0 \end{aligned}$$

by **R4**. It remains to note, that $V : L_{-1-} \rightarrow L_{1+}$ is bounded for $\beta > 2$. \square

Lemma 3.2. *Let (??) hold with some $\beta > 3$ and $N \geq 1$. Then in the regular case,*

$$\begin{aligned} \|\Pi_N^\pm(\lambda)\|_{L_{\sigma_1}^2 \rightarrow L_{-\sigma_2}^2} + \|\partial_\lambda \Pi_N^\pm(\lambda)\|_{L_{\frac{3}{2}+}^2 \rightarrow L_{-\frac{3}{2}-}^2} + \|\partial_\lambda^2 \Pi_N^\pm(\lambda)\|_{L_{\frac{5}{2}+}^2 \rightarrow L_{-\frac{5}{2}-}^2} \\ \leq C(1+\lambda)^{-N}, \quad \sigma_1, \sigma_2 > 1/2, \quad \sigma_1 + \sigma_2 > 2. \end{aligned} \quad (19)$$

Proof. For $N = 1$, (??) follows by (??). In the case $N = 2$, we get

$$\begin{aligned} \|\Pi_2^\pm(\lambda)\|_{L_{\sigma_1}^2 \rightarrow L_{-\sigma_2}^2} &\leq \|R_0^\pm(\lambda^2)\|_{L_{\frac{3}{2}+}^2 \rightarrow L_{-\sigma_2}^2} \|V\|_{L_{-\frac{3}{2}-}^2 \rightarrow L_{\frac{3}{2}+}^2} \|R^\pm(\lambda^2)\|_{L_{\sigma_1}^2 \rightarrow L_{-\frac{3}{2}-}^2} \\ &\leq C(1+\lambda)^{-2} \end{aligned}$$

by (??). Further, (??) implies

$$\begin{aligned} &\|\partial_\lambda \Pi_2^\pm(\lambda)\|_{L_{\frac{3}{2}+}^2 \rightarrow L_{-\frac{3}{2}-}^2} \\ &\leq \|\partial_\lambda R_0^\pm(\lambda^2)\|_{L_{\frac{3}{2}+}^2 \rightarrow L_{-\frac{3}{2}-}^2} \|V\|_{L_{-\frac{3}{2}-}^2 \rightarrow L_{\frac{3}{2}+}^2} \|R^\pm(\lambda^2)\|_{L_{\frac{3}{2}+}^2 \rightarrow L_{-\frac{3}{2}-}^2} \\ &\quad + \|R_0^\pm(\lambda^2)\|_{L_{\frac{3}{2}+}^2 \rightarrow L_{-\frac{3}{2}-}^2} \|V\|_{L_{-\frac{3}{2}-}^2 \rightarrow L_{\frac{3}{2}+}^2} \|\partial_\lambda R^\pm(\lambda^2)\|_{L_{\frac{3}{2}+}^2 \rightarrow L_{-\frac{3}{2}-}^2} \leq C(1+\lambda)^{-2}, \\ &\|\partial_\lambda^2 \Pi_2^\pm(\lambda)\|_{L_{\frac{5}{2}+}^2 \rightarrow L_{-\frac{5}{2}-}^2} \\ &\leq \|\partial_\lambda^2 R_0^\pm(\lambda^2)\|_{L_{\frac{5}{2}+}^2 \rightarrow L_{-\frac{5}{2}-}^2} \|V\|_{L_{-\frac{5}{2}-}^2 \rightarrow L_{\frac{5}{2}+}^2} \|R^\pm(\lambda^2)\|_{L_{\frac{5}{2}+}^2 \rightarrow L_{-\frac{5}{2}-}^2} \\ &\quad + \|\partial_\lambda R_0^\pm(\lambda^2)\|_{L_{\frac{3}{2}+}^2 \rightarrow L_{-\frac{5}{2}-}^2} \|V\|_{L_{-\frac{3}{2}-}^2 \rightarrow L_{\frac{3}{2}+}^2} \|\partial_\lambda R^\pm(\lambda^2)\|_{L_{\frac{5}{2}+}^2 \rightarrow L_{-\frac{3}{2}-}^2} \\ &\quad + \|R_0^\pm(\lambda^2)\|_{L_{\frac{1}{2}+}^2 \rightarrow L_{-\frac{5}{2}-}^2} \|V\|_{L_{-\frac{5}{2}-}^2 \rightarrow L_{\frac{1}{2}+}^2} \|\partial_\lambda^2 R^\pm(\lambda^2)\|_{L_{\frac{5}{2}+}^2 \rightarrow L_{-\frac{5}{2}-}^2} \leq C(1+\lambda)^{-2}. \end{aligned}$$

For $N \geq 2$, (??) follows by induction. \square

Lemma 3.3. *Let (??) hold with some $\beta > 3$. Then for $N \geq 2$, in the regular case,*

$$\|\Lambda_N^+(\lambda) - \Lambda_N^-(\lambda)\|_{L^1 \rightarrow L^\infty} \rightarrow 0, \quad \|\nabla(\Lambda_N^+(\lambda) - \Lambda_N^-(\lambda))\|_{L^1 \rightarrow L^\infty} \rightarrow 0, \quad \lambda \rightarrow 0. \quad (20)$$

Proof. 1) Let us prove the first limit (??). We split $\Lambda_N^+ - \Lambda_N^-$ as

$$\Lambda_N^+ - \Lambda_N^- = (R_0^+ - R_0^-)V\Pi_N^+VR_0^+ + R_0^-V(\Pi_N^+ - \Pi_N^-)VR_0^+ + R_0^-V\Pi_N^-V(R_0^+ - R_0^-).$$

Then, for any $f, g \in L^1$, we obtain

$$\begin{aligned} &|\langle f, (\Lambda_N^+(\lambda) - \Lambda_N^-(\lambda))g \rangle| \leq |\langle V(R_0^-(\lambda^2) - R_0^+(\lambda^2))f, \Pi_N^+(\lambda)VR_0^+(\lambda^2)g \rangle| \\ &\quad + |\langle VR_0^+(\lambda^2)f, (\Pi_N^+(\lambda) - \Pi_N^-(\lambda))VR_0^+(\lambda^2)g \rangle| \\ &\quad + |\langle VR_0^+(\lambda^2)f, \Pi_N^-(\lambda)V(R_0^+(\lambda^2) - R_0^-(\lambda^2))g \rangle| \\ &\leq \|V(R_0^-(\lambda^2) - R_0^+(\lambda^2))f\|_{L_{1+}^2} \|\Pi_N^+(\lambda)\|_{L_{1+}^2 \rightarrow L_{-1-}^2} \|VR_0^+(\lambda^2)g\|_{L_{1+}^2} \\ &\quad + \|VR_0^+(\lambda^2)f\|_{L_{1+}^2} \|(\Pi_N^+(\lambda) - \Pi_N^-(\lambda))\|_{L_{1+}^2 \rightarrow L_{-1-}^2} \|VR_0^+(\lambda^2)g\|_{L_{1+}^2} \\ &\quad + \|VR_0^+(\lambda^2)f\|_{L_{1+}^2} \|\Pi_N^-(\lambda)\|_{L_{1+}^2 \rightarrow L_{-1-}^2} \|V(R_0^+(\lambda^2) - R_0^-(\lambda^2))g\|_{L_{1+}^2} \end{aligned}$$

since $(R_0^\pm)^* = R_0^\mp$. For any $0 \leq \sigma \leq \beta - 1/2$, we have

$$\begin{aligned} \|VR_0^\pm(\lambda^2)f\|_{L_x^2}^2 &\leq \int V^2(x)\langle x \rangle^{2\sigma} \left| \int R_0^\pm(\lambda^2, x, y)f(y)dy \right|^2 dx \\ &\leq C \iint |f(y_1)||f(y_2)| \left(\int \frac{\langle x \rangle^{2\sigma-2\beta}}{|x-y_1||x-y_2|} dx \right) dy_1 dy_2 \leq C_1 \|f\|_{L^1}^2. \end{aligned} \tag{21}$$

Similarly, for any $0 \leq \sigma \leq \beta - 3/2$, we have

$$\begin{aligned} \|V(R_0^\pm(\lambda^2) - R_0^\mp(\lambda^2))f\|_{L_x^2}^2 &\leq C\lambda^2 \iiint \langle x \rangle^{2\sigma-2\beta} |f(y_1)||f(y_2)| dx dy_1 dy_2 \\ &\leq C_1 \lambda^2 \|f\|_{L^1}^2 \end{aligned} \tag{22}$$

since

$$\sup_{x,y \in \mathbb{R}^3} |R_0^+(\lambda^2, x, y) - R_0^-(\lambda^2, x, y)| \leq C\lambda. \tag{23}$$

Hence, the first limit (??) follows by Lemma ??.

2) Now we prove the second limit (??). We have

$$\sup_{x,y \in \mathbb{R}^3} |\nabla_x(R_0^+(\lambda^2, x, y) - R_0^-(\lambda^2, x, y))| \leq C\lambda^2.$$

Hence, similarly to (??), we obtain for any $0 \leq \sigma \leq \beta - 3/2$

$$\|V\nabla(R_0^\pm(\lambda^2) - R_0^\mp(\lambda^2))f\|_{L_x^2} \leq C\lambda^2 \|f\|_{L^1}. \tag{24}$$

Further,

$$\begin{aligned} \|V\nabla R_0^\pm(\lambda^2)f\|_{L^1} &\leq C \int |V(x)| \left(\int \left(\frac{\lambda}{|x-y|} + \frac{1}{|x-y|^2} \right) |f(y)| dy \right) dx \\ &= \int |f(y)| \left(\int \left(\frac{\lambda}{|x-y|} + \frac{1}{|x-y|^2} \right) |V(x)| dx \right) dy \\ &\leq C(1+\lambda) \|f\|_{L^1} \end{aligned} \tag{25}$$

Hence, (??) implies for any $0 \leq \sigma < \beta - 1/2$

$$\|VR_0^\pm(\lambda^2)V\nabla R_0^\pm(\lambda^2)f\|_{L_x^2} \leq C \|V\nabla R_0^\pm(\lambda^2)f\|_{L^1} \leq C_1(1+\lambda) \|f\|_{L^1}. \tag{26}$$

Similarly, (??) and (??) imply for any $0 \leq \sigma \leq \beta - 3/2$

$$\begin{aligned} \|V(R_0^-(\lambda^2) - R_0^+(\lambda^2))V\nabla R_0^\pm(\lambda^2)f\|_{L_x^2} &\leq C\lambda \|V\nabla R_0^\pm(\lambda^2)f\|_{L^1} \\ &\leq C_1\lambda(1+\lambda) \|f\|_{L^1}. \end{aligned} \tag{27}$$

Taking into account (??), (??)-(??), we obtain for any $f, g \in L^1$

$$\begin{aligned} &|\langle f, \nabla(\Lambda_N^+(\lambda) - \Lambda_N^-(\lambda))g \rangle| \\ &\leq \|V\nabla(R_0^-(\lambda^2) - R_0^+(\lambda^2))f\|_{L_{1+}^2} \|\Pi_N^+(\lambda)\|_{L_{1+}^2 \rightarrow L_{2-}^2} \|VR_0^+(\lambda^2)g\|_{L_{1+}^2} \\ &\quad + \|V(R_0^-(\lambda^2) - R_0^+(\lambda^2))V\nabla R_0^+(\lambda^2)f\|_{L_{1+}^2} \|\Pi_{N-1}^+(\lambda)\|_{L_{1+}^2 \rightarrow L_{2-}^2} \|VR_0^+(\lambda^2)g\|_{L_{1+}^2} \\ &\quad + \|VR_0^+(\lambda^2)V\nabla R_0^+(\lambda^2)f\|_{L_{1+}^2} \|\Pi_{N-1}^+(\lambda) - \Pi_{N-1}^-(\lambda)\|_{L_{1+}^2 \rightarrow L_{2-}^2} \|VR_0^+(\lambda^2)g\|_{L_{1+}^2} \\ &\quad + \|VR_0^+(\lambda^2)V\nabla R_0^+(\lambda^2)f\|_{L_{1+}^2} \|\Pi_{N-1}^-(\lambda)\|_{L_{1+}^2 \rightarrow L_{2-}^2} \|V(R_0^+(\lambda^2) - R_0^-(\lambda^2))g\|_{L_{1+}^2} \\ &\leq C(1+\lambda) \|f\|_{L^1} \|g\|_{L^1} \left(\lambda + \|\Pi_{N-1}^+ - \Pi_{N-1}^-(\lambda)\|_{L_{1+}^2 \rightarrow L_{2-}^2} \right) \rightarrow 0, \quad \lambda \rightarrow 0. \end{aligned}$$

□

4. **Decay of $\Lambda_N^\pm(\lambda)$.** In this and the next sections we omit the signs \pm not to overburden the exposition. For example, $R_0(\lambda^2)$ means $R_0^+(\lambda^2)$ or $R_0^-(\lambda^2)$, $\Lambda_N(\lambda) = \Lambda_N^\pm(\lambda)$, etc.

Lemma 4.1. *Let (??) hold with some $\beta > 3$. Then for $N \geq 1$, in the regular case,*

$$\|\partial_\lambda^k \Lambda_N(\lambda)\|_{L^1 \rightarrow L^\infty} \leq C(1 + \lambda)^{-N}, \quad \lambda \geq 0, \quad k = 0, 1. \quad (28)$$

Proof. The estimates (??) and (??) imply for any $f, g \in L^1$ and $k = 0, 1$

$$\begin{aligned} & |\langle f, R_0(\lambda^2) V \partial_\lambda^k \Pi_N(\lambda) V R_0(\lambda^2) g \rangle| = |\langle V R_0^*(\lambda^2) f, \partial_\lambda^k \Pi_N(\lambda) V R_0(\lambda^2) g \rangle| \\ & \leq \|V R_0^*(\lambda^2) f\|_{L^2_{\frac{3}{2}+}} \|\partial_\lambda^k \Pi_N(\lambda)\|_{L^2_{\frac{3}{2}+} \rightarrow L^2_{-\frac{3}{2}-}} \|V R_0(\lambda^2) g\|_{L^2_{\frac{3}{2}+}} \\ & \leq C(1 + \lambda)^{-N} \|f\|_{L^1} \|g\|_{L^1}. \end{aligned} \quad (29)$$

Further, for any $0 \leq \sigma \leq \beta - 3/2$, we obtain

$$\|V \partial_\lambda R_0(\lambda^2) f\|_{L^2_\sigma}^2 \leq C \int \langle x \rangle^{2\sigma-2\beta} \left(\int |f(x_1)| dx_1 \right)^2 dx \leq C_1 \|f\|_{L^1}^2. \quad (30)$$

Hence, (??) and (??) imply for any $f, g \in L^1$

$$\begin{aligned} & |\langle f, \partial_\lambda R_0(\lambda^2) V \Pi_N(\lambda) V R_0(\lambda^2) g \rangle| = |\langle V \partial_\lambda R_0^*(\lambda^2) f, \Pi_N(\lambda) V R_0(\lambda^2) g \rangle| \\ & \leq \|V \partial_\lambda R_0^*(\lambda^2) f\|_{L^2_{1+}} \|\Pi_N(\lambda)\|_{L^2_{1+} \rightarrow L^2_{-1-}} \|V R_0(\lambda^2) g\|_{L^2_{1+}} \\ & \leq C(1 + \lambda)^{-N} \|f\|_{L^1} \|g\|_{L^1}. \end{aligned}$$

Similarly,

$$|\langle f, R_0(\lambda^2) V \Pi_N(\lambda) V \partial_\lambda R_0(\lambda^2) g \rangle| \leq C(1 + \lambda)^{-N} \|f\|_{L^1} \|g\|_{L^1}.$$

Then (??) follows by definition (??) of Λ_N . \square

Lemma 4.2. *Let (??) hold with some $\beta > 3$. Then for $N \geq 2$, in the regular case,*

$$\|\partial_\lambda^k \nabla \Lambda_N(\lambda)\|_{L^1 \rightarrow L^\infty} \leq C(1 + \lambda)^{-N+2}, \quad \lambda \geq 0, \quad k = 0, 1. \quad (31)$$

Proof. First, we prove some auxiliary estimates. The estimates (??), (??) and (??) imply

$$\begin{aligned} & |\langle f, \nabla R_0(\lambda^2) V \partial_\lambda^k \Pi_N(\lambda) V R_0(\lambda^2) g \rangle| \\ & = |\langle V R_0^*(\lambda^2) V \nabla R_0^*(\lambda^2) f, \partial_\lambda^k \Pi_{N-1}(\lambda) V R_0(\lambda^2) g \rangle| \\ & \leq C \|V R_0^*(\lambda^2) V \nabla R_0^*(\lambda^2) f\|_{L^2_{\frac{3}{2}+}} \|\partial_\lambda^k \Pi_{N-1}(\lambda)\|_{L^2_{\frac{3}{2}+} \rightarrow L^2_{-\frac{3}{2}-}} \|V R_0(\lambda^2) g\|_{L^2_{\frac{3}{2}+}} \\ & \leq C(1 + \lambda)^{-N+2} \|f\|_{L^1} \|g\|_{L^1}, \quad k = 0, 1, 2. \end{aligned} \quad (32)$$

Similarly, (??), (??) and (??) imply

$$\begin{aligned} & |\langle f, \nabla R_0(\lambda^2) V \partial_\lambda^k \Pi_N(\lambda) V \partial_\lambda R_0(\lambda^2) g \rangle| \\ & \leq C \|V R_0^*(\lambda^2) V \nabla R_0^*(\lambda^2) f\|_{L^2_{\frac{3}{2}+}} \|\partial_\lambda^k \Pi_{N-1}(\lambda)\|_{L^2_{\frac{3}{2}+} \rightarrow L^2_{-\frac{3}{2}-}} \|V \partial_\lambda R_0(\lambda^2) g\|_{L^2_{\frac{3}{2}+}} \\ & \leq C(1 + \lambda)^{-N+2} \|f\|_{L^1} \|g\|_{L^1}, \quad k = 0, 1. \end{aligned} \quad (33)$$

Further, for any $0 < \sigma \leq \beta - 3/2$,

$$\begin{aligned} & \|V \partial_\lambda \nabla R_0(\lambda^2) f\|_{L^2_\sigma}^2 \leq C \int V^2(x) \langle x \rangle^{2\sigma} \left(\int \left(\lambda + \frac{1}{|x-y|} \right) |f(y)| dy \right)^2 dx \\ & \leq C(1 + \lambda)^2 \iint |f(y_1)| |f(y_2)| \left(\int \langle x \rangle^{2\sigma-2\beta} \left(1 + \frac{1}{|x-y_1|} \right) \left(1 + \frac{1}{|x-y_2|} \right) dx \right) dy_1 dy_2 \\ & \leq C(1 + \lambda)^2 \|f\|_{L^1}^2. \end{aligned} \quad (34)$$

Then, (??), (??) and (??) imply

$$\begin{aligned} & |\langle f, \partial_\lambda \nabla R_0(\lambda^2) V \Pi_N(\lambda) V R_0(\lambda^2) g \rangle| \\ & \leq C \|V \partial_\lambda \nabla R_0^*(\lambda^2) f\|_{L^2_{\frac{3}{2}+}} \|\partial_\lambda^k \Pi_N(\lambda)\|_{L^2_{\frac{3}{2}+} \rightarrow L^2_{-\frac{3}{2}-}} \|V R_0(\lambda^2) g\|_{L^2_{\frac{3}{2}+}} \\ & \leq C(1 + \lambda)^{-N+1} \|f\|_{L^1} \|g\|_{L^1}, \quad k = 0, 1. \end{aligned} \tag{35}$$

It remains to note that (??) for $\nabla \Lambda_N$ follows from (??) with $k = 0$, and (??) for $\partial_\lambda \nabla \Lambda_N$ follows from (??) with $k = 1$ and (??), (??) with $k = 0$. \square

Now we prove the decay of some related functions which we will use below. For $\lambda \geq 0$, denote

$$\begin{aligned} \tilde{\Lambda}_N^\pm(\lambda) &= R_0^\pm(\lambda^2) V \partial_\lambda \Pi_{N-1}^\pm(\lambda) V R_0^\pm(\lambda^2), \\ \hat{\Lambda}_N^\pm(\lambda) &= (G^\mp(\lambda))^* V \Pi_N^\pm(\lambda) V R_0^\pm(\lambda^2), \\ \check{\Lambda}_N^\pm(\lambda) &= R_0^\pm(\lambda^2) V \Pi_N^\pm(\lambda) V G^\pm(\lambda), \end{aligned} \tag{36}$$

where $G^\pm(\lambda)$ is the operator with the kernel

$$G^\pm(\lambda, x, y) = \mp \frac{e^{\pm i\lambda(|x-y|-|y|)}}{4\pi i}, \quad \lambda \geq 0.$$

Lemma 4.3. *Let (??) hold with some $\beta > 3$. Then for $N \geq 1$, in the regular case, $\|\partial_\lambda^k \tilde{\Lambda}_N(\lambda)\|_{L^1 \rightarrow L^\infty} + \|\partial_\lambda^k \hat{\Lambda}_N(\lambda)\|_{L^1 \rightarrow L^\infty} + \|\partial_\lambda^k \check{\Lambda}_N(\lambda)\|_{L^1 \rightarrow L^\infty} \leq C(1 + \lambda)^{-N}$, $k = 0, 1$.*

Proof. 1) First consider the first summand. The required estimate for $\tilde{\Lambda}_N$ is exactly (??) with $k = 1$, hence it remains to consider $\partial_\lambda \tilde{\Lambda}_N$. The estimates (??), (??) and (??) imply

$$\begin{aligned} |\langle f, \partial_\lambda \tilde{\Lambda}_N(\lambda) g \rangle| &\leq \|V \partial_\lambda R_0^*(\lambda^2) f\|_{L^2_{\frac{3}{2}+}} \|\partial_\lambda \Pi_N(\lambda)\|_{L^2_{\frac{3}{2}+} \rightarrow L^2_{-\frac{3}{2}-}} \|V R_0(\lambda^2) g\|_{L^2_{\frac{3}{2}+}} \\ &+ \|V R_0^*(\lambda^2) f\|_{L^2_{\frac{3}{2}+}} \|\partial_\lambda \Pi_N(\lambda)\|_{L^2_{\frac{3}{2}+} \rightarrow L^2_{-\frac{3}{2}-}} \|V \partial_\lambda R_0(\lambda^2) g\|_{L^2_{\frac{3}{2}+}} \\ &+ \|V R_0^*(\lambda^2) f\|_{L^2_{\frac{3}{2}+}} \|\partial_\lambda^2 \Pi_N(\lambda)\|_{L^2_{\frac{3}{2}+} \rightarrow L^2_{-\frac{5}{2}-}} \|V R_0(\lambda^2) g\|_{L^2_{\frac{3}{2}+}} \\ &\leq C(1 + \lambda)^{-N} \|f\|_{L^1} \|g\|_{L^1}. \end{aligned}$$

2) Now consider $\hat{\Lambda}_N$ and $\check{\Lambda}_N$. Note that

$$|\partial_\lambda^k G(\lambda, x, y)| \leq |x|^k / (4\pi), \quad k = 0, 1, 2, \dots$$

Then similarly to (??), we obtain for $k = 0, 1$ and $0 \leq \sigma \leq \beta - k - 3/2$

$$\|V \partial_\lambda^k G(\lambda) f\|_{L^2_\sigma} \leq C \|f\|_{L^1}. \tag{37}$$

Hence, (??), (??) and (??) imply

$$\begin{aligned} |\langle f, \hat{\Lambda}_N(\lambda) g \rangle| &\leq \|V G(\lambda^2) f\|_{L^2_{1+}} \|\Pi_N(\lambda)\|_{L^2_{1+} \rightarrow L^2_{-1-}} \|V R_0(\lambda^2) g\|_{L^2_{1+}} \\ &\leq C(1 + \lambda)^{-N} \|f\|_{L^1} \|g\|_{L^1}. \\ |\langle f, \partial_\lambda \hat{\Lambda}_N(\lambda) g \rangle| &\leq \|V \partial_\lambda G(\lambda^2) f\|_{L^2_{\frac{1}{2}+}} \|\Pi_N(\lambda)\|_{L^2_{\frac{3}{2}+} \rightarrow L^2_{-\frac{1}{2}-}} \|V R_0(\lambda^2) g\|_{L^2_{\frac{3}{2}+}} \\ &+ \|V G(\lambda^2) f\|_{L^2_{1+}} \|\Pi_N(\lambda)\|_{L^2_{1+} \rightarrow L^2_{-1-}} \|V \partial_\lambda R_0(\lambda^2) g\|_{L^2_{1+}} \\ &+ \|V G(\lambda^2) f\|_{L^2_{\frac{3}{2}+}} \|\partial_\lambda \Pi_N(\lambda)\|_{L^2_{\frac{3}{2}+} \rightarrow L^2_{-\frac{3}{2}-}} \|V R_0(\lambda^2) g\|_{L^2_{\frac{3}{2}+}} \\ &\leq C(1 + \lambda)^{-N} \|f\|_{L^1} \|g\|_{L^1}. \end{aligned}$$

The estimates for $\check{\Lambda}_N$ can be obtained similarly. \square

Corollary 1. *Under the conditions of Lemma ??, the integral kernels $\partial_\lambda^k \check{\Lambda}_N(\lambda, x, y)$, $\partial_\lambda^k \hat{\Lambda}_N(\lambda, x, y)$ and $\partial_\lambda^k \tilde{\Lambda}_N(\lambda, x, y)$ of the operators $\partial_\lambda^k \check{\Lambda}_N(\lambda)$, $\partial_\lambda^k \hat{\Lambda}_N(\lambda)$ and $\partial_\lambda^k \tilde{\Lambda}_N(\lambda)$ respectively, belong to $L^\infty(\mathbb{R}^6)$, and*

$$\|\partial_\lambda^k \check{\Lambda}_N(\lambda, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^6)} + \|\partial_\lambda^k \hat{\Lambda}_N(\lambda, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^6)} + \|\partial_\lambda^k \tilde{\Lambda}_N(\lambda, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^6)} \leq C(1+\lambda)^{-N}.$$

Proof. The distributional kernel $A(x, y)$ of any bounded linear operator $A: L^1(\mathbb{R}^3) \rightarrow L^\infty(\mathbb{R}^3)$ belongs to $L^\infty(\mathbb{R}^6)$, and

$$\|A(\cdot, \cdot)\|_{L^\infty(\mathbb{R}^6)} = \|A\|_{L^1(\mathbb{R}^3) \rightarrow L^\infty(\mathbb{R}^3)}.$$

This follows from the estimate $|\langle A, \phi \rangle| \leq C\|\phi\|_{L^1(\mathbb{R}^6)}$ for $\phi \in L^1(\mathbb{R}^6)$ and from the duality $(L^1(\mathbb{R}^6))^* = L^\infty(\mathbb{R}^6)$. \square

Lemma 4.4. *Let (??) hold with some $\beta > 3$. Then for $N \geq 2$, in the regular case,*

$$\begin{aligned} & \|\partial_\lambda^k \nabla \tilde{\Lambda}_N(\lambda)\|_{L^1 \rightarrow L^\infty} + \|\partial_\lambda^k \nabla \hat{\Lambda}_N(\lambda)\|_{L^1 \rightarrow L^\infty} + \|\partial_\lambda^k \nabla \check{\Lambda}_N(\lambda)\|_{L^1 \rightarrow L^\infty} \\ & \leq C(1+\lambda)^{-N+2}, \quad k = 0, 1. \end{aligned} \quad (38)$$

Proof. 1) The estimate for the first summand in (??) follows from (??) with $k = 1, 2$ and (??), (??) with $k = 1$.

3) Consider the second summand in (??). Note that, $|\nabla_x G(\lambda, x, y)| \leq |\lambda|/(4\pi)$, and then $\|V\nabla G(\lambda)f\|_{L_\sigma^2} \leq C\lambda\|f\|_{L^1}$ for $0 \leq \sigma \leq \beta - 3/2$. Hence, (??) and (??) imply

$$\begin{aligned} & |\langle f, \nabla G_*(\lambda)V\partial_\lambda^k \Pi_N(\lambda)VR_0(\lambda^2)g \rangle| \\ & \leq C\|V\nabla G(\lambda)f\|_{L_{\frac{3}{2}+}^2} \|\partial_\lambda^k \Pi_N(\lambda)\|_{L_{\frac{3}{2}+}^2 \rightarrow L_{-\frac{3}{2}-}^2} \|VR_0(\lambda^2)g\|_{L_{\frac{3}{2}+}^2} \\ & \leq C_1(1+\lambda)^{-N+1}\|f\|_{L^1}\|g\|_{L^1}, \quad k = 0, 1. \end{aligned}$$

Here we denote $G_*^\pm = (G^\mp)^*$. Similarly, (??) and (??) imply

$$\begin{aligned} & |\langle f, \nabla G_*(\lambda)V\Pi_N(\lambda)V\partial_\lambda R_0(\lambda^2)g \rangle| \\ & \leq C\|V\nabla G(\lambda)f\|_{L_{1+}^2} \|\Pi_N(\lambda)\|_{L_{1+}^2 \rightarrow L_{-1-}^2} \|V\partial_\lambda R_0(\lambda^2)g\|_{L_{1+}^2} \\ & \leq C_1(1+\lambda)^{-N+1}\|f\|_{L^1}\|g\|_{L^1}, \quad k = 0, 1. \end{aligned}$$

Further, $|\partial_\lambda \nabla_x G(\lambda, x, y)| \leq C\lambda|x|$. Then for any $0 < \sigma \leq \beta - 5/2$, we have

$$\|V\partial_\lambda \nabla G(\lambda)f\|_{L_\sigma^2}^2 \leq C\lambda^2 \int \langle x \rangle^{2\sigma+2-2\beta} \left(\int |f(y)|dy \right)^2 dx \leq C_1\lambda^2\|f\|_{L^1}^2.$$

Hence,

$$\begin{aligned} & |\langle f, \partial_\lambda \nabla G_*(\lambda)V\Pi_N(\lambda)VR_0(\lambda^2)g \rangle| \\ & \leq C\|V\partial_\lambda \nabla G(\lambda)f\|_{L_{\frac{3}{2}+}^2} \|\Pi_N(\lambda)\|_{L_{\frac{3}{2}+}^2 \rightarrow L_{-\frac{1}{2}-}^2} \|VR_0(\lambda^2)g\|_{L_{\frac{3}{2}+}^2} \\ & \leq C_1(1+\lambda)^{-N+1}\|f\|_{L^1}\|g\|_{L^1}. \end{aligned}$$

2) It remains to consider the third summand in (??). The estimates (??), (??) and (??) imply

$$\begin{aligned} & |\langle f, \nabla R_0(\lambda^2)V\partial_\lambda^k \Pi_N(\lambda)VG(\lambda)g \rangle| \\ & \leq C\|VR_0^*(\lambda^2)V\nabla R_0^*(\lambda^2)f\|_{L_{\frac{3}{2}+}^2} \|\partial_\lambda^k \Pi_{N-1}(\lambda)\|_{L_{\frac{3}{2}+}^2 \rightarrow L_{-\frac{3}{2}-}^2} \|VG(\lambda)g\|_{L_{\frac{3}{2}+}^2} \\ & \leq C_1(1+\lambda)^{-N+2}\|f\|_{L^1}\|g\|_{L^1}, \quad k = 0, 1. \end{aligned}$$

$$\begin{aligned} & |\langle f, \nabla R_0(\lambda^2) V \Pi_N(\lambda) V \partial_\lambda G(\lambda) g \rangle| \\ & \leq C \|V R_0(\lambda^2) V \nabla R_0(\lambda^2) f\|_{L^2_{\frac{3}{2}+}} \|\Pi_{N-1}(\lambda)\|_{L^2_{\frac{1}{2}+} \rightarrow L^2_{-\frac{3}{2}-}} \|V \partial_\lambda G(\lambda) g\|_{L^2_{\frac{1}{2}+}} \\ & \leq C_1 (1 + \lambda)^{-N+2} \|f\|_{L^1} \|g\|_{L^1}. \end{aligned}$$

Finally, (??) and (??) imply

$$\begin{aligned} & |\langle f, \partial_\lambda \nabla R_0(\lambda^2) V \Pi_N^\pm(\lambda) V G(\lambda) g \rangle| \\ & \leq C \|V \partial_\lambda \nabla R_0(\lambda^2) f\|_{L^2_{1+}} \|\Pi_N(\lambda)\|_{L^2_{1+} \rightarrow L^2_{1-}} \|V G(\lambda^2) g\|_{L^2_{1+}} \\ & \leq C (1 + \lambda)^{-N+1} \|f\|_{L^1} \|g\|_{L^1}. \end{aligned}$$

□

Corollary 2. *Under the conditions of Lemma ??, the integral kernels of the operators $\partial_\lambda^k \nabla \hat{\Lambda}_N(\lambda)$, $\partial_\lambda^k \nabla \tilde{\Lambda}_N(\lambda)$ and $\partial_\lambda^k \nabla \check{\Lambda}_N(\lambda)$ belong to $L^\infty(\mathbb{R}^6)$, and*

$$\begin{aligned} & \|\partial_\lambda^k \nabla_x \tilde{\Lambda}_N(\lambda, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^6)} + \|\partial_\lambda^k \nabla_x \hat{\Lambda}_N(\lambda, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^6)} + \|\partial_\lambda^k \nabla_x \check{\Lambda}_N(\lambda, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^6)} \\ & \leq C (1 + \lambda)^{-N+2}. \end{aligned}$$

5. Dispersion decay for the Klein-Gordon equation. The required dispersion decay for the free Klein-Gordon equation has been obtained in [?, Lemma 18.2] (see also [?, ?]). Namely,

$$\|\mathcal{U}_0(t)\|_{E_\sigma \rightarrow E_{-\sigma}} \leq C(1 + |t|)^{-3/2}, \quad t \in \mathbb{R}, \quad \sigma > 3/2. \tag{39}$$

Now consider the perturbed Klein-Gordon equation. We use the representation

$$\mathcal{U}(t) P_c(\mathcal{K}) = \frac{1}{2\pi i} \int_\Gamma e^{-i\omega t} (\mathcal{R}(\omega + i0) - \mathcal{R}(\omega - i0)) \, d\omega.$$

Substituting the Born series

$$\mathcal{R} = \sum_{j=0}^6 (-1)^j (\mathcal{R}_0 \mathcal{V})^j \mathcal{R}_0 - (\mathcal{R}_0 \mathcal{V})^6 \mathcal{R} \mathcal{V} \mathcal{R}_0, \quad \text{where } \mathcal{V}(x) = \begin{pmatrix} 0 & 0 \\ iV(x) & 0 \end{pmatrix}, \tag{40}$$

we obtain

$$\mathcal{U}(t) P_c(\mathcal{K}) = \sum_{j=0}^6 \mathcal{U}_j(t) - \mathcal{W}(t),$$

where

$$\begin{aligned} \mathcal{U}_j(t) &= \frac{1}{2\pi i} \int_\Gamma e^{-i\omega t} ([(\mathcal{R}_0 \mathcal{V})^j \mathcal{R}_0](\omega + i0) - [(\mathcal{R}_0 \mathcal{V})^j \mathcal{R}_0](\omega - i0)) \, d\omega, \quad 0 \leq j \leq 6, \\ \mathcal{W}(t) &= \frac{1}{2\pi i} \int_\Gamma e^{-i\omega t} ([(\mathcal{R}_0 \mathcal{V})^6 \mathcal{R} \mathcal{V} \mathcal{R}_0](\omega + i0) - [(\mathcal{R}_0 \mathcal{V})^6 \mathcal{R} \mathcal{V} \mathcal{R}_0](\omega - i0)) \, d\omega. \end{aligned}$$

Proposition 1. *Let the conditions of Theorem ?? hold. Then*

$$\|\mathcal{U}_j(t)\|_{E_\sigma \rightarrow E_{-\sigma}} \leq C(1 + |t|)^{-3/2}, \quad t \in \mathbb{R}, \quad \sigma > 3/2, \quad 1 \leq j \leq 6. \tag{41}$$

Proof. Let $\Psi_0 \in E_\sigma$ with $\sigma > 3/2$. Denote $\Psi_{j+1}(t) = \mathcal{U}_j(t) \Psi_0$.

Lemma 5.1. (cf. [?, Lemma 3.8], [?, Lemma 36.6]). *The convolution representation holds*

$$\Psi_{j+1}(t) = i \int_0^t \mathcal{U}_0(t - \tau) \mathcal{V} \Psi_j(\tau) \, d\tau, \quad t \in \mathbb{R}, \quad j \geq 1, \tag{42}$$

where the integral converges in $E_{-\sigma}$ with $\sigma > 3/2$.

We have $\|\Psi_1\|_{E_{-\sigma}} \leq C(1+|t|)^{-\frac{3}{2}}\|\Psi_0\|_{E_\sigma}$ by (??). Applying (??) to the integrand in (??) with $j = 1$, we obtain for $\sigma \in (\frac{3}{2}, \frac{\beta}{2}]$

$$\|\mathcal{U}_0(t-\tau)\mathcal{V}\Psi_1(\tau)\|_{E_{-\sigma}} \leq \frac{C\|\mathcal{V}\Psi_1(\tau)\|_{E_\sigma}}{(1+|t-\tau|)^{\frac{3}{2}}} \leq \frac{C_1\|\Psi_1(\tau)\|_{E_{-\sigma}}}{(1+|t-\tau|)^{\frac{3}{2}}} \leq \frac{C_2\|\Psi_0\|_{E_\sigma}}{(1+|t-\tau|)^{\frac{3}{2}}(1+|\tau|)^{\frac{3}{2}}}$$

Therefore, integrating here in τ , we obtain for $\sigma > 3/2$

$$\|\Psi_2(t)\|_{E_{-\sigma}} \leq C(1+|t|)^{-\frac{3}{2}}\|\Psi_0\|_{E_\sigma}.$$

Further, we get by induction

$$\|\Psi_j(t)\|_{E_{-\sigma}} \leq C(1+|t|)^{-\frac{3}{2}}\|\Psi_0\|_{E_\sigma}, \quad \sigma > 3/2, \quad j \geq 3.$$

Hence (??) follows. \square

It remains to prove the decay of type (??) for $\mathcal{W}(t)$.

Theorem 5.2. *Let the conditions of Theorem ?? hold. Then*

$$\|\mathcal{W}(t)\|_{E_\sigma \rightarrow E_{-\sigma}} \leq C|t|^{-\frac{3}{2}}, \quad |t| \geq 1.$$

We prove this theorem in the next two subsections.

5.1. The decay in $L^1 \rightarrow L^\infty$ norm. Here we prove a decay of $\mathcal{W}(t)$ in $L^1 \oplus L^1 \rightarrow L^\infty \oplus L^\infty$ norm.

Proposition 2. *Let the conditions of Theorem ?? hold. Then*

$$\|\mathcal{W}(t)\|_{L^1 \oplus L^1 \rightarrow L^\infty \oplus L^\infty} \leq C|t|^{-\frac{3}{2}}, \quad |t| \geq 1. \quad (43)$$

Proof. The resolvent $\mathcal{R}(\omega) = (\mathcal{K} - \omega)^{-1}$ can be expressed in terms of the Schrödinger resolvent R as follows

$$\mathcal{R}(\omega) = \begin{pmatrix} \omega R(\omega^2 - m^2) & iR(\omega^2 - m^2) \\ -i(1 + \omega^2 R(\omega^2 - m^2)) & \omega R(\omega^2 - m^2) \end{pmatrix}.$$

Then

$$\begin{aligned} \mathcal{W}(t) &= \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} \begin{pmatrix} \omega & i \\ -i\omega^2 & \omega \end{pmatrix} [(R_0 V)^6 R V R_0] ((\omega - i0)^2 - m^2) \\ &\quad - [(R_0 V)^6 R V R_0] ((\omega + i0)^2 - m^2) d\omega \\ &= \frac{1}{2\pi i} \sum_{\pm} \int_0^\infty e^{\mp i\sqrt{\lambda^2 + m^2} t} M_{\pm}(\lambda) (\Lambda_6^-(\lambda) - \Lambda_6^+(\lambda)) \frac{\lambda d\lambda}{\sqrt{\lambda^2 + m^2}}, \end{aligned} \quad (44)$$

where $\Lambda_6^\pm(\lambda)$ is defined in (??), and

$$M_{\pm}(\lambda) = \begin{pmatrix} \pm\sqrt{\lambda^2 + m^2} & i \\ -i(\lambda^2 + m^2) & \pm\sqrt{\lambda^2 + m^2} \end{pmatrix}.$$

Due to Lemma ??, the integrand in (??) is a differentiable operator function of $\lambda \geq 0$ with values in the space of bounded operators mapping L^1 into L^∞ . Moreover, due to Lemmas ?? and ??, we can integrate by parts in (??):

$$\mathcal{W}(t) = \frac{1}{2\pi t} \sum_{\pm} \int_0^\infty e^{\mp i\sqrt{\lambda^2 + m^2} t} \partial_\lambda \left(M_{\pm}(\lambda) (\Lambda_6^+(\lambda) - \Lambda_6^-(\lambda)) \right) d\lambda. \quad (45)$$

In the case when the derivative falls on $M_{\pm}(\lambda)$, we can integrate by parts one more time and get the factor t^{-2} then. Hence, it suffices to consider the case when the derivative falls on Λ_6^{\pm} . More precisely, it suffices to prove that

$$\|\mathcal{I}(t)\|_{L^1 \rightarrow L^\infty} \leq C|t|^{-\frac{1}{2}}, \quad |t| \geq 1,$$

where

$$\mathcal{I}(t) = \int_0^\infty e^{-i\sqrt{\lambda^2+m^2}t} M_+(\lambda) \partial_\lambda \Lambda_6^+(\lambda) d\lambda.$$

All other combination of signs “+” and “-” in (??) can be considered in the same way. Denote

$$\phi_1(\lambda) = \sqrt{\lambda^2 + m^2}, \quad \phi_2(\lambda) = \sqrt{\lambda^2 + m^2} - \frac{\lambda|x|}{t}, \quad \phi_3(\lambda) = \sqrt{\lambda^2 + m^2} - \frac{\lambda|y|}{t}. \quad (46)$$

Then the integral kernel of $\mathcal{I}(t)$ reads,

$$\mathcal{I}(t, x, y) = \int_0^\infty M_+(\lambda) \left(e^{-i\phi_1(\lambda)t} \check{\Lambda}_6^+(\lambda, x, y) + e^{-i\phi_2(\lambda)t} \hat{\Lambda}_6^+(\lambda, x, y) + e^{-i\phi_3(\lambda)t} \check{\Lambda}_6^+(\lambda, x, y) \right) d\lambda.$$

We will use the following version of Van der Corput lemma

Lemma 5.3. (cf. [?, Chapter VIII, Proposition II and Corollary]) *Consider the oscillatory integral*

$$I(t) = \int_a^b e^{it\phi(\lambda)} f(\lambda) d\lambda,$$

where $\phi(\lambda)$ is real-valued function and $f \in C^1([a, b])$. If $\phi''(\lambda) \neq 0$ for $\lambda \in [a, b]$ then

$$|I(t)| \leq Ct^{-\frac{1}{2}} [g(b)|f(b)| + \int_a^b g(\lambda)|f'(\lambda)|d\lambda], \quad t \geq 1.$$

where $g(\lambda) = [\min_{a \leq k \leq \lambda} |\phi''(k)|]^{-\frac{1}{2}}$.

Note, that the lemma remains valid for $a = -\infty$ and $b = \infty$.

The second derivative of the phase functions $\phi_j(\lambda)$, $j = 1, 2, 3$, defined in (??), satisfies

$$\phi_j''(\lambda) = \frac{m^2}{\sqrt{(\lambda^2 + m^2)^3}} > 0, \quad \lambda \in \mathbb{R}, \quad [\phi_j''(\lambda)]^{-\frac{1}{2}} \leq (2\lambda)^{3/2}/m, \quad \lambda \geq m.$$

Moreover, $|M_{\pm}(\lambda)| + |\partial_\lambda M_{\pm}(\lambda)| \leq C(1 + \lambda)^2$. Then Corollary ?? and Lemma ?? imply

$$\|\mathcal{I}(t)\|_{L^1 \rightarrow L^\infty} = \|\mathcal{I}(t, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^6)} \leq C|t|^{-\frac{1}{2}} \int_0^\infty (1 + \lambda)^{-5/2} d\lambda \leq C_1|t|^{-\frac{1}{2}}, \quad |t| \geq 1.$$

□

5.2. The decay of the derivatives.

Proposition 3. *Let the conditions of Theorem ?? hold. Then*

$$\|\nabla \mathcal{W}^{1j}(t)\|_{L^1 \rightarrow L^\infty} \leq C|t|^{-\frac{3}{2}}, \quad |t| \geq 1, \quad j = 1, 2.$$

Here \mathcal{W}^{ij} denotes the ij entry of the matrix operator \mathcal{W} .

Proof. Similarly to (??)

$$\nabla \mathcal{W}^{1j}(t) = \frac{1}{2\pi i} \sum_{\pm} \int_0^{\infty} e^{\mp i\sqrt{\lambda^2+m^2}t} M_{\pm}^{1j}(\lambda) \nabla(\Lambda_6^-(\lambda) - \Lambda_6^+(\lambda)) \frac{\lambda d\lambda}{\sqrt{\lambda^2+m^2}}.$$

By Lemmas ?? and Lemma ??, we can integrate by parts and obtain

$$\nabla \mathcal{W}^{1j}(t) = \frac{1}{2\pi t} \sum_{\pm} \int_0^{\infty} e^{\mp i\sqrt{\lambda^2+m^2}t} \partial_{\lambda} \left(M_{\pm}^{1j}(\lambda) \nabla(\Lambda_6^+(\lambda) - \Lambda_6^-(\lambda)) \right) d\lambda.$$

In the cases when the derivative falls on $M_{\pm}^{1j}(\lambda)$, we can integrate by parts one more time and get the factor t^{-2} then. Hence, it suffices to consider the case when the derivative falls on $\nabla \Lambda_6^{\pm}$. As before, we consider only the “++” case, and prove the decay $\sim |t|^{-1/2}$ in $L^{\infty}(\mathbb{R}^6)$ for the integral kernels of the operators

$$J_j(t) = \int_0^{\infty} e^{-i\sqrt{\lambda^2+m^2}t} M_+^{1j}(\lambda) \partial_{\lambda} \nabla \Lambda_6^+(\lambda) d\lambda.$$

The corresponding integral kernel reads

$$\begin{aligned} J_j(t, x, y) &= \int_0^{\infty} M_+^{1j}(\lambda) \left(e^{-i\phi_1(\lambda)t} \nabla \tilde{\Lambda}_6^+(\lambda, x, y) + e^{-i\phi_2(\lambda)t} \nabla \hat{\Lambda}_6^+(\lambda, x, y) \right. \\ &\quad \left. + e^{-i\phi_3(\lambda)t} \nabla \check{\Lambda}_6^+(\lambda, x, y) \right) d\lambda. \end{aligned}$$

Applying Corollary ?? and Lemma ??, we obtain

$$\|J_j(t)\|_{L^{\infty}(\mathbb{R}^6)} \leq C|t|^{-\frac{1}{2}} \int_0^{\infty} (1+\lambda)^{-3/2} d\lambda \leq C|t|^{-\frac{1}{2}}, \quad |t| \geq 1.$$

□

Theorem ?? is completely proved.

6. Application to the Schrödinger equation. We apply our technique, giving a short proof of the result of Goldberg and Schlag [?].

Theorem 6.1. *Let condition (??) with $\beta > 3$ hold. Then in the regular case,*

$$\|e^{it\mathcal{H}} P_c(\mathcal{H})\|_{L^1 \rightarrow L^{\infty}} \leq C|t|^{-\frac{3}{2}}, \quad |t| \geq 1.$$

Substituting the Born expansion $R^{\pm} = \sum_{j=0}^2 (R_0^{\pm} V)^j R_0^{\pm} + (R_0^{\pm} V)^2 R^{\pm} V R_0^{\pm}$ in the representation (??), we obtain

$$e^{it\mathcal{H}} P_c(\mathcal{H}) = \sum_{j=0}^2 \mathcal{S}_j(t) + \mathcal{L}_2(t),$$

where $\mathcal{S}_0(t) = e^{-i\Delta t}$, and

$$\begin{aligned} \mathcal{S}_j(t) &= \frac{1}{2\pi i} \int_0^{\infty} e^{-i\omega t} \left((R_0^+(\omega) V)^j R_0^+(\omega) - (R_0^-(\omega) V)^j R_0^-(\omega) \right) d\omega, \quad j = 1, 2, \\ \mathcal{L}_2(t) &= \frac{1}{2\pi i} \int_0^{\infty} e^{-i\omega t} \left((R_0^+(\omega) V)^2 R^+(\omega) V R_0^+(\omega) - (R_0^-(\omega) V)^2 R^-(\omega) V R_0^-(\omega) \right). \end{aligned}$$

We have $\|\mathcal{S}_0(t)\|_{L^1 \rightarrow L^{\infty}} \leq C|t|^{-\frac{3}{2}}$ for $|t| \geq 1$ by (??). Now we prove the decay $\sim |t|^{-3/2}$ for $\mathcal{S}_j(t)$, $j = 1, 2$ and for $\mathcal{L}_2(t)$ separately.

6.1. The decay of $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$.

Lemma 6.2. *Let condition (??) with $\beta > 3$ holds. Then for $j = 1, 2$*

$$\|\mathcal{S}_j(t)\|_{L^1 \rightarrow L^\infty} \leq C|t|^{-\frac{3}{2}}, \quad |t| \geq 1.$$

Proof. Let $\chi(\lambda)$ be the smooth function such that $\chi(\lambda) = 1$ if $|\lambda| \leq 1$ and $\chi(\lambda) = 0$ if $|\lambda| \geq 2$. It suffices to prove that

$$\sup_{L \geq 1} \|\mathcal{S}_j(t, L)\|_{L^1 \rightarrow L^\infty} \leq C|t|^{-\frac{3}{2}}, \quad |t| \geq 1, \quad j = 1, 2,$$

where

$$\mathcal{S}_j(t, L) = \frac{1}{\pi i} \int_0^\infty e^{-i\lambda^2 t} \chi\left(\frac{\lambda}{L}\right) [(R_0^+(\lambda^2)V)^j R_0^+(\lambda^2) - (R_0^-(\lambda^2)V)^j R_0^-(\lambda^2)] \lambda d\lambda. \quad (47)$$

Using (??), we obtain

$$\begin{aligned} \sup_{x, y \in \mathbb{R}^3} |[R_0^+ V R_0^+ - R_0^- V R_0^-](\lambda^2, x, y)| &\leq C\lambda \sup_{x, y \in \mathbb{R}^3} \int \left(\frac{|V(z)|}{|x-z|} + \frac{|V(z)|}{|y-z|} \right) dz \leq C\lambda, \\ \sup_{x, y \in \mathbb{R}^3} |[R_0^+ V]^2 R_0^+ - [R_0^- V]^2 R_0^-](\lambda^2, x, y)| \\ &\leq C\lambda \sup_{x, y \in \mathbb{R}^3} \iint \left(\frac{|V(x_1)V(y_1)|}{|x_1-y_1||y_1-y|} + \frac{|V(x_1)V(y_1)|}{|x-x_1||x_1-y_1|} + \frac{|V(x_1)V(y_1)|}{|x-x_1||y-y_1|} \right) dx_1 dy_1 \leq C\lambda. \end{aligned}$$

Hence, we can integrate by parts in (??):

$$\begin{aligned} \mathcal{S}_j(t, L) &= \frac{1}{2\pi t} \int_0^\infty e^{-i\lambda^2 t} \partial_\lambda \left(\chi\left(\frac{\lambda}{L}\right) [(R_0^-(\lambda^2)V)^j R_0^-(\lambda^2) - (R_0^+(\lambda^2)V)^j R_0^+(\lambda^2)] \right) d\lambda \\ &= \frac{1}{2\pi t} (T_j^-(t, L) - T_j^+(t, L)). \end{aligned}$$

It remains to note that for $|t| \geq 1$,

$$\sup_{L \geq 1} \sup_{x, y} |T_j^\pm(t, L, x, y)| \leq C|t|^{-1/2}, \quad j = 1, 2. \quad (48)$$

Indeed,

$$\begin{aligned} |T_1^\pm(t, L, x, y)| &\leq \frac{C}{L} \left| \int_{\mathbb{R}^3} \frac{V(z)}{|z-y||z-x|} \left(\int_0^\infty e^{-i\psi_1^\pm(\lambda)t} \chi'(\lambda/L) d\lambda \right) dz \right| \\ &\quad + C \left| \int_{\mathbb{R}^3} \left(\frac{V(z)}{|z-y|} + \frac{V(z)}{|z-x|} \right) \left(\int_0^\infty e^{-i\psi_1^\pm(\lambda)t} \chi(\lambda/L) d\lambda \right) dz \right|, \\ |T_2^\pm(t, L, x, y)| &\leq \frac{C}{L} \left| \iint \frac{|V(x_1)||V(y_1)|}{|x-x_1||x_1-y_1||y_1-y|} \left(\int_0^\infty e^{-i\psi_2^\pm(\lambda)t} \chi'(\lambda/L) d\lambda \right) dx_1 dy_1 \right| \\ &\quad + C \left| \iint dx_1 dy_1 \left(\frac{V(x_1)V(y_1)}{|x_1-y_1||y_1-y|} + \frac{V(x_1)V(y_1)}{|x-x_1||x_1-y_1|} + \frac{V(x_1)V(y_1)}{|x-x_1||y-y_1|} \right) \right. \\ &\quad \left. \times \left(\int_0^\infty e^{-i\psi_2^\pm(\lambda)t} \chi(\lambda/L) d\lambda \right) \right|, \end{aligned}$$

where $\psi_1^\pm(\lambda) = \lambda^2 \mp \lambda(|x-z| + |z-y|)/t$, $\psi_2^\pm(\lambda) = \lambda^2 \mp \lambda(|x-x_1| + |x_1-y| + |y_1-y|)/t$ with $\partial_\lambda^2 \psi_j^\pm(\lambda) = 2$, $j = 1, 2$. Then (??) follows from Lemma ?? \square

6.2. The decay of $\mathcal{L}_2(t)$.

Lemma 6.3. *Let the conditions of Theorem ?? hold. Then*

$$\|\mathcal{L}_2(t)\|_{L^1 \rightarrow L^\infty} \leq C|t|^{-\frac{3}{2}}, \quad |t| \geq 1.$$

Proof. Using the definition (??), we represent $\mathcal{L}_2(t)$ as

$$\mathcal{L}_2(t) = \frac{1}{\pi i} \int_0^\infty e^{-i\lambda^2 t} (\Lambda_2^+(\lambda) - \Lambda_2^-(\lambda)) d\lambda.$$

Due to Lemma ??, the integrand is a differentiable operator function of $\lambda \geq 0$ with values in the space of bounded operators mapping L^1 into L^∞ . Moreover, due to Lemmas ?? and ?? with $N = 2$, we can integrate by parts,

$$\mathcal{L}_2(t) = \frac{1}{2\pi t} \int_0^\infty e^{-i\lambda^2 t} \partial_\lambda (\Lambda_2^-(\lambda) - \Lambda_2^+(\lambda)) d\lambda = \frac{1}{2\pi t} (Q^-(t) - Q^+(t)).$$

It remains to prove that

$$\sup_{x,y} |Q^\pm(t, x, y)| \leq C|t|^{-\frac{1}{2}}, \quad |t| \geq 1. \quad (49)$$

We have

$$Q^\pm(t, x, y) = \int_0^\infty \left(e^{-i\varphi_1 t} \tilde{\Lambda}_2^\pm(\lambda, x, y) + e^{-i\varphi_2^\pm t} \hat{\Lambda}_2^\pm(\lambda, x, y) + e^{-i\varphi_3^\pm t} \check{\Lambda}_2^\pm(\lambda, x, y) \right) d\lambda,$$

where $\varphi_1(\lambda) = -\lambda^2$, $\varphi_2^\pm(\lambda) = -\lambda^2 \mp \lambda|x|/t$, $\varphi_3^\pm(\lambda) = -\lambda^2 \mp \lambda|y|/t$, and $\tilde{\Lambda}$, $\hat{\Lambda}$, $\check{\Lambda}$ are defined in (??). Applying Corollary ?? and Lemma ??, we obtain

$$\sup_{x,y} |Q^\pm(t, x, y)| \leq C|t|^{-\frac{1}{2}} \int_0^\infty (1 + |\lambda|)^{-2} d\lambda \leq C|t|^{-\frac{1}{2}}, \quad |t| \geq 1.$$

□

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