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ASYMPTOTIC STABILITY OF STATIONARY STATES IN WAVE EQUATION COUPLED TO NONRELATIVISTIC PARTICLE

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Abstract. We consider the Hamiltonian system consisting of scalar wave field and a single particle coupled in a translation invariant manner. The point particle is subject to an external potential. The stationary solutions of the system are a Coulomb type wave field centered at those particle positions for which the external force vanishes. It is assumed that the charge density satisfies the Wiener condition which is a version of the "Fermi Golden Rule". We prove that in the large time approximation any finite energy solution, with the initial state close to the some stable stationary solution, is a sum of this stationary solution and a dispersive wave which is a solution of the free wave equation.

1. Introduction

Our paper concerns the problem of nonlinear field-particle interaction. We consider a scalar real-valued wave field $\phi(x)$ in $\mathbb{R}^3$ coupled to a nonrelativistic particle with position q and momentum p governed by

$$\left\{\begin{array}{ll}\dot\phi(x,t)=\pi(x,t), & \dot\pi(x,t)=\ \Delta\phi(x,t)-\rho(x-q(t)),\\ \dot q(t)=p(t), & \dot p(t)=-\nabla V(q(t))+\displaystyle\int\phi(x,t)\nabla\rho(x-q(t))\,dx.\end{array}\right. \qquad (1.1)$$

This is a Hamilton system with the Hamilton functional

$$\mathcal{H}(\phi,\pi,q,p)=\frac12\int\Big(|\pi(x)|^2+|\nabla\phi(x)|^2)dx+\int\phi(x)\rho(x-q)dx+\frac12 p^2. \qquad (1.2)$$ hamilq0

The first two equations in (1.1) for the fields are equivalent to the wave equation with the source $\rho(x-q)$. The form of the last two equations is determined by the choice of the nonrelativistic kinetic energy $p^2/2$ in (1.2).

It is easy to find stationary solutions to the system (1.1). We define For $q\in\mathbb{R}^3$ we set

$$s_q(x)=-\int\frac{d^3y}{4\pi|y-x|}\rho(y-q). \qquad (1.3)$$ sq

Let $Z=\{q\in\mathbb{R}^3:\nabla V(q)=0\}$ be the set of critical points for V. Then the set $\mathcal{S}$ of stationary solutions is given by

$$\mathcal{S}=\{(\phi,\pi,q,p)=(s_q,0,q,0)=:S_q|\ q\in Z\}. \qquad (1.4)$$ WPss

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We assume that $V \in C^2(\mathbb{R}^3)$ and set

$$V_0 := \inf_{q\in\mathbb{R}^3} V(q) > -\infty. \tag{1.5}$$ V-as

For the charge distribution ρ we assume that

$$\rho \in C_0^\infty(\mathbb{R}^3), \quad \rho(x) = 0 \text{ for } |x| \geq R_\rho,\ \rho(x) = \rho_r(|x|). \tag{1.6}$$ rho-as

We also assume that the Wiener condition is satisfied:

$$\hat\rho(k) = \int d^3x\, e^{ikx}\rho(x) \neq 0, \quad k \in \mathbb{R}^3. \tag{1.7}$$ FGR

It is an analogue of the Fermi Golden Rule: the coupling term $\rho(x-q)$ is not orthogonal to the eigenfunctions e^{ikx} of the continuous spectrum of the linear part of the equation (cf. [10]).

Finally we assume that some $q^* \in Z$ is a stable critical point of V:

WP1.1 **Definition 1.1.** *A point $q^* \in Z$ is stable if $d^2V(q^*) > 0$ as a quadratic form.*

Our main results are the following:
For solutions to the system (1.1) with initial data close to $S_{q^*} = (s_{q^*}, 0, q^*, 0)$ we prove the asymptotics

$$\|\phi(\cdot,t)-s_{q^*}\|_{\dot H^1_{-\sigma}}+\|\pi(\cdot,t)\|_{L^2_{-\sigma}}+|q(t)-q^*|+|p(t)| = \mathcal{O}(t^{-\sigma}), \quad t\pm\infty, \quad \sigma>1 \tag{1.8}$$ Yconv

in weighted Sobolev norms (see (2.1)). Such asymptotics in global energy norm do not hold in general since the field components may contain a dispersive term, whose energy radiates to infinity as $t \to \pm\infty$ but does not converge to zero. Namely, in global energy norms we obtain the following scattering asymptotics:

$$(\phi(x,t), \pi(x,t)) \sim (s_{q^*}, 0) + W_0(t)\Phi_\pm, \quad t \to \pm\infty. \tag{1.9}$$ Si

Here $W_0(t)$ is the dynamical group of the free wave equation, and $\Phi_\pm$ are the corresponding asymptotic scattering states,

Asymptotics similar to (1.8) in local energy semi-norms was obtained in [7] in the case of compactly supported difference $\phi(x,0) - s_{q^*}(x)$. We get rid of this restriction in present paper.

For the proof we establish long-time decay of the linearized dynamics using our results [8] on the dispersion decay for the wave equation in weighted Sobolev norms. Then we apply the method of majorants.

Let us comment on previous results in these directions. The asymptotic stability of the solitons was proved in [2] for the system of type (1.1) with the Klein-Gordon equation instead of the wave equations. This result was extended in [3, 4, 5, 6] to similar system with the Schrödinger, Dirac, wave and Maxwell equations. The survey of these results can be found in [1].

2. Main results

To formulate our results precisely, we introduce a suitable phase space. Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with scalar product $\langle\cdot,\cdot\rangle$. Denote $\dot H^1$ the completion of real space $C_0^\infty(\mathbb{R}^3)$ with norm $\|\nabla\phi(x)\|_{L^2}$. Equivalently, using Sobolev's embedding theorem (see [9]),

$$\dot H^1 = \{\phi(x) \in L^6(\mathbb{R}^3) :\ |\nabla\phi(x)| \in L^2\}.$$

Introduce the weighted Sobolev spaces L^2_α and $\dot H^1_\alpha$, $\alpha\in\mathbb{R}$ with the norms

$$\|\psi\|_{L^2_\alpha}:=\|(1+|x|)^\alpha\psi\|_{L^2},\quad \|\psi\|_{\dot H^1_\alpha}:=\|(1+|x|)^\alpha\psi\|_{\dot H^1}. \tag{2.1}$$ Sob

Definition 2.1. *i) The phase space $\mathcal{E}$ is the real Hilbert space $\dot H^1\oplus L^2\oplus\mathbb{R}^3\oplus\mathbb{R}^3$ of states $Y=(\psi,\pi,q,p)$ equipped with the finite norm*

$$\|Y\|_{\mathcal{E}}=\|\nabla\psi\|_{L^2}+\|\pi\|_{L^2}+|q|+|p|.$$

ii) $\mathcal{E}_\alpha$ is the space $\dot H^1_\alpha\oplus L^2_\alpha\oplus\mathbb{R}^3\oplus\mathbb{R}^3$ equipped with the norm

$$\|Y\|_{\mathcal{E}_\alpha}=\|\psi\|_{\dot H^1_\alpha}+\|\pi\|_{L^2_\alpha}+|q|+|p|. \tag{2.2}$$ alfa

iii) $\mathcal{F}_\alpha$ is the space $\dot H^1_\alpha\oplus H^0_\alpha$ of fields $F=(\psi,\pi)$ equipped with the finite norm

$$\|\,F\|_{\mathcal{F}_\alpha}=\|\psi\|_{\dot H^1_\alpha}+\|\pi\|_{L^2_\alpha}. \tag{2.3}$$ Falfa

We consider the Cauchy problem for the Hamiltonian system (1.1)

$$\dot Y(t)=F(Y(t)),\ t\in\mathbb{R},\ Y(0)=Y_0. \tag{2.4}$$ WP2.1

All derivatives are understood in the sense of distributions. Here,

$$Y(t)=(\phi(t),\pi(t),q(t),p(t)),\ Y_0=(\phi_0,\pi_0,q_0,p_0)\in\mathcal{E}$$

.

WPexistence **Lemma 2.2.** *(sf.* [7, Lemma 2.1]*) Let (1.5) and (1.6) be satisfied. Then the following assertions hold.*
(i) For every $Y_0\in\mathcal{E}$ the Cauchy problem (2.4) has a unique solution $Y(t)\in C(\mathbb{R},\mathcal{E})$.
(ii) For every $t\in\mathbb{R}$ the map $Y_0\mapsto Y(t)$ is continuous on $\mathcal{E}$.
(iii) The energy is conserved, i.e.,

$$\mathcal{H}(Y(t))=\mathcal{H}(Y_0)\ for\ t\in\mathbb{R}. \tag{2.5}$$ WPEC

(iv) The energy is bounded from below, and

$$\inf_{Y\in\mathcal{E}}\mathcal{H}(Y)=V_0+\frac{1}{2}(\rho,\Delta^{-1}\rho) \tag{2.6}$$ WPHm

Our first result is the following long-time convergence in $\mathcal{E}_{-\sigma}$ to the stationary stable state:

WPB **Theorem 2.3.** *Let conditions (1.5)- (1.7 hold, and let $Y(t)$ be a solution to the Cauchy problem (2.4) with initial state $Y_0\in\mathcal{E}$ close to $S_{q^*}=(s_{q^*},0,q^*,0)$ with stable $q^*\in Z$:*

$$d_0:=\|\nabla(\phi_0-s_{q^*})\|_{L^2_\sigma}+\|\pi_0\|_{L^2_\sigma}+|q_0-q^*|+|p_0|\ll 1, \tag{2.7}$$ close

where $\sigma>1$. Then for sufficiently small d_0

$$\|Y(t)-S_{q^*}\|_{\mathcal{E}_{-\sigma}}\le C(d_0)(1+|t|)^{-\sigma},\quad t\in\mathbb{R}. \tag{2.8}$$ WPEE1

Our second result is the following scattering long-time asymptotics in global energy norms for the field components of the solution:

main **Theorem 2.4.** *Let the assumptions of Theorem 2.3 hold. Then for sufficiently small d_0*

$$(\phi(x,t),\pi(x,t))=(s_{q^*},0)+W_0(t)\Phi_\pm+r_\pm(x,t),\quad t\to\pm\infty, \tag{2.9}$$ S

where $W_0(t)$ is the dynamical group of the free wave equation, $\Phi_\pm\in\dot H^1\oplus L^2$, and

$$\|r_\pm(t)\|_{\dot H^1\oplus L^2}=\mathcal{O}(|t|^{-\sigma+1}),\quad t\to\pm\infty. \tag{2.10}$$ rm

It suffices to prove (2.9) for positive $t \to +\infty$ since the system (1.1) is time reversible.

3. Linearization at a stationary state

For notational simplicity we also assume isotropy in the sense that

$$\partial_i \partial_j V(q^*) = \omega_0^2 \delta_{ij}, \ i,j = 1,2,3, \ \omega_0 > 0\,. \tag{3.1}$$ WPhesse

Without loss of generality we take $q^* = 0$.
Let $S_q = S_0 = (s_0, 0, 0, 0)$ be the stationary state of (1.1) corresponding to $q^* = 0$, and $Y_0 = (\phi_0, \pi_0, q_0, p_0) \in \mathcal{E}$ be an arbitrary initial data satisfying (2.7). Consider $Y(x,t) = (\phi(x,t), \pi(x,t), q(t), p(t)) \in \mathcal{E}$ the solution to (1.1) with $Y(0) = Y_0$.

To linearize (1.1) at S_0, we set $\phi(x,t) = \psi(x,t) + s_0(x)$. Then (1.1) becomes

$$\left\{\begin{array}{rcl} \dot\psi(x,t) &=& \pi(x,t) \\ \dot\pi(x,t) &=& \Delta\psi(x,t) + \rho(x) - \rho(x - q(t)) \\ \dot q(t) &=& p(t) \\ \dot p(t) &=& -\nabla V(q(t)) + \displaystyle\int d^3x\, \psi(x,t)\, \nabla\rho(x - q(t)) \\ && + \displaystyle\int d^3x\, s_0(x)[\nabla\rho(x - q(t)) - \nabla\rho(x)] \end{array}\right. \tag{3.2}$$

Introducing $X(t) = Y(t) - S_0 = (\psi(t), \pi(t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{E})$, we rewrite the nonlinear system (3.2) in the form

$$\dot X(t) = AX(t) + B(X(t)). \tag{3.3}$$ WPAB

Here A is the linear operator defined by

$$A \begin{pmatrix} \psi \\ \pi \\ q \\ p \end{pmatrix} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ \Delta & 0 & \nabla\rho\cdot & 0 \\ 0 & 0 & 0 & E \\ \langle \cdot, \nabla\rho \rangle & 0 & -\omega_0^2 - \omega_1^2 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \pi \\ q \\ p \end{pmatrix} \tag{3.4}$$ AA

with

$$\omega_1^2 \delta_{ij} = \frac{1}{3}\, \|\rho\|_{L^2}^2 \delta_{ij} = -\int d^3x\, \partial_i s_0(x) \partial_j \rho(x)\,. \tag{3.5}$$ WP01

Here, the factor 1/3 is due to a spherical symmetry of $\rho(x)$ (sf. (1.6)).

The nonlinear part is given by

$$B(X) = (0, \pi_1, 0, p_1), \tag{3.6}$$ WPBB

where

$$\pi_1 = \rho(x) - \rho(x - q) - \nabla\rho(x) \cdot q \tag{3.7}$$ pi1

and

$$\begin{array}{rcl} p_1 &=& -\nabla V(q) + \omega_0^2 q + \displaystyle\int d^3x\, \psi(x)[\nabla\rho(x - q) - \nabla\rho(x)] \\ &+& \displaystyle\int d^3x\, \nabla s_0(x)[\rho(x) - \rho(x - q) - \nabla\rho(x) \cdot q]. \end{array} \tag{3.8}$$

Let us consider the Cauchy problem for the linear equation

$$\dot Z(t) = AZ(t), \quad Z = (\Psi, \Pi, Q, P), \quad t \in \mathbb{R}, \tag{3.9}$$ WPlin

with initial condition

$$Z|_{t=0} = Z_0. \tag{3.10}$$ WPic

System (3.9) is a formal Hamiltonian system with the quadratic Hamiltonian

$$\mathcal{H}_0(Z)=\frac{1}{2}\Big(P^2+\omega^2Q^2+\int d^3x\,(|\Pi(x)|^2+|\nabla\Psi(x)|^2-2\Psi(x)\nabla\rho(x)\cdot Q)\Big),$$

which is the formal Taylor expansion of $\mathcal{H}(Y_0+Z)$ up to second order at $Z=0$.

WPexlin **Lemma 3.1.** *Let the condition (1.6) be satisfied. Then the following assertion hold*
(i) For every $Z_0\in\mathcal{E}$ the Cauchy problem (3.9), (3.10) has a unique solution $Z(\cdot)\in C(\mathbb{R},\mathcal{E})$.
(ii) For every $t\in\mathbb{R}$ the map $U(t):Z_0\mapsto Z(t)$ is continuous on $\mathcal{E}$.
(iii) For $Z_0\in\mathcal{E}$ the energy $\mathcal{H}_0$ is finite and conserved, i.e.

$$\mathcal{H}_0(Z(t))=\mathcal{H}_0(Z_0)\ \ for\ t\in\mathbb{R}. \tag{3.11}$$ WPec

iv) For $Z_0\in\mathcal{E}$

$$\|Z(t)\|_{\mathcal{E}}\le C\ \ for\ t\in\mathbb{R} \tag{3.12}$$ WPlinb

with C depends only on the norm $\|Z_0\|_{\mathcal{E}}$.

4. Decay of linearized dynamics

We prove the following long-time decay of the solution $Z(t)$ to (3.9):

TDL **Proposition 4.1.** *Let the conditions (1.6) and (1.7) hold, and let $Z_0\in\mathcal{E}$ be such that*

$$\|\nabla\Psi_0\|_{L^2_\sigma}+\|\Pi_0\|_{L^2_\sigma}<\infty$$

with some $\sigma>1$. Then for $Z(t)=U(t)Z_0$

$$\|Z(t)\|_{\mathcal{E}_{-\sigma}}\le C(\rho,\sigma)(1+|t|)^{-\sigma+1}(\|\nabla\Psi_0\|_{L^2_\sigma}+\|\Pi_0\|_{L^2_\sigma}). \tag{4.1}$$ Z-dec

To prove this assertion we apply the Fourier-Laplace transform

$$\tilde Z(\lambda)=\Lambda Z(t)=\int_0^\infty e^{-\lambda t}Z(t)dt,\quad \operatorname{Re}\lambda>0 \tag{4.2}$$ FL

to (3.9). We expect that the solution $Z(t)$ is bounded in the norm $\|\cdot\|_{\mathcal{E}}$. Then the integral (4.2) converges and is analytic for $\operatorname{Re}\lambda>0$, and

$$\|\tilde Z(\lambda)\|_{\mathcal{E}}\le\frac{C}{\operatorname{Re}\lambda},\quad \operatorname{Re}\lambda>0. \tag{4.3}$$ PW

Applying the Fourier-Laplace transform to (3.9), we obtain that

$$\lambda\tilde Z(\lambda)=A\tilde Z(\lambda)+Z_0,\quad \operatorname{Re}\lambda>0. \tag{4.4}$$ FLA

Hence the solution $Z(t)$ is given by

$$\tilde Z(\lambda)=-(A-\lambda)^{-1}Z_0,\quad \operatorname{Re}\lambda>0 \tag{4.5}$$ FLAs

By (4.3), the resolvent $R(\lambda)=(A-\lambda)^{-1}$ exists and is analytic in $\mathcal{E}$ for $\operatorname{Re}\lambda>0$.

Let us construct the resolvent for $\operatorname{Re}\lambda>0$. Equation (4.4) takes the form

$$\lambda\begin{pmatrix}\tilde\Psi\\ \tilde\Pi\\ \tilde Q\\ \tilde P\end{pmatrix}=\begin{pmatrix}\tilde\Pi\\ \Delta\tilde\Psi+\tilde Q\cdot\nabla\rho\\ \tilde P\\ -\langle\nabla\tilde\Psi,\rho\rangle-\omega^2\tilde Q\end{pmatrix}+\begin{pmatrix}\Psi_0\\ \Pi_0\\ Q_0\\ P_0\end{pmatrix} \tag{4.6}$$ eq1

where $\omega^2=\omega_0^2+\omega_1^2$.

Step i) We consider the first two equations of (4.6):

$$\left\{\begin{array}{rcl} -\lambda\tilde\Psi+\tilde\Pi &=& -\Psi_0\\ \Delta\tilde\Psi-\lambda\tilde\Pi &=& -\Pi_0-\tilde Q\cdot\nabla\rho \end{array}\right. \tag{4.7}$$ F1

A solutions to the system (4.7) admits the convolution representation

$$\left\{\begin{array}{rcl} \tilde\Psi &=& \lambda g_\lambda*\Psi_0+g_\lambda*\Pi_0+(g_\lambda*\nabla\rho)\cdot\tilde Q\\ \tilde\Pi &=& \Delta g_\lambda*\Psi_0+\lambda g_\lambda*\Pi_0+\lambda(g_\lambda*\nabla\rho)\cdot\tilde Q, \end{array}\right. \tag{4.8}$$ Psi

where

$$g_\lambda(z)=(-\Delta+\lambda^2)^{-1}=\frac{e^{-\lambda|z|}}{4\pi|z|}. \tag{4.9}$$ dete

Step ii) We consider the last two equations of (4.6):

$$\left\{\begin{array}{rcl} -\lambda\tilde Q+\tilde P &=& -Q_0\\ -\omega_2\tilde Q-\langle\nabla\tilde\Psi,\rho\rangle-\lambda\tilde P &=& -P_0 \end{array}\right. \tag{4.10}$$ lte

Let us write the first equation of (4.8) in the form $\tilde\Psi(x)=\tilde\Psi_1(\tilde Q)+\tilde\Psi_2(\Psi_0,\Pi_0)$, where

$$\tilde\Psi_1(\tilde Q)=\tilde Q\cdot(g_\lambda*\nabla\rho),\ \ \tilde\Psi_2(\Psi_0,\Pi_0)=\lambda g_\lambda*\Psi_0+g_\lambda*\Pi_0. \tag{4.11}$$ Psi12

Then the second equation in (4.10) becomes

$$-\omega^2\tilde Q-\langle\nabla\tilde\Psi_1,\rho\rangle-\lambda\tilde P=-P_0+\langle\nabla\tilde\Psi_2,\rho\rangle=:-P_0'.$$

Now we compute term $\langle\nabla\tilde\Psi_1,\rho\rangle$:

$$\begin{array}{rcl} \langle\nabla\tilde\Psi_1,\rho\rangle &=& -\langle\tilde\Psi_1,\partial_i\rho\rangle=-\langle\sum\limits_j(g_\lambda*\partial_j\rho)\tilde Q_j,\partial_i\rho\rangle\\ &=& -\sum\limits_j\langle g_\lambda*\partial_j\rho,\partial_i\rho\rangle\tilde Q_j=-\sum\limits_j H_{ij}(\lambda)\tilde Q_j, \end{array}$$

where

$$\begin{array}{rcl} H_{ij}(\lambda): &=& \langle g_\lambda*\partial_j\rho,\partial_i\rho\rangle=\langle i\hat g_\lambda(k)k_j\hat\rho(k),ik_i\hat\rho(k)\rangle\\ &=& \langle\dfrac{ik_j\hat\rho(k)}{k^2+\lambda^2},ik_i\hat\rho(k)\rangle=\displaystyle\int\frac{k_ik_j|\hat\rho(k)|^2dk}{k^2+\lambda^2}. \end{array} \tag{4.12}$$

The matrix H with entries H_{jj}, $1\le j\le 3$, is well defined for $\operatorname{Re}\lambda>0$ since the denominator does not vanish. The matrix H is diagonal; moreover,

$$H_{11}(\lambda)=H_{22}(\lambda)=H_{33}(\lambda)=h(\lambda). \tag{4.13}$$ hii

Finally, the system (4.10) takes the form

$$M(\lambda)\left(\begin{array}{c}\tilde Q\\ \tilde P\end{array}\right)=\left(\begin{array}{c}Q_0\\ P_0'\end{array}\right), \tag{4.14}$$ Mlam

where

$$M(\lambda)=\left(\begin{array}{cc}\lambda E & -E\\ \omega^2E-H(\lambda) & \lambda E\end{array}\right).$$

cmf **Lemma 4.2.** *The matrix-valued function $M(\lambda)$ ($M^{-1}(\lambda)$) admits an analytic (meromorphic) continuation to the entire complex plane $\mathbb{C}$.*

Proof. The Green function g_λ admits an analytic continuation in λ to the entire complex plane $\mathbb{C}$. Then an analytic continuation of $M(\lambda)$ exists in view of (4.12) since the function $\rho(x)$ is compactly supported because of (1.6). Then the inverse matrix is meromorphic since it exists for large $\operatorname{Re}\lambda$. This fact follows from (4.14) since $H(\lambda)\to 0$, $\operatorname{Re}\lambda\to\infty$ in view of (4.12). □

Since the matrix $H(\lambda)$ is diagonal, the matrix $M(\lambda)$ is equivalent to three independent 2×2- matrices. Namely, let us transpose the columns and rows of the matrix $M(\lambda)$ in the order (142536). Then we get the matrix with three 2×2-blocks on the main diagonal. Therefore, the determinant of $M(\lambda)$ is the product of determinants of the three matrices. Namely,

$$\det M(\lambda)=(\lambda^2+\omega^2-h(\lambda))^3=(\lambda^2+\omega_0^2+\omega_1^2-h(\lambda))^3, \tag{4.15}$$ detM

where

$$\omega_1^2=\int\frac{k_1^2|\hat\rho(k)|^2dk}{k^2},\quad h(\lambda)=\int\frac{k_1^2|\hat\rho(k)|^2dk}{k^2+\lambda^2}.$$

Proposition 4.3. *The matrix $M^{-1}(i\nu+0)$ is analytic in $\nu\in\mathbb{R}$.*

Proof. It suffices to prove that the limit matrix $M(i\nu+0)$ is invertible for $\nu\in\mathbb{R}$ if ρ satisfies the Wiener condition (1.7). Formula (4.15) implies $\det M(0)=\omega_0^2>0$. For $\nu\neq 0$, $\nu\in\mathbb{R}$, we consider

$$h(i\nu+\varepsilon)=\int\frac{k_1^2|\hat\rho(k)|^2dk}{k^2-(\nu-i\varepsilon)^2},\quad \varepsilon>0. \tag{4.16}$$ Hjjlim

The denominator $D(\nu,k)=k^2-\nu^2$ vanishes on $T_\nu=\{k:k^2=\nu^2\}$. Denote by dS the surface area element. Then from the Sokhotsky-Plemelj formula for C^1-functions it follows that

$$\Im h(i\nu+0)=-\frac{\nu}{|\nu|}\pi\int_{T_\nu}\frac{k_1^2|\hat\rho(k)|^2}{|\nabla D(\nu,k)|}dS\neq 0, \tag{4.17}$$ ImHjj

since the integrand in (4.17) is positive by the Wiener condition (1.7). Now, the invertibility of $M(i\nu)$ follows from (4.15). □

4.1. **Time decay.** Here we prove Proposition 4.1. First, we obtain decay (4.1) for the vector components $Q(t)$ and $P(t)$ of $Z(t)$. By (4.14), the components are given by the Fourier integral

$$\begin{pmatrix}Q(t)\\P(t)\end{pmatrix}=\frac{1}{2\pi}\int e^{i\nu t}M^{-1}(i\nu+0)\begin{pmatrix}Q_0\\P_0'\end{pmatrix}d\nu=\mathcal{L}(t)\begin{pmatrix}Q_0\\P_0\end{pmatrix}+\mathcal{L}(t)*\begin{pmatrix}0\\f(t)\end{pmatrix}, \tag{4.18}$$ QP1i

where

$$\mathcal{L}(t)=\frac{1}{2\pi}\int e^{i\nu t}M^{-1}(i\nu+0)d\nu=\lambda^{-1}M^{-1}(i\nu+0),$$

and

$$\begin{aligned}f(t)&=\Lambda^{-1}[\langle\Psi_2(\Psi_0,\Pi_0),\nabla\rho\rangle]=\Lambda^{-1}[\langle i\nu g_{i\nu}*\Psi_0+g_{i\nu}*\Pi_0,\nabla\rho\rangle]\\&=\langle W_0(t)[(\Psi_0,\Pi_0)],\nabla\rho\rangle.\end{aligned} \tag{4.19}$$

We write the nonzero entries of the matrix $M(i\nu+0)$:

$$\frac{i\nu}{-\nu^2+\omega^2-h(i\nu+0)},\quad\frac{1}{-\nu^2+\omega^2-h(i\nu+0)},\quad\frac{-\omega^2+h(i\nu)}{-\nu^2+\omega^2-h(i\nu+0)}.$$

Hence,

$$|M^{-1}(i\nu+0)| \le \frac{C}{|\nu|}, \quad |\partial^k M^{-1}(i\nu+0)| \le \frac{C_k}{|\nu|^2}, \quad \nu\in\mathbb{R}, \quad |\nu|\ge 1, \quad k\in\mathbb{N}.$$

Therefore, $\mathcal{L}(t)$ is continuous in $t\in\mathbb{R}$ and

$$\mathcal{L}(t)=\mathcal{O}(|t|^{-N}), \quad t\to\infty, \quad \forall N>0. \tag{4.20}$$

cM-dec

For the solutions of the free wave equation the following dispersion decay holds:

WPIO **Lemma 4.4.** *(sf.* [8, Proposition 2.1]*) Let* $(\Psi_0,\Pi_0)\in\mathcal{F}_0$ *be such that*

$$\|(\Psi_0,\Pi_0)\|_{\mathcal{F}_\sigma}<\infty$$

with some $\sigma>1$*. Then*

$$\|W(t)[(\Psi_0,\Pi_0)]\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-\sigma}\|(\Psi_0,\Pi_0)\|_{\mathcal{F}_\sigma}, \quad t\in\mathbb{R}. \tag{4.21}$$

lins

Lemma 4.4 and the definition (4.19) imply

$$|f(t)| \le C(\sigma,\rho)(1+|t|)^{-\sigma}\|(\Psi_0,\Pi_0)\|_{\mathcal{F}_\sigma}, \quad t\in\mathbb{R}. \tag{4.22}$$

f-dec

Therefore, (4.18), (4.20) and (4.22) imply

$$|Q(t)|+|P(t)| \le C(\sigma,\rho)(1+|t|)^{-\sigma}\|(\Psi_0,\Pi_0)\|_{\mathcal{F}_\sigma} \tag{4.23}$$

QP

Then (4.1) holds for the vector components $Q(t)$ and $P(t)$.

Now we prove (4.1) for the field components of $Z(t)$. The first two equations of (3.9) have the form

$$\begin{pmatrix}\dot\Psi(t)\\ \dot\Pi(t)\end{pmatrix}=\begin{pmatrix}0 & 1\\ \Delta & 0\end{pmatrix}\begin{pmatrix}\Psi(t)\\ \Pi(t)\end{pmatrix}+\begin{pmatrix}0\\ Q(t)\cdot\nabla\rho\end{pmatrix}. \tag{4.24}$$

linf

The integrated version of (4.24) reads

$$(\Psi(t),\Pi(t))=W(t)[(\Psi_0,\Pi_0)]+\int_0^t W(t-s)[0,Q(s)\cdot\nabla\rho]ds, \ t\ge 0. \tag{4.25}$$

Duh

By (4.21) and (4.23),

$$\|(\Psi(t),\Pi(t))\|_{\mathcal{F}_{-\sigma}} \le C(\rho,\sigma)(1+|t|)^{-\sigma+1}(\|\nabla\Psi_0\|_{L^2_\sigma}+\|\Pi_0\|_{L^2_\sigma}), \ t\in\mathbb{R}.$$

Proposition 4.1 is proved. □

5. Asymptotic stability of stationary states

Here we prove Theorem 2.3. First, we obtain bounds for the nonlinear part $B(X(t))=(0,\pi_1,0,p_1)$ defined in (3.6) - (3.8). We have

$$\begin{aligned}\|\pi_1(t)\|_{L^2_\sigma} &\le \mathcal{R}(|q|)|q(t)|^2,\\ |p_1(t)| &\le \mathcal{R}(|q|)[|q(t)|\|\psi(t)\|_{\dot H^1_{-\sigma}}+|q(t)|^2],\end{aligned}$$

where $R(A)$ denotes a positive function that remains bounded for sufficiently small A. Hence

$$\|B(t)\|_{\mathcal{E}_\sigma} \le \mathcal{R}(|q|)\|X(t)\|^2_{\mathcal{E}_{-\sigma}}. \tag{5.1}$$

B-est

We introduce the majorant

$$m(t)=\sup_{0\le s\le t}(1+s)^{-\sigma}\|X(s)\|_{\mathcal{E}_{-\sigma}}. \tag{5.2}$$

maj

We fix $\varepsilon > \|X(0)\|_{\mathcal{E}_{-\sigma}}$ and introduce the *exit time*

$$t_* = \sup\{t > 0 : m(t) \le \varepsilon\}. \tag{5.3}$$

t*

The integrated version of (3.3) is written as

$$X(t) = e^{At}X(0) + \int_0^t ds\, e^{A(t-s)}B(X(s)). \tag{5.4}$$

WPkolb

Proposition 4.1 implies the integral inequality

$$\begin{aligned} \|X(t)\|_{\mathcal{E}_{-\sigma}} &\le C\,(1+|t|)^{-\sigma}\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma} \\ &+ C\int_0^t ds\,(1+|t-s|)^{-\sigma}\|X(s)\|^2_{\mathcal{E}_{-\sigma}}, \end{aligned} \tag{5.5}$$

for $t < t_*$. We multiply both sides of (5.4) by $(1+t)^{-\sigma}$, and take the supremum in $t \in [0, t_*]$. Then

$$m(t) \le C\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma} + C\sup_{t\in[0,t_*]}\int_0^t \frac{(1+t)^\sigma}{(1+|t-s|)^\sigma}\frac{m^2(s)}{(1+s)^{2\sigma}}ds$$

for $t < t_*$. Since $m(t)$ is a monotone increasing function, we get

$$m(t) \le C(\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma} + Cm^2(t)I(t), \quad t \le t_*, \tag{5.6}$$

mest

where

$$I(t) = \int_0^t \frac{(1+t)^\sigma}{(1+|t-s|)^\sigma}\frac{m^2(s)}{(1+s)^{2\sigma}}ds \le \overline{I} < \infty, \quad t \ge 0, \quad \sigma > 1.$$

Therefore, (5.6) takes the form

$$m(t) \le C\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma} + C\overline{I}m^2(t), \quad t \le t_*, \tag{5.7}$$

mest1

which implies that $m(t)$ is bounded for $t < t_*$; moreover,

$$m(t) \le C\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma}, \quad t < t_*, \tag{5.8}$$

m2est

since $m(0) = \|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma}$ is sufficiently small by (2.7).

The constant C in the estimate (5.8) is independent of t_*. We choose d_0 in (2.7) so small that $\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma} < \varepsilon/(2C)$, which is possible due to (2.7). Then the estimate (5.8) implies $t_* = \infty$, and (5.8) holds for all $t > 0$ if d_0 is small enough. □

6. Scattering asymptotics

Here we prove Theorem 2.4. From the first two equations of (1.1) we obtain the inhomogeneous wave equation for the difference $F(x,t) = (\psi(x,t), \pi(x,t)) = (\phi(x,t), \pi(x,t)) - (s_{q^*}, 0)$:

$$\begin{aligned} \dot\psi(x,t) &= \pi(x,t), \\ \dot\pi(x,t) &= \Delta\psi(x,t) + \rho(x-q^*) - \rho(x-q(t)). \end{aligned} \tag{6.1}$$

Then

$$F(t) = W_0(t)F(0) - \int_0^t W_0(t-s)[(0, \rho(x-q^*) - \rho(x-q(s)))]ds. \tag{6.2}$$

eqacc

To obtain the asymptotics (2.9), it suffices to prove that $F(t) = W_0(t)\Phi_+ + r_+(t)$ for some $\Phi_+ \in \dot H^1 \oplus L^2$ and $\|r_+(t)\|_{\dot H^1\oplus L^2} = \mathcal{O}(t^{-\sigma+1})$. This fact is equivalent to the asymptotics

$$W_0(-t)F(t) = \Phi_+ + r'_+(t), \ \|r'_+(t)\|_{dotH^1\oplus L^2} = \mathcal{O}(t^{-\sigma+1}), \tag{6.3}$$

Sme

since $W_0(t)$ is a unitary group on $\dot H^1\oplus L^2$ by the energy conservation for the free wave equation. Finally, the asymptotics (6.3) hold since (6.2) implies

$$W_0(-t)F(t)=F(0)-\int_0^t W_0(-s)R(s)ds,\quad R(s)=(0,\rho(x-q^*)-\rho(x-q(s))\quad (6.4)$$ duhs

We set

$$\Phi_+=F(0)-\int_0^\infty W_0(-s)R(s)ds,\qquad (6.5)$$ Pr

and

$$r'_+(t)=\int_t^\infty W_0(-s)R(s)ds$$

The integral on the right hand side of (6.5) converges in $\dot H^1\oplus L^2$ with the rate $\mathcal{O}(t^{-\sigma+1})$ because $\Vert W_0(-s)R(s)\Vert_{\dot H^1\oplus L^2}=\mathcal{O}(s^{-\sigma})$ by the unitarity of $W_0(-s)$ and the decay rate $\Vert R(s)\Vert_{\dot H^1\oplus L^2}=\mathcal{O}(s^{-\sigma})$ which follows from the conditions (1.6) on ρ and the asymptotics (2.8). Hence, $\Phi_+\in\dot H^1\oplus L^2$ and (2.10) holds. $\square$

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