Scattering of Solitons for Dirac Equation Coupled to a Particle

A. I. Komech [1]
Fakultät für Mathematik, Universität Wien
and Institute for Information Transmission Problems RAS
e-mail: alexander.komech@univie.ac.at

E. A. Kopylova [2]
Institute for Information Transmission Problems RAS
B.Karetnyi 19, Moscow 101447,GSP-4, Russia
e-mail: elena.kopylova@univie.ac.at

H. Spohn
Faculty of Mathematics, TU Munich, Garching, 85747 Germany
email: spohn@ma.tum.de

Abstract

We establish soliton-like asymptotics for finite energy solutions to the Dirac equation coupled to a relativistic particle. Any solution with initial state close to the solitary manifold, converges in long time limit to a sum of traveling wave and outgoing free wave. The convergence holds in global energy norm. The proof uses spectral theory and the symplectic projection onto solitary manifold in the Hilbert phase space.

[1] Supported partly by the Alexander von Humboldt Research Award.
[2] Supported partly by the grants of FWF, DFG and RFBR.

1 Introduction

We prove the long time convergence to the sum of a soliton and dispersive wave for the Dirac equation coupled to a relativistic particle. The convergence holds in global energy norm for finite energy solution with initial state close to the solitary manifold. Our main motivation is to develop the techniques of Buslaev and Perelman [2, 3] in the context of the Dirac equation. The development is not straightforward because of known peculiarities of the Dirac equation: nonpositivity of the energy, algebra of the Dirac matrices, etc. We expect that the result might be extended to nonlinear relativistic Dirac equation relying on an appropriate development of our techniques.

Let $\psi(x) \in \mathbb{C}^4$ be a Dirac spinor field in $\mathbb{R}^3$, coupled to a relativistic particle with position q and momentum p, governed by

$$\left\{\begin{array}{l} i\dot{\psi}(x,t) = [-i\alpha_1\partial_1 - i\alpha_2\partial_2 - i\alpha_3\partial_3 + \beta m]\psi(x,t) + \rho(x-q(t)) \\ \\ \dot{q}(t) = p(t)/\sqrt{1+p^2(t)}, \qquad \dot{p}(t) = \mathrm{Re}\langle\psi(x,t), \nabla\rho(x-q(t))\rangle \end{array}\right| \; x \in \mathbb{R}^3 \qquad (1.1)$$

where $\rho \in C(\mathbb{R}^3, \mathbb{C}^4)$ and $\langle\cdot,\cdot\rangle$ stands for the Hermitian scalar product in $L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$. Here $\partial_j = \partial/\partial x_j$, α_j and β are 4×4 Dirac matrices. The standard representation for the Dirac matrices α_j and β (in 2×2 blocks) is

$$\beta = \alpha_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1,2,3 \qquad (1.2)$$

where I_2 denotes the unit 2×2 matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The matrices α_j, $j = 0,1,2,3$ are Hermitian and satisfy the anticommutation relations

$$\alpha_j^* = \alpha_j, \quad \alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk} \qquad (1.3)$$

We will use the following *real orthogonality relations*

$$\beta\psi\cdot\alpha_j\psi = 0, \quad j = 1,3, \quad \text{and} \quad \alpha_2\psi\cdot\psi = 0, \qquad \psi \in \mathbb{R}^4 \qquad (1.4)$$

The system (1.1) is translation-invariant and admits soliton solutions

$$s_{a,v}(t) = (\psi_v(x - vt - a), vt + a, p_v), \quad p_v = v/\sqrt{1-v^2} \qquad (1.5)$$

for all $a, v \in \mathbb{R}^3$ with $|v| < 1$. The states $S_{a,v} := s_{a,v}(0)$ form the solitary manifold

$$\mathcal{S} := \{S_{a,v} : a, v \in \mathbb{R}^3, |v| < 1\} \qquad (1.6)$$

Our main result is the soliton-type asymptotics

$$\psi(x,t) \sim \psi_{v_\pm}(x - v_\pm t - a_\pm) + W_0(t)\phi_\pm, \quad t \to \pm\infty \qquad (1.7)$$

for solutions to (1.1) with initial data close to the solitary manifold $\mathcal{S}$. Here $W_0(t)$ is the dynamical group of the free Dirac equation, $\phi_\pm$ are the corresponding *asymptotic scattering states*, and the asymptotics hold *in the global* norm of the Hilbert space $L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$. For the particle trajectory we prove that

$$\dot{q}(t) \to v_\pm, \quad q(t) \sim v_\pm t + a_\pm, \quad t \to \pm\infty \qquad (1.8)$$

The results are established under the following conditions on the complex valued charge distribution: for some $\nu > 5/2$

$$(1+|x|)^\nu |\partial^\alpha \rho| \in L^2(\mathbb{R}^3), \quad |\alpha| \le 3 \tag{1.9}$$

We assume $\rho(-x) = \rho(x)$, $x \in \mathbb{R}^3$, for the simplicity of calculations. Finally, we assume the Wiener condition for the Fourier transform $\hat\rho = (2\pi)^{-3/2} \displaystyle\int e^{ikx}\rho(x)dx$

$$\mathcal{B}(k) = m\beta\hat\rho(k)\cdot\hat\rho(k) > 0, \quad k \in \mathbb{R}^3 \tag{1.10}$$

which is the nonlinear version of the Fermi Golden Rule in our case (cf. [4, 13, 14, 15]): the nonlinear perturbation is not orthogonal to the eigenfunctions of the continuous spectrum of the linear part. The examples are easily constructed. Namely, let us rewrite (1.10) in the form

$$\mathcal{B}(k) = m[|\hat\rho_1(k)|^2 + |\hat\rho_2(k)|^2 - |\hat\rho_3(k)|^2 - |\hat\rho_4(k)|^2] > 0, \quad k \in \mathbb{R}^3 \tag{1.11}$$

Therefore, we can take e.g. ρ_1 constructed in [12], and $\rho_2 = \rho_3 = \rho_4 = 0$.

The system (1.1) describes the charged particle interacting with its "own" Dirac field. The asymptotics (1.7)-(1.8) mean asymptotic stability of uniform motion, i.e. "the law of inertia". The stability is caused by "radiative damping", i.e. radiation of energy to infinity appearing analytically as a local energy decay for solutions to the linearized equation. The radiative damping was suggested first by M.Abraham in 1905 in the context of the Maxwell-Lorentz equations, [1].

One could also expect asymptotics (1.7) for small perturbations of the solitons for the relativistic nonlinear Dirac equations and for the coupled nonlinear Maxwell-Dirac equations whose solitons were constructed in [6]. Our result models this situation though the relativistic case is still open problem.

Asymptotics of type (1.7)-(1.8) were obtained previously for the Klein-Gordon and Schrödinger equations coupled to the particle [8, 10]. More weak asymptotics of type (1.7) in the *local energy norms*, and without the dispersive wave, were obtained in [7] and [11] for the Maxwell-Lorentz and wave equations respectively.

Let us comment on our approach. For 1D translation invariant Schrödinger equation, asymptotics of type (1.7) were proved for the first time by Buslaev and Perelman [2, 3, 4], and extended by Cuccagna [5] for higher dimensions. Here we develop the approach [8] where the general Buslaev and Perelman strategy has been developed for the case of the Klein-Gordon equation: i) symplectic orthogonal decomposition of the dynamics near the solitary manifold, ii) modulation equations for the symplectic projection onto the manifold, and iii) the time decay in the transversal directions, etc (see more details in Introduction [8]). We prove the asymptotics (1.7)– (1.8) in Sections 3-11 developing this general strategy. One of difficulties is caused by well known nonpositivity of the Hamiltonian for the Dirac equation. Respectively, the energy conservation does not provide a priori estimate for the solution. We obtain linear in time estimate for L^2 norm of the solution using unitarity of the free Dirac propagator. The main novelty in our case is thorough establishing the appropriate decay of the linearized dynamics in Sections 12-17, and Appendices A, B, and C:

I. Main difficulty lies in the proof of the decay $\sim t^{-3/2}$ in weighted norms for the free Dirac equation. Here we prove the decay for the first time (Lemma 17.1). The proof relies on the "soft version" of the strong Huygens principle for the Dirac equation. Namely, the free Dirac propagator is concentrated mainly near the light cone, while the contribution of the inner zone is a Hilbert-Schmidt operator.

II. Next difficulty lies in the computation of the spectral properties of the linearized equation at the soliton. We do not postulate any spectral properties of the equation in contrast to majority of the works in the field. Namely, we find that under the Wiener condition (1.10), the discrete spectrum consists only from zero point with algebraic multiplicity 6 (Lemma 16.2). The multiplicity is totally due to the translation invariance of the system (1.1).

III. Moreover, we exactly calculate the symplectic orthogonality conditions (16.7) for initial data of the linearized equation. These conditions are necessary for the proof of the decay.

IV. All computations differ significantly from the case of the Klein-Gordon equation [8] because of the algebra of the Dirac matrices. An important role play the *real orthogonality relations* (1.4) for the Dirac matrices.

Our paper is organized as follows. In Section 2, we formulate the main result. In Section 3, we introduce the symplectic projection onto the solitary manifold. The linearized equation is considered in Sections 4 and 5. In Section 6, we split the dynamics in two components: along the solitary manifold, and in transversal directions. The time decay of the transversal component is established in sections 7-10 using the time decay of the linearized dynamics. In Section 11 we prove the main result. In Sections 12 - 16 we justify the time decay of the linearized dynamics relying on the weighted decay for the free Dirac equation in a moving frame which is proved in Section 17. In Appendices A, B and C we collect some technical calculations.

2 Main results

2.1 Existence of dynamics

We consider the Cauchy problem for the system (3.1) which we write as

$$\dot{Y}(t) = F(Y(t)), \quad t \in \mathbb{R}: \quad Y(0) = Y_0 \tag{2.1}$$

Here $Y(t) = (\psi(t), q(t), p(t))$, $Y_0 = (\psi(0), q_0, p_0)$, and all derivatives are understood in the sense of distributions. To formulate our results precisely, we need some definitions. We introduce a suitable phase space for equation (2.1). Let L^2_α, $\alpha \in \mathbb{R}$, denote the weighted Agmon spaces with the norm $\|\psi\|_\alpha = \|\psi\|_{L^2_\alpha} := \|(1+|x|)^\alpha |\psi|\|_{L^2}$, where $L^2 = L^2(\mathbb{R}^3)$.

Definition 2.1. *i) The phase space $\mathcal{E}$ is the Hilbert space $L^2_0 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ of states $Y = (\psi, q, p)$ with the finite norm*

$$\|Y\|_{\mathcal{E}} = \|\psi\|_0 + |q| + |p|$$

ii) $\mathcal{E}_\alpha$ is the space $L^2_\alpha \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with the finite norm

$$\|Y\|_\alpha = \|\,Y\|_{\mathcal{E}_\alpha} = \|\psi\|_\alpha + |q| + |p|$$

Proposition 2.2. *Let (1.9) hold. Then*
(i) For every $Y_0 \in \mathcal{E}$ the Cauchy problem (3.1) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.
(ii) For every $t \in \mathbb{R}$, the map $U(t) : Y_0 \mapsto Y(t)$ is continuous on $\mathcal{E}$.

Proof. Step i) First, let us fix an arbitrary $b > 0$ and prove *(i)-(ii)* for $Y_0 \in \mathcal{E}$ such that $\|\psi_0\|_0 \le b$ and $|t| \le \varepsilon = \varepsilon(b)$ for some sufficiently small $\varepsilon(b) > 0$. Let us rewrite the Cauchy problem (2.1) as

$$\dot{Y}(t) = F_1(Y(t)) + F_2(Y(t)), \quad t \in \mathbb{R}: \quad Y(0) = Y_0 \tag{2.2}$$

where

$$F_1 : Y \mapsto ((-\alpha_j \partial_j - i\beta m)\psi, 0, 0)$$
$$F_2 : Y \mapsto (-i\rho(x-q),\, p/\sqrt{1+p^2},\, \mathrm{Re} \int \psi \cdot \nabla\rho(x-q)dx)$$

The Fourier transform provides the existence and uniqueness of solution $Y_1(t) \in C(\mathbb{R}, \mathcal{E})$ to the linear problem (2.2) with $F_2 = 0$. Let $U_1(t) : Y_0 \mapsto Y_1(t)$ be the corresponding strongly continuous group of bounded linear operators on $\mathcal{E}$. Then (2.2) for $Y(t) \in C(\mathbb{R}, \mathcal{E})$ is equivalent to the integral Duhamel equation

$$Y(t) = U_1(t)Y_0 + \int_0^t ds\; U_1(t-s)F_2(Y(s)) \tag{2.3}$$

because $F_2(Y(\cdot)) \in C(\mathbb{R}, \mathcal{E})$ in this case. The latter follows from local Lipschitz continuity of the map F_2 in $\mathcal{E}$: for each $b > 0$ there exist a $\varkappa = \varkappa(b) > 0$ such that for all $Y_1 = (\psi_1, q_1, p_1), Y_2 = (\psi_2, q_2, p_2) \in \mathcal{E}$ with $\|\psi_1\|_0,\ \|\psi_2\|_0 \le b$,

$$\|F_2(Y_1) - F_2(Y_2)\|_{\mathcal{E}} \le \varkappa \|Y_1 - Y_2\|_{\mathcal{E}}$$

Therefore, by the contraction mapping principle, equation (2.3) has a unique local solution $Y(\cdot) \in C([-\varepsilon, \varepsilon], \mathcal{E})$ with $\varepsilon > 0$ depending only on b.
Step ii) Second we derive *a priori* estimate. Consider $\psi_0 \in C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$. Then

$$\frac{d}{dt}\|\psi\|_0^2 = \int (\overline{\psi} \cdot \dot{\psi} + \psi \cdot \dot{\overline{\psi}})dx = \int (i\overline{\psi} \cdot \rho(x-q) - i\psi \cdot \overline{\rho}(x-q))dx \le C\|\psi\|_0$$

Hence,

$$\|\psi(t)\|_0 \le \frac{1}{2}Ct + \|\psi(0)\|_0$$

Now, the last two equalities (1.1) imply *a priori* estimates for $|\dot{p}|$ and $|\dot{q}|$. The *a priori* estimates for general initial data $\psi_0 \in L_0^2$ follow by approximating initial data by the functions from $C_0^\infty(\mathbb{R}^3) \otimes \mathbb{C}^4$.
Step iii) Properties *(i)-(ii)* for arbitrary $t \in \mathbb{R}$ now follow from the same properties for small $|t|$ and from *a priori* estimate. $\square$

2.2 Solitary manifold and main result

Let us compute the solitons (1.5). The substitution to (1.1) gives the stationary equations

$$\left.\begin{aligned} -iv_j\partial_j\psi_v(y) &= [-i\alpha_j\partial_j + \beta m]\psi_v(y) + \rho(y) \\ v = p_v/\sqrt{1+p_v^2}, &\quad 0 = \mathrm{Re}\int \psi_v(y) \cdot \nabla\rho(y)\, dy \end{aligned}\right| \tag{2.4}$$

Applying Fourier transform to the first equation in (2.4) we obtain

$$(-v_j k_j + \alpha_j k_j - \beta m)\hat{\psi}_v(k) = \hat{\rho}(k)$$

hence

$$\hat{\psi}_v(k) = -\frac{(v_jk_j + \alpha_jk_j - \beta m)\hat{\rho}(k)}{(v_jk_j + \alpha_jk_j - \beta m)(v_jk_j - \alpha_jk_j + \beta m)} = \frac{(v_jk_j + \alpha_jk_j - \beta m)\hat{\rho}(k)}{k^2 + m^2 - (v_jk_j)^2} \tag{2.5}$$

The soliton is given by the formula

$$\psi_v(x) = \frac{i\gamma}{4\pi}(v_j\partial_j + \alpha_j\partial_j + i\beta m)\int \frac{e^{-m|\gamma(y-x)_{\|}+(y-x)_{\perp}|}\rho(y)d^3y}{|\gamma(y-x)_{\|} + (y-x)_{\perp}|}, \quad p_v = \gamma v = \frac{v}{\sqrt{1-v^2}} \tag{2.6}$$

It remains to prove that the last equation of (2.4) holds. Indeed, Parseval identity and equality (2.5) imply

$$\mathrm{Re}\int \psi_v(y)\cdot\partial_j\rho(y)dy = \mathrm{Re}\int ik_j\hat{\psi}_v(k)\cdot\hat{\rho}(k)dk = \mathrm{Re}\int ik_j\frac{(v_jk_j + \alpha_jk_j - \beta m)\hat{\rho}(k)\cdot\hat{\rho}(k)}{k^2 + m^2 - (v_jk_j)^2}dk = 0$$

since the integrand is pure imaginary function. Hence, the soliton solution (1.5) exists and is defined uniquely for any couple (a, v) with $|v| < 1$ and $a \in \mathbb{R}^3$. Let us denote by $V := \{v \in \mathbb{R}^3 : |v| < 1\}$.

Definition 2.3. *A soliton state is $S(\sigma) := (\psi_v(x - b), b, v)$, where $\sigma := (b, v)$ with $b \in \mathbb{R}^3$ and $v \in V$.*

Obviously, the soliton solution admits the representation $S(\sigma(t))$, where

$$\sigma(t) = (b(t), v(t)) = (vt + a, v) \tag{2.7}$$

Definition 2.4. *A solitary manifold is the set $\mathcal{S} := \{S(\sigma) : \sigma \in \Sigma := \mathbb{R}^3 \times V\}$.*

The main result of our paper is the following theorem.

Theorem 2.5. *Let* (1.9), *and the Wiener condition (1.10) hold. Let $\nu > 5/2$ be the number from* (1.9), *and $Y(t)$ be the solution to the Cauchy problem (2.1) with the initial state Y_0 which is sufficiently close to the solitary manifold:*

$$d_0 := \mathrm{dist}_{\mathcal{E}_\nu}(Y_0, \mathcal{S}) \ll 1 \tag{2.8}$$

Then the asymptotics hold for $t \to \pm\infty$,

$$\dot{q}(t) = v_\pm + \mathcal{O}(|t|^{-2}), \quad q(t) = v_\pm t + a_\pm + \mathcal{O}(|t|^{-1}) \tag{2.9}$$

$$\psi(x, t) = \psi_{v\pm}(x - v_\pm t - a_\pm) + W_0(t)\phi_\pm + r_\pm(x, t) \tag{2.10}$$

with

$$\|r_\pm(t)\|_0 = \mathcal{O}(|t|^{-1/2}) \tag{2.11}$$

It suffices to prove the asymptotics (2.9), (2.10) for $t \to +\infty$ since the system (1.1) is time reversible.

3 Symplectic projection

3.1 Hamiltonian structure

Denote $\psi_1 = \operatorname{Re}\psi$, $\psi_2 = \operatorname{Im}\psi$, $\rho_1 = \operatorname{Re}\rho$, $\rho_2 = \operatorname{Im}\rho$, $\tilde{\alpha}_2 = -i\alpha_2$. Then the system (1.1) reads

$$\left\{\begin{array}{l} \dot{\psi}_1(x,t) = -(\alpha_1\partial_1 + \alpha_3\partial_3)\psi_1(x,t) + (\tilde{\alpha}_2\partial_2 + \beta m)\psi_2(x,t) + \rho_2(x-q(t)) \\ \dot{\psi}_2(x,t) = -(\tilde{\alpha}_2\partial_2 + \beta m)\psi_1(x,t) - (\alpha_1\partial_1 + \alpha_3\partial_3)\psi_2(x,t) - \rho_1(x-q(t)) \\ \dot{q}(t) = p(t)/\sqrt{1+p^2(t)} \\ \dot{p}(t) = \displaystyle\int \big(\psi_1(x,t)\cdot\nabla\rho_1(x-q(t)) + \psi_2(x,t)\cdot\nabla\rho_2(x-q(t))\big)dx \end{array}\right| \quad x\in\mathbb{R}^3 \tag{3.1}$$

This is a Hamilton system with the Hamilton functional

$$\begin{aligned}\mathcal{H}(\psi_1,\psi_2,q,p) &= \frac{1}{2}\int(\psi_1\cdot(\tilde{\alpha}_2\partial_2+\beta m)\psi_1 + \psi_2\cdot(\tilde{\alpha}_2\partial_2+\beta m)\psi_2 + 2\psi_1\cdot(\alpha_1\partial_1+\alpha_3\partial_3)\psi_2)dx \\ &+ \int(\psi_1(x)\cdot\rho_1(x-q) + \psi_2(x)\cdot\rho_2(x-q))dx + \sqrt{1+p^2}\end{aligned} \tag{3.2}$$

Equation (3.1) can be written as a Hamilton system

$$\dot{Y} = JD\mathcal{H}(Y), \quad Y = (\psi_1,\psi_2,q,p), \quad J := \begin{pmatrix} 0 & I_4 & 0 & 0 \\ -I_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_3 \\ 0 & 0 & -I_3 & 0 \end{pmatrix} \tag{3.3}$$

where $D\mathcal{H}$ is the Fréchet derivative with respect to ψ_{1k}, ψ_{2k}, $k=1,2,3,4$, p and q of the Hamilton functional.

3.2 Symplectic projection onto solitary manifold

Let us identify the tangent space to $\mathcal{E}$, at every point, with $\mathcal{E}$. Consider the symplectic form Ω defined on $\mathcal{E}$ by $\Omega = \displaystyle\int d\psi_1(x)\wedge d\psi_2(x)\,dx + dq\wedge dp$, i.e.

$$\Omega(Y^1,Y^2) = \langle Y^1, JY^2\rangle, \quad Y^j = (\psi_1^j,\psi_2^j,q^j,p^j)\in\mathcal{E}, \quad j=1,2 \tag{3.4}$$

where

$$\langle Y^1,Y^2\rangle := \langle\psi_1^1,\psi_1^2\rangle + \langle\psi_2^1,\psi_2^2\rangle + q^1\cdot q^2 + p^1\cdot p^2$$

and $\langle\psi_1^1,\psi_1^2\rangle = \displaystyle\int \psi_1^1(x)\cdot\psi_1^2(x)dx$ stands for the scalar product or its different extensions. It is clear that the form Ω is non-degenerate, i.e.

$$\Omega(Y^1,Y^2) = 0 \ \text{ for every } \ Y^2\in\mathcal{E} \implies Y^1 = 0$$

Definition 3.1. *i) $Y^1 \nmid Y^2$ means that $Y^1\in\mathcal{E}$, $Y^2\in\mathcal{E}$, and Y^1 is symplectic orthogonal to Y^2, i.e. $\Omega(Y^1,Y^2)=0$.*

ii) A projection operator $\mathbf{P}:\mathcal{E}\to\mathcal{E}$ is called symplectic orthogonal if $Y^1\nmid Y^2$ for $Y^1\in\operatorname{Ker}\mathbf{P}$ and $Y^2\in\operatorname{Im}\mathbf{P}$.

Let us consider the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ to the manifold $\mathcal{S}$ at a point $S(\sigma)$. The vectors $\tau_j := \partial_{\sigma_j} S(\sigma)$, where $\partial_{\sigma_j} := \partial_{b_j}$ and $\partial_{\sigma_{j+3}} := \partial_{v_j}$ with $j = 1, 2, 3$, form a basis in $\mathcal{T}_\sigma\mathcal{S}$. In detail,

$$\left.\begin{array}{rcl} \tau_j = \tau_j(v) & := & \partial_{b_j} S(\sigma) = (\,-\partial_j \psi_{v1}(y)\,, -\partial_j \psi_{v2}(y)\,,\ e_j\,,\quad 0\quad) \\ \tau_{j+3} = \tau_{j+3}(v) & := & \partial_{v_j} S(\sigma) = (\ \ \partial_{v_j}\psi_{v1}(y)\,,\ \partial_{v_j}\psi_{v2}(y)\,,\ 0\ ,\ \partial_{v_j} p_v\,) \end{array}\right|\quad j = 1, 2, 3 \tag{3.5}$$

where $\psi_{v1} = \mathrm{Re}\,\psi_v$, $\psi_{v2} = \mathrm{Im}\,\psi_v$, $y := x - b$ is the "moving frame coordinate", $e_1 = (1, 0, 0)$ etc. Let us stress that the functions τ_j will be considered always as the functions of y, not of x. Formula (2.6) and condition (1.9) imply that

$$\tau_j(v) \in \mathcal{E}_\alpha, \quad v \in V, \quad j = 1, \ldots, 6, \quad \forall \alpha \in \mathbb{R} \tag{3.6}$$

Lemma 3.2. *The matrix with the elements $\Omega(\tau_l(v), \tau_j(v))$ is non-degenerate $\forall v \in V$.*

Proof. The elements are computed in Appendix A. As the result, the matrix $\Omega(\tau_l, \tau_j)$ has the form

$$\Omega(v) := (\Omega(\tau_l, \tau_j))_{l,j=1,\ldots,6} = \begin{pmatrix} 0 & \Omega^+(v) \\ -\Omega^+(v) & 0 \end{pmatrix} \tag{3.7}$$

where the 3×3-matrix $\Omega^+(v)$ equals

$$\Omega^+(v) = K + (1 - v^2)^{-1/2} E + (1 - v^2)^{-3/2} v \otimes v \tag{3.8}$$

Here K is a symmetric 3×3-matrix with the elements

$$K_{ij} = \int dk k_j k_l \mathcal{B}(k) \frac{k^2 + m^2 + 3(v_j k_j)^2}{(k^2 + m^2 - (v_j k_j)^2)^3} \tag{3.9}$$

where $\mathcal{B}(k) > 0$ is defined in (1.10). The matrix K is the integral of the symmetric nonnegative definite matrix $k \otimes k = (k_i k_j)$ with a positive weight. Hence, the matrix K is nonnegative definite. Since the unite matrix E is positive definite, the matrix $\Omega^+(v)$ is symmetric and positive definite, hence non-degenerate. Then the matrix $\Omega(\tau_l, \tau_j)$ also is non-degenerate. □

Let us introduce the translations $T_a : (\psi(\cdot), q, p) \mapsto (\psi(\cdot - a), q + a, p)$, $a \in \mathbb{R}^3$. Note that the manifold $\mathcal{S}$ is invariant with respect to the translations. Let us denote $v(p) := p/\sqrt{1 + p^2}$ for $p \in \mathbb{R}^3$.

Definition 3.3. *i) For any $\alpha \in \mathbb{R}$ and $\overline{v} < 1$ denote by $\mathcal{E}_\alpha(\overline{v}) = \{Y = (\psi, q, p) \in \mathcal{E}_\alpha : |v(p)| \le \overline{v}\}$. We set $\mathcal{E}(\overline{v}) := \mathcal{E}_0(\overline{v})$.*
ii) For any $\tilde{v} < 1$ denote by $\Sigma(\tilde{v}) = \{\sigma = (b, v) : b \in \mathbb{R}^3, |v| \le \tilde{v}\}$.

The next Lemma provide that in a small neighborhood of the soliton manifold $\mathcal{S}$ a "symplectic orthogonal projection" onto $\mathcal{S}$ is well-defined.

Lemma 3.4. *(cf.[8, Lemma 3.4]) Let (1.9) hold, $\alpha \in \mathbb{R}$ and $\overline{v} < 1$. Then*
i) there exists a neighborhood $\mathcal{O}_\alpha(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{E}_\alpha$ and a map $\mathbf{\Pi} : \mathcal{O}_\alpha(\mathcal{S}) \to \mathcal{S}$ such that $\mathbf{\Pi}$ is uniformly continuous on $\mathcal{O}_\alpha(\mathcal{S}) \cap \mathcal{E}_\alpha(\overline{v})$ in the metric of $\mathcal{E}_\alpha$,

$$\mathbf{\Pi} Y = Y \quad \text{for} \quad Y \in \mathcal{S}, \qquad \text{and} \qquad Y - S \nmid \mathcal{T}_S\mathcal{S}, \quad \text{where} \quad S = \mathbf{\Pi} Y \tag{3.10}$$

ii) $\mathcal{O}_\alpha(\mathcal{S})$ is invariant with respect to the translations T_a, and

$$\mathbf{\Pi} T_a Y = T_a \mathbf{\Pi} Y, \qquad \text{for} \quad Y \in \mathcal{O}_\alpha(\mathcal{S}) \quad \text{and} \quad a \in \mathbb{R}^3$$

iii) For any $\overline{v} < 1$ there exists a $\tilde{v} < 1$ s.t. $\mathbf{\Pi} Y = S(\sigma)$ with $\sigma \in \Sigma(\tilde{v})$ for $Y \in \mathcal{O}_\alpha(\mathcal{S}) \cap \mathcal{E}_\alpha(\overline{v})$.
iv) For any $\tilde{v} < 1$ there exists an $r_\alpha(\tilde{v}) > 0$ s.t. $S(\sigma) + Z \in \mathcal{O}_\alpha(\mathcal{S})$ if $\sigma \in \Sigma(\tilde{v})$ and $\|Z\|_\alpha < r_\alpha(\tilde{v})$.

We will call $\mathbf{\Pi}$ a symplectic orthogonal projection onto $\mathcal{S}$.

Corollary 3.5. *The condition (2.8) implies that* $Y_0 = S + Z_0$ *where* $S = S(\sigma_0) = \mathbf{\Pi} Y_0$, *and*

$$\|Z_0\|_\nu \ll 1 \tag{3.11}$$

Proof. Lemma 3.4 implies that $\mathbf{\Pi} Y_0 = S$ is well defined for small $d_0 > 0$. Furthermore, the condition (2.8) means that there exists a point $S_1 \in \mathcal{S}$ such that $\|Y_0 - S_1\|_\nu = d_0$. Hence, $Y_0, S_1 \in \mathcal{O}_\nu(\mathcal{S}) \cap \mathcal{E}_\nu(\overline{v})$ with a $\overline{v} < 1$ which does not depend on d_0 for sufficiently small d_0. On the other hand, $\mathbf{\Pi} S_1 = S_1$, hence the uniform continuity of the map $\mathbf{\Pi}$ implies that $\|S_1 - S\|_\nu \to 0$ as $d_0 \to 0$. Therefore, finally, $\|Z_0\|_\nu = \|Y_0 - S\|_\nu \le \|Y_0 - S_1\|_\nu + \|S_1 - S\|_\nu \le d_0 + o(1) \ll 1$ for small d_0. $\square$

4 Linearization on solitary manifold

Let us consider a solution to the system (3.1), and split it as the sum

$$Y(t) = S(\sigma(t)) + Z(t) \tag{4.1}$$

where $\sigma(t) = (b(t), v(t)) \in \Sigma$ is an arbitrary smooth function of $t \in \mathbb{R}$. In detail, denote $Y = (\psi, q, p)$ and $Z = (\Psi, Q, P)$. Then (4.1) means that

$$\psi(x,t) = \psi_{v(t)}(x - b(t)) + \Psi(x - b(t), t), \quad q(t) = b(t) + Q(t), \quad p(t) = p_{v(t)} + P(t) \tag{4.2}$$

Let us substitute (4.2) to (1.1), and linearize the equations in Z. Setting $y = x - b(t)$ which is the "moving frame coordinate", we obtain that

$$\left.\begin{array}{rcl}
\dot\psi &=& \dot v \cdot \nabla_v \psi_v(y) - \dot b \cdot \nabla \psi_v(y) + \dot\Psi(y,t) - \dot b \cdot \nabla \Psi(y,t) \\[2ex]
&=& [-\alpha_j \partial_j - i\beta m](\psi_v(y) + \Psi(y,t)) - i\rho(y - Q) \\[2ex]
\dot q &=& \dot b + \dot Q = \dfrac{p_v + P}{\sqrt{1 + (p_v + P)^2}} \\[2ex]
\dot p &=& \dot v \cdot \nabla_v p_v + \dot P = \mathrm{Re}\langle \psi_v(y) + \Psi(y,t), \nabla \rho(y - Q)\rangle
\end{array}\right| \tag{4.3}$$

Let us extract linear terms in Q. First note that $\rho(y - Q) = \rho(y) - Q \cdot \nabla \rho(y) + N_1(Q)$, $\nabla \rho(y - Q) = \nabla \rho(y) - \nabla(Q \cdot \nabla \rho(y)) + \tilde N_1(Q)$.

The condition (1.9) implies that for $N_1(Q)$ and $\tilde N_1(Q)$ the bound holds,

$$\|N_1(Q)\|_\nu + \|\tilde N_1(Q)\|_\nu \le C_\nu(\overline{Q}) Q^2 \tag{4.4}$$

uniformly in $|Q| \le \overline{Q}$ for any fixed $\overline{Q}$, where ν is the parameter from Theorem 2.5. Second, the Taylor expansion gives

$$\frac{p_v + P}{\sqrt{1 + (p_v + P)^2}} = v + \frac{1}{\gamma}(P - v(v \cdot P)) + N_2(v, P)$$

where $1/\gamma = \sqrt{1 - v^2} = (1 + p_v^2)^{-1/2}$, and

$$|N_2(v, P)| \le C(\tilde v) P^2 \tag{4.5}$$

uniformly with respect to $|v| \le \tilde{v} < 1$. Using the equations (2.4), we obtain from (4.3) the following equations for the components of the vector $Z(t)$:

$$\left.\begin{aligned}
\dot\Psi(y,t) &= [-\alpha_j\partial_j - i\beta m]\Psi(y,t) + \dot b\cdot\nabla\Psi(y,t) + iQ\cdot\nabla\rho(y)\\
&\quad + (\dot b - v)\cdot\nabla\psi_v(y) - \dot v\cdot\nabla_v\psi_v(y) - iN_1\\
\dot Q(t) &= \tfrac{1}{\gamma}(E - v\otimes v)P + (v - \dot b) + N_2\\
\dot P(t) &= -\dot v\cdot\nabla_v p_v + \mathrm{Re}\langle\Psi(y,t),\nabla\rho(y)\rangle + \mathrm{Re}\langle\nabla\psi_v(y), Q\cdot\nabla\rho(y)\rangle + N_3(v,Z)
\end{aligned}\right| \tag{4.6}$$

where $N_3(v,Z) = -\mathrm{Re}\langle\nabla\psi_v, N_1(Q)\rangle - \mathrm{Re}\langle\Psi,\nabla(Q\cdot\nabla\rho)\rangle + \mathrm{Re}\langle\Psi,\tilde N_1(Q)\rangle$. Clearly, $N_3(v,Z)$ satisfies the following estimate

$$|N_3(v,Z)| \le C_\nu(\rho,\overline{v},\overline{Q})\Big[Q^2 + \|\Psi\|_{-\nu}|Q|\Big] \tag{4.7}$$

uniformly in $|v| \le \tilde v$ and $|Q| \le \overline{Q}$ for any fixed $\tilde v < 1$. For the vector version $Z = (\Psi_1,\Psi_2,Q,P)$ with $\Psi_1 = \mathrm{Re}\,\Psi$, $\Psi_2 = \mathrm{Im}\,\Psi$ we rewrite the equations (4.6) as

$$\dot Z(t) = A(t)Z(t) + T(t) + N(t),\quad t\in\mathbb{R} \tag{4.8}$$

Here the operator $A(t) = A_{v,w}(t)$ depends on two parameters, $v = v(t)$, and $w = \dot b(t)$ and can be written in the form

$$A_{v,w}\begin{pmatrix}\Psi_1\\ \Psi_2\\ Q\\ P\end{pmatrix} = \begin{pmatrix} -\alpha_1\partial_1-\alpha_2\partial_2+w\cdot\nabla & \tilde\alpha_2\partial_2+\beta m & -\nabla\rho_2\cdot & 0\\ -(\tilde\alpha_2\partial_2+\beta m) & -\alpha_1\partial_1-\alpha_2\partial_2+w\cdot\nabla & \nabla\rho_1\cdot & 0\\ 0 & 0 & 0 & B_v\\ \langle\cdot,\nabla\rho_1\rangle & \langle\cdot,\nabla\rho_2\rangle & \langle\nabla\psi_{vj},\cdot\nabla\rho_j\rangle & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\\ Q\\ P\end{pmatrix} \tag{4.9}$$

where $B_v = \frac{1}{\gamma}(E - v\otimes v)$. Furthermore, $T(t) = T_{v,w}(t)$ and $N(t) = N(t,\sigma,Z)$ in (4.8) stand for

$$T_{v,w} = \begin{pmatrix}(w-v)\cdot\nabla\psi_{v1} - \dot v\cdot\nabla_v\psi_{v1}\\ (w-v)\cdot\nabla\psi_{v2} - \dot v\cdot\nabla_v\psi_{v2}\\ v-w\\ -\dot v\cdot\nabla_v p_v\end{pmatrix},\qquad N(\sigma,Z) = \begin{pmatrix}N_{12}(Z)\\ -N_{11}(Z)\\ N_2(v,Z)\\ N_3(v,Z)\end{pmatrix} \tag{4.10}$$

where $v = v(t)$, $w = w(t)$, $\sigma = \sigma(t) = (b(t),v(t))$, and $Z = Z(t)$. The estimates (4.4), (4.5) and (4.7) imply that

$$\|N(\sigma,Z)\|_\nu \le C(\tilde v,\overline{Q})\|Z\|^2_{-\nu} \tag{4.11}$$

uniformly in $\sigma\in\Sigma(\tilde v)$ and $\|Z\|_{-\nu} \le r_{-\nu}(\tilde v)$ for any fixed $\tilde v < 1$.

Remark 4.1. i) The term $A(t)Z(t)$ in the right hand side of the equation (4.8) is linear in $Z(t)$, and $N(t)$ is a *high order term* in $Z(t)$. On the other hand, $T(t)$ is a zero order term which does not vanish at $Z(t) = 0$ since $S(\sigma(t))$ generally is not a soliton solution if (2.7) does not hold (though $S(\sigma(t))$ belongs to the solitary manifold).
ii) Formulas (3.5) and (4.10) imply:

$$T(t) = -\sum_{l=1}^{3}[(w-v)_l\tau_l + \dot v_l\tau_{l+3}] \tag{4.12}$$

and hence $T(t)\in\mathcal{T}_{S(\sigma(t))}\mathcal{S}$, $t\in\mathbb{R}$.

5 Linearized equation

Here we collect some Hamiltonian and spectral properties of the generator (4.9) of the linearized equation. First, let us consider the linear equation

$$\dot{X}(t) = A_{v,w}X(t), \quad t \in \mathbb{R}, \quad v \in V, \quad w \in \mathbb{R}^3 \tag{5.1}$$

Lemma 5.1. *(cf. Lemma 5.1 [8]) i) For any $v \in V$ and $w \in \mathbb{R}^3$ the equation (5.1) can be written as the Hamilton system (cf. (3.3)),*

$$\dot{X}(t) = JD\mathcal{H}_{v,w}(X(t)), \quad t \in \mathbb{R} \tag{5.2}$$

where $D\mathcal{H}_{v,w}$ is the Fréchet derivative with respect to Ψ_{1k}, Ψ_{2k}, $k = 1, 2, 3, 4$, P and Q of the Hamilton functional

$$\mathcal{H}_{v,w}(X) = \frac{1}{2}\int(\Psi_1 \cdot (\tilde{\alpha}_2\partial_2 + \beta m)\Psi_1 + \Psi_2 \cdot (\tilde{\alpha}_2\partial_2 + \beta m)\Psi_2 + 2\Psi_1 \cdot (\alpha_1\partial_1 + \alpha_3\partial_3)\Psi_2)dy$$
$$+\int \rho_j(y)Q \cdot \nabla\Psi_j dy + \frac{1}{2}P \cdot B_v P - \frac{1}{2}\langle Q \cdot \nabla\psi_{vj}(y), Q \cdot \nabla\rho_j(y)\rangle, \quad X = (\Psi_1, \Psi_2, Q, P) \in \mathcal{E} \tag{5.3}$$

ii) The skew-symmetry relation holds,

$$\Omega(A_{v,w}X_1, X_2) = -\Omega(X_1, A_{v,w}X_2), \quad X_1 \in \mathcal{E}, \quad X_2 \in H^1(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \tag{5.4}$$

Lemma 5.2. *The operator $A_{v,w}$ acts on the tangent vectors $\tau_j(v)$ to the solitary manifold as follows,*

$$A_{v,w}[\tau_j(v)] = (w - v) \cdot \nabla\tau_j(v), \;\; A_{v,w}[\tau_{j+3}(v)] = (w - v) \cdot \nabla\tau_{j+3}(v) + \tau_j(v), \;\; j = 1, 2, 3 \tag{5.5}$$

Proof. In detail, we have to show that

$$A_{v,w}\begin{pmatrix} -\partial_j\psi_{v1} \\ -\partial_j\psi_{v2} \\ e_j \\ 0 \end{pmatrix} = \begin{pmatrix} (v - w) \cdot \nabla\partial_j\psi_{v1} \\ (v - w) \cdot \nabla\partial_j\psi_{v2} \\ 0 \\ 0 \end{pmatrix}$$
$$A_{v,w}\begin{pmatrix} \partial_{v_j}\psi_{v1} \\ \partial_{v_j}\psi_{v2} \\ 0 \\ \partial_{v_j}p_v \end{pmatrix} = \begin{pmatrix} (w - v) \cdot \nabla\partial_{v_j}\psi_{v1} \\ (w - v) \cdot \nabla\partial_{v_j}\psi_{v2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\partial_j\psi_{v1} \\ -\partial_j\psi_{v2} \\ e_j \\ 0 \end{pmatrix} \tag{5.6}$$

Indeed, differentiate the equations (2.4) in b_j and v_j, and obtain that the derivatives of soliton state in parameters satisfy the following equations,

$$\left.\begin{aligned} -v \cdot \nabla\partial_j\psi_v &= [-\alpha \cdot \nabla - i\beta m]\partial_j\psi_v - i\partial_j\rho \\ -\partial_j\psi_v - v \cdot \nabla\partial_{v_j}\psi_v &= [-\alpha \cdot \nabla - i\beta m]\partial_{v_j}\psi_v \\ \partial_{v_j}p_v &= e_j(1 - v^2)^{-1/2} + v\frac{v_j}{(1 - v^2)^{3/2}} \\ 0 &= \langle\partial_{v_j}\psi_{v1}, \nabla\rho_1\rangle + \langle\partial_{v_j}\psi_{v2}, \nabla\rho_2\rangle \end{aligned}\right| \tag{5.7}$$

for $j = 1, 2, 3$. Then (5.6) follows from (5.7) by definition of A in (4.9) $\square$

Corollary 5.3. *Let $w = v \in V$. Then $\tau_j(v)$ are eigenvectors, and $\tau_{j+3}(v)$ are root vectors of the operator $A_{v,v}$, corresponding to zero eigenvalue, i.e.*

$$A_{v,v}[\tau_j(v)] = 0, \ \ A_{v,v}[\tau_{j+3}(v)] = \tau_j(v), \ \ j = 1, 2, 3. \tag{5.8}$$

Remark 5.4. For a soliton solution of the system(3.1) we have $\dot{b} = v$, $\dot{v} = 0$, and hence $T(t) \equiv 0$. Thus, the equation (5.1) is the linearization of the system (3.1) on a soliton solution. In fact, we do not linearize (3.1) on a soliton solution, but on a trajectory $S(\sigma(t))$ with $\sigma(t)$ being nonlinear in t. We will show later that $T(t)$ is quadratic in $Z(t)$ if we choose $S(\sigma(t))$ to be the symplectic orthogonal projection of $Y(t)$. Then (5.1) is again the linearization of (3.1).

6 Symplectic decomposition of dynamics

Here we decompose the dynamics in two components: along the manifold $\mathcal{S}$ and in transversal directions. The equation (4.8) is obtained without any assumption on $\sigma(t)$ in (4.1). We are going to choose $S(\sigma(t)) := \mathbf{\Pi} Y(t)$, but then we need to know that

$$Y(t) \in \mathcal{O}_{-\nu}(\mathcal{S}), \qquad t \in \mathbb{R} \tag{6.1}$$

It is true for $t = 0$ by our main assumption (2.8) with sufficiently small $d_0 > 0$. Then $S(\sigma(0)) = \mathbf{\Pi} Y(0)$ and $Z(0) = Y(0) - S(\sigma(0))$ are well defined. We will prove below that (6.1) holds if d_0 is sufficiently small. Let us choose an arbitrary $\tilde{v}$ such that $|v(0)| < \tilde{v} < 1$ and let $\delta = \tilde{v} - |v(0)|$. Denote by $r_{-\nu}(\tilde{v})$ the positive numbers from Lemma 3.4 iv) which corresponds to $\alpha = -\nu$. Then $S(\sigma) + Z \in \mathcal{O}_{-\nu}(\mathcal{S})$ if $\sigma = (b, v)$ with $|v| < \tilde{v}$ and $\|Z\|_{-\nu} < r_{-\nu}(\tilde{v})$. Note that $\|Z(0)\|_{-\nu} < r_{-\nu}(\tilde{v})$ if d_0 is sufficiently small. Therefore, $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ and $Z(t) = Y(t) - S(\sigma(t))$ are well defined for $t \geq 0$ so small that $|v| < \tilde{v}$ and $\|Z(t)\|_{-\nu} < r_{-\nu}(\tilde{v})$. This is formalized by the following standard definition.

Definition 6.1. *t_* is the "exit time",*

$$t_* = \sup\{t > 0 : \|Z(s)\|_{-\nu} < r_{-\nu}(\tilde{v}), \ \ |v(s) - v(0)| < \delta, \ \ 0 \leq s \leq t\} \tag{6.2}$$

One of our main goals is to prove that $t_* = \infty$ if d_0 is sufficiently small. This would follow if we show that

$$\|Z(t)\|_{-\nu} < r_{-\nu}(\tilde{v})/2, \ \ |v(s) - v(0)| < \delta/2, \quad 0 \leq t < t_* \tag{6.3}$$

Note that

$$|Q(t)| \leq \overline{Q} := r_{-\nu}(\tilde{v}), \qquad 0 \leq t < t_* \tag{6.4}$$

Now $N(t)$ in (4.8) satisfies, by (4.11), the following estimate,

$$\|N(t)\|_{\nu} \leq C_{\nu}(\tilde{v}) \|Z(t)\|^2_{-\nu}, \ \ 0 \leq t < t_* \tag{6.5}$$

6.1 Longitudinal dynamics: modulation equations

From now on we fix the decomposition $Y(t) = S(\sigma(t)) + Z(t)$ for $0 < t < t_*$ by setting $S(\sigma(t)) = \mathbf{\Pi} Y(t)$ which is equivalent to the symplectic orthogonality condition of type (3.10),

$$Z(t) \nmid \mathcal{T}_{S(\sigma(t))} \mathcal{S}, \ \ 0 \leq t < t_* \tag{6.6}$$

This allows us to simplify drastically the asymptotic analysis of the dynamical equations (4.8) for the transversal component $Z(t)$. As the first step, we derive the longitudinal dynamics, i.e. the "modulation equations" for the parameters $\sigma(t)$. Let us derive a system of ordinary differential equations for the vector $\sigma(t)$. For this purpose, let us write (6.6) in the form

$$\Omega(Z(t),\tau_j(t))=0,\ j=1,\dots,6, \qquad 0\le t<t_* \tag{6.7}$$

where the vectors $\tau_j(t)=\tau_j(\sigma(t))$ span the tangent space $\mathcal{T}_{S(\sigma(t))}\mathcal{S}$. Note that $\sigma(t)=(b(t),v(t))$, where

$$|v(t)|\le\tilde v<1, \qquad 0\le t<t_* \tag{6.8}$$

by Lemma 3.4 iii). It would be convenient for us to use some other parameters (c,v) instead of $\sigma=(b,v)$, where

$$c(t)=b(t)-\int_0^t v(\tau)d\tau, \quad \dot c(t)=\dot b(t)-v(t)=w(t)-v(t), \quad 0\le t<t_* \tag{6.9}$$

The following statement can be proved similar to the Lemma 6.2 in [8].

Lemma 6.2. *Let $Y(t)$ be a solution to the Cauchy problem (3.1), and (4.1), (6.7) hold. Then*

$$|\dot c(t)|+|\dot v(t)|\le C(\tilde v)\|Z\|^2_{-\nu} \tag{6.10}$$

6.2 Decay for transversal dynamics

In Section 11 we will show that our main Theorem 2.5 can be derived from the following time decay of the transversal component $Z(t)$:

Proposition 6.3. *Let all conditions of Theorem 2.5 hold. Then $t_*=\infty$, and*

$$\|Z(t)\|_{-\nu}\le\frac{C(\rho,\overline{v},d_0)}{(1+|t|)^{3/2}}, \qquad t\ge 0 \tag{6.11}$$

We will derive (6.11) in Sections 7-10 from our equation (4.8) for the transversal component $Z(t)$. This equation can be specified using Lemma 6.2. Indeed, the lemma implies that

$$\|T(t)\|_\nu\le C(\tilde v)\|Z(t)\|^2_{-\nu}, \qquad 0\le t<t_* \tag{6.12}$$

by (4.10) since $w-v=\dot c$. Thus (4.8) becomes the equation

$$\dot Z(t)=A(t)Z(t)+\tilde N(t), \qquad 0\le t<t_* \tag{6.13}$$

where $A(t)=A_{v(t),w(t)}$, and $\tilde N(t):=T(t)+N(t)$ satisfies the estimate

$$\|\tilde N(t)\|_\nu\le C(\tilde v,\overline{Q})\|Z(t)\|^2_{-\nu}, \qquad 0\le t<t_* \tag{6.14}$$

In all remaining part of our paper we will analyze mainly the **basic equation** (6.13) to establish the decay (6.11). We are going to derive the decay using the bound (6.14) and the orthogonality condition (6.6).

Similarly [8] we reduce the problem to the analysis of the *frozen* linear equation,

$$\dot X(t)=A_1X(t), \quad t\in\mathbb{R} \tag{6.15}$$

where $A_1=A_{v_1,v_1}$ with $v_1=v(t_1)$ and a fixed $t_1\in[0,t_*)$. Then we can apply some methods of scattering theory and then estimate the error by the method of majorants.

Note, that even for the frozen equation (6.15), the decay of type (6.11) for all solutions does not hold without the orthogonality condition of type (6.6). Namely, by (5.8) the equation (6.15) admits the *secular solutions*

$$X(t) = \sum_1^3 C_j \tau_j(v) + \sum_1^3 D_j[\tau_j(v)t + \tau_{j+3}(v)] \tag{6.16}$$

which arise by differentiation of the soliton (1.5) in the parameters a and v in the moving coordinate $y = x - v_1 t$. Hence, we have to take into account the orthogonality condition (6.6) in order to avoid the secular solutions. For this purpose we will apply the corresponding symplectic orthogonal projection which kills the "runaway solutions" (6.16).

Remark 6.4. The solution (6.16) lies in the tangent space $\mathcal{T}_{S(\sigma_1)}\mathcal{S}$ with $\sigma_1 = (b_1, v_1)$ (for an arbitrary $b_1 \in \mathbb{R}$) that suggests an unstable character of the nonlinear dynamics *along the solitary manifold.*

Definition 6.5. *i) For $v \in V$, denote by $\mathbf{\Pi}_v$ the symplectic orthogonal projection of $\mathcal{E}$ onto the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$, and $\mathbf{P}_v = \mathbf{I} - \mathbf{\Pi}_v$.*
ii) Denote by $\mathcal{Z}_v = \mathbf{P}_v\mathcal{E}$ the space symplectic orthogonal to $\mathcal{T}_{S(\sigma)}\mathcal{S}$ with $\sigma = (b, v)$.

Note that by the linearity,

$$\mathbf{\Pi}_v Z = \sum \mathbf{\Pi}_{jl}(v)\tau_j(v)\Omega(\tau_l(v), Z), \qquad Z \in \mathcal{E} \tag{6.17}$$

with some smooth coefficients $\mathbf{\Pi}_{jl}(v)$. Hence, the projector $\mathbf{\Pi}_v$, in the variable $y = x - b$, does not depend on b, and this explains the choice of the subindex in $\mathbf{\Pi}_v$ and $\mathbf{P}_v$.

Now we have the symplectic orthogonal decomposition

$$\mathcal{E} = \mathcal{T}_{S(\sigma)}\mathcal{S} + \mathcal{Z}_v, \qquad \sigma = (b, v), \tag{6.18}$$

and the symplectic orthogonality (6.6) can be written in the equivalent forms,

$$\mathbf{\Pi}_{v(t)}Z(t) = 0, \quad \mathbf{P}_{v(t)}Z(t) = Z(t), \qquad 0 \le t < t_* \tag{6.19}$$

Remark 6.6. The tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ is invariant under the operator $A_{v,v}$ by Lemma 5.3 i), hence the space $\mathcal{Z}_v$ is also invariant by (5.4): $A_{v,v}Z \in \mathcal{Z}_v$ for *sufficiently smooth* $Z \in \mathcal{Z}_v$.

Below in section 12-18 we will prove the following proposition which will be one of the main ingredients for proving (6.11). Let us consider the Cauchy problem for the equation (6.15) with $A = A_{v,v}$ for a fixed $v \in V$. Recall that the parameter $\nu > 5/2$ is also fixed.

Proposition 6.7. *Let the conditions* (1.9)- (1.10) *hold, $|v| \le \tilde{v} < 1$, and $X_0 \in \mathcal{E}$. Then*
i) Equation (6.15), with $A = A_{v,v}$, admits the unique solution $e^{At}X_0 := X(t) \in C(\mathbb{R}, \mathcal{E})$ with the initial condition $X(0) = X_0$.
ii) For $X_0 \in \mathcal{Z}_v \cap \mathcal{E}_\nu$, the decay holds,

$$\|e^{At}X_0\|_{-\nu} \le \frac{C_\nu(\rho, \tilde{v})}{(1+|t|)^{3/2}}\|X_0\|_\nu, \quad t \in \mathbb{R} \tag{6.20}$$

7 Frozen transversal dynamics

Now let us fix an arbitrary $t_1 \in [0, t_*)$, and rewrite the equation (6.13) in a "frozen form"

$$\dot{Z}(t) = A_1 Z(t) + (A(t) - A_1)Z(t) + \tilde{N}(t), \qquad 0 \le t < t_* \tag{7.1}$$

where $A_1 = A_{v(t_1),v(t_1)}$ and

$$A(t)-A_1 = \begin{pmatrix} [w(t)-v(t_1)]\cdot\nabla & 0 & 0 & 0 \\ 0 & [w(t)-v(t_1)]\cdot\nabla & 0 & 0 \\ 0 & 0 & 0 & B_{v(t)}-B_{v_1(t)} \\ 0 & 0 & \langle\nabla(\psi_{v(t)j}-\psi_{v(t_1)j}), \nabla\rho_j\rangle & 0 \end{pmatrix}$$

The next trick is important since it allows us to kill the "bad terms" $[w(t)-v(t_1)]\cdot\nabla$ in the operator $A(t) - A_1$.

Definition 7.1. *Let us change the variables $(y, t) \mapsto (y_1, t) = (y + d_1(t), t)$, where*

$$d_1(t) := \int_{t_1}^t (w(s) - v(t_1))ds, \qquad 0 \le t \le t_1 \tag{7.2}$$

Next define

$$Z_1(t) := (\Psi_1(y_1 - d_1(t), t), \Psi_2(y_1 - d_1(t), t), Q(t), P(t)) \tag{7.3}$$

Then we obtain the final form of the "frozen equation" for the transversal dynamics

$$\dot{Z}_1(t) = A_1 Z_1(t) + B_1(t) Z_1(t) + \tilde{N}_1(t), \;\; 0 \le t \le t_1 \tag{7.4}$$

where $\tilde{N}_1(t) = \tilde{N}(t)$ expressed in terms of $y = y_1 - d_1(t)$, and

$$B_1(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{v(t)} - B_{v_1(t)} \\ 0 & 0 & \langle\nabla(\psi_{v(t)j}-\psi_{v(t_1)j}), \nabla\rho_j\rangle & 0 \end{pmatrix}$$

Let us estimate the "remaining terms" $B_1(t)Z_1(t)$ and $\tilde{N}_1(t)$.

Lemma 7.2. *The bound holds*

$$\|B_1(t)Z_1(t)\|_\nu \le C(\tilde{v})\|Z(t)\|_{-\nu} \int_t^{t_1} \|Z(s)\|^2_{-\nu} ds, \quad 0 \le t \le t_1 \tag{7.5}$$

Proof. Lemma 6.2 implies

$$|B_{v(t)} - B_{v_1(t)}| \le |\int_{t_1}^{t} \dot{v}(s)\cdot\nabla_v B_{v(s)} ds| \le C(\tilde{v}) \int_{t_1}^{t} \|Z(s)\|^2_{-\nu} ds$$

similarly,

$$|\langle\nabla(\psi_{v(t)j} - \psi_{v(t_1)j}), \nabla\rho_j| \le C(\tilde{v}) \int_{t_1}^{t} \|Z(s)\|^2_{-\nu} ds$$

Therefore,

$$\begin{aligned} \|B_1(t)Z_1(t)\|_\nu &= |\langle\nabla(\psi_{v(t)j} - \psi_{v(t_1)j}), \nabla\rho_j\rangle Q_1(t)| + |(B_{v(t)} - B_{v_1(t)})P_1(t)| \\ &\le C(\tilde{v})(|Q(t)| + |P(t)|) \int_{t_1}^{t} \|Z(s)\|^2_{-\nu} ds \le C(\tilde{v})\|Z(t)\|_{-\nu} \int_t^{t_1} \|Z(s)\|^2_{-\nu} ds \end{aligned}$$

□

Lemma 7.3. *The bounds hold*

$$\|\tilde{N}_1(t)\|_\nu \leq C(\tilde{v},\overline{Q})(1+|d_1(t)|)^\nu \|Z(t)\|^2_{-\nu}, \quad 0 \leq t \leq t_1 \tag{7.6}$$

Proof. For any $\Phi \in L^2_\alpha$ and $d \in \mathbb{R}^3$ we have

$$\begin{aligned}\|\Phi(y-d)\|^2_\alpha &= \int |\Phi(y-d)|^2(1+|y|)^{2\alpha}dy = \int |\Phi(y)|^2(1+|y+d|)^{2\alpha}dy \\ &\leq \int |\Phi(y)|^2(1+|y|)^{2\alpha}(1+|d|)^{2\alpha}dy \leq (1+|d|)^{2\alpha}\|\Phi\|^2_\alpha, \quad \alpha \in \mathbb{R}\end{aligned}$$

Hence, the bound (7.6) follows. □

8 Integral inequality

The equation (7.4) can be written in the integral form:

$$Z_1(t) = e^{A_1 t}Z_1(0) + \int_0^t e^{A_1(t-s)}[B_1 Z_1(s) + \tilde{N}_1(s)]ds, \quad 0 \leq t \leq t_1 \tag{8.1}$$

Now we apply the symplectic orthogonal projection $\mathbf{P}_1 := \mathbf{P}_{v(t_1)}$ to both sides of (8.1):

$$\mathbf{P}_1 Z_1(t) = e^{A_1 t}\mathbf{P}_1 Z_1(0) + \int_0^t e^{A_1(t-s)}\mathbf{P}_1[B_1 Z_1(s) + \tilde{N}_1(s)]ds$$

The projector $\mathbf{P}_1$ commutes with the group $e^{A_1 t}$ since the space $\mathcal{Z}_1 := \mathbf{P}_1\mathcal{E}$ is invariant with respect to $e^{A_1 t}$ by Remark 6.6. Applying (6.20) we obtain that

$$\|\mathbf{P}_1 Z_1(t)\|_{-\nu} \leq C\frac{\|\mathbf{P}_1 Z_1(0)\|_\nu}{(1+t)^{3/2}} + C\int_0^t \frac{\|\mathbf{P}_1[B_1 Z_1(s) + \tilde{N}_1(s)]\|_\nu \, ds}{(1+|t-s|)^{3/2}}.$$

The operator $\mathbf{P}_1 = \mathbf{I} - \mathbf{\Pi}_1$ is continuous in $\mathcal{E}_\nu$ by (6.17). Hence, (7.5)-(7.6) imply

$$\begin{aligned}\|\mathbf{P}_1 Z_1(t)\|_{-\nu} &\leq \frac{C(\overline{d}_1(0))}{(1+t)^{3/2}}\|Z(0)\|_\nu \\ &+ C(\overline{d}_1(t))\int_0^t \frac{1}{(1+|t-s|)^{3/2}}\left[\|Z(s)\|_{-\nu}\int_s^{t_1}\|Z(\tau)\|^2_{-\nu}d\tau + \|Z(s)\|^2_{-\nu}\right]ds, \quad 0 \leq t \leq t_1.\end{aligned} \tag{8.2}$$

where $\overline{d}_1(t) := \sup_{0\leq s\leq t}|d_1(s)|$. Let us introduce the "majorant"

$$m(t) := \sup_{s\in[0,t]}(1+s)^{3/2}\|Z(s)\|_{-\nu}, \quad t \in [0,t_*). \tag{8.3}$$

Now we reduce further the exit time. Denote by $\varepsilon < 1$ a fixed positive number which we will specify below.

Definition 8.1. t'_* *is the exit time*

$$t'_* = \sup\{t \in [0,t_*) : m(s) \leq \varepsilon, \quad 0 \leq s \leq t\}. \tag{8.4}$$

To estimate $d_1(t)$, note that

$$w(s)-v(t_1)=w(s)-v(s)+v(s)-v(t_1)=\dot c(s)+\int_s^{t_1}\dot v(\tau)d\tau \qquad (8.5)$$

by (6.9). Hence, (7.2), Lemma 6.2 and the definition (8.3) imply that for $t_1<t_*'$

$$|d_1(t)|=|\int_{t_1}^t(w(s)-v(t_1))ds|\le\int_t^{t_1}\Big(|\dot c(s)|+\int_s^{t_1}|\dot v(\tau)|d\tau\Big)ds$$

$$\le C(\tilde v)m^2(t_1)\int_t^{t_1}\Big(\frac{1}{(1+s)^3}+\int_s^{t_1}\frac{d\tau}{(1+\tau)^3}\Big)ds\le C(\tilde v)m^2(t_1)\le C(\tilde v),\quad 0\le t\le t_1 \qquad (8.6)$$

Now we can to replace $C(\overline{d}_1)$ with $C(\tilde v)$ in (8.2): for $t_1<t_*'$

$$\|\mathbf{P}_1Z_1(t)\|_{-\nu}\le\frac{C(\tilde v)}{(1+t)^{3/2}}\|Z(0)\|_\nu$$
$$+C(\tilde v)\int_0^t\frac{1}{(1+|t-s|)^{3/2}}\Big[\|Z(s)\|_{-\nu}\int_s^{t_1}\|Z(\tau)\|_{-\nu}^2d\tau+\|Z(s)\|_{-\nu}^2\Big]ds,\quad 0\le t\le t_1 \qquad (8.7)$$

9 Symplectic orthogonality

Finally, we are going to change $\mathbf{P}_1Z_1(t)$ by $Z(t)$ in the left hand side of (8.7). We will prove that it is possible using again that $d_0\ll 1$ in (2.8).

Lemma 9.1. *(cf.[8]) For sufficiently small $\varepsilon>0$, we have for $t_1<t_*'$*

$$\|Z(t)\|_{-\nu}\le C\|\mathbf{P}_1Z_1(t)\|_{-\nu},\qquad 0\le t\le t_1, \qquad (9.1)$$

where C depends only on ρ and $\overline{v}$.

Proof. Since $|d_1(t)|\le C$ for $t\le t_1<t_*'$ then $\|Z(t)\|_{-\nu}\le C\|Z_1(t)\|_{-\nu}$, and it suffices to prove that

$$\|Z_1(t)\|_{-\nu}\le 2\|\mathbf{P}_1Z_1(t)\|_{-\nu},\quad 0\le t\le t_1. \qquad (9.2)$$

Recall that $\mathbf{P}_1Z_1(t)=Z_1(t)-\mathbf{\Pi}_{v(t_1)}Z_1(t)$. Then estimate (9.2) will follow from

$$\|\mathbf{\Pi}_{v(t_1)}Z_1(t)\|_{-\nu}\le\frac12\|Z_1(t)\|_{-\nu},\quad 0\le t\le t_1. \qquad (9.3)$$

Symplectic orthogonality (6.19) implies

$$\mathbf{\Pi}_{v(t),1}Z_1(t)=0,\qquad t\in[0,t_1], \qquad (9.4)$$

where $\mathbf{\Pi}_{v(t),1}Z_1(t)$ is $\mathbf{\Pi}_{v(t)}Z(t)$ expressed in terms of the variable $y_1=y+d_1(t)$. Hence, (9.3) follows from (9.4) if the difference $\mathbf{\Pi}_{v(t_1)}-\mathbf{\Pi}_{v(t),1}$ is small uniformly in t, i.e.

$$\|\mathbf{\Pi}_{v(t_1)}-\mathbf{\Pi}_{v(t),1}\|<1/2,\qquad 0\le t\le t_1. \qquad (9.5)$$

It remains to justify (9.5) for small enough $\varepsilon>0$. Formula (6.17) implies

$$\mathbf{\Pi}_{v(t),1}Z_1(t)=\sum\mathbf{\Pi}_{jl}(v(t))\tau_{j,1}(v(t))\Omega(\tau_{l,1}(v(t)),Z_1(t)), \qquad (9.6)$$

where $\tau_{j,1}(v(t))$ are the vectors $\tau_j(v(t))$ expressed in the variables y_1. Since $|d_1(t)| \le C$ and $\nabla\tau_j$ are smooth and fast decaying at infinity functions, then

$$\|\tau_{j,1}(v(t)) - \tau_j(v(t))\|_\nu \le C|d_1(t)|^\nu \le C, \quad 0 \le t \le t_1 \tag{9.7}$$

for all $j = 1, 2, \dots, 6$. Furthermore,

$$\tau_j(v(t)) - \tau_j(v(t_1)) = \int_t^{t_1} \dot v(s) \cdot \nabla_v \tau_j(v(s))ds$$

and therefore

$$\|\tau_j(v(t)) - \tau_j(v(t_1))\|_\nu \le C\int_t^{t_1} |\dot v(s)|ds, \quad 0 \le t \le t_1 \tag{9.8}$$

Similarly,

$$|\mathbf{\Pi}_{jl}(v(t)) - \mathbf{\Pi}_{jl}(v(t_1))| = |\int_t^{t_1} \dot v(s) \cdot \nabla_v \mathbf{\Pi}_{jl}(v(s))ds| \le C\int_t^{t_1} |\dot v(s)|ds, \quad 0 \le t \le t_1 \tag{9.9}$$

since $|\nabla_v \mathbf{\Pi}_{jl}(v(s))|$ is uniformly bounded by (6.8). Hence, the bounds (9.5) will follow from (6.17), (9.6) and (9.7)-(9.9) if we establish that the integral in the right hand side of (9.8) can be made as small as we please by choosing $\varepsilon > 0$ small enough. Indeed,

$$\int_t^{t_1} |\dot v(s)|ds \le Cm^2(t_1)\int_t^{t_1} \frac{ds}{(1+s)^3} \le C\varepsilon^2, \quad 0 \le t \le t_1 \tag{9.10}$$

□

10 Decay of transversal component

Here we prove Proposition 6.3.
Step i) We fix $0 < \varepsilon < 1$ and $t_*' = t_*'(\varepsilon)$ for which Lemma 9.1 holds. Then the bound of type (8.7) holds with $\|\mathbf{P}_1 Z_1(t)\|_{-\nu}$ in the left hand side replaced by $\|Z(t)\|_{-\nu}$:

$$\begin{aligned}\|Z(t)\|_{-\nu} &\le \frac{C}{(1+t)^{3/2}}\|Z(0)\|_\nu \\ &+ C\int_0^t \frac{1}{(1+|t-s|)^{3/2}}\left[\|Z(s)\|_{-\nu}\int_s^{t_1}\|Z(\tau)\|^2_{-\nu}d\tau + \|Z(s)\|^2_{-\nu}\right]ds, \quad 0 \le t \le t_1\end{aligned} \tag{10.1}$$

for $t_1 < t_*'$. This implies an integral inequality for the majorant $m(t)$ defined in (8.3). Namely, multiplying both sides of (10.1) by $(1+t)^{3/2}$, and taking the supremum in $t \in [0, t_1]$, we get

$$m(t_1) \le C\|Z(0)\|_\nu + C\sup_{t\in[0,t_1]}\int_0^t \frac{(1+t)^{3/2}}{(1+|t-s|)^{3/2}}\left[\frac{m(s)}{(1+s)^{3/2}}\int_s^{t_1}\frac{m^2(\tau)d\tau}{(1+\tau)^3} + \frac{m^2(s)}{(1+s)^3}\right]ds$$

for $t_1 \le t_*'$. Taking into account that $m(t)$ is a monotone increasing function, we get

$$m(t_1) \le C\|Z(0)\|_\nu + C[m^3(t_1) + m^2(t_1)]I(t_1), \quad t_1 \le t_*' \tag{10.2}$$

where

$$I(t_1) = \sup_{t\in[0,t_1]}\int_0^t \frac{(1+t)^{3/2}}{(1+|t-s|)^{3/2}}\left[\frac{1}{(1+s)^{3/2}}\int_s^{t_1}\frac{d\tau}{(1+\tau)^3} + \frac{1}{(1+s)^3}\right]ds \le \overline{I} < \infty$$

Therefore, (10.2) becomes

$$m(t_1) \le C\|Z(0)\|_\nu + C\overline{I}[m^3(t_1) + m^2(t_1)], \qquad t_1 < t'_* \tag{10.3}$$

This inequality implies that $m(t_1)$ is bounded for $t_1 < t'_*$, and moreover,

$$m(t_1) \le C_1\|Z(0)\|_\nu, \qquad\qquad t_1 < t'_* \tag{10.4}$$

since $m(0) = \|Z(0)\|_\nu$ is sufficiently small by (3.11).
Step ii) The constant C_1 in the estimate (10.4) does not depend on t_* and t'_* by Lemma 9.1. We choose d_0 in (2.8) so small that $\|Z(0)\|_\nu < \varepsilon/(2C_1)$. It is possible due to (3.11). Then the estimate (10.4) implies that $t'_* = t_*$ and therefore (10.4) holds for all $t_1 < t_*$. Further,

$$|v(t) - v(0)| \le \int\limits_0^t |\dot v(s)|ds \le Cm^2(t)\int\limits_0^t \frac{ds}{(1+s)^3} \le Cm^2(t)$$

Hence the both inequalities (6.3) also holds if $\|Z(0)\|_\nu$ is sufficiently small by (8.3). Finally, this implies that $t_* = \infty$, hence also $t'_* = \infty$ and (10.4) holds for all $t_1 > 0$ if d_0 is small enough. It complete the proof of Proposition 6.3.

11 Soliton asymptotics

Here we prove our main Theorem 2.5 under the assumption that the decay (6.11) holds. First we will prove the asymptotics (2.9) for the vector components, and afterwards the asymptotics (2.10) for the fields.
Asymptotics for the vector components. From (4.3) we have $\dot q = \dot b + \dot Q$, and from (6.13), (6.14), (4.9) it follows that $\dot Q = P + \mathcal{O}(\|Z\|^2_{-\nu})$. Thus,

$$\dot q = \dot b + \dot Q = v(t) + \dot c(t) + P(t) + \mathcal{O}(\|Z\|^2_{-\nu}) \tag{11.1}$$

Bounds (6.10) and (6.11) imply that

$$|\dot c(t)| + |\dot v(t)| \le \frac{C_1(\rho, \overline{v}, d_0)}{(1+t)^3}, \qquad t \ge 0 \tag{11.2}$$

Therefore, $c(t) = c_+ + \mathcal{O}(t^{-2})$ and $v(t) = v_+ + \mathcal{O}(t^{-2})$, $t \to \infty$. Since $|P| \le \|Z\|_{-\nu}$, the estimate (6.11), and (11.1)-(11.2), imply that

$$\dot q(t) = v_+ + \mathcal{O}(t^{-3/2}) \tag{11.3}$$

Similarly,

$$b(t) = c(t) + \int_0^t v(s)ds = v_+ t + a_+ + \mathcal{O}(t^{-1}) \tag{11.4}$$

hence the second part of (1.8) follows:

$$q(t) = b(t) + Q(t) = v_+ t + a_+ + \mathcal{O}(t^{-1}) \tag{11.5}$$

since $Q(t) = \mathcal{O}(t^{-3/2})$ by (6.11).
Asymptotics for the fields. For the field part of the solution $\psi(x,t)$ let us define the accompanying soliton field as $\psi_{\mathrm{v(t)}}(x - q(t))$, where we define now $\mathrm{v}(t) = \dot q(t)$, cf. (11.1). Then for the difference $z(x,t) = \psi(x,t) - \psi_{\mathrm{v(t)}}(x - q(t))$ we obtain the equation

$$\dot z(x,t) = [-\alpha_j\partial_j - i\beta m]z(x,t) - i\dot{\mathrm{v}}\cdot\nabla_{\mathrm{v}}\psi_{\mathrm{v}(t)}(x - q(t))$$

Then

$$z(t) = W_0(t)z(0) - \int_0^t W_0(t-s)[i\dot{\rm v}(s)\cdot\nabla_{\rm v}\psi_{{\rm v}(s)}(\cdot - q(s))]ds \tag{11.6}$$

To obtain the asymptotics (2.10) it suffices to prove that $z(t) = W_0(t)\phi_+ + r_+(t)$ with some $\phi_+ \in L^2_0$ and $\|r_+(t)\|_0 = \mathcal{O}(t^{-1/2})$. This is equivalent to

$$W_0(-t)z(t) = \phi_+ + r'_+(t) \tag{11.7}$$

where $\|r'_+(t)\|_0 = \mathcal{O}(t^{-1/2})$ since $W_0(t)$ is a unitary group in L^2_0 by the charge conservation for the free Dirac equation. Finally, (11.7) holds since (11.6) implies

$$W_0(-t)z(t) = z(0) - \int_0^t W_0(-s)f(s)ds, \quad f(s) = i\dot{\rm v}(s)\cdot\nabla_{\rm v}\psi_{{\rm v}(s)}(\cdot - q(s))$$

where the integral in the right hand side converges in L^2_0 with the rate $\mathcal{O}(t^{-1/2})$. The latter holds since $\|W_0(-s)f(s)\|_0 = \mathcal{O}(s^{-3/2})$ by the unitarity of $W_0(-s)$ and the decay rate $\|f(s)\|_0 = \mathcal{O}(s^{-3/2})$. Let us prove this rate of decay. It suffices to prove that $|\dot{\rm v}(s)| = \mathcal{O}(s^{-3/2})$, or equivalently $|\dot p(s)| = \mathcal{O}(s^{-3/2})$. Substitute (4.2) to the last equation of (1.1) and obtain

$$\begin{aligned}
\dot p(t) &= \mathrm{Re}\int \left[\psi_{v(t)}(x-b(t)) + \Psi(x-b(t),t)\right]\nabla\rho(x-b(t)-Q(t))dx \\
&= \mathrm{Re}\int \psi_{v(t)}(y)\nabla\rho(y)dy + \mathrm{Re}\int \psi_{v(t)}(y)\left[\nabla\rho(y-Q(t)) - \nabla\rho(y)\right]dy \\
&+ \mathrm{Re}\int \Psi(y,t)\nabla\rho(y-Q(t))dy
\end{aligned}$$

The first integral in the right hand side is zero by the stationary equations (2.4). The second integral is $\mathcal{O}(t^{-3/2})$, since $Q(t) = \mathcal{O}(t^{-3/2})$, and by the conditions (1.9) on ρ. Finally, the third integral is $\mathcal{O}(t^{-3/2})$ by the estimate (6.11). The proof is complete.

12 Decay for linearized dynamics

In remaining sections we prove Proposition 6.7. Here we discuss our general strategy of the proof. We apply the Fourier-Laplace transform

$$\tilde X(\lambda) = \int_0^\infty e^{-\lambda t}X(t)dt, \qquad \mathrm{Re}\,\lambda > 0 \tag{12.1}$$

to (6.15). According to Proposition 6.7, we expect that the solution $X(t)$ is bounded in the norm $\|\cdot\|_{-\nu}$. Then the integral (12.1) converges and is analytic for $\mathrm{Re}\,\lambda > 0$. We will write A and v instead of A_1 and v_1 in all remaining part of the paper. After the Fourier-Laplace transform (6.15) becomes

$$\lambda\tilde X(\lambda) = A\tilde X(\lambda) + X_0, \quad \mathrm{Re}\,\lambda > 0 \tag{12.2}$$

Let us stress that (12.2) is equivalent to the Cauchy problem for the functions $X(t) \in C_b([0,\infty);\mathcal{E}_{-\nu})$. Hence the solution $X(t)$ is given by

$$\tilde X(\lambda) = -(A-\lambda)^{-1}X_0, \qquad \mathrm{Re}\,\lambda > 0 \tag{12.3}$$

if the resolvent $R(\lambda) = (A-\lambda)^{-1}$ exists for $\mathrm{Re}\,\lambda > 0$.

Let us comment on our following strategy in proving the decay (6.11). First, we will construct the resolvent $R(\lambda)$ for $\operatorname{Re}\lambda > 0$ and prove that it is a continuous operator in $\mathcal{E}_{-\nu}$. Then $\tilde{X}(\lambda) \in \mathcal{E}_{-\nu}$ and is an analytic function for $\operatorname{Re}\lambda > 0$. Second, we have to justify that there exist a (unique) function $X(t) \in C([0,\infty);\mathcal{E}_{-\nu})$ satisfying (12.1).

The analyticity of $\tilde{X}(\lambda)$ and Paley-Wiener arguments (see [9]) should provide the existence of a $\mathcal{E}_{-\nu}$ - valued distribution $X(t)$, $t \in \mathbb{R}$, with a support in $[0,\infty)$. Formally,

$$\Lambda^{-1}\tilde{X} = X(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}\tilde{X}(i\omega+0)d\omega, \quad t \in \mathbb{R} \tag{12.4}$$

However, to check the continuity of $X(t)$ for $t \geq 0$, we need additionally a bound for $\tilde{X}(i\omega+0)$ at large $|\omega|$. Finally, for the time decay of $X(t)$, we need an additional information on the smoothness and decay of $\tilde{X}(i\omega+0)$. More precisely, we should prove that the function $\tilde{X}(i\omega+0)$

i) is smooth outside $\omega = 0$ and $\omega = \pm\mu$, where $\mu = \mu(v) > 0$,

ii) decays in a certain sense as $|\omega| \to \infty$.

iii) admits the Puiseux expansion at $\omega = \pm\mu$.

iv) is analytic at $\omega = 0$ if $X_0 \in \mathcal{Z}_v := \mathbf{P}_v\mathcal{E}$ and $X_0 \in \mathcal{E}_\nu$.

Then the decay (6.11) would follow from the Fourier-Laplace representation (12.4).

13 Solving the linearized equation

Here we construct the resolvent as a bounded operator in $\mathcal{E}_{-\nu}$ for $\operatorname{Re}\lambda > 0$. We will write $(\tilde{\Psi}_1, \tilde{\Psi}_2, \tilde{Q}, \tilde{P})$ instead of $(\tilde{\Psi}_1(y,\lambda), \tilde{\Psi}_2(y,\lambda), \tilde{Q}(\lambda), \tilde{P}(\lambda))$ to simplify the notations. Then (12.2) reads

$$(A-\lambda)\begin{pmatrix}\tilde{\Psi}_1\\ \tilde{\Psi}_2\\ \tilde{Q}\\ \tilde{P}\end{pmatrix} = -\begin{pmatrix}\Psi_{01}\\ \Psi_{02}\\ Q_0\\ P_0\end{pmatrix}$$

It is the system of equations

$$\left.\begin{array}{r}
(-\alpha_1\partial_1 - \alpha_3\partial_3 + v\cdot\nabla - \lambda)\tilde{\Psi}_1 + (\beta m + \tilde{\alpha}_2\partial_2)\tilde{\Psi}_2 - \tilde{Q}\cdot\nabla\rho_2 = -\Psi_{01}\\[2ex]
-(\beta m + \tilde{\alpha}_2\partial_2)\tilde{\Psi}_1 + (-\alpha_1\partial_1 - \alpha_3\partial_3 + v\cdot\nabla - \lambda)\tilde{\Psi}_2 + \tilde{Q}\cdot\nabla\rho_1 = -\Psi_{02}\\[2ex]
B_v\tilde{P} - \lambda\tilde{Q} = -Q_0\\[2ex]
-\langle\nabla\tilde{\Psi}_j, \rho_j\rangle + \langle\nabla\psi_{vj}, \tilde{Q}\cdot\nabla\rho_j\rangle - \lambda\tilde{P} = -P_0
\end{array}\right| \tag{13.1}$$

Step i) Let us study the first two equations. First, we compute the matrix integral kernel $G_\lambda(y-y')$ of the Green operator

$$G_\lambda = \begin{pmatrix} -\alpha_1\partial_1 - \alpha_3\partial_3 + v\cdot\nabla - \lambda & \beta m + \tilde{\alpha}_2\partial_2\\ -\beta m - \tilde{\alpha}_2\partial_2 & -\alpha_1\partial_1 - \alpha_3\partial_3 + v\cdot\nabla - \lambda\end{pmatrix}^{-1} \tag{13.2}$$

In Fourier space

$$\hat{G}_\lambda(k) = \begin{pmatrix} i\alpha_1 k_1 + i\alpha_3 k_3 - ivk - \lambda & \beta m - \alpha_2 k_2\\ -\beta m + \alpha_2 k_2 & i\alpha_1 k_1 + i\alpha_3 k_3 - ivk - \lambda\end{pmatrix}^{-1}, \quad vk = \sum_{j=1}^{3} v_j k_j$$

To invert the matrix, let us solve the system

$$\left.\begin{array}{r} af_1 + bf_2 = g_1 \\ -bf_1 + af_2 = g_2 \end{array}\right| \qquad (13.3)$$

where $a = i\alpha_1 k_1 + i\alpha_3 k_3 - ivk - \lambda$, $b = \beta m - \alpha_2 k_2$. Multiplying the first equation of (13.3) by $c = -i\alpha_1 k_1 - i\alpha_3 k_3 - ivk - \lambda$ and the second equation by $-b$, we obtain

$$\left.\begin{array}{rcr} caf_1 + cbf_2 &=& cg_1 \\ b^2 f_1 - cbf_2 &=& -bg_2 \end{array}\right| \qquad (13.4)$$

since $ba = cb$ by the anticommutations (1.3). Further, $b^2 + ac = k^2 + m^2 + (ivk + \lambda)^2$. Therefore, summing up the equations (13.4), we obtain that

$$f_1 = \frac{cg_1 - bg_2}{k^2 + m^2 + (ivk + \lambda)^2}$$

Similarly, we obtain

$$f_2 = \frac{bg_1 + cg_2}{k^2 + m^2 + (ivk + \lambda)^2}$$

Hence

$$\hat{G}_\lambda(k) = \frac{1}{k^2 + m^2 + (ivk + \lambda)^2} \begin{pmatrix} -i\alpha_1 k_1 - i\alpha_3 k_3 - ivk - \lambda & -\beta m + \alpha_2 k_2 \\ \beta m - \alpha_2 k_2 & -i\alpha_1 k_1 - i\alpha_3 k_3 - ivk - \lambda \end{pmatrix} \qquad (13.5)$$

Taking the inverse Fourier transform we obtain

$$G_\lambda(y) = \begin{pmatrix} (\alpha_1 \partial_1 + \alpha_3 \partial_3 + v \cdot \nabla - \lambda) & -(\beta m + \tilde{\alpha}_2 \partial_2) \\ (\beta m + \tilde{\alpha}_2 \partial_2) & (\alpha_1 \partial_1 + \alpha_3 \partial_3 + v \cdot \nabla - \lambda) \end{pmatrix} g_\lambda(y) \qquad (13.6)$$

where

$$g_\lambda(y) = F^{-1}_{k \to y} \frac{1}{k^2 + m^2 + (ivk + \lambda)^2}, \quad y \in \mathbb{R}^3 \qquad (13.7)$$

Note that denominator in RHS (13.7) does not vanish for $\operatorname{Re}\lambda > 0$ since $|v| < 1$. This implies

Lemma 13.1. *The operator G_λ with the integral kernel $G_\lambda(y - y')$, is continuous operator $L^2_0 \oplus L^2_0 \to L^2_0 \oplus L^2_0$ for $\operatorname{Re}\lambda > 0$.*

From now on we use the system of coordinates in y-space in which $v = (|v|, 0, 0)$, hence $vk = |v|k_1$. Let us compute the function $g_\lambda(y)$. One has

$$k^2 + m^2 + (i|v|k_1 + \lambda)^2 = \frac{1}{\gamma^2} k_1^2 + k_2^2 + k_3^2 + 2i|v|k_1\lambda + \lambda^2 + m^2 = \frac{1}{\gamma^2}(k_1 + i\gamma^2|v|\lambda)^2 + k_2^2 + k_3^2 + \varkappa^2$$

where

$$\gamma = 1/\sqrt{1 - v^2}, \quad \varkappa^2 = \frac{v^2\lambda^2}{1 - v^2} + \lambda^2 + m^2 = \frac{\lambda^2}{1 - v^2} + m^2 = \gamma^2(\lambda^2 + \mu^2), \quad \mu := m/\gamma \quad (13.8)$$

Hence formally,

$$\begin{aligned} g_\lambda(y) &= \frac{1}{(2\pi)^{3/2}} \int \frac{e^{-iky} dk}{\frac{1}{\gamma^2}(k_1 + i\gamma^2|v|\lambda)^2 + k_2^2 + k_3^2 + \kappa^2} = \frac{e^{-\gamma^2|v|\lambda y_1}}{(2\pi)^{3/2}} \int \frac{e^{-iky} dk}{\frac{1}{\gamma^2} k_1^2 + k_2^2 + k_3^2 + \kappa^2} \\ &= \frac{\gamma e^{-\gamma|v|\lambda \tilde{y}_1}}{(2\pi)^{3/2}} \int \frac{e^{-ik\tilde{y}} dk}{k_1^2 + k_2^2 + k_3^2 + \kappa^2} = \gamma e^{-\gamma|v|\lambda \tilde{y}_1} R(\tilde{y}, -\kappa^2) \end{aligned} \qquad (13.9)$$

Here $\tilde{y}_1 = \gamma y_1$, $\tilde{y} = (\gamma y_1, y_2, y_3)$, and $R(y-y',\zeta)$ is the integral kernel of the operator $R(\zeta) = (-\Delta - \zeta)^{-1}$. It is well known that $R_0(y,\zeta) = e^{i\sqrt{\zeta}|y|}/4\pi|y|$. Therefore,

$$g_\lambda(y) = \frac{e^{-\varkappa|\tilde{y}| - \varkappa_1 \tilde{y}_1}}{4\pi|\tilde{y}|} \tag{13.10}$$

where $\varkappa_1 := \gamma|v|\lambda$. We choose $\mathrm{Re}\,\kappa > 0$ for $\mathrm{Re}\,\lambda > 0$. Let us note that for $0 < |v| < 1$

$$0 < \mathrm{Re}\,\varkappa_1 < \mathrm{Re}\,\varkappa, \qquad \mathrm{Re}\,\lambda > 0 \tag{13.11}$$

Let us state the result which we have got above.

Lemma 13.2. *i) The function $g_\lambda(y)$ decays exponentially in y for* $\mathrm{Re}\,\lambda > 0$.
ii) The formulas (13.10) and (13.8) imply that for every fixed y, the function $g_\lambda(y)$ admits an analytic continuation in λ to the Riemann surface of the algebraic function $\sqrt{\lambda^2+\mu^2}$ with the branching points $\lambda = \pm i\mu$.

Thus, from (13.1) and (13.2) we obtain the representation

$$\begin{aligned}\tilde{\Psi}_1 &= -G_\lambda^{11}\Psi_{01} - G_\lambda^{12}\Psi_{02} - (G_\lambda^{12}\nabla\rho_1)\cdot\tilde{Q} + (G_\lambda^{11}\nabla\rho_2)\cdot\tilde{Q} \\ \tilde{\Psi}_2 &= -G_\lambda^{21}\Psi_{01} - G_\lambda^{22}\Psi_{02} - (G_\lambda^{22}\nabla\rho_1)\cdot\tilde{Q} + (G_\lambda^{21}\nabla\rho_2)\cdot\tilde{Q}\end{aligned} \tag{13.12}$$

Step ii) Now we proceed to the last two equations (13.1):

$$-\lambda\tilde{Q} + B_v\tilde{P} = -Q_0, \qquad \langle\nabla\psi_{vj}, \tilde{Q}\cdot\nabla\rho_j\rangle - \langle\nabla\tilde{\Psi}_j, \rho_j\rangle - \lambda\tilde{P} = -P_0. \tag{13.13}$$

Let us rewrite equations (13.12) as $\tilde{\Psi}_j = \tilde{\Psi}_j(\tilde{Q}) + \tilde{\Psi}_j(\Psi_0)$, where

$$\tilde{\Psi}_1(\Psi_0) = -G_\lambda^{11}\Psi_{01} - G_\lambda^{12}\Psi_{02}, \quad \tilde{\Psi}_2(\Psi_0) = -G_\lambda^{21}\Psi_{01} - G_\lambda^{22}\Psi_{02} \tag{13.14}$$

$$\tilde{\Psi}_1(\tilde{Q}) = (-G_\lambda^{12}\nabla\rho_1 + G_\lambda^{11}\nabla\rho_2)\cdot\tilde{Q}, \quad \tilde{\Psi}_2(\tilde{Q}) = (-G_\lambda^{22}\nabla\rho_1 + G_\lambda^{21}\nabla\rho_2)\cdot\tilde{Q} \tag{13.15}$$

Then $\langle\nabla\tilde{\Psi}_j, \rho_j\rangle = \langle\nabla\tilde{\Psi}_j(\tilde{Q}), \rho_j\rangle + \langle\nabla\tilde{\Psi}_j(\Psi_0), \rho_j\rangle$, and the last equation (13.13) becomes

$$\langle\nabla\psi_{vj}, \tilde{Q}\cdot\nabla\rho_j\rangle - \langle\nabla\tilde{\Psi}_j(\tilde{Q}), \rho_j\rangle - \lambda\tilde{P} = -P_0 + \langle\nabla\tilde{\Psi}_j(\Psi_0), \rho_j\rangle =: -P_0 - \Phi(\lambda)$$

where

$$\Phi(\lambda) = \langle\tilde{\Psi}_j(\Psi_0), \nabla\rho_j\rangle \tag{13.16}$$

First we compute the term

$$\langle\nabla\psi_{vj}, \tilde{Q}\cdot\nabla\rho_j\rangle = \sum_{lj}\langle\nabla\psi_{vj}, \tilde{Q}_l\partial_l\rho_j\rangle = \sum_{lj}\langle\nabla\psi_{vj}, \partial_l\rho_j\rangle\tilde{Q}_l$$

Applying the Fourier transform $F_{y\to k}$, we have by the Parseval identity and (A.20) that

$$\begin{aligned}\sum_j\langle\partial_i\psi_{vj}, \partial_l\rho_j\rangle &= \sum_j\langle -ik_i\hat{\psi}_{vj}, -ik_l\hat{\rho}_j\rangle = \int k_ik_l(\hat{\psi}_{v1}\cdot\hat{\rho}_1 + \hat{\psi}_{v2}\cdot\hat{\rho}_2)dk \\ &= -\int k_ik_lm\frac{\beta\hat{\rho}_1\cdot\hat{\rho}_1 + \beta\hat{\rho}_2\cdot\hat{\rho}_2}{k^2+m^2-(|v|k_1)^2}dk = -\int\frac{k_ik_l\mathcal{B}(k)dk}{k^2+m^2-(|v|k_1)^2} =: -L_{il}\end{aligned} \tag{13.17}$$

As the result, $\langle\nabla\psi_{vj}, \tilde{Q}\cdot\nabla\rho_j\rangle = -L\tilde{Q}$, where L is the 3×3 matrix with the matrix elements L_{il}. The matrix L is diagonal and positive defined by (1.10).
Now let us compute the term $-\langle\nabla\tilde{\Psi}_j(\tilde{Q}), \rho_j\rangle = \langle\tilde{\Psi}_j(\tilde{Q}), \nabla\rho_j\rangle$. One has

$$\langle\tilde{\Psi}_j(\tilde{Q}), \partial_i\rho_j\rangle = \sum_l\Big(\langle -G_\lambda^{12}\partial_l\rho_1 + G_\lambda^{11}\partial_l\rho_2, \partial_i\rho_1\rangle - \langle G_\lambda^{22}\partial_l\rho_1 - G_\lambda^{21}\partial_l\rho_2, \partial_i\rho_2\rangle\Big)\tilde{Q}_l = \sum_l H_{il}(\lambda)\tilde{Q}_l$$

and by the Parseval identity and (1.3)-(1.4) we have

$$\begin{aligned} H_{il}(\lambda): &= \langle -G_\lambda^{12}\partial_l\rho_1 + G_\lambda^{11}\partial_l\rho_2, \partial_i\rho_1\rangle - \langle G_\lambda^{22}\partial_l\rho_1 - G_\lambda^{21}\partial_l\rho_2, \partial_i\rho_2\rangle \\ &= \langle[(\beta m-\alpha_2k_2)\hat\rho_1 - (i\alpha_1k_1+i\alpha_3k_3+i|v|k_1+\lambda)\hat\rho_2]\hat g_\lambda k_l, k_i\hat\rho_1\rangle \\ &+ \langle[(i\alpha_1k_1+i\alpha_3k_3+i|v|k_1+\lambda)\hat\rho_1 + (\beta m-\alpha_2k_2)\hat\rho_2]\hat g_\lambda k_l, k_i\hat\rho_2\rangle \qquad (13.18) \\ &= \int k_ik_lm\frac{\beta\hat\rho_1\cdot\hat\rho_1+\beta\hat\rho_2\cdot\hat\rho_2}{k^2+m^2-(|v|k_1-i\lambda)^2}dk = \int\frac{k_ik_l\mathcal{B}(k)dk}{k^2+m^2-(|v|k_1-i\lambda)^2} \end{aligned}$$

The matrix H is well defined for $\operatorname{Re}\lambda>0$ since the denominator does not vanish. The matrix H is diagonal. Indeed, if $i\neq l$, then at least one of these indices is not equal to one, and the integrand in (13.17) is odd with respect to the corresponding variable. Thus, $H_{il}=0$. As the result, $\langle\tilde\Psi_j(\tilde Q),\nabla\rho_j\rangle = H\tilde Q$, where H is the matrix with matrix elements H_{il}. Finally the equations (13.13) become

$$M(\lambda)\begin{pmatrix}\tilde Q\\ \tilde P\end{pmatrix} = \begin{pmatrix}Q_0\\ P_0+\Phi(\lambda)\end{pmatrix}, \text{ where } M(\lambda)=\begin{pmatrix}\lambda E & -B_v\\ L-H(\lambda) & \lambda E\end{pmatrix} \qquad (13.19)$$

Assume for a moment that the matrix $M(\lambda)$ is invertible (later we will prove this). Then we obtain

$$\begin{pmatrix}\tilde Q\\ \tilde P\end{pmatrix} = M^{-1}(\lambda)\begin{pmatrix}Q_0\\ P_0+\Phi(\lambda)\end{pmatrix}, \qquad \operatorname{Re}\lambda>0 \qquad (13.20)$$

Finally, formula (13.20) and formulas (13.12), where $\tilde Q$ is expressed from (13.20), give the expression of the resolvent $R(\lambda)=(A-\lambda)^{-1}$, $\operatorname{Re}\lambda>0$.

Lemma 13.3. *The matrix function $M(\lambda)$ (respectively, $M^{-1}(\lambda)$), $\operatorname{Re}\lambda>0$ admits an analytic (respectively meromorphic) continuation to the Riemann surface of the function $\sqrt{\mu^2+\lambda^2}$, $\lambda\in\mathbb{C}$.*

Proof. The analytic continuation of $M(\lambda)$, exists by Lemma 13.1 ii) and the convolution expressions in (13.18) by (1.9). The inverse matrix is then meromorphic since it exists for large $\operatorname{Re}\lambda$. The latter follows from (13.19) since $H(\lambda)\to0$, $\operatorname{Re}\lambda\to\infty$, by (13.18). □

14 Regularity on imaginary axis

Let us describe the continuous spectrum of the operator $A=A_{v,v}$ on the imaginary axis. By definition, the continuous spectrum corresponds to $\omega\in\mathbb{R}$, such that the resolvent $R(i\omega+0)$ is not a bounded operator in $\mathcal{E}$. By the formulas (13.12), this is the case when the Green function $G_\lambda(y-y')$ loses the exponential decay. This is equivalent to the condition $\operatorname{Re}\varkappa=0$. Thus, $i\omega$ belongs to the continuous spectrum if $|\omega|\geq\mu=m\sqrt{1-v^2}$. By Lemma 13.3, the limit matrix

$$M(i\omega):=M(i\omega+0)=\begin{pmatrix}i\omega E & -B_v\\ L-H(i\omega+0) & i\omega E\end{pmatrix}, \quad \omega\in\mathbb{R} \qquad (14.1)$$

exists, and its entries are continuous functions of $\omega\in\mathbb{R}$, smooth for $|\omega|<\mu$ and $|\omega|>\mu$. Recall that the point $\lambda=0$ belongs to the discrete spectrum of the operator A by Lemma 5.3, hence $M(i\omega+0)$ (probably) also is not invertible at $\omega=0$.

Proposition 14.1. *(cf. [8, Proposition 15.1]) Let ρ satisfy the conditions (1.9)- (1.10), and $|v|<1$. Then the limit matrix $M(i\omega+0)$ is invertible for $\omega\neq0$, $\omega\in\mathbb{R}$.*

Corollary 14.2. *The matrix $M^{-1}(i\omega)$ is smooth in $\omega\in\mathbb{R}$ outside three points $\omega=0,\pm\mu$.*

15 Singular spectral points

The components $Q(t)$ and $P(t)$ are given by the Fourier integral

$$\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \frac{1}{2\pi} \int e^{i\omega t} M^{-1}(i\omega+0) \begin{pmatrix} Q_0 \\ P_0 + \Phi(i\omega) \end{pmatrix} d\omega \tag{15.1}$$

if it converges in the sense of distributions. Corollary 14.2 alone is not sufficient for the proof of the convergence and decay of the integral. Namely, we need an additional information about behavior of the matrix $M^{-1}(i\omega)$ near its singular points $\omega = 0, \pm\mu$, and asymptotics at $|\omega| \to \infty$. We will analyze all the points separately.

I. First we consider the points $\pm\mu$.

Lemma 15.1. *The matrix $M^{-1}(i\omega)$ admits the asymptotics in a vicinity of $\pm\mu$:*

$$M^{-1}(i\omega) = C^{\pm} + \mathcal{O}((\omega\mp\mu)^{\frac{1}{2}}), \quad \partial_\omega M^{-1}(i\omega) = \mathcal{O}((\omega\mp\mu)^{-\frac{1}{2}}), \quad \partial^2_\omega M^{-1}(i\omega) = \mathcal{O}((\omega\mp\mu)^{-\frac{3}{2}}) \tag{15.2}$$

Proof. It suffices to prove similar asymptotics for $M(i\omega)$. Then (15.2) holds also for $M^{-1}(i\omega)$, since the matrices $M(\pm i\mu)$ are invertible. The asymptotics for $M(i\omega)$ holds by the convolution representation (13.18)

$$H_{jj}(\lambda) = \langle mg_\lambda \beta * \partial_j \rho_1, \partial_j \rho_1 \rangle + \langle mg_\lambda \beta * \partial_j \rho_2, \partial_j \rho_2 \rangle, \quad j = 1, 2, 3 \tag{15.3}$$

since g_λ admits the corresponding asymptotics by the formula (13.10). Namely

$$g_\lambda(y) = \frac{1}{4\pi|\tilde{y}|} + r_\pm(\lambda, y), \quad \lambda \to \pm i\mu, \quad \operatorname{Re}\lambda > 0$$

where

$$r_\pm(\lambda,y) = \mathcal{O}((\lambda \mp i\mu)^{\frac{1}{2}}), \quad \partial_\lambda r_\pm(\lambda,y) = \mathcal{O}((\lambda \mp i\mu)^{-\frac{1}{2}}), \quad \partial^2_\lambda r_\pm(\lambda,y) = \mathcal{O}((1+|y|)(\lambda \mp i\mu)^{-\frac{3}{2}})$$

The condition (1.9) provides the convergence of all integrals arising in $\partial^k_\lambda H_{jj}$. □

II. Second, we study the asymptotic behavior of $M^{-1}(\lambda)$ at infinity.

Lemma 15.2. *There exist a matrix D_0 and a matrix-function $D_1(\omega)$, such that*

$$M^{-1}(i\omega) = \frac{D_0}{\omega} + D_1(\omega), \quad |\omega| \geq \mu + 1, \quad \omega \in \mathbb{R} \tag{15.4}$$

where, for $k = 0, 1, 2$

$$|\partial^k_\omega D_1(\omega)| \leq \frac{C(k)}{|\omega|^2}, \quad |\omega| \geq \mu + 1, \quad \omega \in \mathbb{R} \tag{15.5}$$

Proof. The structure (14.1) of the matrix $M(i\omega)$ provides that it suffices to prove the following estimate for the elements of the matrix $H(i\omega) := H(i\omega + 0)$:

$$|\partial^k_\lambda H_{jj}(\lambda)| \leq C(k), \quad \lambda \in \mathbb{C}, \quad |\lambda| \geq \mu + 1, \quad j = 1, 2, 3, \quad k = 0, 1, 2 \tag{15.6}$$

The estimate (15.6) follows from the representation (15.3) and the bounds

$$|g_\lambda(y)| \leq \frac{C_1}{|y|}, \quad |\partial_\lambda g_\lambda(y)| \leq \frac{C_2}{|y|} + C_3, \quad |\partial^2_\lambda g_\lambda(y)| \leq \frac{C_4}{|y|} + C_5|y|, \quad \operatorname{Re}\lambda > 0$$

□

III. Finally, we consider the point $\omega = 0$ which is an isolated pole of a finite degree by Lemma 13.3. In Appendix B we prove that the matrix $M^{-1}(i\omega)$ can be written in the form

$$M^{-1}(i\omega) = \begin{pmatrix} \frac{1}{\omega}\mathcal{M}_{11}(\omega) & \frac{1}{\omega^2}\mathcal{M}_{12}(\omega) \\ \\ \mathcal{M}_{21}(\omega) & \frac{1}{\omega}\mathcal{M}_{22}(\omega) \end{pmatrix} \tag{15.7}$$

where $\mathcal{M}_{ij}(\omega)$, $i,j = 1,2$ are the diagonal matrices, smooth for the $\omega \in (-\mu, \mu)$. Moreover,

$$\mathcal{M}_{22} = \mathcal{M}_{11}, \quad \mathcal{M}_{11} = i\mathcal{M}_{12}B_\nu^{-1} \tag{15.8}$$

16 Transversal decay for the linearized equation

Here we prove Proposition (6.7).

16.1 Decay of vector components

First, we establish the decay (6.20) for the components $Q(t)$ and $P(t)$.

Lemma 16.1. *Let $X_0 \in \mathcal{Z}_v \cap \mathcal{E}_\nu$. Then $Q(t)$, $P(t)$ are continuous and*

$$|Q(t)| + |P(t)| \le C_\nu(\rho, \tilde{v})(1 + |t|)^{-3/2}, \qquad t \ge 0. \tag{16.1}$$

Proof. The expansions (15.2), (15.4) and (15.7) imply the convergence of the Fourier integral (15.1) in the sense of distributions to a continuous function of $t \ge 0$. Let us prove (16.1). First let us note that the condition $X_0 \in \mathcal{Z}_v$ implies that the whole trajectory $X(t)$ lies in $\mathcal{Z}_v$. This follows from the invariance of the space $\mathcal{Z}_v$ under the generator $A_{v,v}$ (cf. Remark 6.6). If $X_0 \notin \mathcal{Z}_v$, then the components $Q(t)$ and $P(t)$ may contain non-decaying terms which correspond to the singular point $\omega = 0$ since the linearized dynamics admits the secular solutions without decay, see (6.16). We will show that the symplectic orthogonality condition leads to (16.1). Let us split the Fourier integral (15.1) into three terms using the partition of unity $\zeta_1(\omega) + \zeta_2(\omega) + \zeta_3(\omega) = 1$, $\omega \in \mathbb{R}$:

$$\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \frac{1}{2\pi}\int e^{i\omega t}(\zeta_1(\omega) + \zeta_2(\omega) + \zeta_3(\omega))M^{-1}(i\omega + 0)\begin{pmatrix} Q_0 \\ P_0 + \Phi(i\omega) \end{pmatrix} d\omega = \sum_{j=1}^{3} I_j(t)$$

where the functions $\zeta_j(\omega) \in C^\infty(\mathbb{R})$ are supported by

$$\left.\begin{array}{rcl} \operatorname{supp}\zeta_1 & \subset & \{\omega \in \mathbb{R} : \varepsilon_0/2 < |\omega| < \mu + 2\} \\ \\ \operatorname{supp}\zeta_2 & \subset & \{\omega \in \mathbb{R} : |\omega| > \mu + 1\} \\ \\ \operatorname{supp}\zeta_3 & \subset & \{\omega \in \mathbb{R} : |\omega| < \varepsilon_0\} \end{array}\right| \tag{16.2}$$

i) Let us represent $I_j(t)$, $j = 1,2$ as

$$\begin{aligned} I_j(t) &= \frac{1}{2\pi}\int e^{i\omega t}\zeta_j(\omega)\Big[M^{-1}(i\omega + 0)\begin{pmatrix} Q_0 \\ P_0 \end{pmatrix} + M^{-1}(i\omega + 0)\begin{pmatrix} 0 \\ \Phi(i\omega) \end{pmatrix}\Big]d\omega \\ &= s_j(t)\begin{pmatrix} Q_0 \\ P_0 \end{pmatrix} + s_j(t) * \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \end{aligned} \tag{16.3}$$

where

$$s_j(t) = \Lambda^{-1}\zeta_j(\omega)M^{-1}(i\omega + 0), \quad f(t) = \Lambda^{-1}\Phi(i\omega)$$

By (13.14)

$$\tilde{\Psi}_1(\Psi_0) = -\Lambda\,\mathrm{Re}\,W_v^+(t)(\Psi_{10}+i\Psi_{20}),\quad \tilde{\Psi}_2(\Psi_0) = -\Lambda\,\mathrm{Im}\,W_v^+(t)(\Psi_{10}+i\Psi_{20})$$

where $W_v^+(t)$ is the dynamical group of the equation

$$\dot{\Psi}(x,t) = [\alpha\cdot\nabla + i\beta m + v\cdot\nabla]\Psi(x,t) \tag{16.4}$$

Evidently, for the group the $W_v^+(t)$ the bound (17.5) obtained in Lemma 17.1 for the group $W_v(t)$ also holds. Hence, (13.16) implies

$$|f(t)| = |\,\mathrm{Re}\langle W_v^+(t)(\Psi_{10}+i\Psi_{20}),\nabla\rho\rangle| \le C_\nu(\rho,v)(1+t)^{-3/2} \tag{16.5}$$

Further, the function $s_1(t)$ decays as $(1+|t|)^{-3/2}$ by asymptotics (15.2), and the function $s_2(t)$ decays as $(1+|t|)^{-2}$ due to Proposition 15.2. Hence, formula (16.3) implies the decay $(1+|t|)^{-3/2}$ for $I_1(t)$ and $I_2(t)$.

iii) Finally, the function $I_3(t)$ decays as $t^{-\infty}$ if $Z_0\in\mathcal{Z}_v$. It follows from next lemma

Lemma 16.2. *If $Z_0\in\mathcal{Z}_v$ then*

$$\begin{pmatrix}\tilde{Q}(i\omega)\\ \tilde{P}(i\omega)\end{pmatrix} = M^{-1}(i\omega)\begin{pmatrix}Q_0\\ P_0+\Phi(i\omega)\end{pmatrix}\in C^\infty(-\mu,\mu) \tag{16.6}$$

Proof. In Appendix C we prove that the symplectic orthogonality conditions (6.7) at $t=0$ imply

$$P_0+\Phi(0)=0,\quad B_v^{-1}Q_0+\Phi'(0)=0 \tag{16.7}$$

Then

$$P_0+\Phi(i\omega) = \Phi(i\omega)-\Phi(0) = i\omega\Upsilon_1(\omega)$$
$$B_v^{-1}Q_0+\Upsilon_1(\omega) = \frac{\Phi(i\omega)-\Phi(0)}{i\omega}-\Phi'(0) = i\omega\Upsilon_2(\omega)$$

where $\Upsilon_j(\omega)\in C^\infty(-\mu,\mu)$, since $\tilde{\Psi}_j(\Psi_0)\in C^\infty(-\mu,\mu)$, by (13.14) and (13.6). Therefore, representations (15.7)-(15.8) imply

$$\begin{aligned}\tilde{P}(i\omega) &= \mathcal{M}_{21}(\omega)Q_0 + i\mathcal{M}_{22}(\omega)\Upsilon_1(\omega)\in C^\infty(-\mu,\mu)\\ \tilde{Q}(i\omega) &= \frac{1}{\omega}\mathcal{M}_{11}(\omega)Q_0+\frac{i}{\omega}\mathcal{M}_{12}(\omega)\Upsilon_1(\omega) = \frac{i}{\omega}\mathcal{M}_{12}(B_v^{-1}Q_0+\Upsilon_1(\omega))\\ &= -\mathcal{M}_{12}\Upsilon_2(\omega)\in C^\infty(-\mu,\mu)\end{aligned}$$

□

16.2 Decay of fields

Now we prove the decay of the field components $\Psi_1(x,t),\Psi_2(x,t)$ corresponding to (6.20). The first two equations of (6.15) may be written as one equation:

$$\dot{\Psi}(x,t) = [-\alpha\cdot\nabla - i\beta m + v\cdot\nabla]\Psi(x,t) - iQ(t)\cdot\nabla\rho,\quad x\in\mathbb{R}^3,\quad t\in\mathbb{R} \tag{16.1}$$

where $\Psi(t)=\Psi_1(\cdot,t)+i\Psi_2(\cdot,t))$. Applying the Duhamel representation, we obtain

$$\Psi(t) = W_v(t)\Psi_0 - \int_0^t W_v(t-s)Q(s)\cdot\nabla\rho\,ds,\quad t\ge 0 \tag{16.2}$$

where $W_v(t)$ the dynamical group (propagator) of the "modified" free Dirac equation

$$\dot{\Psi}(x,t) = [-\alpha\cdot\nabla - i\beta m + v\cdot\nabla]\Psi(x,t) \tag{16.3}$$

Lemma 17.1 on the weighted decay for the group $W_v(t)$, the decay of Q from (16.1), and representation (16.2) yield

$$\|\Psi(t)\|_{-\nu} \le C_\nu(\rho,\tilde{v})\|\Psi_0\|_\nu(1+|t|)^{-3/2},\quad t\ge 0 \tag{16.4}$$

for any $\Psi_0\in\mathcal{Z}_v\cap\mathcal{E}_\nu$. It completes the proof of Proposition 6.7.

17 Weighted decay for free Dirac equation

Lemma 17.1. *For any $\Phi \in L^2_\nu$ with $\nu > 3/2$ the bound holds*

$$\|W_v(t)\Phi\|_{-\nu} \le \frac{C_\nu(v)\|\Phi\|_\nu}{(1+|t|)^{3/2}}, \quad t \ge 0 \tag{17.5}$$

Proof. Step i) Note, that

$$(\partial_t + \alpha\cdot\nabla + i\beta m + v\cdot\nabla)\,(\partial_t - \alpha\cdot\nabla - i\beta m + v\cdot\nabla) = (\partial_t^2 + 2\partial_t v\cdot\nabla - \Delta(1-v^2) + m^2)$$

Hence the integral kernel $W_v(x-y,t)$ of the operator $W_v(t)$ has the form

$$W_v(z,t) = (\partial_t + \alpha\cdot\nabla + i\beta m + v\cdot\nabla)\,G_v(z,t), \tag{17.6}$$

where $G_v(z,t)$ is a fundamental solution of the "modified" Klein-Gordon operator

$$(\partial_t^2 + 2\partial_t v\cdot\nabla - \Delta(1-v^2) + m^2)G_v(z,t) = \delta(z)\delta(t)$$

Let $G_v(t)$, $t\ge 0$ be the operator with the integral kernel $G_v(x-y,t)$. It is easy to see that

$$[G_v(t)\Phi](x) = [G_0(t)\Phi](x-vt), \quad x\in\mathbb{R}^3, \quad t\ge 0$$

Then

$$G_v(z,t) = G_0(z-vt,t) = \frac{\delta(t-|z-vt|)}{4\pi t} - \frac{m}{4\pi}\frac{\theta(t-|z-vt|)J_1(m\sqrt{t^2-|z-vt|^2})}{\sqrt{t^2-|z-vt|^2}}, \quad t>0$$

where J_1 is the Bessel function of order 1, and θ is the Heavyside function. Let us fix an arbitrary $\varepsilon\in(|v|,1)$. Well known asymptotics of the Bessel function imply that

$$|\partial_t G_v(z,t)|,\ |\partial_{z_j} G_v(z,t)| \le C(\varepsilon)(1+t)^{-3/2}, \quad |z-vt|\le\varepsilon t, \quad t\ge 1, \quad j=1,2,3 \tag{17.7}$$

Step ii) Now we consider an arbitrary $t\ge 1$. Denote $\varepsilon_1 = \varepsilon - |v|$. We split the function Φ in two terms, $\Phi = \Phi_{1,t} + \Phi_{2,t}$ such that

$$\|\Phi_{1,t}\|_{L^2_\nu} + \|\Phi_{2,t}\|_{L^2_\nu} \le C\|\Phi\|_{L^2_\nu}, \quad t\ge 1 \tag{17.8}$$

and

$$\Phi_{1,t}(x) = 0 \ \text{ for } |x| > \frac{\varepsilon_1 t}{2}, \qquad \text{and} \qquad \Phi_{2,t}(x) = 0 \ \text{ for } |x| < \frac{\varepsilon_1 t}{4} \tag{17.9}$$

The estimate (16.4) for $W_v(t)\Phi_{2,t}$ follows by charge conservation for Dirac equation, (17.9) and (17.8):

$$\|W_v(t)\Phi_{2,t}\|_{L^2_{-\nu}} \le \|W_v(t)\Phi_{2,t}\|_{L^2_0} = \|\Phi_{2,t}\|_{L^2_0} \le \frac{C(\varepsilon)\|\Phi_{2,t}\|_{L^2_\nu}}{(1+t)^\nu} \le \frac{C_1(\varepsilon)\|\Phi\|_{L^2_\nu}}{(1+t)^{3/2}}, \quad t\ge 1 \tag{17.10}$$

since $\nu > 3/2$.
Step iii) Next we consider $W_v(t)\Phi_{1,t}$. Now we split the operator $W_v(t)$ in two terms:

$$W_v(t) = (1-\zeta)W_v(t) + \zeta W_v(t), \qquad t\ge 1$$

where ζ is the operator of multiplication by the function $\zeta(|x|/t)$ such that $\zeta = \zeta(s) \in C_0^\infty(\mathbb{R})$, $\zeta(s) = 1$ for $|s| < \varepsilon_1/4$, $\zeta(s) = 0$ for $|s| > \varepsilon_1/2$. Since $1 - \zeta(|x|/t) = 0$ for $|x| < \varepsilon_1 t/4$, then applying the charge conservation and (17.8), we have for $t \geq 1$

$$\|(1-\zeta)W_v(t)\Phi_{1,t}\|_{L^2_{-\nu}} \leq \frac{C(\varepsilon)\|W_v(t)\Phi_{1,t}\|_{L^2_0}}{(1+t)^\nu} = \frac{C(\varepsilon)\|\Phi_{1,t}\|_{L^2_0}}{(1+t)^\nu} \leq \frac{C_1(\varepsilon)\|\Phi_{1,t}\|_{L^2_\nu}}{(1+t)^\nu} \leq \frac{C_2(\varepsilon)\|\Phi\|_{L^2_\nu}}{(1+t)^{3/2}} \tag{17.11}$$

since $\nu > 3/2$.

Step iv) It remains to estimate $\zeta W_v(t)\Phi_{1,t}$. Let χ_t be the characteristic function of the ball $|x| \leq \varepsilon_1 t/2$. We will use the same notation for the operator of multiplication by this characteristic function. By (17.9), we have

$$\zeta W_v(t)\Phi_{1,t} = \zeta W_v(t)\chi_t\Phi \tag{17.12}$$

The matrix kernel of the operator $\zeta W_v(t)\chi_t$ is equal to

$$W_v'(x-y,t) = \zeta(|x|/t)W_v(x-y,t)\chi_t(y)$$

Since $\zeta(|x|/t) = 0$ for $|x| > \varepsilon_1 t/2$ and $\chi_t(y) = 0$ for $|y| > \varepsilon_1 t/2$. Therefore, $W_v'(x-y,t) = 0$ for $|x-y| > \varepsilon_1 t$. On the other hand, $|x-y| \leq \varepsilon_1 t$ implies $|x-y-vt| \leq \varepsilon t$, since $\varepsilon_1 + |v| = \varepsilon$ by definition of ε_1. Hence, equality (17.6) and bounds (17.7) yield

$$|W_v'(x-y,t)| \leq C(1+t)^{-3/2}, \qquad t \geq 1 \tag{17.13}$$

The norm of the operator $\zeta W_v(t)\chi_t : L^2_\nu \to L^2_{-\nu}$ is equivalent to the norm of the operator

$$\langle x\rangle^{-\nu}\zeta W_v(t)\chi_t(y)\langle y\rangle^{-\nu} : L^2_0 \to L^2_0 \tag{17.14}$$

Therefore, (17.13) implies that operator (17.14) is Hilbert-Schmidt operator since $\nu > 3/2$, and its Hilbert-Schmidt norm does not exceed $C(1+t)^{-3/2}$. Hence, by (17.12) and (17.8)

$$\|\zeta W_v(t)\Phi_{1,t}\|_{L^2_{-\nu}} \leq C(1+t)^{-3/2}\|\Phi\|_{L^2_\nu}, \quad t \geq 1 \tag{17.15}$$

Finally, the estimates (17.15), (17.11) and (17.10) imply (16.4). □

A Computing $\Omega(\tau_i, \tau_j)$

Here we justify the formulas (3.7)-(3.9) for the matrix Ω.

1) First, the Parseval identity implies

$$\Omega(\tau_j,\tau_l) = \langle\partial_j\psi_{v1},\partial_l\psi_{v2}\rangle - \langle\partial_j\psi_{v2},\partial_l\psi_{v1}\rangle = \int k_j k_l\, dk(\hat\psi_{v1}\cdot\hat\psi_{v2} - \hat\psi_{v2}\cdot\hat\psi_{v1}) = 0, \quad j,l = 1,2,3$$

since the integrand is odd function.

2) Second, we consider

$$\Omega(\tau_{j+3},\tau_{l+3}) = \langle\partial_{v_j}\psi_{v1},\partial_{v_l}\psi_{v2}\rangle - \langle\partial_{v_j}\psi_{v2},\partial_{v_l}\psi_{v1}\rangle \tag{A.16}$$

Let us derive the formulas for ψ_{v1} and ψ_{v2}. The first equation of (2.4) implies

$$[(v_j\partial_j)^2 - \Delta + m^2]\psi_v = [iv_j\partial_j + i\alpha_j\partial_j - \beta m]\rho_1$$

Hence

$$[(v_j\partial_j)^2 - \Delta + m^2]\psi_{v1} = -[v_j\partial_j + \alpha_1\partial_1 + \alpha_3\partial_3]\rho_2 - [\tilde\alpha_2\partial_2 + \beta m]\rho_1$$

$$[(v_j\partial_j)^2-\Delta+m^2]\psi_{v2}=[v_j\partial_j+\alpha_1\partial_1+\alpha_3\partial_3]\rho_1-[\tilde\alpha_2\partial_2+\beta m]\rho_2$$

Applying the Fourier transform, we obtain

$$\left.\begin{aligned}\hat\psi_{v1}&=\frac{[ivk+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_2+[\alpha_2k_2-\beta m]\hat\rho_1}{-(vk)^2+k^2+m^2}\\ \hat\psi_{v2}&=\frac{-[ivk+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_1+[\alpha_2k_2-\beta m]\hat\rho_2}{-(vk)^2+k^2+m^2}\end{aligned}\right| \tag{A.17}$$

Differentiating, we get

$$\left.\begin{aligned}\partial_{v_j}\hat\psi_{v1}&=\frac{ik_j\hat\rho_2}{-(vk)^2+k^2+m^2}+\frac{2k_jvk\hat\psi_{v1}}{-(vk)^2+k^2+m^2}\\ \partial_{v_l}\hat\psi_{v2}&=\frac{-ik_l\hat\rho_1}{-(vk)^2+k^2+m^2}+\frac{2k_lvk\hat\psi_{v2}}{-(vk)^2+k^2+m^2}\end{aligned}\right| \tag{A.18}$$

Hence, (A.16) implies

$$\begin{aligned}\Omega(\tau_{j+3},\tau_{l+3})\ =\ &\int\frac{k_jk_l[\hat\rho_1\cdot\hat\rho_2-\hat\rho_2\cdot\hat\rho_1]dk}{(k^2+m^2-(vk)^2)^2}+\int\frac{4k_jk_l(vk)^2[\hat\psi_{v1}\cdot\hat\psi_{v2}-\hat\psi_{v2}\cdot\hat\psi_{v1}]dk}{(k^2+m^2-(vk)^2)^2}\\ +\ &\int\frac{2ik_jk_lvk[\hat\rho_2\cdot\hat\psi_{v2}+\hat\psi_{v2}\cdot\hat\rho_2+\hat\rho_1\cdot\hat\psi_{v1}+\hat\psi_{v1}\cdot\hat\rho_1]}{(k^2+m^2-(vk)^2)^2}=0\end{aligned}$$

since all integrands are odd functions.

3) Finally, (A.18) implies

$$\begin{aligned}\Omega(\tau_j,\tau_{l+3})\ &=\ -\langle\partial_j\psi_{v1},\partial_{v_l}\psi_{v2}\rangle+\langle\partial_j\psi_{v2},\partial_{v_l}\psi_{v1}\rangle+e_j\cdot\partial_{v_l}p_v \qquad (A.19)\\ &=\int\frac{ik_j\hat\psi_{v1}\cdot[-ik_l\hat\rho_1+2k_lvk\hat\psi_{v2})]-ik_j\hat\psi_{v2}\cdot[ik_l\hat\rho_2+2k_lvk\hat\psi_{v1})]}{k^2+m^2-(vk)^2}dk+e_j\cdot\partial_{v_l}p_v\\ &=\int k_jk_l\frac{-[\hat\psi_{v1}\cdot\hat\rho_1+\hat\psi_{v2}\cdot\hat\rho_2]+2ivk[\hat\psi_{v1}\cdot\hat\psi_{v2}-\hat\psi_{v2}\cdot\hat\psi_{v1}]}{k^2+m^2-(vk)^2}dk+e_j\cdot\partial_{v_l}p_v\end{aligned}$$

Recall, that $\rho_j(x)$ are even, then $\hat\rho_j(k)$ are real. Hence (1.3)-(1.4) and (A.17) imply

$$\begin{aligned}(k^2+m^2-(vk)^2)(\hat\psi_{v1}\cdot\hat\rho_1+\hat\psi_{v2}\cdot\hat\rho_2)=[\alpha_2k_2-\beta m]\hat\rho_1\cdot\hat\rho_1+[\alpha_2k_2-\beta m]\hat\rho_2\cdot\hat\rho \quad (A.20)\\ +[ivk+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_2\cdot\hat\rho_1-[ivk+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_1\cdot\hat\rho_2=-\mathcal{B}\hat\rho\cdot\hat\rho\end{aligned}$$

$$\begin{aligned}(k^2+m^2-(vk)^2)^2(\hat\psi_{v1}\cdot\hat\psi_{v2}-\hat\psi_{v2}\cdot\hat\psi_{v1})=2i(k^2+m^2-(vk)^2)^2\,\mathrm{Im}(\hat\psi_{v1}\cdot\hat\psi_{v2}) \quad (A.21)\\ =-2\beta m\hat\rho_1\cdot[ivk+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_1-2[ivk+i\alpha_1k_1+i\alpha_3k_3]\hat\rho_2\cdot\beta m\hat\rho_2=-2ivk\mathcal{B}\hat\rho\cdot\hat\rho\end{aligned}$$

Substituting (A.20) and (A.21) into the right hand site of (A.19), we obtain

$$\Omega(\tau_j,\tau_{l+3})=\int k_jk_l\Big(\frac{\mathcal{B}(k)}{(k^2+m^2-(vk)^2)^2}+\frac{4(vk)^2\mathcal{B}(k)}{(k^2+m^2-(vk)^2)^3}\Big)dk+e_j\cdot\partial_{v_l}p_v$$

that correspond to (3.7) - (3.9).

B Computing $M^{-1}(i\omega)$

Here we derive formula (15.7). Denote $F(\omega) := -L + H(i\omega + 0)$ which is diagonal. Then by (14.1) for $\omega \in \mathbb{R}$ we obtain

$$\det M(i\omega) = \det \begin{pmatrix} i\omega E & -B_v \\ -F(\omega) & i\omega E \end{pmatrix} = -\Big(\omega^2 + \frac{F_{11}(\omega)}{\gamma^3}\Big)\Big(\omega^2 + \frac{F_{22}(\omega)}{\gamma}\Big)\Big(\omega^2 + \frac{F_{33}(\omega)}{\gamma}\Big) \quad \text{(B.22)}$$

where

$$F_{jj}(\omega) = \int k_j^2 \mathcal{B} dk \left(\frac{1}{m^2 + k^2 - (|v|k_1 + \omega)^2} - \frac{1}{m^2 + k^2 - (|v|k_1)^2} \right), \quad j = 1, 2, 3 \quad \text{(B.23)}$$

Formula (B.22) is obvious since both matrices $F(\omega)$ and B_v are diagonal, hence the matrix $M(i\omega)$ is equivalent to three independent matrices 2×2. Namely, let us transpose the columns and rows of the matrix $M(i\omega)$ in the order (142536). Then we get the matrix with three 2×2 blocks on the main diagonal. Therefore, the determinant of $M(i\omega)$ is product of the determinants of the three matrices. Further,

$$M^{-1}(i\omega) = \begin{pmatrix} M_{11}(\omega) & M_{12}(\omega) \\ M_{21}(\omega) & M_{22}(\omega) \end{pmatrix} \quad \text{(B.24)}$$

where

$$M_{11}(\omega) = M_{22}(\omega) = \begin{pmatrix} \frac{-i\omega\gamma^3}{\omega^2\gamma^3 + F_{11}(\omega)} & 0 & 0 \\ 0 & \frac{-i\omega\gamma}{\omega^2\gamma + F_{22}(\omega)} & 0 \\ 0 & 0 & \frac{-i\omega\gamma}{\omega^2\gamma + F_{33}(\omega)} \end{pmatrix}$$

$$M_{12} = \begin{pmatrix} \frac{-1}{\omega^2\gamma^3 + F_{11}} & 0 & 0 \\ 0 & \frac{-1}{\omega^2\gamma + F_{22}} & 0 \\ 0 & 0 & \frac{-1}{\omega^2\gamma + F_{33}} \end{pmatrix}, \quad M_{21} = \begin{pmatrix} \frac{-\gamma F_{11}}{\omega^2\gamma^3 + F_{11}} & 0 & 0 \\ 0 & \frac{-\gamma F_{22}}{\omega^2\gamma + F_{22}} & 0 \\ 0 & 0 & \frac{-\gamma F_{33}}{\omega^2\gamma + F_{33}} \end{pmatrix}$$

Let us prove that for $\omega \in (-\mu, \mu)$

$$F_{jj}(\omega) = \omega^2 f_{jj}(\omega), \quad f_{jj}(\omega) \in \mathbb{C}^\infty(-\mu, \mu), \quad f_{jj}(0) > 0 \quad \text{(B.25)}$$

Indeed, formula (B.23) implies that $F_{jj}(0) = 0$. Differentiating (B.23), we obtain

$$F'_{jj}(0) = 2 \int k_j^2 \mathcal{B}(k) dk \frac{|v|k_1}{(k^2 + m^2 - (|v|k_1)^2)^2} = 0$$

since integrand is odd function in respect to k_1, and

$$F''_{jj}(0) = 2 \int k_j^2 \mathcal{B}(k) dk \frac{k^2 + m^2 + 3(|v|k_1)^2}{(k^2 + m^2 - (|v|k_1)^2)^3} > 0$$

By (B.25) we can represent the matrices $M_{ij}(\omega)$ as

$$M_{11}(\omega) = M_{22}(\omega) = \frac{1}{\omega} \begin{pmatrix} \frac{-i\gamma^3}{\gamma^3 + f_{11}(\omega)} & 0 & 0 \\ 0 & \frac{-i\gamma}{\gamma + f_{22}(\omega)} & 0 \\ 0 & 0 & \frac{-i\gamma}{\gamma + f_{33}(\omega)} \end{pmatrix} = \frac{1}{\omega} \mathcal{M}_{11}(\omega)$$

$$M_{12}(\omega) = \frac{1}{\omega^2} \begin{pmatrix} \frac{-1}{\gamma^3 + f_{11}(\omega)} & 0 & 0 \\ 0 & \frac{-1}{\gamma + f_{22}(\omega)} & 0 \\ 0 & 0 & \frac{-1}{\gamma + f_{33}(\omega)} \end{pmatrix} = \frac{1}{\omega^2} \mathcal{M}_{12}(\omega) \quad \text{(B.26)}$$

$$M_{21}(\omega) = \begin{pmatrix} \frac{-\gamma^3 f_{11}(\omega)}{\gamma^3 + f_{11}(\omega)} & 0 & 0 \\ 0 & \frac{-\gamma f_{22}(\omega)}{\gamma + f_{22}(\omega)} & 0 \\ 0 & 0 & \frac{-\gamma f_{33}(\omega)}{\gamma + f_{33}(\omega)} \end{pmatrix} = \mathcal{M}_{21}(\omega)$$

where $\mathcal{M}_{ij}(\omega) \in C^\infty(-\mu, \mu)$.

C Symplectic orthogonality conditions

Here we derive conditions (16.7) from the symplectic orthogonality conditions (6.7). First let us compute $\Phi(0)$. Formulas (13.14) and (13.16) imply

$$(\Phi(0))_j = \langle \hat G_0^{11}\hat\Psi_{01} + \hat G_0^{12}\hat\Psi_{02}, ik_j\hat\rho_1\rangle + \langle \hat G_0^{11}\hat\Psi_{02} - \hat G_0^{12}\hat\Psi_{01}, ik_j\hat\rho_2\rangle, \quad j=1,2,3$$

On the other hand, by (13.5) formulas (A.17) read

$$\hat\psi_{v1} = -\hat G_0^{11}\hat\rho_2 + \hat G_0^{12}\hat\rho_1, \quad \hat\psi_{v2} = \hat G_0^{11}\hat\rho_1 + \hat G_0^{12}\hat\rho_2$$

Hence, for $j=1,2,3$

$$\begin{aligned} 0 &= -\Omega(Z_0,\tau_j) = \langle\Psi_{01},\partial_j\psi_{v2}\rangle - \langle\Psi_{02},\partial_j\psi_{v1}\rangle + P_0\cdot e_j \\ &= -\langle\Psi_{01}, ik_j(\hat G_0^{11}\hat\rho_1 + \hat G_0^{12}\hat\rho_2)\rangle + \langle\Psi_{02}, ik_j(\hat G_0^{12}\hat\rho_1 - \hat G_0^{11}\hat\rho_2)\rangle + P_0\cdot e_j = (\Phi(0)+P_0)_j \end{aligned}$$

since $(\hat G_0^{11})^* = -\hat G_0^{11}$, $(\hat G_0^{12})^* = \hat G_0^{12}$. Hence the first condition (16.7) follows. Further,

$$\partial_\lambda \hat G_\lambda^{11}\Big|_{\lambda=0} = \frac{-1-2ivk\hat G_0^{11}}{k^2+m^2-(vk)^2}, \quad \partial_\lambda \hat G_\lambda^{12}\Big|_{\lambda=0} = \frac{-2ivk\hat G_0^{12}}{k^2+m^2-(vk)^2}$$

Then (13.14) and (13.16) imply for $j=1,2,3$

$$(\Phi'(0))_j = -\Big\langle \frac{\hat\Psi_{01}+2ivk(\hat G_0^{11}\hat\Psi_{01}+\hat G_0^{12}\hat\Psi_{02})}{k^2+m^2-(vk)^2}, ik_j\hat\rho_1\Big\rangle - \Big\langle \frac{\hat\Psi_{02}+2ivk(\hat G_0^{11}\hat\Psi_{02}-G_0^{12}\hat\Psi_{01})}{k^2+m^2-(vk)^2}, ik_j\hat\rho_2\Big\rangle$$

On the other hand, from (A.17) and (A.18) it follows that for $j=1,2,3$

$$\partial_{v_j}\hat\psi_{v1} = \frac{ik_j\hat\rho_2 + 2k_jvk(-\hat G_0^{11}\hat\rho_2 + \hat G_0^{12}\hat\rho_1)}{k^2+m^2-(vk)^2}, \quad \partial_{v_j}\hat\psi_{v2} = \frac{-ik_j\hat\rho_1 + 2k_jvk(\hat G_0^{11}\hat\rho_1 + \hat G_0^{12}\hat\rho_2)}{k^2+m^2-(vk)^2}$$

Hence,

$$\begin{aligned} 0 &= \Omega(Z_0,\tau_{j+3}) = \langle\Psi_{01},\partial_{v_j}\psi_{v2}\rangle - \langle\Psi_{02},\partial_{v_j}\psi_{v1}\rangle + Q_0\cdot\partial_{v_j}p_v \\ &= \Big\langle\Psi_{01}, \frac{-ik_j\hat\rho_1 + 2k_jvk(\hat G_0^{11}\hat\rho_1 + \hat G_0^{12}\hat\rho_2)}{k^2+m^2-(vk)^2}\Big\rangle - \Big\langle\Psi_{02}, \frac{ik_j\hat\rho_2 + 2k_jvk(-\hat G_0^{11}\hat\rho_2 + \hat G_0^{12}\hat\rho_1)}{k^2+m^2-(vk)^2}\Big\rangle \\ &+ Q_0\cdot\partial_{v_j}p_v = (\Phi'(0) + B_v^{-1}Q_0)_j, \quad j=1,2,3 \end{aligned}$$

since $Q_0\cdot\partial_{v_j}p_v = Q_0\cdot B_v^{-1}e_j = B_v^{-1}Q_0\cdot e_j$. Hence the second condition (16.7) follows.

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