To the memory of Vladimir Buslaev

On global attractors and radiation damping for nonrelativistic particle coupled to scalar field

A. Komech [1]
Faculty of Mathematics Vienna University
and Institute for Information Transmission Problems RAS
Email: alexander.komech@univie.ac.at

E. Kopylova [1]
Faculty of Mathematics Vienna University
and Institute for Information Transmission Problems RAS
Email: elena.kopylova@univie.ac.at

H. Spohn
Faculty of Mathematics, TU Munich.
Email: spohn@ma.tum.de

Abstract We consider the Hamiltonian system of scalar wave field and a single nonrelativistic particle coupled in a translation invariant manner. The particle is also subject to a confining external potential. The stationary solutions of the system are a Coulomb type wave field centered at those particle positions for which the external force vanishes. We prove that solutions of finite energy converge, in suitable local energy seminorms, to the set $\mathscr{S}$ of all stationary states in the long time limit $t \to \pm\infty$. Further we show that the rate of relaxation to a stable stationary state is determined by spatial decay of initial data. The convergence is followed by the radiation of the dispersion wave which is a solution to the free wave equation.

Similar relaxation has been proved previously for the case of relativistic particle when the speed of the particle is less than the speed of light. Now we extend these results to nonrelativistic particle with arbitrary superlight velocity. However, we restrict ourselves by the plane particle trajectories in $\mathbb{R}^3$. The extension to general case remains an open problem.

[1] The research was carried out at the IITP RAS at the expense of the Russian Foundation for Sciences (project 14-50-00150).

1 Introduction

We consider the Hamiltonian system of a real scalar field $\varphi(x)$ on $\mathbb{R}^3$, and an extended nonrelativistic particle with the center position $q \in \mathbb{R}^3$ and with the charge density $\rho(x-q)$. The field is governed by the wave equation with a source. The particle is subject to the wave field and also to an external potential V, which is confining in the sense of (1.9). The interaction between the particle and the scalar field is local, translation invariant, and linear in the field. We study the long-time behavior of the coupled system. Our main results are the asymptotics

$$\dot{q}(t) \to 0, \qquad \ddot{q}(t) \to 0, \quad t \to \pm\infty, \tag{1.1}$$

and the convergence of the field to the corresponding Coulombic potential. Moreover, we establish the rate of the convergence in the case when $q_\pm$ is a nondegenerate local minimum of the potential V.

Let $\pi(x)$ be the canonically conjugate field to $\varphi(x)$ and let p be the momentum of the particle. The Hamiltonian (energy functional) reads then

$$\mathscr{H}(\varphi, q, \pi, p) \equiv \frac{1}{2}p^2 + V(q) + \frac{1}{2}\int (|\pi(x)|^2 + |\nabla\varphi(x)|^2)\,dx + \int \varphi(x)\rho(x-q)\,dx. \tag{1.2}$$

Taking formally variational derivatives in (1.2), the coupled dynamics becomes

$$\left.\begin{array}{ll} \dot{\varphi}(x,t) = \pi(x,t), & \dot{\pi}(x,t) = \Delta\varphi(x,t) - \rho(x-q(t)), \\[2ex] \dot{q}(t) = p(t), & \dot{p}(t) = -\nabla V(q(t)) + \displaystyle\int \varphi(x,t)\nabla\rho(x-q(t))dx. \end{array}\right| \tag{1.3}$$

For smooth $\varphi(x)$ vanishing at infinity the Hamiltonian can be rewritten as

$$\mathscr{H}(\varphi, q, \pi, p) \equiv \frac{1}{2}p^2 + V(q) + \frac{1}{2}\int (|\pi(x)|^2 + |\nabla[\varphi(x) - \Delta^{-1}\rho(x-q)]|^2)\,dx + \frac{1}{2}\langle\rho, \Delta^{-1}\rho\rangle, \tag{1.4}$$

where

$$\frac{1}{2}\langle\rho, \Delta^{-1}\rho\rangle = -\frac{1}{8\pi}\int\int \frac{\rho(x)\rho(y)}{|x-y|}\,dxdy \le 0. \tag{1.5}$$

Thus the energy (1.4) is bounded from below if $|\langle\rho, \Delta^{-1}\rho\rangle| < \infty$ which provides a priori estimates for solutions to (1.3), and hence guarantees the existence of global solutions. Otherwise, the dynamics is not well defined. For example, $\langle\rho, \Delta^{-1}\rho\rangle = -\infty$ for the point particle with $\rho(x) = \delta(x)$:

$$\langle\delta, \Delta^{-1}\delta\rangle = -(2\pi)^{-3}\int \frac{1}{k^2}dk = -\infty. \tag{1.6}$$

This “ultraviolet divergence” was discovered first for the point particle in classical electrodynamics, where $-\langle\rho, \Delta^{-1}\rho\rangle$ is proportional to the energy of the particle in its own electrostatic field. Respectively, the infinite energy (1.6) for the point particle is not satisfactory since it also means its infinite mass. This infinity inspired the introduction of the “extended electron” by Abraham [1]. Our system (1.3) is a scalar analog of the Abraham electrodynamics with the extended electron [23, 28].

The stationary solutions for (1.3) are easily determined. Denote

$$s_q(x) = -\int \frac{\rho(y-q)}{4\pi|y-x|}dy, \quad x, q \in \mathbb{R}^3. \tag{1.7}$$

Let $Z=\{q\in\mathbb{R}^3:\nabla V(q)=0\}$ be the set of critical points for V. Then the set of all stationary states is given by

$$\mathscr{S}=\{(\varphi,\pi,q,p)=(s_q,0,q,0):=S_q|\ \ q\in Z\}. \tag{1.8}$$

One natural goal is to investigate the domain of attraction for $\mathscr{S}$ and in particular to prove that each finite energy solution of (1.3) converges to some stationary states $S_{q_\pm}=(s_{q_\pm},0,q_\pm,0)\in\mathscr{S}$ in the long time limit $t\to\pm\infty$.

To state our main results we need some assumptions on V and ρ. We assume that

$$V\in C^2(\mathbb{R}^3),\qquad \lim_{|q|\to\infty}V(q)=\infty. \tag{1.9}$$

$$\rho\in C_0^\infty(\mathbb{R}^3),\quad \rho(x)=0\ \text{ for }\ |x|\ge R_\rho,\quad \rho(x)=\rho_r(|x|). \tag{1.10}$$

Moreover, we suppose that the following Wiener condition holds:

$$\hat\rho(k)\ne 0\qquad\text{for}\quad k\in\mathbb{R}^3. \tag{1.11}$$

It is an analogue of the Fermi Golden Rule: the coupling term $\rho(x-q)$ is not orthogonal to the eigenfunctions e^{ikx} of the continuous spectrum of the linear part of the equation (cf. [26, 27]). As we will see, the Wiener condition (1.11) is very essential for our asymptotic analysis.

For technical reasons, we restrict ourselves to the case when the particle moves in the plane, i.e. we suppose that $q(t)=(q^1(t),q^2(t),q^3(t))\in R^3$ such that

$$q^3(t)=0,\qquad t\in\mathbb{R}. \tag{1.12}$$

For example, this condition holds if initial fields $\varphi_0(x)=\varphi(x,0)$ and $\pi_0(x)=\pi(x,0)$ are symmetric in x^3, and

$$q^3(0)=p^3(0)=0\ \text{ and }\qquad \partial_{x^3}V(x^1,x^2,0)=0,\ \text{ for }\ (x^1,x^2)\in\mathbb{R}^2. \tag{1.13}$$

In the first part of the paper we prove that the set $\mathscr{S}$ is an attracting set for each trajectory $Y(t)=(\varphi(t),\pi(t),q(t),p(t))$. Namely, we consider initial data $Y(0)=(\varphi_0,\pi_0,q_0,p_0)$ with

$$\varphi_0\in C^2(\mathbb{R}^3),\quad \pi_0\in C^1(\mathbb{R}^3) \tag{1.14}$$

such that

$$|\nabla\varphi_0(x)|+|\pi_0(x)|+|x|(|\nabla\nabla\varphi_0(x)|+|\nabla\pi_0(x)|)=\mathscr{O}(|x|^{-\sigma}),\quad |x|\to\infty,\ \text{ where }\ \sigma>3/2, \tag{1.15}$$

which guarantees the finiteness of the energy (1.2). First, we prove the relaxation (1.1). Further, we prove the long-time attraction

$$Y(t)\to\mathscr{S},\quad t\longrightarrow\pm\infty. \tag{1.16}$$

where the convergence of the fields holds in local energy seminorms. If additionally, the set $\mathscr{S}$ is discrete, then (1.16) implies

$$Y(t)\to S_{q_\pm},\quad t\longrightarrow\pm\infty, \tag{1.17}$$

where the stationary states $S_{q_\pm}\in\mathscr{S}$ depend on the solution $Y(t)$ considered.

In the second part of the paper we specify the rate of convergence in (1.17) to a stationary state S_{q+} in the case where $q_* \in Z$ is a non-degenerate minimum of the potential, i.e.,

$$d^2V(q_+) > 0. \tag{1.18}$$

where $d^2V(q_+)$ is the Hessian. We suppose that the initial fields belong to the weighted space $øH^1_\alpha \oplus L^2_\alpha$ with some $\alpha > 1$ (see Definition 2.1). Then for any $\varepsilon > 0$

$$\dot{q}(t) = \mathscr{O}(|t|^{-\alpha+\varepsilon}),\ \ q(t) = q_+ + \mathscr{O}(|t|^{-\alpha+\varepsilon}),\ \ \|(\varphi(t),\pi(t)) - (s_{q_+},0)\|_{øH^1_{-\alpha}\oplus L^2_{-\alpha}} = \mathscr{O}(t^{-\alpha+\varepsilon}), \quad t\to\infty. \tag{1.19}$$

Moreover, in this case the scattering asymptotics hold,

$$(\varphi(x,t),\pi(x,t)) = (s_{q_+},0) + W(t)\Phi_+ + r(x,t). \tag{1.20}$$

Here $W(t)$ is the dynamical group of the free wave equation, $\Phi \in øH^1 \oplus L^2$ is the corresponding asymptotic state, and

$$\|r(t)\|_{øH^1\oplus L^2} = \mathscr{O}(|t|^{-\alpha+1+\varepsilon}), \quad t\to\infty. \tag{1.21}$$

The investigation is inspired by fundamental problems of the field theory and quantum mechanics. Namely, the relaxation of the acceleration (1.1) is known as *radiation damping* in classical electrodynamics since Lorentz and Abraham [1], however it was proved for the first time in [23, 22] for the case of relativistic particle with $\dot{q} = p/\sqrt{p^2+1}$. Second, the asymptotics (1.17) give a dynamical model of Bohr's transitions to quantum stationary states, see the details in [17, 18].

Our extension to the nonrelativistic particle is not straightforward and important in connection with the Cherenkov radiation. The main difficulty is due to the singular nature of the radiation for $|\dot{q}(t)| \geq 1$.

Traditionally the classical Larmor and Liénard formulas [6, (14.22)] and [6, (14.24)] are accepted for the power of radiation of a point particle. These formulas contain the factor $(1-\beta\cdot\omega)^{-3}$ (cf. our formula (5.4)) where $\beta = v/c$ and ω is the direction of the radiation. Here $v = \dot{q}(\tau)$ is the particle velocity at the "retarded time" τ and c is the propagation speed of the wave field in the dispersive medium. These formulas are deduced from the Liénard-Wiechert expressions for the retarded potentials neglecting the initial fields. Moreover, these formulas neglect the back fieldreaction though it should be the key reason for the relaxation. The main problem is that this back field-reaction is infinite for the point particles. In (1.3) we set $c = 1$. Generally c is less than the speed of light in vacuum, so the particle velocities $\dot{q}(t) > 1$ are possible. Then the factor $(1-\beta\cdot\omega)^{-3}$ in the Larmor formula becomes infinite for some directions ω.

A rigorous meaning to these calculations for relativistic particle has been suggested first in [23, 22] for the Abraham model of the "extended electron" under the Wiener condition (1.11). The survey can be found in [28].

For the nonrelativistic Abraham type model (1.3) with the "extended electron" the radiation remains finite due to the smoothing by the coupling function ρ. Nevertheless, the case $|\dot{q}(t)| > 1$ rises many open questions.

Our main novelties in present paper are the following.

I. Global attraction of finite energy solutions to stationary states for the case of nonrelativistic particle.

II. Asymptotics (1.19)–(1.21) in the weighted Sobolev norms for the case of nonrelativistic particle.

Let us comment on previous results in these directions. The global attractions (1.16) and (1.17) were proved in [22, 23] for the system of type (1.3) with relativistic particle and for the similar Maxwell-Lorentz system. In [20] the global attraction to solitons was proved for the system (1.3) without external potential under the Wiener condition (1.11). In [11] this result was extended to similar Maxwell-Lorentz system. In [7]–[10] the global attraction to solitons is proved for the system (1.3) and similar systems with the Klein-Gordon and Maxwell equations with small ρ. In [12]–[16] the global attraction to solitary waves is proved for the Klein–Gordon and Dirac equations coupled to $U(1)$-invariant nonlinear oscillators.

The asymptotics of type (1.19)–(1.21) were established in [22] for the case of relativistic particle in local energy seminorms for initial fields with compact support. In [21] we have proved the asymptotic stability of the stationary states for the system (1.3) in the weighted Sobolev norms.

In a series of papers, Egli, Fröhlich, Gang, Sigal, and Soffer have established the convergence to a soliton for the system of type (1.3) with the Schrödinger equation instead of the wave equation. The main result of [5] is the long time convergence to a soliton with a subsonic speed for initial solitons with supersonic speeds. The convergence is considered as a reason for the Cherenkov radiation, see [5] and the references therein.

The asymptotics of type (1.20) were proved by Soffer and Weinstein for nonlinear Schrödinger equations with a potential [29, 30], and for translation invariant nonlinear Schrödinger equations by Buslaev, Perelman and Sulem [2, 3, 4].

Now let us comment on our methods. For the proof of (1.1) we estimate the energy dissipation by decomposing φ into a near and far field. Energy is radiated in the far field. Since the Hamiltonian is bounded from below, such radiation cannot go on forever and a certain "energy radiation functional" has to be bounded. This radiation functional can be written as a convolution. By a Wiener Tauberian Theorem, using (1.11), we conclude (1.1) for $\ddot{q}$. Therefore (1.1) also holds for $\dot{q}$ since $|q(t)|$ is bounded by some $\overline{q}_0 < \infty$ due to (1.9). Finally, we deduce (1.16) and (1.17) from (1.1) and integral representations for the fields. This strategy is close to [22, 23, 28], however, the singularity of the radiation at $|\dot{q}(t)| \geq 1$ requires suitable modifications in application of the Wiener Tauberian Theorem. We suggest the modification for the plane particle trajectories (1.12). The extension to general case remains an open problem.

We prove the asymptotics (1.19)–(1.21) by a development of the methods of [22] and controlling the nonlinear part of (1.3) by the dispersion decay for the linearized equation which we established in [21]. Let us emphasize however, that the asymptotics (1.19)–(1.21) are quite different from the asymptotic stability proved in [21].

The plan of our paper is as follows. In §2 we introduce appropriate functional spaces and formulate our main results. In §3 we refine known results on the long range asymptotics of the Liénard-Wiehert potentials. In §4 we calculate the energy radiation integral. We use this formula in §5 to prove the velocity relaxation. In §6 we prove the attraction to stationary states. In §7 we consider the linearization at stationary state. In §8 we prove a version of strong Huygens principle for nonlinear system (1.3). In §§9–10 we deduce the asymptotics (1.19)–(1.21).

2 Existence of dynamics and main results

We consider the Cauchy problem for the Hamiltonian system (1.3) which can be written as

$$\dot{Y}(t) = F(Y(t)),\ \ t \in \mathbb{R},\ \ Y(0) = Y_0. \tag{2.1}$$

Here $Y(t) = (\varphi(t), \pi(t), q(t), p(t))$, $Y_0 = (\varphi_0, \pi_0, q_0, p_0)$, and all derivatives are understood in the sense of distributions.

Now we introduce a suitable phase space. Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_{L^2}$, and let H^1 denote the Sobolev space $H^1 = \{\psi \in L^2 : |\nabla\psi| \in L^2\}$ with the norm $\|\psi\|_{H^1} = \|\nabla\psi\|_{L^2} + \|\psi\|_{L^2}$. For $\alpha \in \mathbb{R}$ let us define by L^2_α the weighted Sobolev spaces L^2_α with the norms $\|\psi\|_{L^2_\alpha} := \|(1+|x|)^\alpha\psi\|_{L^2}$.

Denote by $øH^1$ the completion of real space $C_0^\infty(\mathbb{R}^3)$ with the norm $\|\nabla\varphi(x)\|_{L^2}$. Equivalently, using Sobolev's embedding theorem, $øH^1 = \{\varphi(x) \in L^6(\mathbb{R}^3) :\ \ |\nabla\varphi(x)| \in L^2\}$. Denote by $øH^1_\alpha$ the completion of real space $C_0^\infty(\mathbb{R}^3)$ with the norm $\|(1+|x|)^\alpha\nabla\varphi(x)\|_{L^2}$.

For any $R > 0$ denote by $\|\varphi\|_{L^2(B_R)}$ the norm in $L^2(B_R)$, where $B_R = \{x \in \mathbb{R}^3 : |x| \le R\}$. Then the seminorms $\|\varphi\|_{H^1(B_R)} = \|\nabla\varphi\|_{L^2(B_R)} + \|\varphi\|_{L^2(B_R)}$ are continuous on $øH^1$.

Definition 2.1. *i) The phase space $\mathscr{E}$ is the real Hilbert space $øH^1 \oplus L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ of states $Y = (\psi, \pi, q, p)$ with the finite norm*

$$\|Y\|_{\mathscr{E}} = \|\nabla\psi\|_{L^2} + \|\pi\|_{L^2} + |q| + |p|.$$

ii) $\mathscr{E}_F$ is the space $\mathscr{E}$ endowed with the Fréchet topology defined by the local energy seminorms

$$\|Y\|_R = \|\varphi\|_{H^1(B_R)} + \|\pi\|_{L^2(B_R)} + |q| + |p|, \quad \forall R > 0. \tag{2.2}$$

iii) $\mathscr{E}_\alpha$ with $\alpha \in \mathbb{R}$ is the Hilbert space $øH^1_\alpha \oplus L^2_\alpha \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ with the norm

$$\|Y\|_\alpha = \|Y\|_{\mathscr{E}_\alpha} = \|\nabla\psi\|_{L^2_\alpha} + \|\pi\|_{L^2_\alpha} + |q| + |p|. \tag{2.3}$$

iv) $\mathscr{F}_\alpha$ is the space $øH^1_\alpha \oplus L^2_\alpha$ of fields $F = (\psi, \pi)$ with the finite norm

$$\|F\|_\alpha = \|F\|_{\mathscr{F}_\alpha} = \|\nabla\psi\|_{L^2_\alpha} + \|\pi\|_{L^2_\alpha}. \tag{2.4}$$

Note that we use the same notation for the norms in the space $\mathscr{F}_\alpha$ as in the space $\mathscr{E}_\alpha$ defined in (2.3). We hope it will not create misunderstandings since it will be always clear from the context if we deal with fields only, and therefore with the space $\mathscr{F}_\alpha$, or with fields-particles, and therefore with elements of the space $\mathscr{E}_\alpha$.

Note that both spaces $\mathscr{E}_F$ and $\mathscr{E}$ are metrisable, $øH^1$ is not contained in L^2 and for instance $\|S_q\|_{L^2} = \infty$. On the other hand, $S_q \in \mathscr{E}$. Therefore, $\mathscr{E}$ is the space of finite energy states. The Hamiltonian functional (1.4) is continuous on the space $\mathscr{E}$ and is bounded from below. In the point charge limit the lower bound tends to $-\infty$ by (1.6).

Lemma 2.2. *(see [22, Lemma 2.1]) Let conditions (1.9) and (1.10) hold. Then*
(i) For every $Y_0 \in \mathscr{E}$ the Cauchy problem (2.1) has a unique solution $Y(t) \in C(\mathbb{R}, \mathscr{E})$.

(ii) For every $t\in\mathbb{R}$ the map $U(t): Y_0 \mapsto Y(t)$ is continuous both on $\mathscr{E}$ and on $\mathscr{E}_F$.
(iii) The energy is conserved, i.e.

$$\mathscr{H}(Y(t)) = \mathscr{H}(Y_0) \quad for \;\; t\in\mathbb{R}. \tag{2.5}$$

(iv) The following a priori estimates hold

$$\|Y(t)\|_{\mathscr{E}} \le C(Y_0), \qquad t\in\mathbb{R}. \tag{2.6}$$

(v) The time derivatives $q^{(k)}(t)$, $k=0,1,2,3$, are uniformly bounded, i.e. there are constants $\overline{q}_k>0$, depending only on the initial data, such that

$$|q^{(k)}(t)| \le \overline{q}_k \quad for \;\; t\in\mathbb{R}. \tag{2.7}$$

Our first main result is the following theorem.

Theorem 2.3. *Let conditions (1.9)–(1.12) and (1.14)-(1.15) hold. Then for the corresponding solution $Y(t)\in\mathscr{E}$ to the Cauchy problem (2.1)*
i) The attraction holds

$$Y(t) \xrightarrow{\mathscr{E}_F} \mathscr{S}, \quad t\longrightarrow \pm\infty. \tag{2.8}$$

ii) If additionally, the set $\mathscr{S}$ is discrete, then (2.8) implies similar convergence

$$Y(t) \xrightarrow{\mathscr{E}_F} S_{\pm}, \quad t\longrightarrow \pm\infty. \tag{2.9}$$

Our second main result refine the asymptotics (2.8)– (2.9) for initial fields from the Sobolev weighted spaces.

Theorem 2.4. *Let conditions (1.9)–(1.11) hold, and let $Y(t)\in C(\mathbb{R},\mathscr{E})$ be a solution to the Cauchy problem (2.1) with $Y_0\in\mathscr{E}_\alpha$, where $\alpha>1$. Suppose that*

$$Y(t) \xrightarrow{\mathscr{E}_F} S_{q_+}, \qquad t\to\infty \tag{2.10}$$

where the limit point $q_+\in Z$ satisfies (1.18). Then
i) For every $\varepsilon>0$

$$\|Y(t)-S_{q_+}\|_{-\alpha} = \mathscr{O}(t^{-\alpha+\varepsilon}), \quad t\to\infty. \tag{2.11}$$

ii) For every $\varepsilon>0$ the scattering asymptotics hold,

$$(\varphi(x,t),\pi(x,t)) = (s_{q_+},0) + W(t)\Phi_+ + r(x,t), \tag{2.12}$$

where $\Phi_+\in øH^1\oplus L^2$, and

$$\|r(t)\|_{øH^1\oplus L^2} = \mathscr{O}(|t|^{-\alpha+1+\varepsilon}), \quad t\to\infty. \tag{2.13}$$

3 Liénard-Wiechert asymptotics

The solution to the non-homogeneous wave equation from the system (1.3) is the sum of two terms. The first is the retarded Liénard-Wiechert potential (3.1) which is the solution to the non-homogeneous wave equation with zero initial data. The second term is the solution to the homogeneous equation with the initial data of the total field. This term is given by the Kirchhoff formula (3.13).

The second term does not does not affect the long-time asymptotics of the solution due to the strong Huygens principle. Thus, exactly the retarded Liénard-Wiechert potential is responsible for the long-time asymptotics.

In this section we refine the results [22, 23] on the long time and long range asymptotics of the Liénard-Wiechert potentials

$$\varphi_r(x,t) = -\frac{1}{4\pi}\int \frac{dy\ \theta(t-|x-y|)}{|x-y|}\rho(y-q(t-|x-y|)), \qquad \pi_r(x,t) = \dot{\varphi}_r(x,t). \tag{3.1}$$

These asymptotics will play the key role in subsequent calculation of the energy radiation which is used in the proof of the relaxation (1.1). Furthermore, we estimate the energy radiation corresponding to the Kirchhoff integral (3.13).

First, we prove asymptotics of the retarded potentials in the wave zone $|x| \sim t \to \infty$. These asymptotics and their proofs are similar to that of Lemma 3.2 of [23].

Lemma 3.1. *Let conditions (1.9) and (1.10) hold. Then there exists $T_r > 0$ such that the following asymptotics hold uniformly in $t \in [T_r, T]$ for every fixed $T > T_r$,*

$$\pi_r(x,|x|+t) = \overline{\pi}(\omega(x),t)|x|^{-1} + \mathscr{O}(|x|^{-2}), \tag{3.2}$$
$$\nabla\varphi_r(x,|x|+t) = -\omega(x)\overline{\pi}(\omega(x),t)|x|^{-1} + \mathscr{O}(|x|^{-2}) \tag{3.3}$$

as $|x| \to \infty$ with a function $\overline{\pi}(\omega,t)$. Here $\omega(x) = x/|x|$.

Proof. The integrand of (3.1) vanishes for $|y| > T_r := \overline{q}_0 + R_\rho$. Then for $t - |x| > T_r$ one has

$$|x-y| \le |x| + |y| \le t - T_r + T_r \le t,$$

and hence (3.1) implies that

$$\pi_r(x,t) = -\int dy\ \frac{1}{4\pi|x-y|}\nabla\rho(y-q(\tau))\cdot\dot{q}(\tau), \tag{3.4}$$

where $\tau = t - |x-y|$. Similarly, for $t - |x| > T_r$

$$\begin{aligned} \nabla\varphi_r(x,t) &= \int dy\ \frac{1}{4\pi|x-y|} n\nabla\rho(y-q(\tau))\cdot\dot{q}(\tau) + \mathscr{O}(|x|^{-2}) \\ &= -\omega(x)\pi_r(x,t) + \mathscr{O}(|x|^{-2}), \end{aligned} \tag{3.5}$$

since $n = \dfrac{x-y}{|x-y|} = \omega(x) + \mathscr{O}(|x|^{-1})$ for bounded y. Now we substitute $|x|+t$ instead of t in representations (3.4), (3.5) to get asymptotics (3.2), (3.3) for $t > T_r$. Then τ becomes

$$\tau = |x| + t - |x-y| = t + \omega(x)\cdot y + \mathscr{O}(|x|^{-1}) = \overline{\tau} + \mathscr{O}(|x|^{-1}), \qquad \overline{\tau} = t + \omega\cdot y, \tag{3.6}$$

since

$$|x|-|x-y|=|x|-\sqrt{|x|^2-2x\cdot y+|y|^2}\sim|x|\Big(\frac{x\cdot y}{|x|^2}+\frac{|y|^2}{2|x|^2}\Big)=\omega(x)\cdot y+\mathscr{O}(|x|^{-1}).$$

Hence (3.4) implies (3.2) with

$$\overline{\pi}(\omega,t)=-\frac{1}{4\pi}\int dy\,\nabla\rho(y-q(\overline{\tau}))\cdot\dot q(\overline{\tau}). \tag{3.7}$$

Then (3.5) gives (3.3) immediately. □

Note that asymptotics (3.2) - (3.3) hold without condition (1.12). However, this condition allows us to represent (3.7) in a more efficient way for ω close to $(0,0,\pm1)$, see next lemma. Namely, let us denote

$$\Theta=\left\{\begin{array}{rl}0, & \overline{q}_1<1\\ \varepsilon+\sqrt{1-(\overline{q}_1)^{-2}}, & \overline{q}_1\ge 1\end{array}\right. \tag{3.8}$$

with an arbitrary small $0<\varepsilon<1-\sqrt{1-(\overline{q}_1)^{-2}}$. Then for $\omega=(\omega^1,\omega^2,\omega^3)$ with $|\omega^3|\ge\Theta$ we obtain

$$|\omega\cdot\dot q|=|\dot q||\cos(\omega,\dot q)|\le\overline{q}_1\sqrt{1-(\omega^3)^2}\le\overline{q}_1\sqrt{1-\Theta^2}<1. \tag{3.9}$$

Lemma 3.2. *Let conditions (1.9), (1.10) and (1.12) hold. Then for any ω with $|\omega^3|\ge\Theta$ one has*

$$\overline{\pi}(\omega,t)=\frac{1}{4\pi}\int dy\,\rho(y-q(\overline{\tau}))\frac{\omega\cdot\ddot q(\tau)}{(1-\omega\cdot\dot q(\tau))^2} \tag{3.10}$$

Proof. We observe that

$$\nabla_y\rho(y-q(\overline{\tau}))\cdot\dot q(\overline{\tau})=\nabla\rho(y-q(\overline{\tau}))\cdot\dot q(\overline{\tau})\,(1-\omega\cdot\dot q(\overline{\tau})).$$

Then (3.9) implies

$$\begin{aligned}\int dy\,\nabla\rho(y-q(\overline{\tau}))\cdot\dot q(\overline{\tau}) &= \int dy\,\nabla_y\rho(y-q(\overline{\tau}))\cdot\dot q(\overline{\tau})\frac{1}{1-\omega\cdot\dot q(\overline{\tau})}\\ &= -\int dy\,\rho(y-q(\overline{\tau}))\sum_{j=1}^{2}\frac{\partial}{\partial y^j}\frac{\dot q^j(\overline{\tau})}{1-\omega\cdot\dot q(\overline{\tau})}.\end{aligned} \tag{3.11}$$

Differentiating, we get

$$\sum_{j=1}^{2}\frac{\partial}{\partial y^j}\frac{\dot q^j}{1-\omega\cdot\dot q}=\frac{\omega\cdot\ddot q}{(1-\omega\cdot\dot q)^2}. \tag{3.12}$$

Then (3.7) agrees evidently with (3.10). □

Denote $(\varphi_K(t),\pi_K(t)):=W(t)[(\varphi_0,\pi_0)]$, where $\varphi_K(x,t)$ is the Kirchhoff integral

$$\varphi_K(x,t)=\frac{1}{4\pi t}\int_{S_t(x)}d^2y\,\pi_0(y)+\frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{S_t(x)}d^2y\,\varphi_0(y)\right), \tag{3.13}$$

and $\pi_K(x,t)=\dot\varphi_K(x,t)$. Here $S_t(x)$ denotes the sphere $\{y:\ |y-x|=t\}$ and d^2y is the corresponding surface area element. Below we will use the following lemma:

Lemma 3.3. *Let (φ_0,π_0) satisfies (1.14) and (1.15). Then there exist $I_0<\infty$ such that for every $R>0$ and every $T>T_0\geq 0$*

$$\int_{R+T_0}^{R+T} dt \int_{S_R} d^2x \left(|\pi_K(x,t)|^2+|\nabla\varphi_K(x,t)|^2\right)\leq I_0. \tag{3.14}$$

Here and below $S_R=S_R(0)$.

Proof. Formula (3.13) implies

$$\varphi_K(x,t)=\frac{t}{4\pi}\int_{S_1} d^2z\ \pi_0(x+tz)+\frac{1}{4\pi}\int_{S_1} d^2z\ \varphi_0(x+tz)+\frac{t}{4\pi}\int_{S_1} d^2z\ \nabla\varphi_0(x+tz)\cdot z.$$

Therefore

$$\nabla\varphi_K(x,t)=\frac{t}{4\pi}\int_{S_1} d^2z\ \nabla\pi_0(x+tz)+\frac{1}{4\pi}\int_{S_1} d^2z\ \nabla\varphi_0(x+tz)+\frac{t}{4\pi}\int_{S_1} d^2z\ \nabla_x(\nabla\varphi_0(x+tz)\cdot z).$$

Here all derivatives are understood in the classical sense. A similar representation holds for $\pi_K(x,t)$. Hence, taking into account the assumption (1.15), we obtain

$$|\pi_K(x,t)|,\ |\nabla\varphi_K(x,t)|\leq C\sum_{s=0}^{1} t^s\int_{S_1} d^2z\ |x+tz|^{-\sigma-s},\quad \sigma>3/2. \tag{3.15}$$

Further, for $\sigma\neq 2$ we have

$$\int_{S_1} d^2z\ |x+tz|^{-\sigma-s}=\frac{2\pi}{(\sigma+s-2)|x|t}\Big((t-|x|)^{2-\sigma-s}-(t+|x|)^{2-\sigma-s}\Big),\quad s=0,1.$$

Therefore,

$$\begin{aligned}\int\limits_{R+T_0}^{R+T} dt\int\limits_{S_R} d^2x\Big(|\pi_K(x,t)|^2+|\nabla\varphi_K(x,t)|^2\Big)\ &\leq\ C\int\limits_{R+T_0}^{R+T}\Big[\frac{(t+R)^{4-2\sigma}+(t-R)^{4-2\sigma}}{t^2}+(t-R)^{2-2\sigma}\Big]dt\\ &\leq\ C_1\int\limits_{R+T_0}^{R+T} dt\Big[\Big(1+\frac{R}{t}\Big)^2+\Big(1-\frac{R}{t}\Big)^2+1\Big](t-R)^{2-2\sigma}<\infty.\end{aligned}$$

□

4 Scattering of energy to infinity

In this section we establish a lower bound on the total energy radiated to infinity in terms of a ”radiation integral”. Since the energy is bounded a priori, this integral has to be finite, which is then our main input for proving Theorem 2.3.

Proposition 4.1. *Let conditions (1.9), (1.10), (1.14), (1.15) hold, and let $Y(t)=(\varphi(t),\pi(t),q(t),p(t))\in C(\mathbb{R},\mathscr{E})$ be the solution to (1.2) with initial data $Y(0)=(\varphi_0,\pi_0,q_0,p_0)$. Then*

$$\int_0^\infty dt\int_{S_1} d^2\omega|\overline{\pi}(\omega,t)|^2<\infty. \tag{4.1}$$

Proof. Step i). The energy $\mathscr{H}_R(t)$ at time $t \in \mathbb{R}$ in the ball B_R with a radius $R > \overline{q}_0 + R_\rho$ is defined by

$$\mathscr{H}_R(t) = \frac{1}{2}\int_{B_R} dx\, \left(|\pi(x,t)|^2 + |\nabla\varphi(x,t)|^2\right) + \frac{1}{2}p^2(t) + V(q(t)) + \int_{\mathbb{R}^3} dx\, \varphi(x,t)\rho(x-q(t)). \qquad (4.2)$$

Let us fix a $R > 0$ and consider a total radiated energy $\mathscr{H}_R(R+T_0) - \mathscr{H}_R(R+T)$ from the ball B_R during the time interval $[R+T_0, R+T]$, where $T > T_0 \geq 0$. This quantity is bounded a priori, because $\mathscr{H}_R(R+T_0)$ and $\mathscr{H}_R(R+T)$ are bounded by (2.6). Hence,

$$\mathscr{H}_R(R+T_0) - \mathscr{H}_R(R+T) \leq I < \infty, \qquad (4.3)$$

where I does not depend on T_0, T and R.

Step ii). Note that the function $\varphi(x,t) = \varphi_r(x,t) + \varphi_K(x,t)$ is C^1 differentiable in the region $t > |x|$ by (1.15), (3.1) and (3.13). Hence, differentiating (4.2) in t and integrating by parts, we get

$$\frac{d}{dt}\mathscr{H}_R(t) = \int_{S_R} d^2x\, \omega(x)\cdot\pi(x,t)\nabla\varphi(x,t), \quad t > R. \qquad (4.4)$$

Now (4.4) and (4.3) imply

$$-\int_{R+T_0}^{R+T} dt\, \int_{S_R} d^2x\, \omega(x)\cdot\pi(x,t)\nabla\varphi(x,t) \leq I. \qquad (4.5)$$

Step iii). Let us show that 4.5 leads to (4.1) in the limits $R \to \infty$ and then $T \to \infty$. Indeed, substituting

$$\pi = \pi_r + \pi_K, \quad \varphi = \varphi_r + \varphi_K \qquad (4.6)$$

into (4.5), we obtain

$$-\int_{R+T_0}^{R+T} dt\, \int_{S_R} d^2x\, \omega(x)\cdot(\pi_r\nabla\varphi_r + \pi_K\nabla\varphi_r + \pi_r\nabla\varphi_K + \pi_K\nabla\varphi_K) \leq I. \qquad (4.7)$$

Then Lemmas 3.1 and 3.3 imply for every fixed $T > T_0 := T_r$,

$$\int_{T_r}^{T} dt\, \int_{S_1} d^2\omega\, |\overline{\pi}(\omega,t)|^2 \leq I_1 + T\,\mathscr{O}(R^{-1}), \qquad (4.8)$$

where $I_1 < \infty$ does not depend on T and R. This follows by the Cauchy-Schwarz inequality. Taking the limit $R \to \infty$ and then $T \to \infty$ we obtain (4.1). □

5 Relaxation of the particle acceleration and velocity

In this section we deduce the relaxation $\dot{q}(t) \to 0$, $\ddot{q}(t) \to 0$ as $t \to \infty$ using Proposition 4.1. First, the function

$$\overline{\pi}(\omega,t) = \frac{1}{4\pi}\int dy\, \rho(y - q(t+\omega\cdot y))\frac{\omega\cdot\ddot{q}(t+\omega\cdot y)}{(1-\omega\cdot\dot{q}(t+\omega\cdot y))^2} \qquad (5.1)$$

is globally Lipschitz-continuous in ω and t for $|\omega^3| \geq \Theta$ due to (3.9) and the bounds (2.7) with $k = 2,3$. Hence, Proposition 4.1 implies that

$$\lim_{t\to\infty}\overline{\pi}(\omega,t) = 0 \qquad (5.2)$$

uniformly in $\omega \in \Omega(\Theta) := \{\omega \in S_1\colon |\omega^3| \geq \Theta\}$. Denote $r(t) = \omega \cdot q(t) \in \mathbb{R}$, $s = \omega \cdot y$, and $\rho_a(q^3) = \int dq^1 dq^2 \rho(q^1, q^2, q^3)$, and decompose in (5.1) the y-integration along and transversal to ω. Then

$$\begin{aligned} \overline{\pi}(\omega,t) &= \int ds\, \rho_a(s - r(t+s))\, \frac{\ddot{r}(t+s)}{(1-\dot{r}(t+s))^2} \\ &= \int d\tau\, \rho_a(t - (\tau - r(\tau)))\, \frac{\ddot{r}(\tau)}{(1-\dot{r}(\tau))^2} = \int d\theta\, \rho_a(t-\theta) g_\omega(\theta) = \rho_a * g_\omega(t). \end{aligned} \tag{5.3}$$

Here we substituted $\theta = \theta(\tau) = \tau - r(\tau)$, which is a nondegenerate diffeomorphism since $\dot{r} \leq \overline{r} < 1$ due to (3.9), and we set

$$g_\omega(\theta) = \frac{\ddot{r}(\tau(\theta))}{(1-\dot{r}(\tau(\theta)))^3}, \qquad \omega \in \Omega(\Theta). \tag{5.4}$$

Now we extend $q(t)$ smoothly to zero for $t < 0$. Then $\tilde{\rho} * g_\omega(t)$ is defined for all t and agrees with $\overline{\pi}(\omega,t)$ for sufficiently large t. Hence (5.2) reads as a convolution limit

$$\lim_{t\to\infty} \rho_a * g_\omega(t) = 0, \qquad \omega \in \Omega(\Theta). \tag{5.5}$$

Now note that (2.7) with $k = 2,3$ imply that $g'_\omega(\theta)$ is bounded. Hence (5.5) and (1.11) imply by Pitt's extension to Wiener's Tauberian Theorem, cf. [25, Thm. 9.7(b)],

$$\lim_{\theta\to\infty} g_\omega(\theta) = 0, \qquad \omega \in \Omega(\Theta). \tag{5.6}$$

Lemma 5.1. *Let conditions (1.9)–(1.12) and (1.14)-(1.15) hold, and let $Y(t) \in \mathscr{E}$ be the corresponding solution to the Cauchy problem (2.1). Then*

$$\lim_{t\to\infty} \ddot{q}(t) = 0. \tag{5.7}$$

Proof. The limit (5.6) holds for any $\omega \in S_1$ with $|\omega^3| \geq \Theta$ (see (3.8)). Moreover, $\theta(t) \to \infty$ as $t \to \infty$. Hence, $\ddot{r}(t) = \omega \cdot \ddot{q}(t) \to 0$ as $t \to \infty$ for any $\omega \in \Omega(\Theta)$. □

Remarks 5.2. *(i) For a point charge $\rho(x) = \delta(x)$ we have $\rho_a(s) = \delta(s)$. Hence, (5.5) implies (5.6) directly, without the application of the Wiener Tauberian Theorem.*
(ii) Condition (1.11) is necessary for the implication (5.6)⇒(5.7). Indeed, if (1.11) is violated, then $\hat{\rho}_a(\xi) = 0$ for some $\xi \in \mathbb{R}$, and with the choice $g(\theta) = \exp(i\xi\theta)$ we have $\rho_a * g(t) = 0$ whereas g does not decay to zero.

Corollary 5.3. *Let conditions of Lemma 5.1 hold. Then*

$$\lim_{t\to\infty} \dot{q}(t) = 0. \tag{5.8}$$

Proof. (5.7) implies (5.8) since $|q(t)| \leq \overline{q}_0$ due to (2.7) with $k = 0$. □

6 Transitions to stationary states

Here we prove our main Theorem 2.3. First we show that the set

$$\mathscr{A} = \{S_q : \ q = (q^1, q^2, 0) \in \mathbb{R}^3, \ |q| \le \overline{q}_0)\} \tag{6.1}$$

is an attracting subset. It is compact in $\mathscr{E}_F$ since $\mathscr{A}$ is homeomorphic to a closed ball in $\mathbb{R}^3$.

Lemma 6.1. *Let conditions of Theorem 2.3 hold. Then*

$$Y(t) \xrightarrow{\mathscr{E}_F} \mathscr{A}, \quad t \longrightarrow \pm\infty. \tag{6.2}$$

Proof. It suffices to verify that for every $R > 0$

$$\|Y(t) - S_{q(t)}\|_R = \|\varphi(t) - s_{q(t)}\|_{H^1(B_R)} + \|\pi(t)\|_{L^2(B_R)} + |p(t)| \to 0 \text{ as } t \to \infty. \tag{6.3}$$

Let us estimate each term separately.

i) Convergence (5.8) implies that $|p(t)| \to 0$ as $t \to \infty$.

ii) The integral representation (3.4) implies that for $|x| < R$ and $t > R + T_r$, $T_r = \overline{q}_0 + R_\rho$, we have

$$|\pi_r(x,t)| \le \ C \max_{\tau \in [t-R-T_r,t]} |\dot{q}(\tau)| \int_{|y|<T_r} dy \frac{1}{|x-y|} |\nabla \rho(y - q(t - |x-y|))|.$$

Here the integral is bounded uniformly in $t > R + T_r$ for $x \in B_R$, and therefore (5.8) implies that $\|\pi_r(t)\|_{L^2(B_R)} \to 0$ as $t \to \infty$. Hence, $\|\pi(t)\|_{L^2(B_R)} \to 0$ by (4.6) and (3.15).

iii) The integral representation (3.1) implies for $t > R + T_r$ and $|x| < R$ that

$$\varphi_r(x,t) - s_{q(t)}(x) = -\int_{|y|<T_r} dy \frac{1}{4\pi|x-y|} \Big(\rho(y - q(t - |x-y|)) - \rho(y - q(t))\Big).$$

The difference $q(t - |x-y|) - q(t)$ may be written as an integral depending only on $\dot{q}(\tau)$ for $\tau \in [t - R - T_r, t]$, which tends to zero as $t \to \infty$ uniformly in $x \in B_R$ due to (5.8). Hence $\|\varphi_r(t) - \varphi_{q(t)}\|_{L^2(B_R)} \to 0$ as $t \to \infty$. Then $\|\varphi(t) - \varphi_{q(t)}\|_{L^2(B_R)} \to 0$ by (4.6) and (3.15). This proves the claim, since $\|\nabla(\varphi(t) - \varphi_{q(t)})\|_{L^2(B_R)}$ may be estimated in a similar way. $\square$

Now we prove the convergences (2.8).

Lemma 6.2. *Under conditions of Theorem 2.3 the convergence holds*

$$Y(t) \xrightarrow{\mathscr{E}_F} \mathscr{S}, \quad t \longrightarrow \pm\infty. \tag{6.4}$$

Proof. Lemma 6.1 implies that the orbit $O(Y) := \{Y(t) : t \in \mathbb{R}\}$ is precompact in $\mathscr{E}_F$ since $\mathscr{A}$ is the compact set in $\mathscr{E}_F$. Let us denote by Ω the set of all omega-limit points of the orbit in $\mathscr{E}_F$: $\overline{Y} \in \Omega$ means by definition that

$$Y(t_k) \xrightarrow{\mathscr{E}_F} \overline{Y}, \qquad t_k \to \infty. \tag{6.5}$$

It suffices to prove that $\Omega \subset \mathscr{S}$, i.e. that any omega-limit point $\overline{Y} = S_{q_+}$ with some $q_+ \in Z$.

First, Lemma 6.1 implies that $\overline{Y} \in \mathscr{A}$. Further, Ω is invariant with respect to the dynamical group $U(t)$ with $t \in \mathbb{R}$ due to the continuity of $U(t)$ in $\mathscr{E}_F$. Hence, there exists a C^2-curve $t \mapsto Q(t) \in \mathbb{R}^3$ such that $U(t)\overline{Y} = S_{Q(t)}$, according to Definition (6.1). However, for $S_{Q(t)}$ to be a solution of (1.3) we must have $\dot{Q}(t) \equiv 0$, and hence $Q(t) \equiv q_+ \in Z$. Therefore, $\overline{Y} = S_{q_+} \in \mathscr{S}$. $\square$

At last, we formalize the implication (2.8) $\Rightarrow$ (2.9) by the following definition. Let $\mathscr{T}$ be a subset of a metrisable space $\mathscr{F}$.

Definition 6.3. *$\mathscr{T}$ is a trapping set in $\mathscr{F}$, if for every continuous curve $Y(t) \in C(\mathbb{R}, \mathscr{F})$ with a precompact orbit $O(Y)$ the convergence $Y(t) \xrightarrow{\mathscr{F}} \mathscr{T}$ as $t \to \infty$ implies the convergence $Y(t) \xrightarrow{\mathscr{F}} T$ as $t \to \infty$ to some point $T \in \mathscr{T}$.*

For example every discrete set in $\mathbb{R}^3$ is a trapping set in $\mathbb{R}^3$.

Lemma 6.4. *Let the conditions of of Theorem 2.3 hold and let Z be a trapping set in $\mathbb{R}^3$. Then there exist stationary states $S^{\pm} \in \mathscr{S}$ depending on Y_0 such that (2.9) holds.*

Proof. The set Z is the image of the set $\mathscr{S}$ under the map $I: \ (\varphi, \pi, q, p) \mapsto q$. This map is continuous $\mathscr{E}_F \to \mathbb{R}^3$ and it is injection on $\mathscr{S}$. Therefore $\mathscr{S}$ is a trapping set in $\mathscr{E}_F$, because Z is a trapping set in $\mathbb{R}^3$. Hence (2.8) implies (2.9). $\square$

7 Linearization at stationary state

In the rest of the paper we prove Theorem 2.4. If the particle is close to a stable minimum of V, we expect the nonlinear evolution to be dominated by the linearized dynamics. In this case the rate of the convergence (2.9) corresponds to the decay rate of initial fields. For notational simplicity we assume isotropy in the following sense

$$\partial_i \partial_j V(q_+) = v_0^2 \delta_{ij}, \ \ i, j = 1, 2, 3 \ \ v_0 > 0. \tag{7.1}$$

Without loss of generality we take $q_+ = 0$. Let $S_0 = (s_0, 0, 0, 0)$ be the stationary state of (1.3) corresponding to $q_+ = 0$. To linearize (1.3) at S_0, we set $\varphi(x,t) = s_0(x) + \psi(x,t)$. Then (1.3) becomes

$$\left.\begin{array}{llcl} \dot{\psi}(x,t) = \pi(x,t), & \dot{\pi}(x,t) & = & \Delta\psi(x,t) + \rho(x) - \rho(x - q(t)), \\ \dot{q}(t) = p(t), & \dot{p}(t) & = & -\nabla V(q(t)) + \displaystyle\int d^3x \ \psi(x,t) \nabla\rho(x - q(t)) \\ & & & + \displaystyle\int d^3x \, s_0(x)[\nabla\rho(x - q(t)) - \nabla\rho(x)]. \end{array}\right| \tag{7.2}$$

We denote $X(t) = Y(t) - S_0 = (\psi(t), \pi(t), q(t), p(t)) \in C(\mathbb{R}, \mathscr{E})$ and rewrite the nonlinear system (7.2) in the form

$$\dot{X}(t) = AX(t) + B(X(t)). \tag{7.3}$$

Here the linear operator A reads

$$A: (\psi, \pi, q, p) \mapsto (\pi, \ \Delta\psi + \nabla\rho \cdot q, \ p, \ -(v_0^2 + v_1^2)q + \int d^3x \, \psi(x) \nabla\rho(x)),$$

with

$$v_1^2 \delta_{ij} = \frac{1}{3} \|\rho\|_{L^2}^2 \delta_{ij} = -\int d^3x \, \partial_i s_0(x) \partial_j \rho(x). \tag{7.4}$$

The factor $1/3$ is due to a spherical symmetry of $\rho(x)$ (see (1.10)). The nonlinear part is given by

$$\begin{aligned} B(X) \ = \ \Big(0, \ \rho(x) - \rho(x-q) - \nabla\rho(x) \cdot q, \ 0, \ -\nabla V(q) + v_0^2 q + \int d^3x \, \psi(x)[\nabla\rho(x-q) - \nabla\rho(x)] \\ + \int d^3x \nabla s_0(x)[\rho(x) - \rho(x-q) - \nabla\rho(x) \cdot q]\Big). \end{aligned} \tag{7.5}$$

Consider the Cauchy problem for the linear equation

$$\dot{Z}(t) = AZ(t), \quad Z = (\Psi, \Pi, Q, P), \quad t \in \mathbb{R}, \tag{7.6}$$

with initial condition

$$Z|_{t=0} = Z_0 = (\Psi_0, \Pi_0, Q_0, P_0). \tag{7.7}$$

System (7.6) is a formal Hamiltonian system with the quadratic Hamiltonian

$$\mathscr{H}_0(Z) = \frac{1}{2}\Big(P^2 + (v_0^2 + v_1^2)Q^2 + \int d^3x\,(|\Pi(x)|^2 + |\nabla\Psi(x)|^2 - 2\Psi(x)\nabla\rho(x)\cdot Q)\Big), \tag{7.8}$$

which is the formal Taylor expansion of $\mathscr{H}(Y_0 + Z)$ up to second order at $Z = 0$.

Lemma 7.1. *Let condition (1.10) holds and $Z_0 \in \mathscr{E}$. Then*
(i) The Cauchy problem (7.6), (7.7) has a unique solution $Z(t) \in C(\mathbb{R}, \mathscr{E})$.
(ii) For every t, the map $U_0(t) : Z_0 \mapsto Z(t)$ is continuous both on $\mathscr{E}$ and $\mathscr{E}_F$.
(iii) The energy $\mathscr{H}_0$ is conserved, i.e.

$$\mathscr{H}_0(Z(t)) = \mathscr{H}_0(Z_0), \quad t \in \mathbb{R}. \tag{7.9}$$

iv) The estimate holds

$$\|Z(t)\|_{\mathscr{E}} \le C, \quad t \in \mathbb{R} \tag{7.10}$$

with C depending only on the norm $\|Z_0\|_{\mathscr{E}}$ of the initial state.

The key role in the proof is played the positivity of the Hamiltonian (7.8):

$$2\mathscr{H}_0(Z) = P^2 + v_0^2 Q^2 + \int d^3x\,(|\Pi(x)|^2 + |\nabla\Psi(x) + \rho(x)Q)|^2 \ge 0.$$

Thus (7.10) follows from (7.9) because of $v_0 > 0$. The positivity of $\mathscr{H}_0$ is also obvious from (1.4).

In [21] we proved the following long-time decay of the linearized dynamics in the weighted Sobolev norms.

Proposition 7.2. *Let conditions (1.10)–(1.11) hold, and let $Z_0 \in \mathscr{E}_\alpha$ with some $\alpha > 1$. Then*

$$\|U_0(t)Z_0\|_{-\alpha} \le C(\rho, \alpha)(1 + |t|)^{-\alpha}\|Z_0\|_\alpha, \qquad t \in \mathbb{R}. \tag{7.11}$$

Similar decay also holds for the dynamical group $W(t)$ of 3D free wave equation.

Proposition 7.3. *(cf. [24, Proposition 2.1] and [19]. Let $(\varphi_0, \pi_0) \in \mathscr{F}_\alpha$ with some $\alpha > 1$. Then*

$$\|W(t)(\varphi_0, \pi_0)\|_{-\alpha} \le C(\alpha)(1 + |t|)^{-\alpha}\|(\varphi_0, \pi_0)\|_\alpha, \qquad t \in \mathbb{R}. \tag{7.12}$$

We will use both these decays in the next section.

8 A nonlinear Huygens principle

The following lemma is a version of strong Huygens principle for the nonlinear system (1.3). Let M_* be a fixed number, $M_* > 3R_\rho + 1$.

Lemma 8.1. *Let conditions of Theorem 2.4 hold and let $\delta > 0$ be an arbitrary fixed number. Then for sufficiently large $t_* > 0$ there exists a solution*

$$Y_*(t) = (\varphi_*(x,t), \pi_*(x,t), q_*(t), p_*(t)) \in C([t_*, \infty), \mathscr{E})$$

to the system (1.3) such that
(i) $Y_*(t)$ coincides with $Y(t)$ in some future cone,

$$\begin{array}{rll} \varphi_*(x,t) & = \varphi(x,t) & for\ \ |x| < t - t_*, \\ q_*(t) & = q(t) & for\ \ t > t_*. \end{array} \tag{8.1}$$

(ii) $Y_*(t_*)$ admits a decomposition $Y_*(t_*) = S_0 + K_0 + Z_0$, where $Z_0 = (\Psi_0, \Pi_0, Q_0, P_0)$ satisfies

$$\Psi_0(x) = \Pi_0(x) = 0 \ \ for\ \ |x| \ge M_*\ , \tag{8.2}$$

$$\|Z_0\|_\alpha \le \delta, \tag{8.3}$$

and K_0 satisfies

$$\|U_0(\tau)K_0\|_{-\alpha} \le C(1 + t_* + \tau)^{-\alpha}, \quad \tau > 0, \tag{8.4}$$

where $C = C(\alpha)$ does not depend on δ.

Proof. The convergence (2.10) with $q_+ = 0$ implies that for every $\varepsilon > 0$ there exist t_ε such that

$$|q(t)| + |\dot{q}(t)| < \varepsilon \ \ \text{for}\ \ t > t_\varepsilon. \tag{8.5}$$

We may assume that $t_\varepsilon > 1/\varepsilon$. Denote

$$t_{0,\varepsilon} = t_\varepsilon + R_\rho,\ \ t_{1,\varepsilon} = t_{0,\varepsilon} + 1,\ \ t_{2,\varepsilon} = t_{1,\varepsilon} + \varepsilon + R_\rho,\ \ t_{3,\varepsilon} = t_{2,\varepsilon} + \varepsilon + R_\rho. \tag{8.6}$$

Then there exist a function $q_\varepsilon(\cdot) \in C^1(\mathbb{R})$ such that

$$q_\varepsilon(t) = \left\{ \begin{array}{rl} q(t), & t > t_{1,\varepsilon}, \\ 0, & t < t_{0,\varepsilon}, \end{array} \right. \quad \text{and}\ \ |q_\varepsilon(t)| + |\dot{q}_\varepsilon(t)| < \varepsilon \ \ \text{for all}\ \ t \in \mathbb{R} \tag{8.7}$$

by suitable interpolation. Now we define the modification $\varphi_\varepsilon(x,t)$ of the solution $\varphi(x,t) = \varphi_r(x,t) + \varphi_K(x,t)$:

$$\varphi_\varepsilon(x,t) = \varphi_{r,\varepsilon}(x,t) + \varphi_K(x,t) \ \ \text{for}\ \ x \in \mathbb{R}^3 \ \ \text{and}\ \ t > 0, \tag{8.8}$$

where

$$\varphi_{r,\varepsilon}(x,t) = -\int d^3y\, \frac{1}{4\pi|x-y|} \rho(y - q_\varepsilon(t - |x-y|))\ . \tag{8.9}$$

For $|x| < t - t_{2,\varepsilon}$ and $|y| \le R_\rho + \varepsilon$, we have

$$t - |x-y| > t - (|x| + |y|) > t - (t - t_{2,\varepsilon} + R_\rho + \varepsilon) = t_{1,\varepsilon}.$$

Then (8.9), (3.1), and (8.7) imply

$$\varphi_{r,\varepsilon}(x,t)=\varphi_r(x,t) \text{ for } |x|<t-t_{2,\varepsilon}. \tag{8.10}$$

Further, for $|x|>t-t_\varepsilon$ and $|y|\le R_\rho$, we obtain

$$t-|x-y|<t-(|x|-|y|)<t-(t-t_\varepsilon-R_\rho)=t_{0,\varepsilon}.$$

Then $q_\varepsilon(t-|x-y|)=0$ by (8.7), and hence

$$\varphi_{r,\varepsilon}(x,t)=s_0(x) \text{ for } |x|>t-t_\varepsilon. \tag{8.11}$$

Moreover, $\varphi_{r,\varepsilon}(\cdot,\cdot)\in C^1(\mathbb{R}^4)$, and (8.7) implies

$$\sup_{x\in\mathbb{R}^3,t\in\mathbb{R}}(|\dot\varphi_{r,\varepsilon}(x,t)|+|\nabla\varphi_{r,\varepsilon}(x,t)-\nabla s_0(x)|+|\varphi_{r,\varepsilon}(x,t)-s_0(x)|)=\mathscr{O}(\varepsilon). \tag{8.12}$$

Now we define $t_*:=t_{3,\varepsilon}$, and

$$Y_*(t)=(\varphi_\varepsilon(t),\dot\varphi_\varepsilon(t),q(t),p(t)),\ \ K_0=(\varphi_K(t_*),\dot\varphi_K(t_*),0,0),\ \ Z_0=(\varphi_{r,\varepsilon}(t_*)-s_0,\dot\varphi_{r,\varepsilon}(t_*),q(t_*),p(t_*)). \tag{8.13}$$

It is easy to check that t_* and $Y_*(t)$, K_0, Z_0 satisfy all requirements of Lemma 8.1, provided $\varepsilon>0$ be sufficiently small.

First, $Y_*(x,t)$ is a solution to (1.3) for $t>t_*$. Indeed, for $|x|<\varepsilon+R_\rho$ one has $t-|x-y|>t_{3,\varepsilon}-2\varepsilon-2R_\rho=t_{1,\varepsilon}$. Since, (8.6) implies that $q_\varepsilon(t-|x-y|)=q(t-|x-y|)$ and $\varphi_\varepsilon(x,t)=\varphi(x,t)$ then. Hence, $Y_*(t)=Y(t)$ in the region $|x|<\varepsilon+R_\rho$. On the other hand, (8.5) and (8.7) imply

$$\rho(x-q_\varepsilon(t))=\rho(x-q(t))=0 \text{ for } |x|>\varepsilon+R_\rho \text{ and } t>t_\varepsilon.$$

Hence, $\varphi_{r,\varepsilon}(x,t)$ satisfies the equation

$$\ddot\varphi(x,t)=\Delta\varphi(x,t) \text{ for } |x|>\varepsilon+R_\rho \text{ and } t>t_\varepsilon. \tag{8.14}$$

Therefore, $Y_*(t)$ satisfies (1.3) in the region $|x|>\varepsilon+R_\rho$. Now (8.1) follows from (8.7) and (8.10), (8.2) for $M_*=3R_\rho+2\varepsilon+1$ follows from (8.11), and (8.3) follows from (8.2) and (8.12).

It remains to prove (8.4). We deduce the estimate from the decay (7.12) for the linearized dynamics $U(t)$ and decay (7.11) for $W(t)$. Denote $U(\tau)K_0=(\Psi(x,\tau),\Pi(x,\tau),Q(\tau),P(\tau))$. From [21, formulas (4.18), (4.19), (4.25)] it follows that

$$\begin{pmatrix} Q(\tau)\\ P(\tau)\end{pmatrix}=\mathscr{L}*\begin{pmatrix}0\\ f_k\end{pmatrix}(\tau)$$

where

$$f_k(\tau)=\langle W(\tau)[\phi_k(t_*),\dot\phi_k(t_*)],\nabla\rho\rangle=\langle W(\tau+t_*)[\phi_0,\pi_0],\nabla\rho\rangle,$$

and

$$(\Psi(\tau),\Pi(\tau))=W(\tau+t_*)[\phi_0,\pi_0]+\int_0^\tau W(\tau-s)[0,Q(s)\cdot\nabla\rho]ds$$

Moreover, according to [21, formula (4.20)] for $\mathscr{L}(t)$ the decay holds

$$\mathscr{L}(t)=\mathscr{O}(|t|)^{-N},\quad t\to\infty,\quad \forall N>0.$$

Then the decay (8.4) follows. □

9 The rate of convergence

Here we prove Theorem 2.4 i). Due to (8.1) it suffices to prove that for any $\varepsilon > 0$

$$\|Y_*(t) - S_0\|_{-\alpha} = \mathscr{O}(t^{-\alpha+\varepsilon}), \quad t \to \infty. \tag{9.1}$$

Denote $X(\tau) = Y_*(t_* + \tau) - S_0$. Then $X(0) = K_0 + Z_0$ and the integrated version of (7.3) reads

$$X(\tau) = U_0(\tau)K_0 + U_0(\tau)Z_0 + \int_0^\tau ds U_0(\tau - s)B(X(s)), \quad \tau > 0. \tag{9.2}$$

Further, (7.11), (7.5), (8.2) and (8.4) imply

$$\|X(\tau)\|_{-\alpha} \le C\Big((t_* + \tau + 1)^{-\alpha} + (1+\tau)^{-\alpha}\|Z_0\|_\alpha + \int_0^\tau ds\,(1+\tau-s)^{-\alpha}\|X(s)\|^2_{-\alpha}\Big), \quad \tau > 0. \tag{9.3}$$

We fix an arbitrary $\varepsilon \in (0, 1/2)$ and introduce the majorant

$$m(t) = \sup_{0\le s\le t} (1+s)^{\alpha-\varepsilon}\|X(s)\|_{-\alpha}. \tag{9.4}$$

Let μ be any fixed positive number, and let T_μ be the exit time:

$$T_\mu = \sup\{t > 0 : m(t) \le \mu\}. \tag{9.5}$$

Multiplying both sides of (9.3) by $(1+\tau)^{\alpha-\varepsilon}$, and taking the supremum in $\tau \in [0, T_\mu]$, we get

$$m(\tau) \le C\Big(\frac{(1+\tau)^{\alpha-\varepsilon}}{(1+t_*+\tau)^\alpha} + \delta + \int_0^\tau ds \frac{(1+\tau)^{\alpha-\varepsilon}}{(1+\tau-s)^\alpha}\frac{m^2(s)}{(1+s)^{2\alpha-2\varepsilon}}\Big), \quad \tau \le T_\mu. \tag{9.6}$$

Note that for every $\varepsilon > 0$

$$\sup_{\tau>0} \frac{(1+\tau)^{\alpha-\varepsilon}}{(1+t_*+\tau)^\alpha} \to 0, \quad t_* \to \infty. \tag{9.7}$$

Hence taking into account that $m(t)$ is a monotone increasing function, we get for sufficiently large t_* that

$$m(\tau) \le C(\delta + Cm^2(\tau)), \quad \tau \le T_\mu. \tag{9.8}$$

This inequality implies that $m(\tau)$ is bounded for $\tau \le T_\mu$, and moreover,

$$m(\tau) \le C_1\delta, \quad \tau \le T \tag{9.9}$$

if δ is sufficiently small. The constant C_1 in (9.9) does not depend on T. Due to Lemma 8.1 we can choose t_* so large that $\delta < \mu/(2C_1)$. Then (9.9) implies that $T = \infty$ and (9.9) holds for all $\tau > 0$ if t_* is sufficiently large. □

10 Scattering asymptotics

Here we prove Theorem 2.4 ii). We prove asymptotics (2.12)–(2.13) for $t \to +\infty$ only since system (1.3) is time reversible. Denote $\Phi(x,t) = (\Phi_1(x,t), \Phi_2(x,t)) = (\varphi(x,t), \pi(x,t)) - (s_{q_+}, 0)$. Then asymptotics (2.12)– (2.13) are equivalent to

$$\Phi(t) = W(t)\Phi_+ + r(t), \quad \|r(t)\|_{øH^1 \oplus L^2} = \mathscr{O}(t^{-\alpha+1+\varepsilon}), \quad t \to +\infty,$$

This is equivalent to

$$W(-t)\Phi(t) = \Phi_+ + r_1(t), \quad \|r_1(t)\|_{øH^1 \oplus L^2} = \mathscr{O}(t^{-\alpha+1+\varepsilon}), \quad t \to +\infty \tag{10.1}$$

due to the unitarity of $W(t)$ on $øH^1 \oplus L^2$. The first two equations of (1.3) imply

$$\dot{\Phi}_1(x,t) = \Phi_2(x,t), \quad \dot{\Phi}_2(x,t) = \Delta\Phi_1(x,t) + \rho(x-q_+) - \rho(x-q(t)).$$

Then

$$\Phi(t) = W(t)\Phi(0) - \int_0^t W(t-s)[(0, \rho(x-q_+) - \rho(x-q(s)))]ds. \tag{10.2}$$

Therefore,

$$W(-t)\Phi(t) = \Phi(0) - \int_0^t W(-s)R(s)ds, \qquad R(s) = (0, \rho(x-q_+) - \rho(x-q(s))), \tag{10.3}$$

where the integral converges in $øH^1 \oplus L^2$ with the rate $\mathscr{O}(t^{-\alpha+1+\varepsilon})$. Indeed,

$$\|W(-s)R(s)\|_{øH^1 \oplus L^2} = \mathscr{O}(s^{-\alpha+\varepsilon}), \quad 0 < \varepsilon < \alpha - 1$$

by the unitarity of $W(-s)$ and the decay rate $\|R(s)\|_{øH^1 \oplus L^2} = \mathscr{O}(s^{-\alpha+\varepsilon})$ which follows from the conditions (1.10) on ρ and the asymptotics (2.11). Setting

$$\Phi_+ = \Phi(0) - \int_0^\infty W(-s)R(s)ds, \quad r_1(t) = \int_t^\infty W(-s)R(s)ds,$$

we obtain (10.1).

References

[1] M. Abraham, Prinzipien der Dynamik des Elektrons, Annalen der Physik **10** (1903), 105-179.

[2] V.S. Buslaev, G.S. Perelman, Scattering for the nonlinear Schrödinger equation: states that are close to a soliton, *Algebra i Analiz*, **4** (1992), 63-102.

[3] V.S. Buslaev, G.S. Perelman, On the stability of solitary waves for nonlinear Schrödinger equations, *Nonlinear evolution equations*, vol. 164 of Amer. Math. Soc. Transl. Ser. 2, Amer. Math. Soc., Providence, RI, 1995, 75-98.

[4] V.S. Buslaev, C. Sulem, On asymptotic stability of solitary waves for nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **20** (2003), 419-475.

[5] J. Fröhlich, Z. Gang, Emission of Cherenkov radiation as a mechanism for Hamiltonian friction, *Advances in Mathematics* **264** (2014), 183-235

[6] J. D. Jackson, Classical Electrodynamics, John Wiley & Sons, New York, NY, 1962.

[7] V. Imaikin, A.Komech, H. Spohn, Scattering theory for a particle coupled to a scalar field, *Journal of Discrete and Continuous Dynamical Systems* **10** (2004), no. 1-2, 387-396.

[8] V. Imaikin, A.Komech, P. Markowich, Scattering of solitons of the Klein-Gordon equation coupled to a classical particle, *Journal of Mathematical Physics* **44** (2003), no. 3, 1202-1217.

[9] V. Imaikin, A. Komech, H. Spohn, Soliton-like asymptotics and scattering for a particle coupled to Maxwell field, *Russian Journal of Mathematical Physics* **9** (2002), no. 4, 428-436.

[10] V. Imaikin, A.Komech, H. Spohn, Rotating charge coupled to the Maxwell field: scattering theory and adiabatic limit, *Monatshefte für Mathematik* **142** (2004), no. 1-2, 143-156.

[11] V. Imaikin, A. Komech, N. Mauser, Soliton-type asymptotics for the coupled Maxwell-Lorentz equations, *Ann. Inst. Poincaré, Phys. Theor.* **5** (2004), 1117-1135.

[12] A.I. Komech, On attractor of a singular nonlinear $U(1)$-invariant Klein-Gordon equation, p. 599-611 in: Proc. 3^{rd} ISAAC Congress, Berlin, 2003.

[13] A.I. Komech, A.A. Komech, Global attractor for a nonlinear oscillator coupled to the Klein-Gordon field, *Arch. Rat. Mech. Anal.* **185** (2007), 105-142.

[14] A.I. Komech, A.A. Komech, On global attraction to solitary waves for the Klein-Gordon field coupled to several nonlinear oscillators, *J. Math. Pures App.*, **93** (2010), 91-111.

[15] A.I. Komech, A.A. Komech, Global attraction to solitary waves for Klein- Gordon equation with mean field interaction, *Annales de l'IHP-ANL* **26** (2009), no. 3, 855-868.

[16] A.I. Komech, A.A. Komech, Global attraction to solitary waves for nonlinear Dirac equation with mean field interaction, *SIAM J. Math. Analysis*, **42** (2010), no. 6, 2944-2964.

[17] A.I. Komech, Quantum Mechanics: Genesis and Achievements, Springer, Dordrecht, 2013.

[18] A.I. Komech, Attractors of nonlinear Hamilton PDEs, *Discrete and Continuous Dynamical Systems A* **36** (2016), no. 11, 6201-6256.

[19] A. I. Komech, E. A. Kopylova, Dispersion Decay and Scattering Theory, Wiley & Sons, Hoboken, 2012.

[20] A. Komech, H. Spohn, Soliton-like asymptotics for a classical particle interacting with a scalar wave field, *Nonlinear Analysis* **33** (1998), no. 1, 13-24.

[21] E. A. Kopylova, A. I. Komech, Asymptotic atability of stationary states in the wave equation coupled to a nonrelativistic particle, *Russian J. Math. Phys.* **23** (2016), no. 1, 93-100.

[22] A. Komech, H. Spohn, M. Kunze, Long-time asymptotics for a classical particle interacting with a scalar wave field, *Comm. in PDEs* **22** (1997), no. 1-2, 307-335.

[23] A. Komech, H. Spohn, Long-time asymptotics for the coupled Maxwell-Lorentz equations, *Comm. PDE* **25** (2000), no. 3-4, 558-585.

[24] E. Kopylova, Weighted energy decay for 3D wave equation, *Asymptotic Anal.* **65** (2009), no. 1-2, 1-16.

[25] W. Rudin, “Functional Analysis”, McGraw Hill, New York, 1977.

[26] I. M. Sigal, Nonlinear wave and Schrödinger equations. I. Instability of periodic and quasiperiodic solutions, *Comm. Math. Phys.* **153** (1993), no. 2, 297-320.

[27] A. Soffer, M. I. Weinstein, Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations, *Invent. Math.* **136** (1999), no. 1, 9-74.

[28] H. Spohn, Dynamics of Charged Particles and Their Radiation Field, Cambridge University Press, 2004.

[29] A. Soffer, M.I. Weinstein, Multichannel nonlinear scattering for nonintegrable equations, *Comm. Math. Phys.*, **133** (1990), 119-146.

[30] A. Soffer, M.I. Weinstein, Multichannel nonlinear scattering for nonintegrable equations. II. The case of anisotropic potentials and data, *J. Differential Equations*, **98** (1992), 376-390.