On nonlinear wave equations with parabolic potentials

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Abstract

We introduce a new class of piece-wise quadratic potentials for nonlinear wave equations with a kink solutions. The potentials allow an exact description of the spectral properties for the linearized equation at the kink. This description is necessary for the study of the stability properties of the kinks.

In particular, we construct examples of the potentials of Ginzburg-Landau type providing the asymptotic stability of the kinks [4, 5].

Keywords: Relativistic invariant nonlinear wave equations, asymptotic stability, soliton, kink, Fermi Golden Rule.

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1 Introduction

Last two decades there was an outstanding activity in the field of asymptotic stability of solitary waves for nonlinear Schrödinger equations [1, 2, 9, 10, 11, 13, 14, 15, 16], nonlinear Klein-Gordon equations [3, 12], relativistic Ginzburg-Landau equations [4, 5], and other Hamiltonian PDEs [6, 8]. All these results rely on different assumptions on the spectral properties of the corresponding linearized equations. On the other hand, the examples were mostly unknown. Here we construct a model nonlinear wave equations, providing different spectral properties: given number of the eigenvalues, absence of the resonances, and Fermi Golden Rule.

In particular, we construct the examples of relativistic Ginzburg-Landau equations providing all properties assumed in [4, 5].

We consider real solutions to 1D nonlinear Ginzburg-Landau equations

$$\ddot{\psi}(x,t)=\psi''(x,t)+F(\psi(x,t)),\qquad x\in\mathbb{R} \tag{1.1}$$

where $F(\psi)=-U'(\psi)$. We assume the following condition.

Condition U1 *For some $K>3$ and $m>0$ the potential $U(\psi)$ is smooth even function satisfying*

$$\begin{aligned} &U(\psi)>0 \quad \text{for} \quad \psi\neq a,\\ &U(\psi)=\frac{m^2}{2}(\psi\mp a)^2+\mathcal{O}(|\psi\mp a|^K),\quad \psi\to\pm a. \end{aligned} \tag{1.2}$$

The corresponding stationary equation reads

$$s''(x)-U'(s(x))=0,\qquad x\in\mathbb{R}. \tag{1.3}$$

Constant stationary solutions are: $\psi(x)\equiv 0$ and $\psi(x)\equiv\pm a$. There are also the "kinks", i.e. nonconstant finite energy solutions $s(x)$ to (1.3) such that

$$s(x)\to\pm a,\qquad x\to\pm\infty$$

Condition **U1** implies that $(s(x)\mp a)''\sim m^2(s(x)\mp a)$ for $x\to\pm\infty$, hence

$$|s(x)\mp a|\sim Ce^{-m|x|},\qquad x\to\pm\infty. \tag{1.4}$$

Due to relativistic invariance of equation (1.1) the moving kinks

$$s_{q,v}(x,t)=s(\kappa(x-vt-q)),\quad q,v\in\mathbb{R},\quad |v|<1,\quad \kappa=1/\sqrt{1-v^2}$$

also are the solutions to (1.1). Let us linearize equation (1.1) at the kink $s(x)$. Substituting $\psi(x,t)=s(x)+\phi(x,t)$, we obtain formally

$$\ddot{\phi}(x,t)=-H\phi(x,t)+\mathcal{O}(|\phi(x,t)|^2),$$

where H is the Schrödinger operator

$$H:=-\frac{d^2}{dx^2}+m^2+V(x)$$

with the potential

$$V(x) = -F'(s(x)) - m^2 = U''(s(x)) - m^2.$$

The condition **U1** and the asymptotics (1.4) imply that

$$|V(x)| = \mathcal{O}(|s(x) \mp a|^{K-1}) \sim Ce^{-(K-1)m|x|}, \qquad x \to \pm\infty.$$

The next properties of H are valid:

H1. The continuous spectrum of H is $\sigma_c H = [m^2, \infty)$.

H2. The point $\lambda_0 = 0$ belongs to the discrete spectrum, and corresponding eigenfunction is $s'(x)$.

H3. Since $s'(x) > 0$, the point $\lambda_0 = 0$ is the groundstate, and all remaining discrete spectrum is contained in $(0, m^2]$.

To establish an asymptotic stability of the kinks $s_{q,v}(x,t)$ one need certain spectral properties of H (cf. [4], [5]):

Condition U2 *The edge point $\lambda = m^2$ of the continuous spectrum is neither eigenvalue nor resonance.*

Condition U3 *The discrete spectrum of H consists of two points: $\lambda_0 = 0$ and $\lambda_1 \in (0, m^2)$ satisfying*

$$4\lambda_1 > m^2. \tag{1.5}$$

We assume also a non-degeneracy condition known as "Fermi Golden Rule" meaning the strong coupling of the nonlinear term to the continuous spectrum. This coupling provides the energy radiation to infinity (cf. condition (10.0.11) in [2] and condition (1.11) in [5]).

Condition U4 *The inequality holds*

$$\int \varphi_{4\lambda_1}(x) F''(s(x)) \varphi^2_{\lambda_1}(x) dx \neq 0. \tag{1.6}$$

where $\varphi_{4\lambda_1}$ is the nonzero odd solution to $H\varphi_{4\lambda_1} = 4\lambda_1 \varphi_{4\lambda_1}$.

Note that the known quartic double well Ginzburg-Landau potential $U_{GL}(\psi) = (\psi^2 - a^2)^2/(4a^2)$ satisfies condition **U1** with $m^2 = 2$ and $K = 3$ as well as conditions **U3-U4**. However, there exist the resonance for the corresponding operator H at the edge point $\lambda = m^2$. Hence, the asymptotic stability of the kinks for U_{GL} is the open problem.

Our main result is the following theorem.

Theorem 1.1. *There exist potentials $U(\psi)$ satisfying conditions* **U1-U4**.

2 Piece wise parabolic potentials

As a first step, we will consider the class of the potentials which are piece-wise second order polynomials.

$$U_0(\psi) = \begin{cases} \dfrac{1}{2} - \dfrac{b}{2}\psi^2, & |\psi| \le \gamma \\ \\ \dfrac{d}{2}(\psi \mp 1)^2, & \pm\psi \ge \gamma \end{cases} \tag{2.1}$$

with some constants $b, d, \gamma > 0$. Let us find the parameters γ, b, d providing $U_0(\psi) \in C^1(\mathbb{R})$. We have

$$U_0(\gamma) = \frac{1}{2} - \frac{b}{2}\gamma^2 = \frac{d}{2}(\gamma-1)^2, \qquad U_0'(\gamma) = -b\gamma = d(\gamma-1).$$

Solving the equations, we obtain

$$b = \frac{1}{\gamma}, \qquad d = \frac{1}{1-\gamma}, \qquad\qquad 0 < \gamma < 1. \tag{2.2}$$

Then the functions $U_0''(\psi)$ are piece-wise constant with the jumps at the points $\psi = \pm\gamma$. Thus, the potentials $U_0 \in C^1(\mathbb{R})$ form one-dimensional manifold parametrized by $\gamma \in (0,1)$.

2.1 Kinks

Let us solve the equation of type (1.3) for the kink in the case of potential (2.1):

$$s_0''(x) - U_0'(s_0(x)) = 0, \qquad x \in \mathbb{R}. \tag{2.3}$$

We search an odd solution to

$$s_0''(x) = \begin{cases} -bs_0(x), & 0 < s_0(x) \le \gamma, \\ d(s_0(x)-1), & s_0(x) > \gamma. \end{cases}$$

We have

$$s_0(x) = \begin{cases} C\sin\sqrt{b}x, & 0 < x \le q, \\ Ae^{-\sqrt{d}x} + 1, & x > q, \end{cases} \tag{2.4}$$

where $C > \gamma$, $A < 0$, $q = \dfrac{1}{\sqrt{b}}\arcsin\dfrac{\gamma}{C}$. Equating the values of $s(x)$ and its derivative at $x = q$ we obtain

$$\begin{cases} Ae^{-\sqrt{d}q} + 1 = C\sin\sqrt{b}q = \gamma, \\ -\sqrt{d}Ae^{-\sqrt{d}q} = \sqrt{b}C\cos\sqrt{b}q. \end{cases} \tag{2.5}$$

The first line of (2.5) implies $Ae^{-\sqrt{d}q} = \gamma - 1$. Hence the second line of (2.5) becomes

$$\sqrt{d}(1-\gamma) = \sqrt{b}C\cos\sqrt{b}q.$$

The both side of the last equality is positive. Hence it is equivalent to

$$d(1-\gamma)^2 = b(C^2 - \gamma^2).$$

Substituting (2.2) we obtain $1 - \gamma = C^2/\gamma - \gamma$. Then

$$C = \sqrt{\gamma}, \quad A = (\gamma-1)e^{\sqrt{\gamma/(1-\gamma)}\arcsin\sqrt{\gamma}}$$

and

$$q = \sqrt{\gamma}\arcsin\sqrt{\gamma}. \tag{2.6}$$

2.2 Linearized equation

Let us linearize equation (1.1) with $F(\psi) = F_0(\psi) = -U_0(\psi)$ at the kink $s_0(x)$ splitting the solution as the sum

$$\psi(t) = s_0 + \phi(t), \tag{2.7}$$

Substituting (2.7) to (1.1), we obtain

$$\ddot{\phi}(x,t) = \phi''(x,t) - U_0'(s_0(x) + \phi(x,t)) + U_0'(s_0(x)). \tag{2.8}$$

By (2.1) we can write equations (2.8) as

$$\ddot{\phi}(t) = -H_0\phi(t) + \mathcal{N}(\phi(t)), \quad t \in \mathbb{R}$$

where $\mathcal{N}(\phi)$ is at least quadratic in ϕ. and

$$H_0 = -\frac{d^2}{dx^2} + W_0(x), \qquad W_0(x) = U_0''(s_0(x)) = \begin{cases} -b, & |x| \le q \\ \\ d, & |x| > q \end{cases} \tag{2.9}$$

(see Fig. 1).

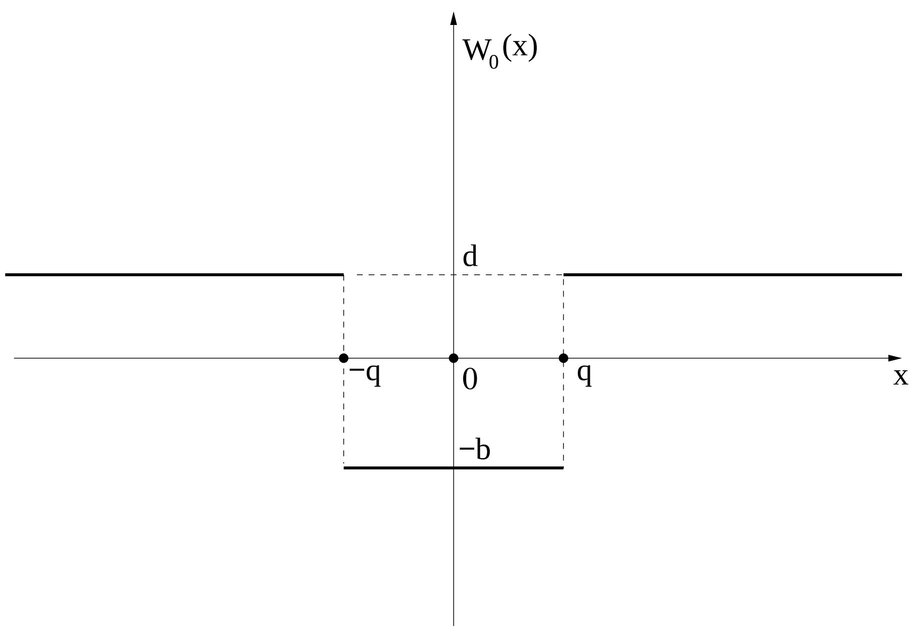

Figure 1: Potential W_0

3 Spectrum of linearized equation

The continuous spectrum $\sigma_c H_0 = [d, \infty)$. The point $\lambda_0 = 0$ is the groundstate since it corresponds to the symmetric positive eigenfunction $\varphi_0(x) = s_0'(x)$:

$$H_0\varphi_0 = -s_0'''(x) + U_0''(s_0(x))s_0'(x) = 0,$$

which follows by differentiation of (2.3). Therefore, the discrete spectrum $\sigma_d H_0 \subset [0, d]$, and the next eigenfunction $\varphi_1(x)$ should be antisymmetric.

3.1 Antisymmetric eigenfunctions

The eigenfunction $\varphi(x)$ corresponding to eigenvalue λ should satisfy the equation

$$\begin{cases} -\varphi''(x) - b\varphi(x) = \lambda\varphi(x), & |x| \le q, \\ \\ -\varphi''(x) + d\varphi(x) = \lambda\varphi(x), & |x| > q. \end{cases} \tag{3.1}$$

Equations (3.1) imply that the antisymmetric eigenfunctions have the form

$$\varphi(x) = \begin{cases} B \sin \beta x \qquad , & |x| \le q, \\ \\ A \operatorname{sgn} x \; e^{-\alpha |x|}, & |x| > q. \end{cases} \tag{3.2}$$

where $\alpha > 0$, $\beta \ge 0$, and $\alpha^2 = d - \lambda$, $\beta^2 = b + \lambda$. Let us calculate the corresponding eigenvalues λ. First, equating the values of the eigenfunction and its first derivatives at $x = q$, we obtain

$$Ae^{-\alpha q} = B \sin \beta q, \qquad - A\alpha e^{-\alpha q} = B\beta \cos \beta q. \tag{3.3}$$

The system admits nonzero solutions if and only if its determinant vanishes:

$$- \alpha = \beta \cot \beta q. \tag{3.4}$$

At last, multiplying by q, and denoting $\xi = \beta q$ and $\eta = \alpha q$, we obtain the system

$$- \eta = \xi \cot \xi, \qquad \xi^2 + \eta^2 = R^2, \tag{3.5}$$

where $R = q\sqrt{b+d}$ is the radius of the circle. Substituting b, d and q from (2.2) and (2.6) respectively, we obtain

$$R = q\sqrt{\frac{1}{\gamma} + \frac{1}{1-\gamma}} = \frac{q}{\sqrt{\gamma(1-\gamma)}} = \frac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}}. \tag{3.6}$$

Finally, the solutions to (3.5) can be found grafically (see Fig. 1). Taking into account that $\eta > 0$, we obtain that

$$\left.\begin{array}{ll} R \in (0,\ \frac{\pi}{2}] & : \text{ no solution to (3.5)} \\ \\ R \in (\frac{\pi}{2},\ \frac{3\pi}{2}] & : \text{ exactly one solution to (3.5)} \\ \\ R \in (\frac{3\pi}{2}, \frac{5\pi}{2}] & : \text{ exactly two solution to (3.5)} \\ \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right| \tag{3.7}$$

Let us note that $R(0) = 0$ and $R(1) = \infty$, and the radius $R(\gamma)$ is monotone increasing on $[0, 1]$. Denote by γ_k, $k \in \mathbb{N}$ the solution to the equation

$$\frac{\arcsin\sqrt{\gamma_k}}{\sqrt{1-\gamma_k}} = \frac{k\pi}{2}, \quad k \in \mathbb{N}. \tag{3.8}$$

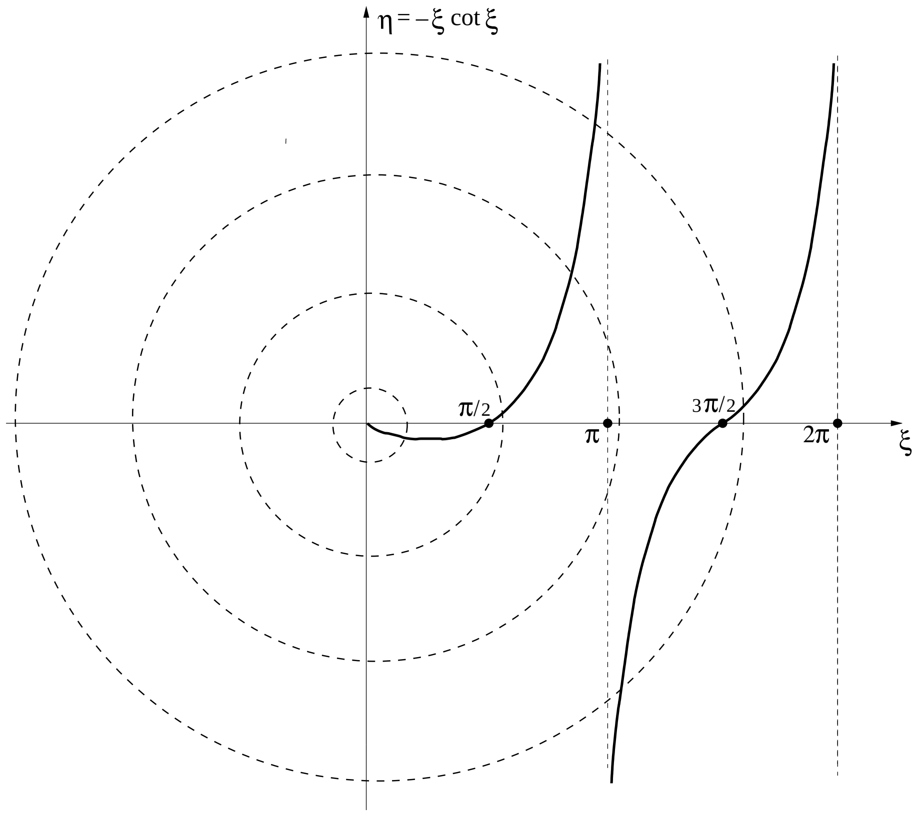

Figure 2:

Numerical calculations gives

$$\gamma_1 \sim 0.64643, \quad \gamma_2 \sim 0.8579, \quad \gamma_3 \sim 0.92472, \quad \gamma_4 \sim 0.95359, \quad \gamma_5 \sim 0.96856\ldots . \tag{3.9}$$

We have $\gamma_k \sim 1 - \frac{4}{(k\pi)^2}$, for large k. Further, (3.7) implies that

$$\left.\begin{array}{rl} \gamma \in (0,\ \gamma_1] & : \text{ no nonzero antisymmetric eigenfunctions} \\ \gamma \in (\gamma_1,\ \gamma_3] & : \text{ exactly one linearly independent antisymmetric eigenfunctions} \\ \gamma \in (\gamma_3,\ \gamma_5] & : \text{ exactly two linearly independent antisymmetric eigenfunctions} \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right| \tag{3.10}$$

In particular, for $\gamma \in (\gamma_1, \gamma_3]$ we obtain the eigenvalue $\lambda_1 \in (0, d)$ corresponding to the antisymmetric eigenfunction:

$$\lambda_1 = \lambda_1(\gamma) = \beta^2 - b = \frac{\xi^2}{q^2} - b = \frac{1}{\gamma}\Big(\frac{\xi^2}{\arcsin^2\sqrt{\gamma}} - 1\Big) = \frac{1}{\gamma}\Big(\frac{\sin^2\xi}{1-\gamma} - 1\Big), \tag{3.11}$$

where ξ is the solution to

$$\frac{\xi^2}{\sin^2\xi} = \frac{\arcsin^2\sqrt{\gamma}}{1-\gamma}. \tag{3.12}$$

3.2 Symmetric eigenfunctions

Now we consider symmetric eigenfunctions. Equations (3.1) imply that the symmetric eigenfunctions have the form

$$\varphi(x) = \begin{cases} B\cos\beta x, & |x| \le q, \\ \\ A\ e^{-\alpha|x|}\ , & |x| > q, \end{cases} \tag{3.13}$$

where $\alpha > 0$, $\beta \ge 0$, and $\alpha^2 = d - \lambda$, $\beta^2 = b + \lambda$. Let us calculate the corresponding eigenvalues λ. Similarly (3.3)-(3.5), denoting $\xi = \beta q$ and $\eta = \alpha q$, we obtain the system

$$\eta = \xi \tan\xi, \qquad \xi^2 + \eta^2 = R^2, \tag{3.14}$$

where $R = \dfrac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}}$. The solutions to system (3.5) can be found grafically (see Fig. 2). We

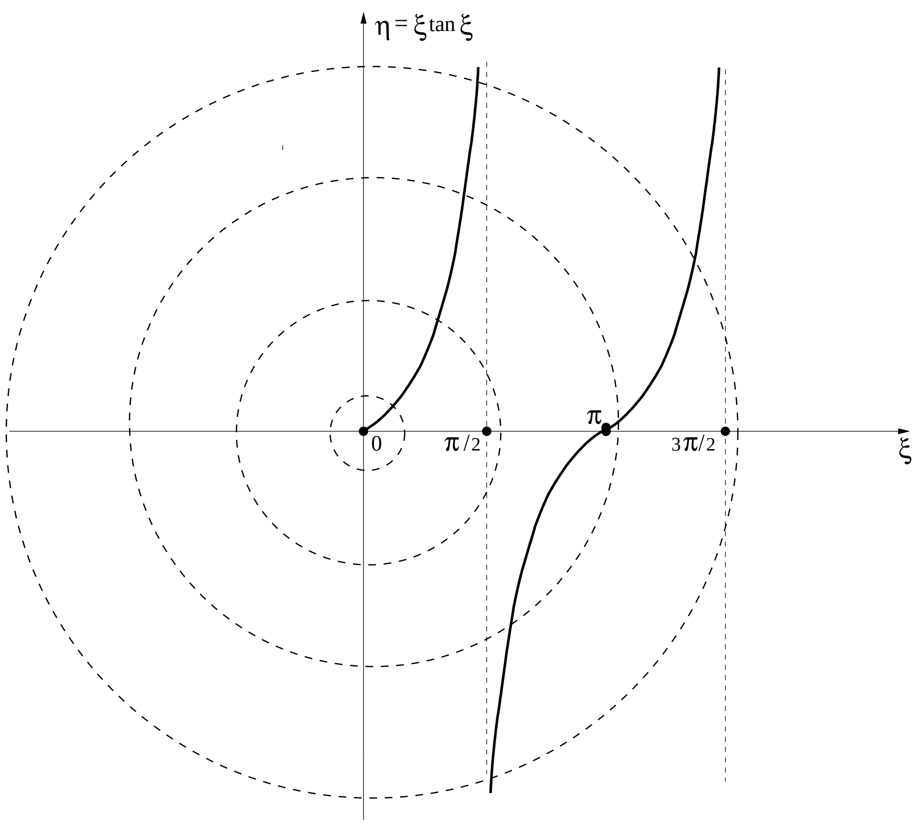

Figure 3:

have

$$\left.\begin{array}{ll} R \in (0,\ \pi] : & \text{exactly one solution to (3.14)} \\ \\ R \in (\pi, 2\pi] : & \text{exactly two linearly independent symmetric eigenfunctions} \\ \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right| \tag{3.15}$$

Note that for any $\gamma \in (0, 1)$ equation (3.14) has the solution $\xi = \arcsin \sqrt{\gamma} \in (0, \pi/2)$. The solution corresponds to eigenvalue $\lambda = 0$ and the first symmetric eigenfunction. Moreover, (3.15) implies that

$$\left.\begin{array}{ll} \gamma \in (0,\ \gamma_2] & : \text{ exactly one linearly independent symmetric eigenfunctions} \\ \\ \gamma \in (\gamma_2,\ \gamma_4] & : \text{ exactly two linearly independent symmetric eigenfunctions} \\ \\ & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right| \tag{3.16}$$

where γ_i are defined in (3.8).
Conclusion:
1) There is exactly one eigenvalue $\lambda_0 = 0$ for $\gamma \in (0, \gamma_1]$.
2) There are exactly two eigenvalues $\lambda_0 = 0$ and $0 < \lambda_1 < d$ for $\gamma \in (\gamma_1, \gamma_2]$.
etc.

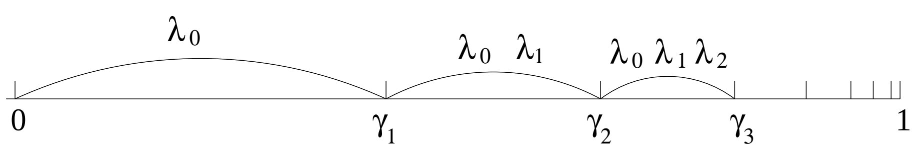

Figure 4: Spectrum

4 Spectral conditions

We deduce Theorems 1.1 in Section 5 below from the following proposition.

Proposition 4.1. *For any $\gamma \in (\gamma_1, \gamma_2)$ the piece wise parabolic potentials U_0, defined in (2.1), satisfy conditions* **U1**- **U3** *except for the smoothness condition at the points $\psi = \pm\gamma$. Condition* **U4** *holds for any $\gamma \in (\gamma_1, \gamma_2)$ except for one point γ_*.*

Proof. Step i) Obviously, for $U_0(\psi)$ condition **U1** with $a = 1$, $m^2 = d$ and any integer $K \geq 3$ holds except the smoothness at the points $\psi = \pm\gamma$.

Consider condition **U2**. Note that the solutions to (3.5) or (3.14) with $\eta = 0$ and $R = k\pi/2$, $k \in \mathbb{N}$ correspond to $\alpha = 0$ i.e. $\lambda = d$. Then the functions (3.2) and (3.13) with $A \neq 0$ is a nonzero constant for $|x| \geq \gamma$. Hence, the function is the *resonance* corresponding to the edge point $\lambda = d$ of the continuous spectrum. Thus, the resonances exist only for the discrete set of parameters $\gamma_k \in (0, 1)$ defined in (3.8). Evidently, the set has just one limit point 1. Hence, conditions **U2** holds if $\gamma \in (\gamma_1, \gamma_2)$.

Step ii) For any $\gamma \in (\gamma_1, \gamma_2)$ the operator H_0 defined in (2.9) has exactly two eigenvalues $\lambda_0 = 0$ and $\lambda_1 \in (0, d)$. For condition **U3** it remains to verify (1.5) with $m^2 = d$. Namely, due to (3.11)-(3.12) we must prove that for any $\gamma \in (\gamma_1, \gamma_2)$ the inequality holds

$$\frac{4}{\gamma}\Big(\frac{\sin^2 \xi(\gamma)}{1-\gamma} - 1\Big) > \frac{1}{1-\gamma},$$

where $\xi(\gamma) \in (\pi/2, \pi)$ is the solution to (3.12), After the simple transformations we obtain

$$4\cos^2\xi(\gamma) < 3\gamma, \tag{4.1}$$

and then

$$\frac{\pi}{2} < \xi(\gamma) < \pi - \arccos\frac{\sqrt{3\gamma}}{2}.$$

Since $\dfrac{\xi}{\sin\xi}$ is monotonically increasing function for $\xi \in (\pi/2, \pi)$, then

$$\frac{\pi}{2} < \frac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}} < \frac{2(\pi - \arccos\frac{\sqrt{3\gamma}}{2})}{\sqrt{4-3\gamma}}.$$

Finally, we obtain

$$\gamma_1 < \gamma < \alpha,$$

where α is the solution to

$$\frac{\arcsin\sqrt{\alpha}}{\sqrt{1-\gamma_0}} = \frac{2(\pi - \arccos\frac{\sqrt{3\alpha}}{2})}{\sqrt{4-3\alpha}}.$$

Numerical calculation gives

$$\alpha = 0.921485 > \gamma_2.$$

Therefore, condition **U3** holds for any $\gamma \in (\gamma_1, \gamma_2)$.

Step iii) Finally, consider condition **U4** (Fermi Golden Rule). The condition can be rewritten as

$$\int U_0'''(s_0(x))\varphi_{4\lambda_1}(x)\varphi_{\lambda_1}^2(x)dx = \int \frac{d}{dx}U_0''(s_0(x))\frac{\varphi_{4\lambda_1}(x)\varphi_{\lambda_1}^2(x)}{s_0'(x)}dx \neq 0. \tag{4.2}$$

By (2.9) we have that $U_0''(s_0(x)) = W_0(x)$ is the piece wise constant function. Hence,

$$\frac{d}{dx}U_0''(s_0(x)) = (b+d)\delta(x-q) - (b+d)\delta(x+q),$$

and (4.2) becomes

$$\varphi_{4\lambda_1}(q)\varphi_{\lambda_1}^2(q) \neq 0.$$

Formula (3.2) yields that $\varphi_{\lambda_1}(q) = Ae^{-\alpha q} \neq 0$. Hence it is sufficient to verify that

$$\varphi_{4\lambda_1}(q) \neq 0.$$

The eigenfunction $\varphi_{4\lambda_1}$ satisfies the equations

$$\begin{cases} -\varphi_{4\lambda_1}''(x) - b\varphi_{4\lambda_1}(x) = 4\lambda_1\varphi_{4\lambda_1}(x), & |x| \le q, \\ -\varphi_{4\lambda_1}''(x) + d\varphi_{4\lambda_1}(x) = 4\lambda_1\varphi_{4\lambda_1}(x), & |x| > q. \end{cases} \tag{4.3}$$

For the odd solution to (4.3) we have $\varphi_{4\lambda_1}(x) = \sin\beta x$, $|x| \le q$, where $\beta^2 = b + 4\lambda_1 > 0$. Therefore,

$$\varphi_{4\lambda_1}(q) = \sin\beta q = 0$$

if either $\beta q = k\pi$, $k = 0, 1, 2, ..,$ or

$$\sqrt{1 + 4\gamma\lambda_1(\gamma)}\arcsin\sqrt{\gamma} = k\pi, \quad k = 0, 1, 2, ... \tag{4.4}$$

where $\lambda_1(\gamma)$ is defined in (3.11)-(3.12). Substituting $\lambda_1(\gamma)$ into (4.4) we obtain from (3.11)-(3.12)

$$\begin{cases} \dfrac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}}\sqrt{4\sin^2\xi - 3(1-\gamma)} = k\pi \\[2ex] \dfrac{\xi^2}{\sin^2\xi} = \dfrac{\arcsin^2\sqrt{\gamma}}{1-\gamma}. \end{cases} \tag{4.5}$$

For $\gamma \in (\gamma_1, \gamma_2)$ this system has a solution only for $k = 1$ since

$$0 < \frac{\arcsin\sqrt{\gamma}}{\sqrt{1-\gamma}}\sqrt{4\sin^2\xi - 3(1-\gamma)} < 2\pi, \quad \gamma_1 < \gamma < \gamma_2$$

Let us prove that (4.5) with $k = 1$ has a unique solution. Denote

$$\theta = \arcsin\sqrt{\gamma} \in (\pi\sqrt{1-\gamma_1}/2, \pi\sqrt{1-\gamma_2}). \tag{4.6}$$

Then (4.5) with $k = 1$ is equivalent to

$$\begin{cases} 4\xi^2 - 3\theta^2 = \pi^2 \\[2ex] \dfrac{\sin\xi}{\xi} = \dfrac{\cos\theta}{\theta} \end{cases} \tag{4.7}$$

The function $\theta_1(\xi) := \dfrac{1}{\sqrt{3}}\sqrt{4\xi^2 - \pi^2}$ increases for $\xi(\gamma_1) < \xi < \xi(\gamma_2)$, and

$$\theta_1'(\xi) = \frac{1}{\sqrt{3}}\frac{4\xi}{\sqrt{4\xi^2 - \pi^2}} > \frac{1}{\sqrt{3}}\frac{4(\pi/2)}{\sqrt{4(3\pi/4)^2 - \pi^2}} = \frac{4}{\sqrt{15}} > 1, \quad \xi(\gamma_2) < \xi < \xi(\gamma_2) \tag{4.8}$$

since $\xi(\gamma_1) = \pi/2$ and $\xi(\gamma_2) \sim 2.3137 < 3\pi/4$.
On the other hand, denote $\theta_2 := \theta_2(\xi)$ the solution of $\dfrac{\sin\xi}{\xi} = \dfrac{\cos\theta}{\theta}$. We have

$$\theta_2'(\xi) = \frac{\sin\xi - \xi\cos\xi}{\xi^2}\frac{\theta^2}{\cos\theta + \theta\sin\theta} > 0, \quad \pi/2 < \xi < \xi(\gamma_2). \tag{4.9}$$

Moreover, by (4.6) and (4.7) we obtain

$$\theta_2'(\xi) = \frac{\theta}{\xi}\,\frac{\frac{\sin\xi}{\xi} - \cos\xi}{\frac{\cos\theta}{\theta} + \sin\theta} < \frac{\frac{\sin\xi}{\xi} - \cos\xi}{\frac{\sin\xi}{\xi} + \sin\theta} < 1, \quad \pi/2 < \xi < \xi(\gamma_2) \tag{4.10}$$

since $|\cos\xi| < |\cos\xi(\gamma_2)| < \sqrt{2}/2$, and $\sin\theta = \sqrt{\gamma} > \sqrt{\gamma_1} > \sqrt{2}/2$ by (3.9). Finally,

$$\theta_2(\pi/2) > \theta_1(\pi/2) = 0, \quad \theta_2(\xi(\gamma_2)) \sim 1.1843 < \theta_1(\xi(\gamma_2)) \sim 1.9616. \tag{4.11}$$

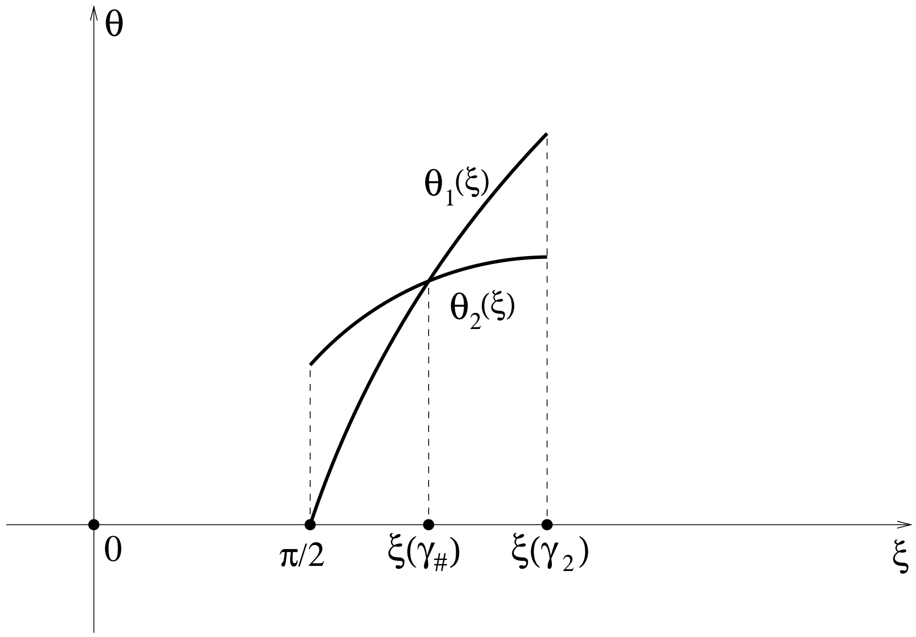

Figure 5: Functions θ_1 and θ_2

Therefore, (4.8)-(4.11) imply that $\theta_1(\theta) = \theta_2(\theta)$ for a single value $\xi(\gamma_*) \in (\pi/2, \xi(\gamma_2))$ (see. Figure 5). Numerical calculation gives $\gamma_* \sim 0.7925$. Hence, system (4.5) on the interval (γ_1, γ_2) has the solution only for $\gamma = \gamma_*$. Thus, the Fermi Golden Rule holds for any $\gamma \in (\gamma_1, \gamma_2)$ except the only point γ_*.

Conclusion:
The potential $U_0(\psi)$ satisfies conditions **U1-U4** except the smoothness condition at the points $\psi = \pm\gamma$ for any $\gamma \in (\gamma_1, \gamma_*) \cup (\gamma_*, \gamma_2)$. □

5 Smooth potentials

We deduce Theorem 1.1 from Proposition 4.1 by an approximation of the potential (2.9) with a smooth functions satisfying conditions **U1-U4**. Namely, let $h(\psi) \in C_0^\infty(\mathbb{R})$ be an even mollifying function with the following properties:

$$h(\psi) \geq 0, \quad \text{supp } h \subset [-1, 1], \qquad \int h(\psi) d\psi = 1. \tag{5.1}$$

For $\varepsilon \in (0, 1]$ we define the approximations

$$\tilde{U}_\varepsilon(\psi) := \frac{1}{\varepsilon} \int h(\frac{\psi - \psi'}{\varepsilon}) U_0(\psi') d\psi'. \tag{5.2}$$

This is a smooth even function, and it is positive and symmetric w.r.t. points $\psi = \pm 1$ in a small neighborhood of these points for $\varepsilon < \gamma$. More precisely, the difference

$$\tilde{U}_\varepsilon(\psi) - U_0(\psi) = \begin{cases} \mu_\varepsilon > 0, & |\psi| \geq \gamma + \varepsilon, \\ -\nu_\varepsilon < 0, & |\psi| \leq \gamma - \varepsilon, \end{cases}$$

where $\mu_\varepsilon, \nu_\varepsilon = \mathcal{O}(\varepsilon^2)$. Let us set

$$U_\varepsilon(\psi) = \tilde{U}_\varepsilon(\psi) - \mu_\varepsilon. \tag{5.3}$$

Then

$$U_\varepsilon(\psi) = \begin{cases} U_0(\psi), & |\psi| \geq \gamma + \varepsilon, \\ \\ U_0(\psi) - \mu_\varepsilon - \nu_\varepsilon, & |\psi| \leq \gamma - \varepsilon. \end{cases} \tag{5.4}$$

Obviously,

$$\sup_{\psi\in\mathbb{R}} |U_\varepsilon(\psi) - U_0(\psi)| \leq C\varepsilon \tag{5.5}$$

with some constant C. Moreover,

$$U_\varepsilon'''(\psi) \leq 0 \quad \text{for} \ \ \psi \leq 0, \qquad U_\varepsilon'''(\psi) \geq 0 \quad \text{for} \ \ \psi \geq 0. \tag{5.6}$$

The corresponding kinks are the odd solutions to the equation

$$s_\varepsilon''(x) - U_\varepsilon'(s_\varepsilon(x)) = 0, \qquad x \in \mathbb{R}.$$

The equation can be integrated using the "energy conservation"

$$\frac{|s_\varepsilon'(x)|^2}{2} - U_\varepsilon(s_\varepsilon(x)) = \text{const}, \qquad x \in \mathbb{R}$$

with const $= 0$:

$$\int_0^{s_\varepsilon(x)} \frac{ds}{\sqrt{2U_\varepsilon(s)}} = x, \qquad x \in \mathbb{R}. \tag{5.7}$$

Hence, $s_\varepsilon(x)$ is a monotone increasing function, and

$$s_\varepsilon(x) \to \pm 1, \qquad x \to \pm\infty.$$

Moreover, (5.4), (5.5) and (5.7) imply that

$$\sup_{x\in\mathbb{R}} |s_\varepsilon(x) - s_0(x)| \leq C_1\varepsilon.$$

Therefore,

$$||s_\varepsilon(x)| - \gamma| \geq \varepsilon \qquad \text{for} \qquad ||x| - q| \geq \delta$$

where

$$\delta \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{5.8}$$

Hence,

$$W_\varepsilon(x) := U_\varepsilon''(s_\epsilon(x)) = W_0(x) \qquad \text{for} \qquad ||x| - q| \geq \delta$$

and

$$W_\varepsilon'(x) \leq 0 \quad \text{for} \ \ x \leq 0, \qquad W_\varepsilon'(x) \geq 0 \quad \text{for} \ \ x \geq 0$$

by (5.6) (see Fig. 6).

As a result,

$$W_\varepsilon(x) - W_0(x) = 0 \quad \text{for} \ \ ||x| - q| \geq \delta, \qquad |W_\varepsilon(x) - W_0(x)| \leq b + d \quad \text{for } x \in \mathbb{R}. \tag{5.9}$$

Hence, denoting $w_\varepsilon(x) = W_\varepsilon(x) - W_0(x)$, we obtain

$$\|w_\varepsilon\|_{L^2(\mathbb{R})} \to 0, \qquad \varepsilon \to 0 \tag{5.10}$$

by (5.9) and (5.8).

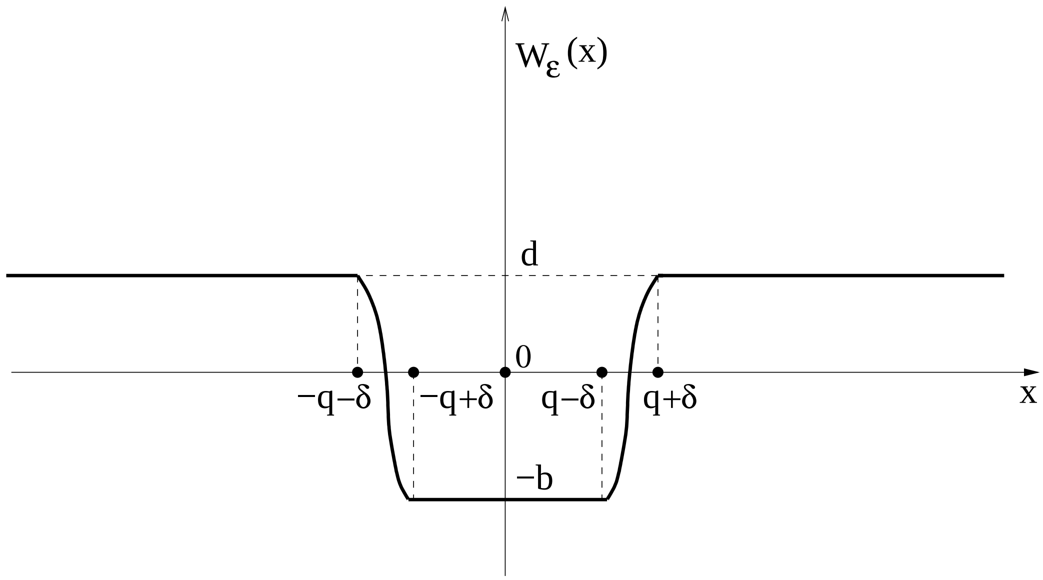

Figure 6: Potential W_ε

Lemma 5.1. *The eigenvalues of the Schrödinger operator*

$$H_\varepsilon = -\frac{d^2}{dx^2} + W_\varepsilon(x) \tag{5.11}$$

converge to the ones of H_0 *as* $\varepsilon \to 0$.

Proof. The eigenvalues of H_0 and H_ε are the poles of the resolvents $R_0(\omega) = (H_0 - \omega)^{-1}$ and $R_\varepsilon(\omega) = (H_\varepsilon - \omega)^{-1}$ respectively. Hence, the lemma follows from (5.10) due to the relation

$$R_\varepsilon(\omega) = (H_0 - \omega + w_\varepsilon)^{-1} = R_0(\omega)(1 + w_\varepsilon R_0(\omega))^{-1}. \tag{5.12}$$

□

Proof of Theorem 1.1 Consider the potential $U(\psi) = U_\varepsilon(\psi)$ defined in (5.2)-(5.3). Let us prove that there exist $\varepsilon_0 > 0$ such that for any $\gamma \in (\gamma_1, \gamma_2)$, and $0 < \varepsilon < \varepsilon_0$ the potential U_ε satisfies conditions **U1- U4**.

Step i) Condition **U1** with $a = 1$, $m^2 = d$ and any integer $K \geq 3$ obviously holds.

Step ii) For $\sigma \in \mathbb{R}$, and $s = 0, 1, 2, ...$ denote by $\mathcal{H}^s_\sigma = \mathcal{H}^s_\sigma(\mathbb{R})$ the weighted Sobolev spaces with the finite norms

$$\|\psi\|_{\mathcal{H}^s_\sigma} = \sum_{k=0}^{s} \|(1 + |x|)^\sigma \psi^{(k)}\|_{L^2(\mathbb{R})} < \infty,$$

By [7, Theorem 7.2], the absence of the resonance at the point $\omega = d$ for the Schrödinger operator H_ε is equivalent to the boundedness of the corresponding resolvent $R_\varepsilon(\omega) : \mathcal{H}^0_\sigma \to \mathcal{H}^2_{-\sigma}$ at $\omega = d$ for any $\sigma > 1/2$. Hence, the resolvent $R_0(d) : \mathcal{H}^0_\sigma \to \mathcal{H}^2_{-\sigma}$ is bounded by Proposition 4.1. Further, (5.9) imply

$$\|w_\varepsilon\|_{\mathcal{H}^0_{-\sigma} \to \mathcal{H}^0_\sigma} \to 0, \quad \varepsilon \to 0$$

Hence, for sufficiently small ε the operator $R_\varepsilon(d) : \mathcal{H}^0_\sigma \to \mathcal{H}^2_{-\sigma}$ is bounded by (5.12). Then condition **U2** holds for U_ε.

Step iii) Lemma 5.1 implies that for $\gamma \in (\gamma_1, \gamma_2)$ and sufficiently small ε the operator H_ε has exactly two eigenvalues $\lambda_0 = 0$ and $0 < \lambda_1(\varepsilon) < d$. Moreover, $\lambda_1(\varepsilon) \to \lambda_1(0) = \lambda_1$ as $\varepsilon \to 0$ and then $4\lambda_1(\varepsilon) > d$ for sufficiently small ε. Hence, condition **U3** holds.

Step iv) It remains to check condition **U4**. Denote $\varphi^\varepsilon_{\lambda_1(\varepsilon)}$ and $\varphi^\varepsilon_{4\lambda_1(\varepsilon)}$ the corresponding odd eigenfunctions of H_ε. Then we have

$$\int U'''_\varepsilon(s_\varepsilon(x))\varphi^\varepsilon_{4\lambda_1(\varepsilon)}(x)(\varphi^\varepsilon_{\lambda_1(\varepsilon)}(x))^2dx = \int\limits_{|x-q|\le\delta} \frac{d}{dx}W_\varepsilon(x)\frac{\varphi^\varepsilon_{4\lambda_1(\varepsilon)}(x)(\varphi^\varepsilon_{\lambda_1(\varepsilon)}(x))^2}{s'_\varepsilon(x)}dx$$

$$= \sum_\pm \Big(d\frac{\varphi^\varepsilon_{4\lambda_1(\varepsilon)}(\pm q+\delta)(\varphi^\varepsilon_{\lambda_1(\varepsilon)}(\pm q+\delta))^2}{s'_\varepsilon(\pm q+\delta)} + b\frac{\varphi^\varepsilon_{4\lambda_1(\varepsilon)}(\pm q-\delta)(\varphi^\varepsilon_{\lambda_1(\varepsilon)}(\pm q-\delta))^2}{s'_\varepsilon(\pm q-\delta)}\Big)$$

$$- \int\limits_{|x-q|\le\delta} W_\varepsilon(x)\frac{d}{dx}\frac{\varphi_{4\lambda^\varepsilon_1(\varepsilon)}(x)(\varphi^\varepsilon_{\lambda_1(\varepsilon)}(x))^2}{s'_\varepsilon(x)}dx$$

$$\underset{\varepsilon\to 0}{\longrightarrow} \; 2(d+b)\frac{\varphi_{4\lambda_1}(q)\varphi^2_{\lambda_1}(q)}{s'_0(q)} = \int U'''_0(s_0(x))\varphi_{4\lambda_1}(x)\varphi^2_{\lambda_1}(x)dx \ne 0$$

since $\delta \to 0$ as $\varepsilon \to 0$. Hence, condition **U4** holds for sufficiently small ε. □

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