On long-time decay for modified Klein-Gordon equation

E. A. Kopylova [1]
Institute for Information Transmission Problems RAS
B.Karetnyi 19, Moscow 101447,GSP-4, Russia
e-mail: elena.kopylova@univie.ac.at

Abstract

We obtain a dispersive long-time decay in weighted energy norms for solutions of the Klein-Gordon equation in a moving frame. The decay extends the results of Jensen, Kato and Murata for the equations of the Schrödinger type. We modify the approach to make it applicable to relativistic equations.

Keywords: Klein-Gordon equation, relativistic equations, resolvent, spectral representation, weighted spaces, Born series, convolution.

2000 Mathematics Subject Classification: 35L10, 34L25, 47A40, 81U05.

[1]Supported partly by FWF, DFG and RFBR grants.

1 Introduction

In this paper, we establish a dispersive long time decay in weighted energy norms for the solutions to 1D Klein-Gordon equation in a moving frame with the velocity v

$$\dot{\Psi}(t) = \mathcal{A}\Psi(t) \tag{1.1}$$

where

$$\Psi(t) = \begin{pmatrix} \psi(t) \\ \pi(t) \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 - V & v\nabla \end{pmatrix}, \quad \nabla = \frac{d}{dx}, \quad \Delta = \frac{d^2}{dx^2}$$

with $m > 0$, and $|v| < 1$. For $s, \sigma \in \mathbb{R}$, we denote by $H^s_\sigma = H^s_\sigma(\mathbb{R})$ the weighted Agmon-Sobolev spaces [1], with the finite norms

$$\|\psi\|_{H^s_\sigma} = \|\langle x\rangle^\sigma \langle\nabla\rangle^s \psi\|_{L^2(\mathbb{R})} < \infty, \qquad \langle x\rangle = (1 + |x|^2)^{1/2}$$

Denote $L^2_\sigma = H^0_\sigma$. We assume that $V(x)$ is a real function, and

$$|V(x)| + |V'(x)| \le C\langle x\rangle^{-\beta}, \qquad x \in \mathbb{R} \tag{1.2}$$

for some $\beta > 5$. Then the multiplication by $V(x)$ is bounded operator $H^1_s \to H^1_{s+\beta}$ for any $s \in \mathbb{R}$.

We consider the "nonsingular case" in the terminology of [9], when the truncated resolvent of the operator $-\Delta + \gamma^2 V(x)$, $\gamma = 1/\sqrt{1 - v^2}$ is bounded at the edge point $\zeta = 0$ of the continuous spectrum. In other words,

$$\textit{the point } \zeta = 0 \textit{ is neither eigenvalue nor resonance for the operator } -\Delta + \gamma^2 V(x) \tag{1.3}$$

By definition (see [9, page 18]) the point $\zeta = 0$ is the resonance if there exists a nonzero solution $\psi \in L^2_{-1/2-0} \setminus L^2$ to the equation $(-\Delta + \gamma^2 V(x))\psi = 0$.

Definition 1.1. *$\mathcal{F}_\sigma$ is the complex Hilbert space $H^1_\sigma \oplus H^0_\sigma$ of vector-functions $\Psi = (\psi, \pi)$ with the norm*

$$\|\,\Psi\|_{\mathcal{F}_\sigma} = \|\psi\|_{H^1_\sigma} + \|\pi\|_{H^0_\sigma} < \infty$$

Our main result is the following long time decay of the solutions to (1.1): in the nonsingular case, the asymptotics hold

$$\|\mathcal{P}_c\Psi(t)\|_{\mathcal{F}_{-\sigma}} = \mathcal{O}(|t|^{-3/2}), \quad t \to \pm\infty \tag{1.4}$$

for initial data $\Psi_0 = \Psi(0) \in \mathcal{F}_\sigma$ with $\sigma > 5/2$, where $\mathcal{P}_c$ is a Riesz projection onto the continuous spectrum of the operator $\mathcal{A}$. The decay is desirable for the study of asymptotic stability and scattering for the solutions to nonlinear hyperbolic equations.

Let us comment on previous results in this direction. The decay of type (1.4) in weighted norms has been established first by Jensen and Kato [6] for the Schrödinger equation in the dimension $n = 3$. The result has been extended to all other dimensions by Jensen and Nenciu [4, 5, 7], and to more general PDEs of the Schrödinger type by Murata [9].

The Jensen-Kato-Murata approach is not applicable directly to the relativistic equations. The difference reflects distinct character of wave propagation in the relativistic and nonrelativistic equations (see the discussion in [8, Introduction]).

In [8] the decay of type (1.4) in the weighted energy norms has been proved for the 1D Klein-Gordon equation with $v = 0$. The approach develops the Jensen-Kato-Murata techniques to make it applicable to the relativistic equations. Namely, we apply the finite Born series and convolution. Here we extend the result [8] to the case $v \neq 0$.

Our paper is organized as follows. In Section 2 we obtain the time decay for the solution to the free modified Klein-Gordon equation and state the spectral properties of the free resolvent.. In Section 3 we obtain spectral properties of the perturbed resolvent and prove the decay (1.4).

2 Free equation

Here we consider the free equation with zero potential $V(x) = 0$:

$$\dot{\Psi}(t) = \mathcal{A}_0 \Psi(t) \tag{2.1}$$

where

$$\mathcal{A}_0 = \begin{pmatrix} v\nabla & 1 \\ \Delta - m^2 & v\nabla \end{pmatrix}$$

2.1 Spectral properties

For $t > 0$ and $\Psi_0 = \Psi(0) \in \mathcal{F}_0$, the solution $\Psi(t)$ to (2.1) admits the spectral Fourier-Laplace representation

$$\theta(t)\Psi(t) = -\frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{(i\omega+\varepsilon)t} \mathcal{R}_0(i\omega + \varepsilon)\Psi_0 \, d\omega, \qquad t \in \mathbb{R} \tag{2.2}$$

with any $\varepsilon > 0$, where $\theta(t)$ is the Heaviside function, $\mathcal{R}_0(\lambda) = (\mathcal{A}_0 - \lambda)^{-1}$ for $\mathrm{Re}\,\lambda > 0$ is the resolvent of the operator $\mathcal{A}_0$. The representation follows from the stationary equation $\lambda\tilde{\Psi}^+(\lambda) = \mathcal{A}_0\tilde{\Psi}^+(\lambda) + \Psi_0$ for the Fourier-Laplace transform $\tilde{\Psi}^+(\lambda) := \int_{\mathbb{R}} \theta(t)e^{-\lambda t}\Psi(t)dt$, $\mathrm{Re}\,\lambda > 0$. The solution $\Psi(t)$ is continuous bounded function of $t \in \mathbb{R}$ with the values in $\mathcal{F}_0$ by the energy conservation for the equation (2.1). Hence, $\tilde{\Psi}^+(\lambda) = -\mathcal{R}_0(\lambda)\Psi_0$ is analytic function in $\mathrm{Re}\,\lambda > 0$ with the values in $\mathcal{F}_0$, and bounded for $\mathrm{Re}\,\lambda > \varepsilon$. Therefore, the integral (2.2) converges in the sense of distributions of $t \in \mathbb{R}$ with the values in $\mathcal{F}_0$. Similarly to (2.2),

$$\theta(-t)\Psi(t) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{(i\omega-\varepsilon)t} \mathcal{R}_0(i\omega - \varepsilon)\Psi_0 \, d\omega, \qquad t \in \mathbb{R} \tag{2.3}$$

Let us calculate the resolvent $\mathcal{R}_0(\lambda)$. We have

$$\mathcal{R}_0(\lambda) = (\mathcal{A}_0 - \lambda)^{-1} = \begin{pmatrix} v\nabla - \lambda & 1 \\ \Delta - m^2 & v\nabla - \lambda \end{pmatrix}^{-1}, \quad \mathrm{Re}\,\lambda > 0$$

In the Fourier space we obtain

$$\begin{pmatrix} -(ivk + \lambda) & 1 \\ -(k^2 + m^2) & -(ivk + \lambda) \end{pmatrix}^{-1} = [(ivk + \lambda)^2 + k^2 + m^2]^{-1} \begin{pmatrix} -(ivk + \lambda) & -1 \\ k^2 + m^2 & -(ivk + \lambda) \end{pmatrix}$$

Taking the inverse Fourier transform, we obtain the resolvent

$$\mathcal{R}_0(\lambda)=\left(\begin{array}{cc} v\nabla-\lambda & -1\\ -\Delta+m^2 & v\nabla-\lambda\end{array}\right)R_0(\lambda)=\left(\begin{array}{cc}(v\nabla-\lambda)R_0(\lambda) & -R_0(\lambda)\\ 1-(v\nabla-\lambda)^2R_0(\lambda) & (v\nabla-\lambda)R_0(\lambda)\end{array}\right) \quad (2.4)$$

where $R_0(\lambda)$ is the operator with the integral kernel

$$R_0(\lambda,x,y)=F^{-1}_{k\to x-y}\frac{1}{k^2+m^2+(ivk+\lambda)^2},\quad x,y\in\mathbb{R} \quad (2.5)$$

which is well defined since the denominator in (2.5) does not vanish for $\mathrm{Re}\,\lambda>0$. Denote $\mathcal{H}_0=-(1-v^2)\Delta+m^2=-\frac{1}{\gamma^2}\Delta+m^2$. Since

$$(\mathcal{H}_0+\lambda^2-2v\lambda\nabla)\psi(x)=e^{-\gamma^2v\lambda x}(\mathcal{H}_0+\gamma^2\lambda^2)e^{\gamma^2v\lambda x}\psi(x) \quad (2.6)$$

we have

$$R_0(\lambda)=(\mathcal{H}_0+\lambda^2-2v\lambda\nabla)^{-1}=e^{-\gamma^2v\lambda x}\gamma^2\tilde{R}_0(\gamma^2m^2+\gamma^4\lambda^2)e^{\gamma^2v\lambda y} \quad (2.7)$$

where

$$\tilde{R}_0(\zeta)==(-\Delta+\zeta)^{-1}=\mathrm{Op}\Big[\frac{e^{-\sqrt{\zeta}|z|}}{2\sqrt{\zeta}}\Big]$$

is the Schrödinger resolvent. Finally,

$$R_0(\lambda,x,y)=\frac{e^{-\gamma^2(\sqrt{\lambda^2-\mu^2}\,|x-y|+v\lambda(x-y))}}{2\sqrt{\lambda^2-\mu^2}},\quad \mu=\frac{im}{\gamma} \quad (2.8)$$

Denote $\Gamma:=(-i\infty,-\mu,)\cup(\mu,\ i\infty)$. We choose $\mathrm{Re}\sqrt{\lambda^2-\mu^2}>0$ for $\lambda\in\mathbb{C}\setminus\overline{\Gamma}$. Then

$$0<\mathrm{Re}\,(v\lambda)<\mathrm{Re}\sqrt{\lambda^2-\mu^2},\qquad \lambda\in\mathbb{C}\setminus\overline{\Gamma} \quad (2.9)$$

Denote by $\mathcal{L}(B_1,B_2)$ the Banach space of bounded linear operators from a Banach space B_1 to a Banach space B_2. Formulas (2.8) implies the following properties of $R_0(\lambda)$:

Lemma 2.1. *(cf. [1, 9])*
i) The operator $R_0(\lambda)$ is analytic function of $\lambda\in\mathbb{C}\setminus\overline{\Gamma}$ with the values in $\mathcal{L}(H^0_0,H^1_0)$.
ii) For $\lambda\in\Gamma$, the convergence (limiting absorption principle) holds

$$R_0(\lambda\pm\varepsilon)\to R_0(\lambda\pm 0),\quad \varepsilon\to 0+ \quad (2.10)$$

in $\mathcal{L}(H^0_\sigma,H^1_{-\sigma})$ with $\sigma>1/2$, uniformly in $|\lambda|\ge|\mu|+r$ for any $r>0$.
iii) The asymptotics hold

$$R_0(\lambda)=B_0^\pm\frac{1}{\sqrt{\nu}}+B_1^\pm+\mathcal{O}(|\nu|^{1/2}),\quad \nu=\lambda\mp\mu\to 0,\quad \lambda\in\mathbb{C}\setminus\overline{\Gamma} \quad (2.11)$$

in $\mathcal{L}(H^0_\sigma,H^1_{-\sigma})$ with $\sigma>5/2$, where

$$B_0^\pm = \mathrm{Op}\Big[\frac{e^{\mp\gamma^2v\mu(x-y)}}{2\sqrt{\pm 2\mu}}\Big]\in\mathcal{L}(H^0_\sigma,H^1_{-\sigma}),\quad \sigma>1/2 \quad (2.12)$$

$$B_1^\pm = \mathrm{Op}\Big[-\frac{\gamma^2e^{\mp\gamma^2v\mu(x-y)}|x-y|}{2}\Big]\in\mathcal{L}(H^0_\sigma,H^1_{-\sigma}),\quad \sigma>3/2$$

iv) The asymptotics (2.11) *can be differentiated two times:*

$$R_0'(\lambda)=-B_0^{\pm}\frac{1}{2\nu\sqrt{\nu}}+\mathcal{O}\big(|\nu|^{-1/2}\big),\quad R_0''(\lambda)=\mathcal{O}\big(|\nu|^{-5/2}\big),\quad \nu=\lambda\mp\mu\to 0,\quad \lambda\in\mathbb{C}\setminus\overline{\Gamma}\quad (2.13)$$

in $\mathcal{L}(H^0_\sigma,H^1_{-\sigma})$ *with* $\sigma>5/2$.
v) For $s\in\mathbb{R}$, $l=-1,0,1,2$, $k=0,1,2,...$ *and* $\sigma>1/2+k$ *the decay holds*

$$\|R_0^{(k)}(\lambda)\|_{\mathcal{L}(H^s_\sigma,H^{s+l}_{-\sigma})}=\mathcal{O}(|\lambda|^{-(1-l)}),\quad |\lambda|\to\infty,\quad \lambda\in\mathbb{C}\setminus\Gamma \tag{2.14}$$

Proof. We prove the properties ii) and v) since other properties follow directly from (2.8).
Step i) First, we prove the convergence (2.10). The norm of the operator $R_0(\lambda):H^0_\sigma\to H^1_{-\sigma}$ is equivalent to the norm of the operator

$$\langle x\rangle^{-\sigma}R_0(\lambda)\langle y\rangle^{-\sigma}:L^2\to H^1$$

The norm of the latter operator does not exceed the sum in k, $k=0,1$, of the norms of operators

$$\partial_x^k[\langle x\rangle^{-\sigma}R_0(\lambda,x,y)\langle y\rangle^{-\sigma}]:L^2\to L^2 \tag{2.15}$$

According (2.8) and (2.9),

$$|\partial_x^k R_0(\lambda,x,y)|\le C(\lambda),\quad k=0,1,\quad x,y\in\mathbb{R},\quad \lambda\in\mathbb{C}\setminus\overline{\Gamma}$$

Hence for $\sigma>1/2$ we have

$$\sum_k\int|\partial_x^k[\langle x\rangle^{-\sigma}R_0(\lambda,x,y)\langle y\rangle^{-\sigma}]|^2dxdy\le C(\lambda)\int\langle x\rangle^{-2\sigma}\langle y\rangle^{-2\sigma}dxdy\le C_1(\lambda)$$

The estimate implies that Hilbert-Schmidt norms of operators (2.15) is finite. For $\lambda\in\Gamma$ and $x,y\in\mathbb{R}$, there exists the pointwise limit

$$R_0(\lambda\pm\varepsilon,x,y)\to\ R_0(\lambda\pm 0,x,y),\quad \varepsilon\to 0+$$

Therefore,

$$\sum_k\int|\partial_x^k[\langle x\rangle^{-\sigma}R_0(\lambda\pm\varepsilon,x,y)\langle y\rangle^{-\sigma}-\langle x\rangle^{-\sigma}R_0(\lambda\pm\varepsilon,x,y)\langle y\rangle^{-\sigma}]|^2dxdy\to 0,\quad \varepsilon\to 0+$$

by the Lebesgue dominated convergence theorem, hence (2.10) is proved.
Step ii) Now we prove the decay (2.14). It suffices to verify the case $s=0$ since $R_0(\lambda)$ commutes with the operators $\langle\nabla\rangle^s$ with arbitrary $s\in\mathbb{R}$. For $k=0$ and $l=0,1,2$ the decay (2.14) follows obviously from (2.8). In the case $k=0$ and $l=-1$ the decay follows from the identity

$$R_0(\lambda)=\frac{1}{m^2+\lambda^2}\big(1+\frac{\Delta R_0(\lambda)}{\gamma^2}+2v\lambda\nabla R_0(\lambda)\big) \tag{2.16}$$

Namely, using (2.14) with $l=0$ and $l=1$, we obtain

$$\|\nabla R_0(\lambda)\|_{\mathcal{L}(H^0_\sigma,H^{-1}_{-\sigma})}=\mathcal{O}(|\lambda|^{-1}),\quad \|\Delta R_0(\lambda)\|_{\mathcal{L}(H^0_\sigma,H^{-1}_{-\sigma})}=\mathcal{O}(1)$$

hence (2.16) implies

$$\|R_0(\lambda)\|_{\mathcal{L}(H^0_\sigma,H^{-1}_{-\sigma})}=\mathcal{O}(|\lambda|^{-2})$$

In the case $k\ne 0$ the bounds (2.14) follow similarly by differentiating (2.8). □

Formula (2.4) and Lemma 2.1 imply

Corollary 2.2. *i) The resolvent $\mathcal{R}_0(\lambda)$ is analytic function of $\lambda\in\mathbb{C}\setminus\overline{\Gamma}$ with the values in $\mathcal{L}(\mathcal{F}_0,\mathcal{F}_0)$.*
ii) For $\lambda\in\Gamma$, the convergence (limiting absorption principle) holds

$$\mathcal{R}_0(\lambda\pm\varepsilon)\to\mathcal{R}_0(\lambda\pm 0),\quad \varepsilon\to 0+ \tag{2.17}$$

in $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>1/2$.

iii) The asymptotics hold

$$\mathcal{R}_0(\lambda)=\mathcal{B}_0^{\pm}\frac{1}{\sqrt{\nu}}+\mathcal{B}_1^{\pm}+\mathcal{O}(|\nu|^{1/2}),\quad \nu=\lambda\mp\mu\to 0,\quad \lambda\in\mathbb{C}\setminus\overline{\Gamma} \tag{2.18}$$

in $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>5/2$, where

$$\mathcal{B}_0^{\pm}=B_0^{\pm}\begin{pmatrix}\mp i\gamma m & -1\\ \gamma^2m^2 & \mp i\gamma m\end{pmatrix}\in\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})\ \text{ with }\ \sigma>1/2 \tag{2.19}$$

and $\mathcal{B}_1^{\pm}\in\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>3/2$.
iv) The asymptotics (2.18) can be differentiated two times:

$$\mathcal{R}_0'(\lambda)=-\mathcal{B}_0^{\pm}\frac{1}{2\nu\sqrt{\nu}}+\mathcal{O}(|\nu|^{-1/2}),\quad \mathcal{R}_0''(\lambda)=\mathcal{O}(|\nu|^{-5/2}),\quad \nu=\lambda\mp\mu\to 0,\quad \lambda\in\mathbb{C}\setminus\overline{\Gamma} \tag{2.20}$$

in $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>5/2$.
v) For $k=0,1,2,...$ and $\sigma>1/2+k$ the asymptotics hold

$$\|\mathcal{R}_0^{(k)}(\lambda)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}=\mathcal{O}(1),\quad |\lambda|\to\infty,\quad \lambda\in\mathbb{C}\setminus\Gamma \tag{2.21}$$

Denote by $\mathcal{G}_v(t)$ the dynamical group of equation (2.1).

Corollary 2.3. *For $t\in\mathbb{R}$ and $\Psi_0\in\mathcal{F}_\sigma$ with $\sigma>1/2$, the group $\mathcal{G}_v(t)$ admits the integral representation*

$$\mathcal{G}_v(t)\Psi_0=\frac{1}{2\pi i}\int\limits_{\Gamma}e^{\lambda t}\Big[\mathcal{R}_0(\lambda-0)-\mathcal{R}_0(\lambda+0)\Big]\Psi_0\,d\lambda \tag{2.22}$$

where the integral converges in the sense of distributions of $t\in\mathbb{R}$ with the values in $\mathcal{F}_{-\sigma}$.

Proof. Summing up the representations (2.2) and (2.3), and sending $\varepsilon\to 0+$, we obtain (2.22) by the Cauchy theorem and Corollary 2.2. □

2.2 Time decay

For the integral kernel of the operator $\mathcal{G}_v(t)$ we have

$$\mathcal{G}_v(x-y,t)=\mathcal{G}_0(x-y-vt,t),\quad x,y\in\mathbb{R},\quad t\in\mathbb{R} \tag{2.23}$$

Here

$$\mathcal{G}_0(z,t)=\left(\begin{array}{cc}\dot G_0(z,t) & G_0(z,t)\\ \ddot G_0(z,t) & \dot G_0(z,t)\end{array}\right),\quad G_0(z,t)=\frac{1}{2}\theta(t-|z|)J_0(m\sqrt{t^2-z^2}),\quad z=x-y \quad (2.24)$$

where J_0 is the Bessel function. The relation (2.23) implies the Huygen's principle for the group $\mathcal{G}_v(t)$, i.e.

$$\mathcal{G}_v(x-y,t)=0,\qquad |x-y-vt|>t$$

Also, the relation (2.23) implies the energy conservation for the group $\mathcal{G}_v(t)$. Namely, for $\Psi(t)=(\psi(\cdot,t),\pi(\cdot,t))=\mathcal{G}_v(t)\Psi_0$ we have

$$\int[|\pi(x,t)+v\cdot\nabla\psi(x,t)|^2+|\nabla\psi(x,t)|^2+m^2|\psi(x,t)|^2]dx=\text{const},\qquad t\in\mathbb{R}$$

In particular, this gives that

$$\|\Psi(t)\|_{\mathcal{F}_0}\le C\|\Psi_0\|_{\mathcal{F}_0},\ \ t\in\mathbb{R}$$

We represent $\mathcal{G}_v(z,t)$ as

$$\mathcal{G}_v(z,t)=\mathcal{G}_b(z,t)+\mathcal{G}_r(z,t),\quad z\in\mathbb{R},\quad t\ge 0$$

where

$$\mathcal{G}_b(z,t):=\frac{1}{\sqrt{2m\pi t/\gamma}}\left(\begin{array}{cc}-\frac{m}{\gamma}\sin[m(\frac{t}{\gamma}+\gamma vz)-\frac{\pi}{4}] & \cos[m(\frac{t}{\gamma}+\gamma vz)-\frac{\pi}{4}]\\ -\frac{m^2}{\gamma^2}\cos[m(\frac{t}{\gamma}+\gamma vz)-\frac{\pi}{4}] & -\frac{m}{\gamma}\sin[m(\frac{t}{\gamma}+\gamma vz)-\frac{\pi}{4}]\end{array}\right) \quad (2.25)$$

The entries of the matrix $\mathcal{G}_b(z,t)$ admit the bounds

$$|\mathcal{G}_b^{ij}(z,t)|\le C(v)/\sqrt{t},\quad i,j=1,2,\quad z\in\mathbb{R},\quad t\ge 1 \quad (2.26)$$

The group $\mathcal{G}_v(t)$ slow decays, like $t^{-1/2}$. We will show that $\mathcal{G}_b(t)=\text{Op}[\mathcal{G}_b(x-y,t)]$ is only term responsible for the slow decay. More exactly, in the next section we will prove the following basic proposition

Proposition 2.4. *The decay holds*

$$\mathcal{G}_r(t)=\text{Op}[\mathcal{G}_r(x-y,t)]=\mathcal{O}(t^{-3/2}),\quad t\to\infty \quad (2.27)$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ *with* $\sigma>5/2$.

The following key observation is that (2.25) contains just two frequencies $\pm\mu$ which are the edge points of the continuous spectrum. This suggests that the term $\mathcal{G}_b(t)$ with "bad decay" $t^{-1/2}$ should not contribute to the high energy component of the group $\mathcal{G}_v(t)$ and the high energy component of the group $\mathcal{G}_v(t)$ decays like $t^{-3/2}$.

More precisely, let us introduce the *low energy* and *high energy* components of $\mathcal{G}_v(t)$:

$$\mathcal{G}_l(t)=\frac{1}{2\pi i}\int\limits_{\Gamma}e^{\lambda t}l(i\lambda)\Big[\mathcal{R}_0(\lambda-0)-\mathcal{R}_0(\lambda+0)\Big]\,d\lambda \quad (2.28)$$

$$\mathcal{G}_h(t)=\frac{1}{2\pi i}\int\limits_{\Gamma}e^{\lambda t}h(i\lambda)\Big[\mathcal{R}_0(\lambda-0)-\mathcal{R}_0(\lambda+0)\Big]\,d\lambda \quad (2.29)$$

where $l(\omega)\in C_0^\infty(\mathbb{R})$ is an even function, $l(\omega)=0$ if $|\omega|>|\mu|+2\varepsilon$, and $l(\omega)=1$ if $|\omega|\le|\mu|+\varepsilon$ with an $\varepsilon>0$, and $h(\omega)=1-l(\omega)$.

Theorem 2.5. *In $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ with $\sigma > 5/2$ the decay holds*

$$\mathcal{G}_h(t) = \mathcal{O}(t^{-3/2}), \quad t \to \infty \tag{2.30}$$

Proof. We deduce asymptotics (2.30) from Proposition 2.4.
Step i) Let $\Psi_0 \in \mathcal{F}_\sigma$. Denote

$$\Psi^+(t) = \theta(t)\mathcal{G}_v(t)\Psi_0,\ \Psi_b^+(t) = \theta(t)\mathcal{G}_b(t)\Psi_0,\ \Psi_h^+(t) = \theta(t)\mathcal{G}_h(t)\Psi_0,\ \Psi_r^+(t) = \theta(t)\mathcal{G}_r(t)\Psi_0$$

Then

$$\begin{aligned}\Psi_h^+(t) &= -\frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{i\omega t}h(\omega)\mathcal{R}_0(i\omega+0)\Psi_0 d\omega \\ &= \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{i\omega t}h(\omega)\tilde\Psi^+(i\omega)d\omega = \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{i\omega t}h(\omega)\Big[\tilde\Psi_b^+(i\omega) + \tilde\Psi_r^+(i\omega)\Big]\, d\omega \\ &= \Psi_r^+(t) + \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{i\omega t}h(\omega)\tilde\Psi_b^+(i\omega)d\omega - \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{i\omega t}l(\omega)\tilde\Psi_r^+(i\omega)d\omega \end{aligned}\tag{2.31}$$

where $\tilde\Psi^+(\lambda) = \int\limits_0^\infty e^{-\lambda t}\Psi^+(t)dt$ and so on. By (2.27)

$$\|\Psi_r^+(t)\|_{\mathcal{F}_{-\sigma}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty \tag{2.32}$$

Step ii) Let us consider the second summand in the last line of (2.31). By (2.25) the vector function $\tilde\Psi_b^+(i\omega)$ is a smooth function for $|\omega| > |\mu| + \varepsilon$, and $\partial_\omega^k \tilde\Psi_b^+(i\omega) = \mathcal{O}(|\omega|^{-1/2-k})$, $k = 0, 1, 2...$, $\omega \to \infty$. Hence partial integration implies that

$$\Big\|\int\limits_{\mathbb{R}} e^{i\omega t}h(\omega)\tilde\Psi_b^+(i\omega)d\omega\Big\|_{\mathcal{F}_{-\sigma}} = \mathcal{O}(t^{-N}), \quad \forall N \in \mathbb{N}, \quad t \to \infty \tag{2.33}$$

Step iii) Finally, let us consider the third summand in the last line of (2.31). Introducing the function $L(t)$ such that $\tilde L(\lambda) = l(i\lambda)$, we obtain

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{i\omega t}l(\omega)\tilde\Psi_r^+(i\omega)d\omega = [L \star \Psi_r^+](t) = \mathcal{O}(t^{-3/2}), \quad t \to \infty \tag{2.34}$$

in the norm of $\mathcal{F}_{-\sigma}$, since $L(t) = \mathcal{O}(t^{-N})$, $t \to \infty$ for any $N \in \mathbb{N}$, and $\|\Psi_r^+(t)\|_{\mathcal{F}_{-\sigma}} = \mathcal{O}(t^{-3/2})$ by (2.27). Finally, (2.31)- (2.34) imply (2.30). □

2.3 Proof of Proposition 2.4

Let us fix an arbitrary $\varepsilon \in (|v|, 1)$. Denote $\varepsilon_1 = \varepsilon - |v|$. For any $t \geq 1$ we split the initial function $\Psi_0 \in \mathcal{F}_\sigma$ in two terms, $\Psi_0 = \Psi'_{0,t} + \Psi''_{0,t}$, $\Psi'_{0,t} = (\psi'_{0,t}, \pi'_{0,t})$, $\Psi''_{0,t} = (\psi''_{0,t}, \pi''_{0,t})$, such that

$$\|\Psi'_{0,t}\|_{\mathcal{F}_\sigma} + \|\Psi''_{0,t}\|_{\mathcal{F}_\sigma} \leq C\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geq 1 \tag{2.35}$$

$$\Psi'_{0,t}(x)=0\quad\text{for}\quad |x|>\frac{\varepsilon_1 t}{2},\quad\text{and}\quad \Psi''_{0,t}(x)=0\quad\text{for}\quad |x|<\frac{\varepsilon_1 t}{4}\tag{2.36}$$

We estimate $\mathcal{G}_r(t)\Psi'_{0,t}$ and $\mathcal{G}_r(t)\Psi''_{0,t}$ separately.

Step i) First we consider $\mathcal{G}_r(t)\Psi''_{0,t}=\mathcal{G}_v(t)\Psi''_{0,t}-\mathcal{G}_b(t)\Psi''_{0,t}$. Using energy conservation and properties (2.35)- (2.36) we obtain

$$\|\mathcal{G}_v(t)\Psi''_{0,t}\|_{\mathcal{F}_{-\sigma}}\le\|\mathcal{G}_v(t)\Psi''_{0,t}\|_{\mathcal{F}_0}\le C\|\Psi''_{0,t}\|_{\mathcal{F}_0}\le C(\varepsilon)t^{-\sigma}\|\Psi''_{0,t}\|_{\mathcal{F}_\sigma}\le C_1(\varepsilon)t^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\ge 1\tag{2.37}$$

since $\sigma>5/2$. Further, (2.26) and the Cauchy inequality imply

$$\begin{aligned}|(\mathcal{G}_b^{22}(t)\pi''_{0,t})(y)| &\le \frac{C}{\sqrt{t}}\Big|\int\pi''_{0,t}(x)dx\Big|\le\frac{C}{\sqrt{t}}\Big(\int|\pi''_{0,t}(x)|^2(1+x^2)^\sigma dx\Big)^{1/2}\Big(\int\limits_{\varepsilon_1 t/4}^{\infty}\frac{dx}{(1+x^2)^\sigma}\Big)^{1/2}\\ &\le \frac{C(\varepsilon)}{\sqrt{t}}t^{-\sigma+1/2}\|\pi''_{0,t}\|_{H^0_\sigma}\le C(\varepsilon)t^{-5/2}\|\pi''_{0,t}\|_{H^0_\sigma},\quad t\ge 1\end{aligned}\tag{2.38}$$

Hence $\|\mathcal{G}_b^{22}(t)\pi''_{0,t}\|_{H^0_{-\sigma}}\le C(\varepsilon)t^{-5/2}\|\pi''_{0,t}\|_{H^0_\sigma}$. The functions $\mathcal{G}_b^{12}(t)\pi''_{0,t}$ and $\mathcal{G}_b^{i1}(t)\psi''_{0,t}$, $i=1,2$ can be estimated similarly. Therefore,

$$\|\mathcal{G}_b(t)\Psi''_{0,t}\|_{\mathcal{F}_{-\sigma}}\le C(\varepsilon)t^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\ge 1\tag{2.39}$$

and (2.37)- (2.39) imply that

$$\|\mathcal{G}_r(t)\Psi''_{0,t}\|_{\mathcal{F}_{-\sigma}}\le C(\varepsilon)t^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\ge 1\tag{2.40}$$

Step ii) Denote by ζ the operator of multiplication by the function $\zeta(|x|/t)$, where $\zeta=\zeta(s)\in C_0^\infty(\mathbb{R})$, $\zeta(s)=1$ for $|s|<\varepsilon_1/4$, $\zeta(s)=0$ for $|s|>\varepsilon_1/2$. Obviously, for any k, we have

$$|\partial_x^k\zeta(|x|/t)|\le C(\varepsilon)<\infty,\qquad t\ge 1$$

Since $1-\zeta(|x|/t)=0$ for $|x|<\varepsilon_1 t/4$, then by the energy conservation and (2.35), we obtain

$$||(1-\zeta)\mathcal{G}_v(t)\Psi'_{0,t}||_{\mathcal{F}_{-\sigma}}\le C(\varepsilon)t^{-\sigma}||\mathcal{G}_v(t)\Psi'_{0,t}||_{\mathcal{F}_0}\le C_1(\varepsilon)t^{-\sigma}||\Psi'_{0,t}||_{\mathcal{F}_0}\le C_2(\varepsilon)t^{-5/2}||\Psi_0||_{\mathcal{F}_\sigma},\ \ t\ge 1\tag{2.41}$$

Further, (2.26) and the Cauchy inequality imply, similarly (2.38), that

$$|(\mathcal{G}_b^{22}(t)\pi'_{0,t})(y)|\le\frac{C}{\sqrt{t}}\Big|\int\pi'_{0,t}(x)dx\Big|\le\frac{C}{\sqrt{t}}\|\pi'_{0,t}\|_{H^0_\sigma}$$

Hence, we obtain

$$\|(1-\zeta)\mathcal{G}_b^{22}(t)\pi'_{0,t}\|_{H^0_{-\sigma}}\le\frac{C}{\sqrt{t}}\|\pi'_{0,t}\|_{H^0_\sigma}\Big(\int\limits_{\varepsilon_1 t/4}^{\infty}\frac{dy}{(1+y^2)^\sigma}\Big)^{1/2}\le C(\varepsilon)t^{-5/2}\|\pi'_{0,t}\|_{H^0_\sigma}$$

The functions $(1-\zeta)\mathcal{G}_b^{12}(t)\pi'_{0,t}$ and $(1-\zeta)\mathcal{G}_b^{i1}(t)\psi'_{0,t}$, $i=1,2$ can be estimated similarly. Hence,

$$||(1-\zeta)\mathcal{G}_b(t)\Psi'_{0,t}||_{\mathcal{F}_{-\sigma}}\le C(\varepsilon)t^{-5/2}||\Psi_0||_{\mathcal{F}_\sigma},\qquad t\ge 1\tag{2.42}$$

and (2.41) - (2.42) imply

$$||(1-\zeta)\mathcal{G}_r(t)\Psi'_{0,t}||_{\mathcal{F}_{-\sigma}} \le C(\varepsilon)t^{-5/2}||\Psi_0||_{\mathcal{F}_\sigma}, \qquad t\ge 1 \tag{2.43}$$

Step iii) Finally, let us estimate $\zeta\mathcal{G}_r(t)\Psi'_{0,t}$. Let χ_t be the characteristic function of the ball $|x|\le \varepsilon_1 t/2$. We will use the same notation for the operator of multiplication by this characteristic function. By (2.36), we have

$$\zeta\mathcal{G}_r(t)\Psi'_{0,t} = \zeta\mathcal{G}_r(t)\chi_t\Psi'_{0,t} \tag{2.44}$$

The matrix kernel of the operator $\zeta\mathcal{G}_r(t)\chi_t$ is equal to

$$\mathcal{G}'_r(x-y,t) = \zeta(|x|/t)\mathcal{G}_r(x-y,t)\chi_t(y)$$

Well known asymptotics of the Bessel function [10] imply the following lemma, which we prove in Appendix.

Lemma 2.6. *For any $\varepsilon\in(|v|,1)$ the bounds hold*

$$|\partial_z^k\mathcal{G}_r(z,t)| \le C(\varepsilon)(1+z^2)t^{-3/2}, \quad |z|\le(\varepsilon-|v|)t, \quad t\ge 1, \quad k=0,1 \tag{2.45}$$

Since $\zeta(|x|/t)=0$ for $|x|>\varepsilon_1 t/2$ and $\chi_t(y)=0$ for $|y|>\varepsilon_1 t/2$ then $\mathcal{G}'_r(x-y,t)=0$ for $|x-y|>\varepsilon_1 t=(\varepsilon-|v|)t$. Hence, (2.45) imply that

$$|\partial_x^k\mathcal{G}'_r(x-y,t)| \le C(\varepsilon)(1+(x-y)^2)t^{-3/2}, \quad k=0,1, \quad t\ge 1 \tag{2.46}$$

The norm of the operator $\zeta\mathcal{G}_r(t)\chi_t : \mathcal{F}_\sigma\to\mathcal{F}_{-\sigma}$ is equivalent to the norm of the operator

$$\langle x\rangle^{-\sigma}\zeta\mathcal{G}_r(t)\chi_t(y)\langle y\rangle^{-\sigma} : \mathcal{F}_0\to\mathcal{F}_0$$

The norm of the later operator does not exceed the sum in k, $k=0,1$ of the norms of operators

$$\partial_x^k[\langle x\rangle^{-\sigma}\zeta\mathcal{G}_r(t)\chi_t(y)\langle y\rangle^{-\sigma}] : L^2(\mathbb{R})\oplus L^2(\mathbb{R})\to L^2(\mathbb{R})\oplus L^2(\mathbb{R}) \tag{2.47}$$

The bounds (2.46) imply that the Hilbert-Schmidt norms of operators (2.47) do not exceed $C(\varepsilon)t^{-3/2}$ since $\sigma>5/2$. Hence, (2.35) and (2.44) imply that

$$||\zeta\mathcal{G}_r(t)\Psi'_{0,t}||_{\mathcal{F}_{-\sigma}} \le C(\varepsilon)t^{-3/2}||\Psi'_{0,t}||_{\mathcal{F}_\sigma} \le C_1(\varepsilon)t^{-3/2}||\Psi_0||_{\mathcal{F}_\sigma}, \quad t\ge 1 \tag{2.48}$$

Finally, (2.43) and (2.48) imply

$$||\mathcal{G}_r(t)\Psi'_{0,t}||_{\mathcal{F}_{-\sigma}} \le C(\varepsilon)t^{-3/2}||\Psi_0||_{\mathcal{F}_\sigma}, \quad t\ge 1$$

Proposition 2.4 is proved.

3 Perturbed equation

3.1 Perturbed resolvent

Now we consider the resolvent of the perturbed equation. We use the formula

$$\mathcal{R}(\lambda) = (1 + \mathcal{R}_0(\lambda)\mathcal{V})^{-1}\mathcal{R}_0(\lambda), \quad \mathcal{V} = \begin{pmatrix} 0 & 0 \\ -V & 0 \end{pmatrix} \tag{3.1}$$

By (2.4) we have

$$(1 + \mathcal{R}_0(\lambda)\mathcal{V})^{-1} = \begin{pmatrix} 1 + R_0(\lambda)V & 0 \\ -(v\nabla - \lambda)R_0(\lambda)V & 1 \end{pmatrix}^{-1} = \begin{pmatrix} (1 + R_0(\lambda)V)^{-1} & 0 \\ (v\nabla - \lambda)\Big(1 - (1 + R_0(\lambda)V)^{-1}\Big) & 1 \end{pmatrix} \tag{3.2}$$

Let us denote

$$\mathcal{H} = -(1 - v^2)\Delta + m^2 + V, \qquad R(\lambda) = (\mathcal{H} + \lambda^2 - 2v\lambda\nabla)^{-1} = (1 + R_0(\lambda)V)^{-1}R_0(\lambda)$$

Substituting (3.2) into (3.1) we obtain

$$\mathcal{R}(\lambda) = \begin{pmatrix} (1 + R_0(\lambda)V)^{-1} & 0 \\ (v\nabla - \lambda)\Big(1 - (1 + R_0(\lambda)V)^{-1}\Big) & 1 \end{pmatrix} \begin{pmatrix} (v\nabla - \lambda)R_0(\lambda) & -R_0(\lambda) \\ (-\Delta + m^2)R_0(\lambda) & (v\nabla - \lambda)R_0(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} R(\lambda)(v\nabla - \lambda) & -R(\lambda) \\ 1 - (v\nabla - \lambda)R(\lambda)(v\nabla - \lambda) & (v\nabla - \lambda)R(\lambda) \end{pmatrix} \tag{3.3}$$

Similarly (2.6)-(2.7), we obtain

$$(\mathcal{H} + \lambda^2 - 2v\lambda\nabla)\psi(x) = e^{-\gamma^2 v\lambda x}(\mathcal{H} + \gamma^2\lambda^2)e^{\gamma^2 v\lambda x}\psi(x) \tag{3.4}$$

$$R(\lambda) = e^{-\gamma^2 v\lambda x}\gamma^2\tilde{R}(\gamma^2 m^2 + \gamma^4\lambda^2)e^{\gamma^2 v\lambda y} \tag{3.5}$$

where $\tilde{R}(\zeta) = (-\Delta + \zeta + V\gamma^2)^{-1}$ is the resolvent of the Schrödinger operator $-\Delta + V\gamma^2$.

3.2 Spectral properties

To prove the long time decay for the perturbed equation, we first establish the spectral properties of the generator.

3.2.1 Limiting absorption principle

Proposition 3.1. *Let the potential V satisfy (1.2). Then*
i)$R(\lambda)$ is meromorphic function of $\lambda \in \mathbb{C} \setminus \overline{\Gamma}$ with the values in $\mathcal{L}(H^0_0, H^1_0)$;
ii) For $\lambda \in \Gamma$, the convergence holds

$$R(\lambda \pm \varepsilon) \to R(\lambda \pm 0), \quad \varepsilon \to 0+ \tag{3.6}$$

in $\mathcal{L}(H^0_\sigma, H^1_{-\sigma})$ with $\sigma > 1/2$, uniformly in $|\lambda| \geq |\mu| + r$ for any $r > 0$.

Proof. *Step i)* The statement i) follows from Lemma 2.1-i), the Born splitting

$$R(\lambda) = R_0(\lambda)(1 + VR_0(\lambda))^{-1} \tag{3.7}$$

and the Gohberg-Bleher theorem [2, 3] since $VR_0(\lambda)$ is a compact operator in L^2 for $\lambda \in \mathbb{C} \setminus \overline{\Gamma}$.
Step ii) The convergence (3.6) follow from (2.10) by the Born splitting (3.7) if

$$[1 + VR_0(\lambda \pm \varepsilon)]^{-1} \to [1 + VR_0(\lambda \pm 0)]^{-1}, \quad \varepsilon \to +0, \quad \lambda \in \Gamma$$

in $\mathcal{L}(H^0_\sigma; H^0_\sigma)$. This convergence holds if and only if both limit operators $1 + VR_0(\lambda \pm 0)$ are invertible in H^0_σ for $\lambda \in \Gamma$. The operators are invertible according to the reversibility of the operators $1 + \gamma^2 V\tilde{R}_0(\zeta \pm i0)$ in H^0_σ for $\zeta < 0$ (see [1, Theorem 3.3 and Lemma 4.2]) and the relations

$$1 + VR_0(\lambda \pm 0) = e^{-\gamma^2 v\lambda x}\big(1 + \gamma^2 V\tilde{R}_0(\gamma^2 m^2 + \gamma^4(\lambda \pm i0)^2)\big)e^{\gamma^2 v\lambda y}$$

which follows from (2.7). □

Formula (3.3) and Proposition 3.1 imply

Corollary 3.2. *Let the conditions* (1.2) *holds. Then*
i) $\mathcal{R}(\lambda)$ *is meromorphic function of* $\lambda \in \mathbb{C} \setminus \overline{\Gamma}$ *with the values in* $\mathcal{L}(\mathcal{F}_0, \mathcal{F}_0)$*;*
ii) For $\lambda \in \Gamma$*, the convergence holds*

$$\mathcal{R}(\lambda \pm \varepsilon) \to \mathcal{R}(\lambda \pm 0), \quad \varepsilon \to 0+ \tag{3.8}$$

in $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ *with* $\sigma > 1/2$.

3.2.2 High energy decay

Lemma 3.3. *For* $k = 0, 1, 2$, $s = 0, 1$ *and* $l = -1, 0, 1$ *with* $s + l \in \{0, 1\}$ *the asymptotics hold*

$$\|R^{(k)}(\lambda \pm 0)\|_{\mathcal{L}(H^s_\sigma, H^{s+l}_{-\sigma})} = \mathcal{O}(|\lambda|^{-(1-l+k)}), \quad |\lambda| \to \infty, \quad \lambda \in \Gamma \tag{3.9}$$

with $\sigma > 1/2 + k$.

Proof. The decay follows from formula (3.5) and the known decay of Schrödinger resolvent $\tilde{R}(\zeta)$ (see [1, 6, 9, 8]). □

Corollary 3.4. *For* $k = 0, 1, 2$ *and* $\sigma > 1/2 + k$ *the asymptotics hold*

$$\|\mathcal{R}^{(k)}(\lambda \pm 0)\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} = \mathcal{O}(1), \quad |\lambda| \to \infty, \quad \lambda \in \Gamma \tag{3.10}$$

The resolvents $\mathcal{R}(\lambda)$ and $\mathcal{R}_0(\lambda)$ are related by the Born perturbation series

$$\mathcal{R}(\lambda) = \mathcal{R}_0(\lambda) - \mathcal{R}_0(\lambda)\mathcal{V}\mathcal{R}_0(\lambda) + \mathcal{R}_0(\lambda)\mathcal{V}\mathcal{R}_0(\lambda)\mathcal{V}\mathcal{R}(\lambda), \quad \lambda \in \mathbb{C} \setminus [\Gamma \cup \Sigma] \tag{3.11}$$

where Σ is the set of eigenvalues of the operator $\mathcal{A}$. An important role in (3.11) plays the product $\mathcal{W}(\lambda) := \mathcal{V}\mathcal{R}_0(\lambda)\mathcal{V}$. Now we obtain the asymptotics of $\mathcal{W}(\lambda)$ for large λ.

Lemma 3.5. *Let* $k = 0, 1, 2$, *and the potential* V *satisfy* (1.2) *with* $\beta > 1/2 + k + \sigma$ *where* $\sigma > 0$. *Then the asymptotics hold*

$$\|\mathcal{W}^{(k)}(\lambda)\|_{\mathcal{L}(\mathcal{F}_{-\sigma}, \mathcal{F}_\sigma)} = \mathcal{O}(|\lambda|^{-2}), \quad |\lambda| \to \infty, \quad \lambda \in \mathbb{C} \setminus \overline{\Gamma} \tag{3.12}$$

Proof. Asymptotics (3.12) follow from the algebraic structure of the matrix

$$\mathcal{W}^{(k)}(\lambda)=\mathcal{V}\mathcal{R}_0^{(k)}(\lambda)\mathcal{V}=\left(\begin{array}{cc}0 & 0\\ -VR_0^{(k)}(\lambda)V & 0\end{array}\right)$$

since (2.14) with $s=1$ and $l=-1$ implies that

$$\|VR_0^{(k)}(\lambda)Vf\|_{H^0_\sigma}\le C\|R_0^{(k)}(\lambda)Vf\|_{H^0_{\sigma-\beta}}=\mathcal{O}(|\lambda|^{-2})\|Vf\|_{H^1_{\beta-\sigma}}=\mathcal{O}(|\lambda|^{-2})\|f\|_{H^1_{-\sigma}}$$

since $\beta-\sigma>1/2+k$. □

3.2.3 Low energy expansions

Proposition 3.6. *The asymptotics hold*

$$\left.\begin{array}{l}\mathcal{R}(\lambda)=\mathcal{B}^\pm+\mathcal{O}(\nu^{1/2})\\ \mathcal{R}'(\lambda)=\mathcal{O}(\nu^{-1/2})\\ \mathcal{R}''(\lambda)=\mathcal{O}(\nu^{-3/2})\end{array}\right|\quad \nu:=\lambda\mp\mu\to 0,\quad \lambda\in\mathbb{C}\setminus\Gamma \tag{3.13}$$

in the norm $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>5/2$, where $\mathcal{B}^\pm\in\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ does not depend on λ.

First we prove the boundedness of the resolvent near the points $\pm\mu$.

Lemma 3.7. *Let the conditions (1.2) and (1.3) hold. Then the families $\{\mathcal{R}(\pm\mu+\varepsilon):\pm\mu+\varepsilon\in\mathbb{C}\setminus\overline{\Gamma},|\varepsilon|<\delta\}$ are bounded in the operator norm of $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ for any $\sigma>3/2$ and sufficiently small δ.*

Proof. Let us consider the equation for eigenfunctions of operator $\mathcal{A}$ with eigenvalues $\lambda=\pm\mu$:

$$\left(\begin{array}{cc}v\nabla & 1\\ \Delta-m^2-V & v\nabla\end{array}\right)\left(\begin{array}{c}\psi\\ \pi\end{array}\right)=\pm\mu\left(\begin{array}{c}\psi\\ \pi\end{array}\right),\quad \Psi=\left(\begin{array}{c}\psi\\ \pi\end{array}\right)\in\mathcal{F}_0$$

From the first equation we have $\pi=-(v\nabla\mp\mu)\psi$. Then the second equation becomes

$$(\mathcal{H}+\mu^2\mp 2v\mu\nabla)\psi=e^{\mp i\gamma vmx}(-\frac{1}{\gamma^2}\Delta+V)e^{\pm i\gamma vmx}\psi=0 \tag{3.14}$$

Hence, the condition (1.3) implies that $\Psi=0$. Similarly, (1.3) implies that the equation $\mathcal{A}\Psi=\pm\mu\Psi$ has no nonzero solutions $\Psi\in\mathcal{F}_{-1/2-0}$. Then the required boundedness of the resolvent near the points $\pm\mu$ follows similarly to [9, Theorem 7.2]. □

This lemma implies that the operators $(1+\mathcal{R}_0(\lambda)\mathcal{V})^{-1}=1-\mathcal{R}(\lambda)\mathcal{V}$ and $(1+\mathcal{V}\mathcal{R}_0(\lambda))^{-1}=1-\mathcal{V}\mathcal{R}(\lambda)$ are bounded in $\mathcal{L}(\mathcal{F}_{-\sigma},\mathcal{F}_{-\sigma})$ and in $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_\sigma)$ respectively for $|\lambda\mp\mu|<\delta$, $\lambda\in\mathbb{C}\setminus\overline{\Gamma}$. Now we prove more detailed asymptotics

Lemma 3.8. *The asymptotics hold*

$$(1+\mathcal{R}_0(\lambda)\mathcal{V})^{-1}\mathcal{B}_0^\pm=\mathcal{O}(\sqrt{\nu}),\quad \mathcal{B}_0^\pm(1+\mathcal{V}\mathcal{R}_0(\lambda))^{-1}=\mathcal{O}(\sqrt{\nu}),\quad \nu=\lambda\mp\mu\to 0,\quad \lambda\in\mathbb{C}\setminus\Gamma \tag{3.15}$$

in $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>3/2$.

Proof. The asymptotics (2.18) implies

$$\left.\begin{array}{l}\mathcal{R}(\lambda)=\big(1+\mathcal{R}_0(\lambda)\mathcal{V}\big)^{-1}\mathcal{R}_0(\lambda)=\big(1+\mathcal{R}_0(\lambda)\mathcal{V}\big)^{-1}\big(\mathcal{B}_0^{\pm}\frac{1}{\sqrt{\nu}}+\mathcal{O}(1)\big)\\ \\ \mathcal{R}(\lambda)=\mathcal{R}_0(\lambda)\big(1+\mathcal{V}\mathcal{R}_0(\lambda)\big)^{-1}=\big(\mathcal{B}_0^{\pm}\frac{1}{\sqrt{\nu}}+\mathcal{O}(1)\big)\big(1+\mathcal{V}\mathcal{R}_0(\lambda)\big)^{-1}\end{array}\right|\ \nu=\lambda\mp\mu\to 0,\ \ \lambda\in\mathbb{C}\backslash\Gamma$$

Hence, the boundedness $\mathcal{R}(\lambda)$, $(1+\mathcal{R}_0(\lambda)\mathcal{V})^{-1}$ and $(1+\mathcal{V}\mathcal{R}_0(\lambda))^{-1}$ at the points $\lambda=\pm\mu$ in corresponding norms imply the asymptotics (3.15). □

Corollary 3.9. *i) The asymptotics hold*

$$\|(1+\mathcal{R}_0(\lambda)\mathcal{V})^{-1}[e^{\mp i\gamma vmx}]\|_{\mathcal{F}_{-\sigma}}=\mathcal{O}(\sqrt{\nu}),\quad \nu=\lambda\mp\mu\to 0,\quad \lambda\in\mathbb{C}\setminus\Gamma,\quad \sigma>3/2 \tag{3.16}$$

ii) For any $f\in\mathcal{F}_\sigma$ with $\sigma>3/2$

$$\int e^{\pm i\gamma vmx}[(1+\mathcal{V}\mathcal{R}_0(\lambda))^{-1}f](x)dx=\mathcal{O}(\sqrt{\nu}),\quad \nu=\lambda\mp\mu\to 0,\quad \lambda\in\mathbb{C}\setminus\Gamma \tag{3.17}$$

Proof of Proposition 3.13. Taking into account the identities

$$\mathcal{R}'=(1+\mathcal{R}_0\mathcal{V})^{-1}\mathcal{R}_0'(1+\mathcal{V}\mathcal{R}_0)^{-1},\quad \mathcal{R}''=\Big[(1+\mathcal{R}_0\mathcal{V})^{-1}\mathcal{R}_0''-2\mathcal{R}'\mathcal{V}\mathcal{R}_0'\Big](1+\mathcal{V}\mathcal{R}_0)^{-1}$$

we obtain from (2.20) and (3.16)-(3.17) the asymptotics (3.13) for the derivatives. The asymptotics (3.13) for $\mathcal{R}(\lambda)$ follows by integration of asymptotics for $\mathcal{R}'(\lambda)$. Proposition 3.13 is proved.

Corollary 3.10. *Let the conditions (1.2) and (1.3) hold. Then the set Σ of eigenvalues of the operator $\mathcal{A}$ is finite, i.e. $\Sigma=\{\lambda_j,\ \ j=1,...,N\}$.*

3.3 Time decay

Our main result is

Theorem 3.11. *Let conditions (1.2) and (1.3) hold. Then*

$$\|e^{t\mathcal{A}}-\sum_{\omega_j\in\Sigma}e^{\lambda_j t}P_j\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}=\mathcal{O}(|t|^{-3/2}),\quad t\to\pm\infty \tag{3.18}$$

with $\sigma>5/2$, where P_j are the Riesz projections onto the corresponding eigenspaces.

Proof. Corollaries 3.2 and 3.4 and Proposition 3.6 imply similarly to (2.22), that

$$\Psi(t)-\sum_{\lambda_j\in\Sigma}e^{\lambda_j t}P_j\Psi_0=\frac{1}{2\pi i}\int\limits_{\Gamma}e^{\lambda t}\Big[\mathcal{R}(\lambda-0)-\mathcal{R}(\lambda+0)\Big]\Psi_0\ d\lambda=\Psi_l(t)+\Psi_h(t)$$

where $P_j\Psi_0:=\dfrac{1}{2\pi i}\displaystyle\int_{|\lambda-\lambda_j|=\delta}\mathcal{R}(\lambda)\Psi_0 d\lambda$ with a small $\delta>0$, and low and high energy components are defined by

$$\Psi_l(t)=\frac{1}{2\pi i}\int\limits_{\Gamma}l(i\lambda)e^{\lambda t}\Big[\mathcal{R}(\lambda-0)-\mathcal{R}(\lambda+0)\Big]\Psi_0\ d\lambda \tag{3.19}$$

$$\Psi_h(t)=\frac{1}{2\pi i}\int\limits_{\Gamma}h(i\lambda)e^{\lambda t}\Big[\mathcal{R}(\lambda-0)-\mathcal{R}(\lambda+0)\Big]\Psi_0\ d\lambda \tag{3.20}$$

where $l(i\lambda)$ and $h(i\lambda)$ are defined in Section 2.2. We analyze $\Psi_l(t)$ and $\Psi_h(t)$ separately.

3.3.1 Low energy component

We prove the desired decay of $\Psi_l(t)$ using a special case of Lemma 10.2 from [6]. We consider only the integral (3.19) over $(\mu, \mu+2i\varepsilon)$. The integral over $(-\mu-2i\varepsilon, -\mu)$ is dealt with in the same way. Denote by $\mathbf{B}$ a Banach space with the norm $\|\cdot\|$.

Lemma 3.12. *Let $F \in C([a,b], \mathbf{B})$, satisfy*

$$F(a) = F(b) = 0, \qquad \|F''(\omega)\| = \mathcal{O}(|\omega - a|^{-3/2}), \quad \omega \to a$$

Then

$$\int_a^b e^{-it\omega} F(\omega) d\omega = \mathcal{O}(t^{-3/2}), \quad t \to \infty$$

Due to (3.13), we can apply Lemma 3.12 with $\omega = -i\lambda$, $F = l(\omega)\big(\mathcal{R}(i\omega - 0) - \mathcal{R}(i\omega + 0)\big)$, $\mathbf{B} = \mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$, $a = |\mu|$, $b = |\mu| + 2\varepsilon$ and $\sigma > 5/2$, to get

$$\|\Psi_l(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}, \quad \sigma > 5/2$$

3.3.2 High energy component

Let us substitute the series (3.11) into the spectral representation (3.20) for $\Psi_h(t)$:

$$\begin{aligned}
\Psi_h(t) &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} h(i\lambda)\Big[\mathcal{R}_0(\lambda - 0) - \mathcal{R}_0(\lambda + 0)\Big]\Psi_0 \, d\lambda \\
&+ \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} h(i\lambda)\Big[\mathcal{R}_0(\lambda - 0)\mathcal{V}\mathcal{R}_0(\lambda - 0) - \mathcal{R}_0(\lambda + 0)\mathcal{V}\mathcal{R}_0(\lambda + 0)\Big]\Psi_0 \, d\lambda \\
&+ \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} h(i\lambda)\Big[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\lambda - 0) - \mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\lambda + 0)\Big]\Psi_0 \, d\lambda \\
&= \Psi_{h1}(t) + \Psi_{h2}(t) + \Psi_{h3}(t), \qquad t \in \mathbb{R}
\end{aligned}$$

We analyze each term Ψ_{hk}, $k = 1, 2, 3$ separately.
Step i) The first term $\Psi_{h1}(t) = \mathcal{G}_h(t)\Psi_0$ by (2.29). Hence, Theorem 2.5 implies that

$$\|\Psi_{h1}(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}, \quad \sigma > 5/2 \tag{3.21}$$

Step ii) Now we consider the second term $\Psi_{h2}(t)$. Denote $h_1(\omega) = \sqrt{h(\omega)}$ (we can assume that $h(\omega) \ge 0$ and $h_1 \in \mathbb{C}_0^\infty(\mathbb{R})$). We set

$$\Phi_{h1} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} h_1(i\lambda)\Big[\mathcal{R}_0(\lambda - 0) - \mathcal{R}_0(\lambda + 0)\Big]\Psi_0 \, d\lambda$$

It is obvious that for Φ_{h1} the inequality (3.21) also holds. Namely,

$$\|\Phi_{h1}(t)\|_{\mathcal{F}_{-\sigma}} \le C(1+|t|)^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}, \quad \sigma > 5/2$$

Further, the second term $\Psi_{h2}(t)$ can be written as a convolution.

Lemma 3.13. *(cf. [8, Lemma 3.11]) The convolution representation holds*

$$\Psi_{h2}(t)=\int\limits_0^t \mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\ d\tau, \qquad t\in\mathbb{R} \tag{3.22}$$

where the integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma>5/2$.

Applying Theorem 2.5 with h_1 instead of h to the integrand in (3.22), we obtain that

$$\|\mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_{-\sigma}}\le\frac{C\|\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_{\sigma'}}}{(1+|t-\tau|)^{3/2}}\le\frac{C\|\Phi_{h1}(\tau)\|_{\mathcal{F}_{\sigma'-\beta}}}{(1+|t-\tau|)^{3/2}}\le\frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{3/2}(1+|\tau|)^{3/2}}$$

where $\sigma'\in(5/2,\beta-5/2)$. Therefore, integrating here in τ, we obtain by (3.22) that

$$\|\Psi_{h2}(t)\|_{\mathcal{F}_{-\sigma}}\le C(1+|t|)^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\in\mathbb{R},\quad \sigma>5/2$$

Step iii) Let us rewrite the last term $\Psi_{h3}(t)$ as

$$\Psi_{h3}(t)=\frac{1}{2\pi i}\int\limits_\Gamma e^{\lambda t}h(i\lambda)\mathcal{N}(\lambda)\Psi_0\ d\lambda,$$

where $\mathcal{N}(\lambda):=\mathcal{M}(\lambda-0)-\mathcal{M}(\lambda+0)$ for $\lambda\in\Gamma$, and

$$\mathcal{M}(\lambda\pm0):=\mathcal{R}_0(\lambda\pm0)\mathcal{V}\mathcal{R}_0(\lambda\pm0)\mathcal{V}\mathcal{R}(\lambda\pm0)=\mathcal{R}_0(\lambda\pm0)\mathcal{W}(\lambda\pm0)\mathcal{R}(\lambda\pm0),\quad\lambda\in\Gamma$$

The asymptotics (2.21), (3.10) and (3.12) for $\mathcal{R}_0^{(k)}(\lambda\pm0)$, $\mathcal{R}^{(k)}(\lambda\pm0)$ and $\mathcal{W}^{(k)}(\lambda\pm0)$ imply

Lemma 3.14. *(cf.[8, Lemma 3.12]) For $k=0,1,2$ the asymptotics hold*

$$\|\mathcal{M}^{(k)}(\lambda\pm0)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}=\mathcal{O}(|\lambda|^{-2}),\quad|\lambda|\to\infty,\quad\lambda\in\Gamma,\quad\sigma>1/2+k$$

Finally, we prove the decay of $\Psi_{h3}(t)$. By Lemma 3.14

$$(h\mathcal{N})''\in L^1((-i\infty,-\mu-i\varepsilon)\cup(\mu+i\varepsilon,i\infty);\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma}))$$

with $\sigma>5/2$. Hence, two times partial integration implies that

$$\|\Psi_{h3}(t)\|_{\mathcal{F}_{-\sigma}}\le C(1+|t|)^{-2}\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\in\mathbb{R}$$

This completes the proof of Theorem 3.11. □

Corollary 3.15. *The asymptotics* (3.18) *imply* (1.4) *with the projection*

$$\mathcal{P}_c=1-\mathcal{P}_d,\quad\mathcal{P}_d=\sum_{\omega_j\in\Sigma}P_j$$

A Proof of Lemma 2.6

Formulas (2.23)- (2.24) imply

$$\mathcal{G}_v(z,t)=\tilde{\mathcal{G}}_b(z,t)+\tilde{\mathcal{G}}_r(z,t)$$

where

$$\tilde{\mathcal{G}}_b(z,t)=\frac{\theta(t-|z-vt|)}{\sqrt{2m\pi}}\begin{pmatrix}-\dfrac{mt\sin(m\sqrt{t^2-(z-vt)^2}-\frac{\pi}{4})}{\sqrt[4]{(t^2-(z-vt)^2)^3}} & \dfrac{\cos(m\sqrt{t^2-(z-vt)^2}-\frac{\pi}{4})}{\sqrt[4]{t^2-(z-vt)^2}}\\ -\dfrac{m^2t^2\cos(m\sqrt{t^2-(z-vt)^2}-\frac{\pi}{4})}{\sqrt[4]{(t^2-(z-vt)^2)^5}} & -\dfrac{mt\sin(m\sqrt{t^2-(z-vt)^2}-\frac{\pi}{4})}{\sqrt[4]{(t^2-(z-vt)^2)^3}}\end{pmatrix}$$

For $\varepsilon\in(|v|,1)$ and $|z|\le(\varepsilon-|v|)t$ we have $|z-vt|\le\varepsilon t$. Hence

$$|\partial_z^k\tilde{\mathcal{G}}_r(z,t)|\le C(\varepsilon)t^{-3/2},\quad |z|\le(\varepsilon-|v|)t,\quad k=0,1$$

by known asymptotics of the Bessel function (see [10], p.195). It remains to prove the bounds of type (2.45) for the difference $Q(z,t)=\tilde{\mathcal{G}}_b(z,t)-\mathcal{G}_b(z,t)$. Let us consider the entry $Q^{12}(t,z)$:

$$Q^{12}(t,z)=\frac{1}{\sqrt{2\pi m}}\Big[\frac{\cos(m\sqrt{t^2-(z-vt)^2}-\frac{\pi}{4})}{\sqrt[4]{t^2-(z-vt)^2}}-\frac{\cos(m(\frac{t}{\gamma}+\gamma vz)-\frac{\pi}{4})}{\sqrt{t/\gamma}}\Big]$$

For $|z|\le(\varepsilon-|v|)t$ we have

$$\Big|\frac{1}{\sqrt[4]{t^2-(z-vt)^2}}-\frac{1}{\sqrt{t/\gamma}}\Big|$$
$$=\frac{|z^2-2vtz|}{\sqrt[4]{t^2-(z-vt)^2}\sqrt{t/\gamma}\big(\sqrt[4]{t^2-(z-vt)^2}+\sqrt{t/\gamma}\big)\big(\sqrt{t^2-(z-vt)^2}+t/\gamma\big)}\le\frac{C(\varepsilon)|z|}{t\sqrt{t}}$$

Further,

$$\Big|\cos\Big(m\sqrt{t^2-(z-vt)^2}-\frac{\pi}{4}\Big)-\cos\Big(\frac{m}{\gamma}(t+\gamma^2vz)-\frac{\pi}{4}\Big)\Big|\le2\Big|\sin\Big(\frac{m}{2}(\sqrt{t^2-(z-vt)^2}-\frac{t+\gamma^2vz}{\gamma}\Big)\Big|$$
$$\le C\Big|\sqrt{t^2-(z-vt)^2}-(t+\gamma^2vz)/\gamma\Big|\le C\frac{z^2(1+\gamma^2v^2)}{|\sqrt{t^2-(z-vt)^2}+(t+\gamma^2vz)/\gamma|}\le\frac{C(\varepsilon)z^2}{t}$$

since $\gamma^2|v||z|\le(1-|v|)t/(1-v^2)\le t/(1+|v|)\le t$. Hence,

$$|Q^{12}(t,z)|\le C(\varepsilon)(1+z^2)t^{-3/2},\quad |z|\le(\varepsilon-|v|)t\tag{A.23}$$

Differentiating $Q^{12}(t,z)$, we obtain for $|z|\le(\varepsilon-|v|)t$

$$\partial_zQ^{12}(t,z)=\frac{z-vt}{\sqrt{2\pi m}}\,\frac{\cos(m\sqrt{t^2-(z-vt)^2}-\frac{\pi}{4})}{2\sqrt[4]{(t^2-(z-vt)^2)^5}}+\sqrt{\frac{m}{2\pi}}\frac{z\sin(m\sqrt{t^2-(z-vt)^2}-\frac{\pi}{4})}{\sqrt[4]{(t^2-(z-vt)^2)^3}}$$
$$+\sqrt{\frac{m}{2\pi}}vt\Big[\frac{-\sin(m\sqrt{t^2-(z-vt)^2}-\frac{\pi}{4})}{\sqrt[4]{(t^2-(z-vt)^2)^3}}+\frac{\sin(m(\frac{t}{\gamma}+\gamma vz)-\frac{\pi}{4})}{\sqrt{(t/\gamma)^3}}\Big]$$

Hence, by the arguments above,

$$|\partial_zQ^{12}(t,z)|\le C(\varepsilon)(1+z^2)\,t^{-3/2},\quad |z|\le(\varepsilon-|v|)t\tag{A.24}$$

Other entries $Q^{ij}(t,z)$ also admit the estimates of type (A.23) and (A.24). Hence, the lemma follows since $\mathcal{G}_r(t)=\tilde{\mathcal{G}}_r(t)+Q(t,z)$.

References

[1] Agmon S., Spectral properties of Schrödinger operator and scattering theory, *Ann. Scuola Norm. Sup. Pisa*, Ser. IV **2**, 151-218 (1975).

[2] Bleher P.M., On operators depending meromorphically on a parameter, *Moscow Univ. Math. Bull.* **24**, 21-26 (1972).

[3] Gohberg I.C., Krein M.G., Introduction to the Theory of Linear Nonselfadjoint Operators, American Mathematical Society, Providence, RI, 1969.

[4] Jensen A., Spectral properties of Schrödinger operators and time-decay of the wave function. Results in $L^2(\mathbb{R}^m)$, $m \geq 5$, *Duke Math. J.* **47**, 57-80 (1980).

[5] Jensen A., Spectral properties of Schrödinger operators and time-decay of the wave function. Results in $L^2(\mathbb{R}^4)$, *J. Math. Anal. Appl* **101**, 491-513 (1984).

[6] Jensen A., Kato T., Spectral properties of Schrödinger operators and time-decay of the wave functions, *Duke Math. J.* **46**, 583-611 (1979).

[7] Jensen A., Nenciu G., A unified approach to resolvent expansions at thresholds, *Rev. Math. Phys.* **13**, no.6, 717-754 (2001).

[8] Komech A., Kopylova E., Weighted energy decay for 1D Klein-Gordon equation, *Comm. PDE* **35**, no.2, 353-374 (2010).

[9] Murata M., Asymptotic expansions in time for solutions of Schrödinger-type equations, *J. Funct. Anal.* **49**, 10-56 (1982).

[10] Watson G.N., Bessel Functions, Cambridge, 1922.