On asymptotic stability of kink for relativistic Ginzburg-Landau equation

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Abstract

We prove the asymptotic stability of standing kink for the nonlinear relativistic wave equations of the Ginzburg-Landau type in one space dimension: for any odd initial condition in a small neighborhood of the kink, the solution, asymptotically in time, is the sum of the kink and dispersive part described by the free Klein-Gordon equation. The remainder converges to zero in a global norm. Crucial role in the proofs play our recent results on the weighted energy decay for the Klein-Gordon equations.

Keywords: Relativistic nonlinear wave equation, asymptotic stability, kink, weighted energy decay, symplectic projection, modulation equations, Fermi Golden Rule.

2000 Mathematics Subject Classification: 35Q51, 37K40.

[1]Supported partly by the grants of DFG, FWF and RFBR.
[2]Supported partly by the Alexander von Humboldt Research Award.

1 Introduction

We prove the asymptotic stability of kinks for relativistic nonlinear wave equations with two-well potentials of Ginzburg-Landau type. The work is inspired by the problem of stability of elementary particles which are modeled as solitary waves of the equations. We consider the equation

$$\ddot{\psi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R} \tag{1.1}$$

where $\psi(x,t)$ is a real solution, and $F(\psi) = -U'(\psi)$. We consider the potentials $U(\psi)$ similar to the Ginzburg-Landau potential $U_0(\psi) = (\psi^2-1)^2/4$ which correpsonds to the cubic equation with $F(\psi) = \psi - \psi^3$.

Condition U1. *The potential $U(\psi)$ is a real smooth even function which satisfies the following conditions with some $a, m > 0$ and sufficienly large $k > 0$,*

$$U(\psi) > 0 \;\text{ for }\; \psi \neq \pm a \tag{1.2}$$

$$U(\psi) = \frac{m^2}{2}(\psi \mp a)^2 + \mathcal{O}(|\psi \mp a|^{2k}), \qquad x \to \pm a \tag{1.3}$$

In the vector form, equation (1.1) reads

$$\begin{cases} \dot{\psi}(x,t) = \pi(x,t) \\ \dot{\pi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R} \end{cases} \tag{1.4}$$

Formally it is a Hamiltonian system with the Hamilton functional

$$\mathcal{H}(\psi,\pi) = \int \Big[\frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + U(\psi(x))\Big]\, dx \tag{1.5}$$

The corresponding stationary equation reads

$$s'' - U'(s) = 0 \tag{1.6}$$

The constant solutions of the stationary equation are: $\psi \equiv \pm a$ - stable stationary solutions, and $\psi \equiv 0$ – unstable stationary solution. There is also a "kink", i.e. an odd nonconstant finite energy solution $s(x)$ to (1.6) such that

$$s(0) = 0, \quad s(x) \to \pm a \;\text{ as }\; x \to \pm\infty \tag{1.7}$$

The condition **U1** implies that $(s(x) \mp a)'' \sim m^2(s(x) \mp a)$ for $x \to \pm\infty$, hence

$$s(x) \mp a \sim Ce^{-m|x|}, \quad x \to \pm\infty \tag{1.8}$$

The generator of linearized equation near the kink reads (see Section 2)

$$A = \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix}$$

where H is the Schrödinger operator

$$H = -\frac{d^2}{dx^2} - F'(s) = -\frac{d^2}{dx^2} + m^2 + V(x), \quad V(x) = -F'(s(x)) - m^2 = U''(s(x)) - m^2 \quad (1.9)$$

By (1.8), we have

$$V(x) \sim C(s(x) \mp a)^{2k-2} \sim Ce^{-(2k-2)m|x|}, \quad x \to \pm\infty. \quad (1.10)$$

The continuous spectrum of the operator H is $\mathrm{Spec}_c H = [m^2, \infty)$. By technical reasons, we restrict in this paper to odd solutions $\psi(-x,t) = -\psi(x,t)$. We assume the following spectral conditions:

Condition U2. *The discrete spectrum of the operator H restricted to the subspace of odd functions consists only of one simple eigenvalue*

$$\lambda_1 < m^2, \qquad 4\lambda_1 > m^2 \quad (1.11)$$

We also consider the edge point $\lambda = m^2$ of the continuous spectrum and assume that

$$\lambda = m^2 \text{ is not eigenvalue nor resonance for the Schrödinger operator } H \quad (1.12)$$

We assume also a non-degeneracy condition "Fermi Golden Rule" introduced by Sigal [25]. The condition provides a strong coupling of the nonlinear term with the eigenfunctions of the continuous spectrum and the energy radiation.

Condition U3. *The non-degeneracy condition holds (cf. condition (1.0.11) in [3])*

$$\int_0^\infty \varphi_{4\lambda_1}(x) F''(s(x)) \varphi^2_{\lambda_1}(x) dx \neq 0 \quad (1.13)$$

where $\varphi_{\lambda_1}(x)$ and $\varphi_{4\lambda_1}(x)$ are the odd eigenfunctions of discrete and continuous spectrum corresponding to λ_1 and $4\lambda_1$ respectively.

The Ginzburg-Landau potential $U_0(\psi) = (\psi^2 - 1)^2/4$ satisfies all the conditions **U1–U3** except (1.3). In Appendix C we construct small perturbations of the Ginzburg-Landau potential which satisfy all the conditions **U1–U3** including (1.3).

Our main results are the following asymptotics

$$(\psi(x,t), \dot\psi(x,t)) \sim (s(x), 0) + W_0(t)\Phi_\pm, \quad t \to \pm\infty \quad (1.14)$$

for solutions to (1.4) with odd initial data close to the kink $S(x) = (s(x), 0)$. Here $W_0(t)$ is the dynamical group of the free Klein-Gordon equation, $\Phi_\pm$ are the corresponding asymptotic states, and the remainder converges to zero $\sim t^{-1/3}$ in the "global energy norm" of the Sobolev space $H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$. We consider the odd initial data to fix the limit standing kink: otherwise, the asymptotic holds with a moving kink that we will consider elsewhere.

Remark 1.1. *We consider the solutions close to the kink, $\psi(x,t) = s(x) + \phi(x,t)$, with small perturbations $\phi(x,t)$. For such solution the condition (1.3) and the asymptotics (1.8) mean that the equation (1.1) is almost linear for large $|x|$. This fact is helpful for application of the dispersive properties of the corresponding linearized equation.*

Let us comment on previous results in this field.

• *The Schrödinger equation* The asymptotics of type (1.14) were established for the first time by Soffer and Weinstein [26, 27] (see also [21]) for nonlinear $U(1)$-invariant Schrödinger equation with a potential for small initial states if the nonlinear coupling constant is sufficiently small.

The results have been extended by Buslaev and Perelman [1] to the translation invariant 1D nonlinear $U(1)$-invariant Schrödinger equation. The initial states are sufficiently close to the solitary waves with the unique eigenvalue $\lambda = 0$ in the dicrete spectrum of the corresponding linearized dynamics. The novel techniques [1] are based on the "separation of variables" along the solitary manifold and in transversal directions. The symplectic projection allows to exclude from the transversal dynamics the unstable directions corresponding to the zero discrete spectrum of the linearized dynamics. The extensions to higher dimensions were obtained in [4, 14, 24, 31].

Similar techniques were developed by Miller, Pego and Weinstein for the 1D modified KdV and RLW equations, [19, 20]. These techniques were motivated by the investigation of soliton asymptotics for integrable equations (a survey can be found in [8] and [9]), and by the methods introduced in [26, 27, 33].

The techniques were developed in [2, 3] for the Schrödinger equations in more complicated spectral situation with presence of a nonzero eigenvalue in the linearized dynamics. In that case the transversal dynamics inherits the nonzero discrete spectrum. Now the decay for the transversal dynamics is obtained by the reduction to the Poincaré normal form which makes obvious that the decay depends on the Fermi Golden Rule condition [18, 25]. The condition states a strong interaction of the nonlinear term with the eigenfunctions of the continuous spectrum which provides the dispersive energy radiation to infinity and the decay for the transversal dynamics. The extension to higher dimensions were obtained in [5, 6, 29]. Tsai [32] developed the techniques in presence of an arbitrary finite number of discrete eigenvalues in the linearized dynamics.

• *Nonrelativistic Klein-Gordon equations* The asymptotics of type (1.14) were extended to the nonlinear 3D Klein-Gordon equations with a potential [28], and for translation invariant system of the 3D Klein-Gordon equation coupled to a particle [13].

• *Wave front of 3D Ginzburg-Landau equation* The asymptotic stability of wave front were proved for 3D relativistic Ginzburg-Landau equation with initial data which differ from the wave front on a compact set [7]. The equation differs from the 1D equation (1.1) by the additional 2D Laplacian. The additional Laplacian improves the dispersive decay for the corresponding linearized Klein-Gordon equation in the continuous spectral space that provides the needed decay for the transversal dynamics.

• *Orbital stability of the kinks* For 1D relativistic nonlinear Ginzburg-Landau equations (1.1) the orbital stability of the kinks has been proved in [12].

The proving of the asymptotic stability of the kinks for relativistic equations remained an open problem till now. Main obstacle was the slow decay $\sim t^{-1/2}$ for the free 1D Klein-Gordon equation (see the discussion in [7, Introduction]).

Let us comment on our approach. We follow general strategy of [1, 2, 3, 7, 4, 5, 6, 13, 28, 31, 32]: linearization of the transversal equations and further Taylor expansion of the nonlinearity, the Poincaré normal forms and Fermie Golden Rule, etc. We develop for relativistic equations general scheme which is common in almost all papers in this area: dispersive and $L^1 - L^\infty$ estimates for the linearized equation, virial estimates for the nonlinear equation, and method of majorants. However, the corresponding statements and their proofs in the context of relativistic equations are completely new. Let us comment on our novel techniques.

i) The slow decay $\sim t^{-1/2}$ for the free 1D Klein-Gordon equation corresponds to the presence of the resonances at the ends of the continuous spectrum. We overcome the difficulty developing our recent result [15] *identifying* the slow decaying component with the contribution of the resonances of the free 1D Klein-Gordon equation. More precisely, we prove that the contribution of the high energy spectrum decays like $\sim t^{-3/2}$, in the weighted energy norms. This result plays the crucial role in our paper, and provides the decay $\sim t^{-3/2}$ for the transversal linearized dynamics since the end points of continuous spectrum are not resonances in our case due to the antisymmetry of the solutions.

ii) The "virial type" estimate (H. 2) for the nonlinear Ginzburg-Landau equation (1.1) is novel relativistic version of the bound [3, (1.2.5)] for the nonlinear Schrödinger equations;

iii) We give the complete proof of the dispersive estimate (3.40);

iv) We establish an appropriate relativistic version (3.36) of $L^1 \to L^\infty$ estimates;

v) We prove novel optimal decay estimate (7.132) for the dynamical group of the free 1D Klein-Gordon equation;

vi) We give the complete proof of the soliton asymptotics (1.14). In the context of the Schrödinger equation, the proof of the corresponding asymptotics were sketched in [3].

vii) Finally, we construct the examples of the potentials satisfying all our spectral conditions including the Fermie Golden Rule. The examples were never constructed in all previous papers in this area.

Our paper is organized as follows. In Section 2 we formulate the main theorem. The linearization at the kink is carried out in Section 3. In Section 4 we derive the dynamical equations for the "discrete" and "continuous" components of the solution. In Section 5 we transform the dynamical equations to a Poincare "normal form". We apply the method of majorants in Section 6. Finally, in Section 7 we obtain the soliton asymptotics (1.14).

In Appendices A and B we prove the key estimates (H. 2) and (3.40). In Appendix C we construct the examples of the potentials.

Acknowledgements The authors thank V.S. Buslaev, H. Spohn, and M.I. Vishik for fruitful discussions.

2 Main results

We consider the Cauchy problem for the Hamilton system (2.15) which we write as

$$\dot{Y}(t) = \mathcal{F}(Y(t)), \quad t \in \mathbb{R}: \quad Y(0) = Y_0. \tag{2.15}$$

Here $Y(t) = (\psi(t), \pi(t))$, $Y_0 = (\psi_0, \pi_0)$, and all derivatives are understood in the sense of distributions. To formulate our results precisely, let us first we introduce a suitable phase space for the Cauchy problem (2.15). We will consider only odd states $Y = (\psi, \pi)$:

$$\psi(-x) = -\psi(x), \qquad \pi(-x) = -\pi(x), \qquad x \in \mathbb{R} \tag{2.16}$$

The space of the odd states is invariant with respect to dynamical equation (1.1) since the potential $U(\psi)$ is even function according to (U), and hence $F(\psi)$ is the odd function.

For $\sigma \in \mathbb{R}$, and $l = 0, 1, 2, ..., p \geq 1$, let us denote by $W^{p,l}_\sigma$, the weighted Sobolev space of the odd functions with the finite norm

$$\|\psi\|_{W^{p,l}_\sigma} = \sum_{k=0}^{l} \|(1 + |x|)^\sigma \psi^{(k)}\|_{L^p} < \infty$$

and $H^l_\sigma := W^{2,l}_\sigma$, so $H^0_\sigma = L^2_\sigma$.

Definition 2.1. *i)* $E_\sigma := H^1_\sigma \oplus L^2_\sigma$ *is the space of the odd states* $Y = (\psi, \pi)$ *with finite norm*

$$\| Y\|_{E_\sigma} = \|\psi\|_{H^1_\sigma} + \|\pi\|_{L^2_\sigma} \tag{2.17}$$

ii) The phase space $\mathcal{E} := S + E$, *where* $E = E_0$ *and* $S = (s(x), 0)$. *The metric in* $\mathcal{E}$ *is defined as*

$$\rho_{\mathcal{E}}(Y_1, Y_2) = \|Y_1 - Y_2\|_E, \quad Y_1, Y_2 \in \mathcal{E} \tag{2.18}$$

iii) $W := W^{1,2}_0 \oplus W^{1,1}_0$ *is the space of the odd states* $Y = (\psi, \pi)$ *with finite norm*

$$\| Y\|_W = \|\psi\|_{W^{1,2}_0} + \|\pi\|_{W^{1,1}_0} \tag{2.19}$$

Obviously, the Hamilton functional (1.5) is continuous on the phase space $\mathcal{E}$. The existence and uniqueness of the solutions to the Cauchy problem (2.15) follows by methods [17, 22, 30]:

Proposition 2.2. *i) For any initial data* $Y_0 \in \mathcal{E}$ *there exists the unique solution* $Y(t) \in C(\mathbb{R}, \mathcal{E})$ *to the problem (2.15).*
(ii) For every $t \in \mathbb{R}$, *the map* $U(t) : Y_0 \mapsto Y(t)$ *is continuous in* $\mathcal{E}$.
(iii) The energy is conserved, i.e.

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0), \quad t \in \mathbb{R} \tag{2.20}$$

The main result of our paper is the following theorem

Theorem 2.3. *Let the potential* U *satisfy the conditions (U1)-(U3) with* $k = 7$, *and let* $Y(t)$ *be the solution to the Cauchy problem (2.15) with any initial state* $Y_0 \in \mathcal{E}$ *which is sufficiently close to the kink:*

$$Y_0 = S + X_0, \quad d_0 := \|X_0\|_{E_\sigma \cap W} \ll 1 \tag{2.21}$$

where $\sigma > 5/2$. *Then the asymptotics hold*

$$Y(x,t) = (s(x), 0) + W_0(t)\Phi_\pm + r_\pm(x,t), \quad t \to \pm\infty \tag{2.22}$$

where $\Phi_\pm \in E$, *and* $W_0(t) = e^{A_0 t}$ *is the dynamical group of the free Klein-Gordon equation (see (3.38), while*

$$\|r_\pm(t)\|_E = \mathcal{O}(|t|^{-1/3}) \tag{2.23}$$

It suffices to prove the asymptotics (2.22) for $t \to +\infty$ since the system (1.4) is time reversible.

3 Linearization at the kink

3.1 Linearized equation

Let us linearize the system (1.4) at the kink $S(x)$ splitting the solution as the sum

$$Y(t) = S + X(t), \tag{3.24}$$

In detail, denote $Y = (\psi, \pi)$ and $X = (\Psi, \Pi)$. Then (3.24) means that

$$\left\{\begin{array}{rcl} \psi(x,t) & = & s(x) + \Psi(x,t) \\ \pi(x,t) & = & \Pi(x,t) \end{array}\right. \tag{3.25}$$

Let us substitute (3.25) to (1.4), and linearize the equations in X. First,

$$\left\{\begin{array}{rcl} \dot\Psi(x,t) & = & \Pi(x,t) \\ \\ \dot\Pi(x,t) & = & s''(x) + \Psi''(x,t) + F(s(x) + \Psi(x,t)) \end{array}\right. \tag{3.26}$$

Second, by (1.6) we can write the equations (3.26) as

$$\dot X(t) = AX(t) + \mathcal{N}(X(t)), \ \ t \in \mathbb{R} \tag{3.27}$$

where $\mathcal{N}(X)$ is at least quadratic in X. The linear operator A is

$$A = \left(\begin{array}{cc} 0 & 1 \\ -H & 0 \end{array}\right) \tag{3.28}$$

where H is the Schrödinger operator

$$H = -\frac{d^2}{dx^2} - F'(s) = -\frac{d^2}{dx^2} + m^2 + V(x), \qquad V(x) = -F'(s(x)) - m^2 = U''(s(x)) - m^2 \tag{3.29}$$

Finally, $\mathcal{N}(X)$ in (3.27) is given by

$$\mathcal{N}(x, X) = \left(\begin{array}{c} 0 \\ N(x,\Psi) \end{array}\right), \quad N(x,\Psi) = F(s(x) + \Psi) - F(s(x)) - F'(s(x))\Psi \tag{3.30}$$

3.2 Spectrum of linearized equation

Let us consider the eigenvalue problem for the operator (3.28):

$$A \left(\begin{array}{c} u_1 \\ u_2 \end{array}\right) = \left(\begin{array}{cc} 0 & 1 \\ -H & 0 \end{array}\right) \left(\begin{array}{c} u_1 \\ u_2 \end{array}\right) = \Lambda \left(\begin{array}{c} u_1 \\ u_2 \end{array}\right)$$

The first equation implies that $u_2 = \Lambda u_1$. Then u_1 satisfies the equation

$$\Big(H + \Lambda^2\Big) u_1 = 0 \tag{3.31}$$

Hence, Condition (U2) implies that the operator A have two purely imaginary eigenvalues $\Lambda = \pm i\mu$ where $\mu = \sqrt{\lambda_1}$. The corresponding eigenvectors

$$u = \left(\begin{array}{c} u_1 \\ u_2 \end{array}\right) = \left(\begin{array}{c} \varphi_{\lambda_1} \\ i\mu\varphi_{\lambda_1} \end{array}\right), \quad \overline{u} = \left(\begin{array}{c} \varphi_{\lambda_1} \\ -i\mu\varphi_{\lambda_1} \end{array}\right)$$

where we choose φ_{λ_1} to be real function. This is possible since H is a differential operator with real coefficients. The continuous spectrum of the operator A coincides with $\mathcal{C} := (-i\infty, -im] \cup [im, i\infty)$. The end points $\Lambda = \pm im$ are not eigenvalues nor resonances for the operator A by Condition **U2**.

3.3 Decay for the linearized dynamics

Let us consider the linearized equation

$$\dot{X}(t) = AX(t), \;\; t \in \mathbb{R} \tag{3.32}$$

where A is given in (3.28) with V is defined in (3.29). Let $\langle \cdot, \cdot \rangle$ be the scalar product in $L^2(\mathbb{R}, \mathbb{C}^2)$. Denote by P^d the symplectic projector onto the eigenspace E^d generated by u and $\overline{u}$:

$$P^d X = \frac{\langle X, ju \rangle}{\langle u, ju \rangle} u + \frac{\langle X, j\overline{u} \rangle}{\langle \overline{u}, j\overline{u} \rangle} \overline{u}, \quad X \in E_\sigma, \quad \sigma \in \mathbb{R} \tag{3.33}$$

where

$$j = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right) \tag{3.34}$$

Note, that $P^d X$ is real for real X. Let $P^c = 1 - P^d$ be the projector onto the continuous spectrum of operator A, and by E^c the continuous spectral subspace.

Next decay estimates will play the key role in our proofs. The first estimate follows from our assumption **U2** by Theorem 3.9 of [15] since the condition of type [15, (3.1)] holds in our case in the class of the odd functions (2.16).

Proposition 3.1. *Let the condition* **U2** *hold, and $\sigma > 5/2$. Then for any $X \in E_\sigma$ the weighted energy decay holds*

$$\|e^{At} P^c X\|_{E_{-\sigma}} \leq C(1+t)^{-3/2} \|X\|_{E_\sigma}, \; t \in \mathbb{R} \tag{3.35}$$

Corollary 3.2. *For $\sigma > 5/2$ we have for $X \in E_\sigma \cap W$*

$$\|(e^{At} P^c X)_1\|_{L^\infty} \leq C(1+t)^{-1/2} (\|X\|_W + \|X\|_{E_\sigma}), \; t \in \mathbb{R} \tag{3.36}$$

Here $(\cdot)_1$ stands for the first component of the vector function.

Proof. Let us apply the projector P^c to both sides of (3.32):

$$P^c \dot{X} = AP^c X = A_0 P^c X - \mathcal{V} P^c X \tag{3.37}$$

where

$$A_0 = \left(\begin{array}{cc} 0 & 1 \\ \frac{d^2}{dx^2} - m^2 & 0 \end{array}\right), \quad \mathcal{V} = \left(\begin{array}{cc} 0 & 0 \\ V & 0 \end{array}\right) \tag{3.38}$$

Hence, the Duhamel representation gives,

$$e^{At}P^cX = e^{A_0t}P^cX - \int_0^t e^{A_0(t-\tau)}\mathcal{V}e^{A\tau}P^cXd\tau, \quad t>0. \tag{3.39}$$

Applying estimate (265) from [23], the Hölder inequality and Proposition 3.1 we obtain

$$\|(e^{At}P^cX)_1\|_{L^\infty} \le C(1+t)^{-1/2}\|P^cX\|_W + C\int_0^t (1+t-\tau)^{-1/2}\|V(e^{A\tau}P^cX)_1\|_{W_0^{1,1}}\, d\tau$$

$$\le C(1+t)^{-1/2}\|X\|_W + C\int_0^t (1+t-\tau)^{-1/2}\|e^{A\tau}P^cX\|_{E_{-\sigma}}\, d\tau$$

$$\le C(1+t)^{-1/2}\|X\|_W + C\int_0^t (1+t-\tau)^{-1/2}(1+\tau)^{-3/2}\|X\|_{E_\sigma}\, d\tau \le C(1+t)^{-1/2}(\|X\|_W + \|X\|_{E_\sigma})$$

□

Proposition 3.3. *For $\sigma > 5/2$ the bounds hold*

$$\|e^{At}(A \mp 2i\mu - 0)^{-1}P^cX\|_{E_{-\sigma}} \le C(1+t)^{-3/2}\|X\|_{E_\sigma},\ t>0 \tag{3.40}$$

We will prove the proposition in Appendix B.

4 Decomposition of the dynamics

We decompose the solution to (2.15) as $Y(t) = S + X(t)$, where $X(t) = w(t) + f(t)$ with $w(t) = z(t)u + \overline{z}(t)\overline{u} \in E^d$ and $f(t) \in E^c$.

Lemma 4.1. *Let $Y(t) = S + w(t) + f(t)$ be a solution to the Cauchy problem (2.15). Then the functions $z(t)$ and $f(t)$ satisfy the equations*

$$(\dot{z} - i\mu z)\langle u, ju\rangle = \langle \mathcal{N}, ju\rangle \tag{4.41}$$

$$\dot{f} = Af + P^c\mathcal{N} \tag{4.42}$$

with $\mathcal{N}$ defined in (3.30).

Proof. Applying the projector P^d to the equation (3.27), we obtain

$$\dot{z}u + \dot{\overline{z}}\,\overline{u} = Aw + P^d\mathcal{N}. \tag{4.43}$$

Using $\langle \overline{u}, ju\rangle = 0$ and $Aw = i\mu(zu - \overline{z}\,\overline{u})$, we get equation (4.41), after taking the scalar product of equation (4.43) with ju since $(P^d)^*j = jP^d$. Applying P^c to (3.27), we obtain (4.42) since P^c commutes with A. □

Remark 4.2. *In the remaining part of the paper we shall prove the following asymptotics*

$$\|f(t)\|_{E_{-\sigma}} \sim t^{-1}, \quad z(t) \sim t^{-1/2}, \quad \|f_1(t)\|_{L^\infty} \sim t^{-1/2}, \quad t \to \infty \tag{4.44}$$

To justify these asymptotics, we will single out leading terms in right hand side of the equations (4.41)-(4.42). Namely, we shall expand the expressions for $\dot{z}$ up to terms of the order $\mathcal{O}(t^{-3/2})$, and for $\dot{f}$ up to $\mathcal{O}(t^{-1})$ keeping in mind the asymptotics (4.44). This choice allow us to obtain the uniform bounds using the method of majorants.

Now let us expand $N(x,\Psi)$ from (3.30) in the Taylor series

$$N(x,\Psi) = N_2(x,\Psi) + ... + N_{12}(x,\Psi) + N_R(x,\Psi) \tag{4.45}$$

where

$$N_j(x,\Psi) = \frac{F^{(j)}(s(x))}{j!}\Psi^j, \quad j = 2,...,12 \tag{4.46}$$

and N_R is the remainder. Condition **U1** implies that $F(\psi) = -m^2(\psi \mp a) + \mathcal{O}(|\psi \mp a|^{2k-2})$ as $\psi \to \pm a$ where $k = 7$ by our assumption. Hence, the functions $N_j(x,\Psi)$ with $j \le 12$ decrease exponentially as $|x| \to \infty$ by (1.8), while for the remainder N_R we have

$$|N_R| = \mathcal{R}(|\Psi|)|\Psi|^{13} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})|\Psi|^{13} \tag{4.47}$$

where $\mathcal{R}(A)$ is a general notation for a positive function which remains bounded as A is sufficiently small.

Let us define $\mathcal{N}_2[X_1,X_2] = (0, N_2[\Psi_1,\Psi_2])$ and $\mathcal{N}_3[X_1,X_2,X_3] = (0, N_3[\Psi_1,\Psi_2,\Psi_3])$ as the symmetric bilinear and trilinear forms with

$$N_2[\Psi_1,\Psi_2] = \frac{F''(s)}{2}\Psi_1\Psi_2, \quad N_3[\Psi_1,\Psi_2,\Psi_3] = \frac{F'''(s)}{6}\Psi_1\Psi_2\Psi_3 \tag{4.48}$$

4.1 Leading term in $\dot{z}$

Let us rewrite (4.41) in the form:

$$\dot{z} - i\mu z = \frac{\langle \mathcal{N}, ju\rangle}{\langle u, ju\rangle} = \frac{\langle \mathcal{N}_2[w,w] + 2\mathcal{N}_2[w,f] + \mathcal{N}_3[w,w,w], ju\rangle}{\langle u, ju\rangle} + Z_R \tag{4.49}$$

where

$$|Z_R| = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^2 + \|f\|_{E_{-\sigma}})^2 \tag{4.50}$$

Note that

$$\mathcal{N}_2[w,w] = (z^2 + 2z\overline{z} + \overline{z}^2)\mathcal{N}_2[u,u], \quad \mathcal{N}_3[w,w,w] = (z^3 + 3z^2\overline{z} + 3z\overline{z}^2 + \overline{z}^3)\mathcal{N}_3[u,u,u] \tag{4.51}$$

Hence, (4.49) reads

$$\dot{z} = i\mu z + Z_2(z^2 + 2z\overline{z} + \overline{z}^2) + Z_3(z^3 + 3z^2\overline{z} + 3z\overline{z}^2 + \overline{z}^3) + (z + \overline{z})\langle f, jZ_1'\rangle + Z_R \tag{4.52}$$

where, by (4.48),

$$Z_2 = \frac{\langle \mathcal{N}_2[u,u], ju\rangle}{\langle u, ju\rangle}, \quad Z_3 = \frac{\langle \mathcal{N}_3[u,u,u], ju\rangle}{\langle u, ju\rangle}, \quad Z_1' = 2\frac{\mathcal{N}_2[u,u]}{\langle u, ju\rangle} \tag{4.53}$$

4.2 Leading term in $\dot{f}$

We now turn to equation (4.42) which we rewrite in the form

$$\dot{f} = Af + P^c\mathcal{N} = Af + P^c\mathcal{N}_2[w,w] + F_R \tag{4.54}$$

The remainder $F_R = F_R(x,t)$ reads

$$F_R = P^c(\mathcal{N}(X) - \mathcal{N}_2[w,w]) = (1-P^d)(\mathcal{N}(X) - \mathcal{N}_2[w,w]) = F_I + F_{II} + F_{III} \tag{4.55}$$

Here

$$F_I = -P^d(\mathcal{N}(X) - \mathcal{N}_2[w,w]), \quad F_{II} = \mathcal{N}(X) - \mathcal{N}_2[w,w] - \mathcal{N}_R, \quad F_{III} = \mathcal{N}_R \tag{4.56}$$

where $\mathcal{N}_R = (0, N_R)$ with N_R defined in (4.45). For F_I and F_{II} the following bound holds

$$\|F_I + F_{II}\|_{E_\sigma \cap W} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^3 + |z|\|f\|_{E_{-\sigma}} + \|f_1\|_{L^\infty}\|f\|_{E_{-\sigma}}) \tag{4.57}$$

Indeed, F_I admits the estimate by (3.33) since the function $u(x)$ decays exponentially. Further,

$$(F_{II})_1 = 0, \quad (F_{II})_2 = 2N_2(w_1, f_1) + N_2(f_1, f_1) + N_3(\Psi) + ... + N_{12}(\Psi)$$

and each summand contains an exponentially decreasing factor by **U1**, (1.8) and (4.46). Similarly,

$$\|P^c\mathcal{N}_2[w,w]\|_{E_\sigma \cap W} \le C|z|^2 \tag{4.58}$$

It remains to estimate the term F_{III}.

Lemma 4.3. *For $0 < \nu < 1/2$, the term $F_{III} = \mathcal{N}_R = (0, N_R)$ admits the estimate*

$$\|F_{III}\|_{E_{5/2+\nu}} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(1+t)^{4+\nu}(|z|^{12} + \|f_1\|_{L^\infty}^{12}) \tag{4.59}$$

Proof. Estimate (4.59) means that

$$\|N_R\|_{L^2_{5/2+\nu}} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(1+t)^{4+\nu}(|z|^{12} + \|f_1\|_{L^\infty}^{12}) \tag{4.60}$$

By (4.47), we have

$$\|N_R\|_{L^2_{5/2+\nu}} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^{12} + \|f_1\|_{L^\infty}^{12})\|\Psi\|_{L^2_{5/2+\nu}}$$

We will prove in Appendix A that

$$\|\Psi(t)\|_{L^2_{5/2+\nu}} \le C(d_0)(1+t)^{4+\nu} \tag{4.61}$$

Then (4.60) follows. □

Lemma 4.4. *The bound holds*

$$\|F_{III}\|_W = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^{10} + \|f_1\|_{L^\infty}^{10}) \tag{4.62}$$

Proof. *Step i)* By the Cauchy formula,

$$\tilde{N}_R(x,t) = N_{12}(x,t) + N_R(x,t) = \frac{\Psi^{12}(x,t)}{(12)!}\int_0^1 (1-\rho)^{11} F^{(12)}(s+\rho\Psi(x,t))d\rho$$

Therefore,

$$\|\tilde{N}_R\|_{L^1} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})\int|\Psi|^{12}dx = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z| + \|f_1\|_{L^\infty})^{10}\|\Psi\|^2_{L^2}$$
$$= \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^{11} + \|f_1\|^{10}_{L^\infty})$$

since $\|\Psi(t)\|_{L^2} \le C(d_0)$ by the results of [12].
Step ii) Further,

$$\tilde{N}'_R = \frac{\Psi^{12}}{(12)!}\int_0^1 (1-\rho)^{11}(s' + \rho\Psi')F^{(13)}(s+\rho\Psi)d\rho + \frac{\Psi^{11}\Psi'}{(11)!}\int_0^1 (1-\rho)^{11}F^{(12)}(s+\rho\Psi)d\rho$$

Therefore,

$$\|\tilde{N}'_R\|_{L^1} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z| + \|f_1\|_{L^\infty})^{10}\int|\Psi(x)\Psi'(x)|dx \le \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^{10} + \|f_1\|^{10}_{L^\infty})$$

since $\int|\Psi(x)\Psi'(x)|dx \le \|\Psi\|_{L^2}\|\Psi'\|_{L^2} \le C(d_0)$. Finally, let us note that

$$\|N_{12}\|_{W_0^{1,1}} \le \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^{10} + \|f_1\|^{10}_{L^\infty})$$

□

5 Poincare normal forms

Our goal is to transform the evolution equations for z and f to a "normal form" removing the "nonresonant terms".

5.1 Normal form for $\dot{f}$

We rewrite (4.54) in a more detailed form as

$$\dot{f} = Af + (z^2 + 2z\overline{z} + \overline{z}^2)F_2 + F_R, \quad F_2 = P^c\mathcal{N}_2[u,u]. \tag{5.63}$$

We want to extract from f the term of order $z^2 \sim t^{-1}$ (see Remark 4.2). For this purpose we expand f as

$$f = h + k + g, \tag{5.64}$$

where

$$g(t) = -e^{At}k(0), \quad k = a_{20}z^2 + 2a_{11}z\overline{z} + a_{02}\overline{z}^2, \tag{5.65}$$

with some $a_{ji} \equiv a_{ij}(x)$ satisfying $a_{ij}(x) = \overline{a}_{ij}(x)$. Note that $h(0) = f(0)$.

Lemma 5.1. *There exist the coefficients* $a_{ij} \in H^s_{-\sigma}$ *with any* $s > 0$ *such that the equation for* h_1 *has the form*

$$\dot{h} = Ah + H_R \qquad (5.66)$$

where

$$H_R = F_R + H_I, \quad \text{with} \quad H_I = \sum a_{ij}(x)\mathcal{R}(|z| + \|f\|_{E_{-\sigma}})|z|(|z| + \|f\|_{E_{-\sigma}})^2$$

Proof. Substituting (5.65) into (5.63), we get

$$\begin{array}{rcl} \dot{h} &=& \dot{f} - (2a_{20}z + 2a_{11}\overline{z})\dot{z} - (2a_{11}z + 2a_{02}\overline{z})\dot{\overline{z}} - \dot{g} \\ &=& Af + (z^2 + 2z\overline{z} + \overline{z}^2)F_2 + F_R \\ &-& (2a_{20}z + 2a_{11}\overline{z})(i\mu z + \mathcal{R}(|z| + \|f\|_{E_{-\sigma}})(|z| + \|f\|_{E_{-\sigma}})^2) \\ &-& (2a_{11}z + 2a_{02}\overline{z})(-i\mu\overline{z} + \mathcal{R}(|z| + \|f\|_{E_{-\sigma}})(|z| + \|f\|_{E_{-\sigma}})^2) - Ag \end{array}$$

On the other hand, (5.66) means that

$$\dot{h} = A(f - a_{20}z^2 - 2a_{11}z\overline{z} - a_{02}\overline{z}^2 - g) + H_R$$

Equating the coefficients of the quadratic powers of z, we get

$$\begin{array}{rcl} F_2 - 2i\mu a_{20} &=& -Aa_{20} \\ F_2 &=& -Aa_{11} \\ F_2 + 2i\mu a_{02} &=& -Aa_{02} \end{array}$$

and

$$H_R = F_R + \sum a_{ij}\mathcal{R}(|z| + \|f\|_{E_{-\sigma}})|z|(|z| + \|f\|_{E_{-\sigma}})^2$$

Notice that $F_2 \in E^c$ is smooth, exponentially decreasing function. Hence, there exists a solution a_{11} in the form

$$a_{11} = -A^{-1}F_2 \qquad (5.67)$$

where A^{-1} stands for regular part of the resolvent $R(\lambda)$ at $\lambda = 0$ since the singular part of $R(\lambda)F_2$ vanishes for $F_2 \in E^c$. The function a_{11} is exponentially decreasing at infinity.

For a_{20} and a_{02} we choose the following inverse operators:

$$a_{20} = -(A - 2i\mu - 0)^{-1}F_2, \quad a_{02} = \overline{a}_{20} = -(A + 2i\mu - 0)^{-1}F_2 \qquad (5.68)$$

This choice is motivated by Lemma 3.3.
The remainder H_I can be written as

$$H_I = \sum_m (A - 2i\mu m - 0)^{-1}C_m, \quad m \in \{-1, 0, 1\} \qquad (5.69)$$

with $C_m \in E^c$, satisfying the estimate

$$\|C_m\|_{E_\sigma} = \mathcal{R}(|z| + \|f\|_{E_{-\sigma}})|z|(|z| + \|f\|_{E_{-\sigma}})^2 \qquad (5.70)$$

□

5.2 Normal form for $\dot{z}$

Let us consider the equation (4.52) for z. Substituting (5.64) into (4.52) and putting the contribution of $f = h + k + g$ into the remainder Z_R, we obtain

$$\dot{z} = i\mu z + Z_2(z^2 + 2z\overline{z} + \overline{z}^2) + Z_3(z^3 + 3z^2\overline{z} + 3z\overline{z}^2 + \overline{z}^3) + Z'_{30}z^3 + Z'_{21}z^2\overline{z} + Z'_{12}z\overline{z}^2 + Z'_{03}\overline{z}^3 + \tilde{Z}_R \quad (5.71)$$

We have by (5.64)-(5.65)

$$Z'_{30} = \langle a_{20}, jZ'_1\rangle, \quad Z'_{21} = \langle a_{11} + a_{20}, jZ'_1\rangle, \quad Z'_{03} = \langle a_{02}, jZ'_1\rangle, \quad Z'_{12} = \langle a_{02} + a_{11}, jZ'_1\rangle \quad (5.72)$$

We are specially interested in the resonance term $Z'_{21}z^2\overline{z} = Z'_{21}|z|^2z$. Formulas (4.53), (5.67), (5.68) imply

$$Z'_{21} = -\langle A^{-1}P^c\mathcal{N}_2[u,u], 2j\frac{\mathcal{N}_2[u,u]}{\langle u, ju\rangle}\rangle - \langle (A - 2i\mu - 0)^{-1}P^c\mathcal{N}_2[u,u], 2j\frac{\mathcal{N}_2[u,u]}{\langle u, ju\rangle}\rangle \quad (5.73)$$

For the $\langle u, ju\rangle$ we get

$$\langle u, ju\rangle = i\delta, \quad \textit{with} \quad \delta > 0 \quad (5.74)$$

Now we can prove

Lemma 5.2. *Let the non-degeneracy condition* **U3** *hold. Then*

$$\operatorname{Re} Z'_{21} < 0 \quad (5.75)$$

Proof. We first notice that the coefficient $\langle A^{-1}P^cj\mathcal{N}_2[u,u], 2\mathcal{N}_2[u,u]\rangle$ that appears in the expression (5.73) for Z'_{21} is real since operator $A^{-1}2P^cj$ is selfadjoint. Hence (5.74) implies that $\operatorname{Re} Z'_{21}$ reduces to

$$\operatorname{Re} Z'_{21} = \operatorname{Re} 2\frac{\langle (A - 2i\mu - 0)^{-1}P^c\mathcal{N}_2[u,u], j\mathcal{N}_2[u,u]\rangle}{i\delta} = \frac{2}{\delta}\operatorname{Im}\langle R(2i\mu + 0)P^c\mathcal{N}_2[u,u], j\mathcal{N}_2[u,u]\rangle$$

where we denote

$$R(\lambda) = (A - \lambda)^{-1}, \ \operatorname{Re}\lambda > 0$$

Using that P^c commutes with $R(2i\mu + 0)$, we have $R(2i\mu + 0)P^c = P^cR(2i\mu + 0)P^c$. We have also that $(P^c)^*j = jP^c$, hence

$$\operatorname{Re} Z'_{21} = \frac{2}{\delta}\operatorname{Im}\langle R(2i\mu + 0)\alpha, j\alpha\rangle$$

with $\alpha = P^c\mathcal{N}_2[u,u]$. Now we use the representation (cf.[3], formula(2.1.9))

$$\langle R(2i\mu + 0)\alpha, j\alpha\rangle = \frac{1}{i}\int_b^\infty \theta(\lambda)d\lambda\Big(\frac{\langle \alpha, ju(i\lambda)\rangle\langle u(i\lambda), j\alpha\rangle}{i\lambda - 2i\mu - 0} + \frac{\langle \alpha, j\overline{u}(i\lambda)\rangle\langle \overline{u}(i\lambda), j\alpha\rangle}{-i\lambda - 2i\mu - 0}\Big) \quad (5.76)$$

$$= \int_b^\infty \theta(\lambda)d\lambda\Big(\frac{\langle u(i\lambda), j\alpha\rangle\overline{\langle u(i\lambda), j\alpha\rangle}}{\lambda - 2\mu + i0} + \frac{\langle \overline{u}(i\lambda), j\alpha\rangle\overline{\langle \overline{u}(i\lambda), j\alpha\rangle}}{\lambda + 2\mu - i0}\Big) \quad (5.77)$$

Using that $\dfrac{1}{\nu + i0} = \text{p.v.}\dfrac{1}{\nu} - i\pi\delta(\nu)$, where p.v. is the Cauchy principal value, we obtain

$$\langle R(2i\mu+0)\alpha, j\alpha\rangle = \int\limits_{\sqrt{2}}^{\infty} \theta(\lambda)d\lambda\Big(\frac{|\langle u(i\lambda), j\alpha\rangle|^2}{\lambda - 2\mu} + \frac{|\langle \overline{u}(i\lambda), j\alpha\rangle|^2}{\lambda + 2\mu}\Big) - i\pi\theta(2\mu)|\langle u(2i\mu), j\alpha\rangle|^2 \quad (5.78)$$

The integral terms in (5.78) is real. Thus,

$$\text{Im}\langle R_T(2i\mu_T + 0)\alpha, j\alpha\rangle = -\pi\theta(2\mu)|\langle u(2i\mu), j\alpha\rangle|^2 < 0$$

since $\theta(2\mu) > 0$, and

$$\langle u(2i\mu), j\alpha\rangle = \langle u(2i\mu), jP^c\mathcal{N}_2[u,u]\rangle = \langle u(2i\mu), j\mathcal{N}_2[u,u]\rangle = -\int u_1(2i\mu)(x)N_2[u,u](x)dx$$

$$= -\frac{1}{2}\int \varphi_{4\lambda_1}(x)F''(s(x))\varphi^2_{\lambda_1}(x)dx \neq 0$$

by **U3**. □

Further we need an estimate for the remainder $\tilde{Z}_R$.

Lemma 5.3. *The remainder $\tilde{Z}_R$ has the form*

$$|\tilde{Z}_R| = \mathcal{R}_1(|z| + \|f\|_{L^\infty})\Big[(|z|^2 + \|f\|_{E_{-\sigma}})^2 + |z|\|g\|_{E_{-\sigma}} + |z|\|h\|_{E_{-\sigma}}\Big] \quad (5.79)$$

Proof The remainder $\tilde{Z}_R$ is given by

$$\tilde{Z}_R = Z_R + (z + \overline{z})\langle f - k, jZ_1'\rangle$$

where Z_R satisfies estimate (4.50). Since $f - k = g + h$, we have

$$|\langle f - k, Z_1'\rangle| \leq C(\|g\|_{E_{-\sigma}} + \|h\|_{E_{-\sigma}})$$

Hence, (5.79) follows. Now we can apply the Poincaré method of normal coordinates to equation (5.71).

Lemma 5.4. *(cf. [3, Proposition 4.9]) There exist coefficients c_{ij} such that the new function $z_1(t)$ defined by*

$$z_1 = z + c_{20}z^2 + c_{11}z\overline{z} + c_{02}\overline{z}^2 + c_{30}z^3 + c_{12}z\overline{z}^2 + c_{03}\overline{z}^3$$

satisfies an equation of the form

$$\dot{z}_1 = i\mu z_1 + iK|z_1|^2 z_1 + \hat{Z}_R \quad (5.80)$$

where $\hat{Z}_R$ satisfies estimates of the same type as $\tilde{Z}_R$, and

$$\text{Re }\ iK = \text{Re}\, Z_{21}' < 0 \quad (5.81)$$

Proof. Substituting z_1 in equation (5.71) for z and equating the coefficients, we get, in particular,

$$c_{20} = \frac{i}{\mu} Z_2, \quad c_{11} = -\frac{2i}{\mu} Z_2, \quad c_{02} = -\frac{i}{3\mu} Z_2 \tag{5.82}$$

and

$$iK = 3Z_3 + Z_{21}' + (4c_{20} - c_{11} - 2c_{20}) Z_2 \tag{5.83}$$

Since the coefficients Z_2 and Z_3 defined in (4.53) are purely imaginary then (5.81) is follow. □

It is easier to deal with $y = |z_1|^2$ rather that z_1 because y decreases at infinity while z_1 is oscillating. The equation satisfied by y is simply obtained by multiplying (5.80) by $\overline{z}_1$ and taking the real part:

$$\dot{y} = 2 \operatorname{Re}(iK) y^2 + Y_R, \tag{5.84}$$

where

$$|Y_R| = \mathcal{R}_1(|z| + \|f\|_{L^\infty}) |z| \Big[(|z|^2 + \|f\|_{E_{-\sigma}})^2 + |z| \|g\|_{E_{-\sigma}} + |z| \|h\|_{E_{-\sigma}} \Big]. \tag{5.85}$$

5.3 Summary of normal forms

We summarize the main formulas of Sections 5.1-5.2. First we recall that

$$f = k + g + h$$

where k and g are defined in (5.65). The equation satisfied by f and h are respectively (see (4.54) and (5.66))

$$\dot{f} = Af + \tilde{F}_R, \tag{5.86}$$

$$\dot{h} = Ah + H_R \tag{5.87}$$

Here $\tilde{F}_R = P^c \mathcal{N}_2[w, w] + F_R$, $F_R = F_I + F_{II} + F_{III}$, $H_R = F_R + H_I$. The remainders F_I, F_{II}, $P^c \mathcal{N}_2[w, w]$ and F_{III} are estimated in (4.57)-(4.59), (4.62). The remainder H_I is estimated in (5.69) and (5.70). Note, that

$$\|f\|_{E_{-\sigma}} \le C(\|g\|_{E_{-\sigma}} + |z|^2 + \|h\|_{E_{-\sigma}}) \tag{5.88}$$

The second equation describes the evolution of z_1 from (5.80):

$$\dot{z}_1 = i\mu z_1 + iK |z_1|^2 z_1 + \hat{Z}_R \tag{5.89}$$

where the remainder $\hat{Z}_R$ admits the estimate (5.79). From (5.71), z and z_1 are related by

$$z_1 - z = \mathcal{O}|z|^2. \tag{5.90}$$

The fourth equation is the dynamics for $y = |z_1|^2$

$$\dot{y} = 2 \operatorname{Re}(iK) y^2 + Y_R, \tag{5.91}$$

where the remainder Y_R admits the estimate (5.85). Here $\operatorname{Re} iK < 0$ by Lemma 5.2 that is the key point.

6 Majorants

6.1 Notations

We define the 'majorants'

$$\mathcal{M}_1(T) = \max_{0\le t\le T} |z(t)| \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-1/2} \tag{6.92}$$

$$\mathcal{M}_2(T) = \max_{0\le t\le T} \|f_1(t)\|_{L^\infty} \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-1/2} \log^{-1}(2+\varepsilon t) \tag{6.93}$$

$$\mathcal{M}_3(T) = \max_{0\le t\le T} \|h(t)\|_{E_{-5/2-\nu}} \left(\frac{\varepsilon}{1+\varepsilon t}\right)^{-3/2} \log^{-1}(2+\varepsilon t) \tag{6.94}$$

and denote by $\mathcal{M}$ the 3-dimensional vector $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$. The goal of this section is to prove that all these majorants are bounded uniformly in T for sufficiently small $\varepsilon > 0$.

6.2 Bound for g

Lemma 6.1. *For the function $g(t)$ defined in (5.65), the following bound holds*

$$\|g(t)\|_{E_{-\sigma}} \le c|z(0)|^2 \frac{1}{(1+t)^{3/2}} \le c\frac{\varepsilon}{(1+t)^{3/2}}, \quad \sigma > 5/2 \tag{6.95}$$

Proof. By (5.65), we have $g = -e^{At}k(0)$ and $k(0) = a_{20}z^2(0) + a_{11}z(0)\overline{z}(0) + a_{02}\overline{z}^2(0)$ with a_{ij} defined in (5.67), (5.68). Therefore, Lemma 3.3 and assumption (6.100) imply (6.95). □

6.3 Estimate of the remainders

Lemma 6.2. *The remainder Y_R defined in (5.85) admits the estimate*

$$|Y_R| = \mathcal{R}(\varepsilon^{1/2}\mathcal{M}) \frac{\varepsilon^{5/2}}{(1+\varepsilon t)^2\sqrt{\varepsilon t}} \log(2+\varepsilon t)(1+|\mathcal{M}|)^5 \tag{6.96}$$

Proof. Using the equality $f = k + g + h$ and estimate (5.85), we obtain

$$|Y_R| = \mathcal{R}_2(|z| + \|f\|_{L^\infty})|z|\Big[(|z|^2 + \|g\|_{E_{-\sigma}} + \|h\|_{E_{-\sigma}})^2 + |z|(\|g\|_{E_{-\sigma}} + \|h\|_{E_{-\sigma}})\Big]$$

$$= \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\mathcal{M}_1\Bigg[\Big(\frac{\varepsilon}{1+\varepsilon t}\mathcal{M}_1^2 + \frac{\varepsilon}{(1+t)^{3/2}} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(2+\varepsilon t)\mathcal{M}_3\Big)^2$$

$$+\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\mathcal{M}_1\Big(\frac{\varepsilon}{(1+t)^{3/2}} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(2+\varepsilon t)\mathcal{M}_3\Big)\Bigg]$$

$$= \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\frac{\varepsilon^{5/2}}{(1+\varepsilon t)^2\sqrt{\varepsilon+\varepsilon t}}\log(2+\varepsilon t)(1+|\mathcal{M}|)^5$$

□

Now let us turn to the remainders $F_R = F_I + F_{II} + F_{III}$, $\tilde{F}_R = P^c\mathcal{N}_2[w,w] + F_R$, and $H_R = F_R + H_I$ in equations (5.86) and (5.87) for f and h respectively.

Lemma 6.3. *For $0 < \nu < 1/2$ the remainder F_R admits the bound*

$$\|F_R\|_{E_{5/2+\nu}} = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(2+\varepsilon t)\Big((\mathcal{M}_1+\mathcal{M}_2)(1+\mathcal{M}_1^2)+\varepsilon^{1/2-\nu}(1+|\mathcal{M}|)^{12}\Big) \tag{6.97}$$

Proof. Step i) (4.57) and (5.88) imply for $\sigma = 5/2+\nu$

$$\|F_I + F_{II}\|_{E_\sigma} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^3 + |z|\|f\|_{E_{-\sigma}} + \|f_1\|_{L^\infty}\|f\|_{E_{-\sigma}})$$

$$= \mathcal{R}(|z| + \|f_1\|_{L^\infty})\Big[|z|^3 + |z|\|g\|_{E_{-\sigma}} + |z|\|h\|_{E_{-\sigma}} + \|f_1\|_{L^\infty}(|z|^2 + \|g\|_{E_{-\sigma}} + \|h\|_{E_{-\sigma}})\Big]$$

$$= \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Bigg(\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathcal{M}_1^3 + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\frac{\varepsilon}{(1+t)^{3/2}}\mathcal{M}_1 + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^2\log(2+\varepsilon t)\mathcal{M}_1\mathcal{M}_3$$

$$+\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\log(2+\varepsilon t)\mathcal{M}_2\Big[\frac{\varepsilon}{1+\varepsilon t}\mathcal{M}_1^2 + \frac{\varepsilon}{(1+t)^{3/2}} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(2+\varepsilon t)\mathcal{M}_3\Big]\Bigg)$$

which implies (6.97) for $F_I + F_{II}$.
Step ii) Let us consider $\|F_{III}\|_{E_\sigma}$. Bound (4.59) implies

$$\|F_{III}\|_{E_{5/2+\nu}} = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(1+t)^{4+\nu}(|z|^{12} + \|f_1\|_{L^\infty}^{12})$$

$$= \mathcal{R}(\varepsilon^{1/2}\mathcal{M})(1+t)^{4+\nu}\log^{12}(2+\varepsilon t)\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^6(M_1^{12} + M_2^{12})$$

and then bound (6.97) for F_{III} follows.

Lemma 6.4. *For $0 < \nu < 1/2$ the remainder $\tilde{F}_R$ admits the bound*

$$\|\tilde{F}_R\|_{E_{5/2+\nu}\cap W} = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\frac{\varepsilon}{1+\varepsilon t}\Big(M_1^2 + \varepsilon^{1/2}(1+|\mathcal{M}|)^{12}\Big) \tag{6.98}$$

For F_I and F_{II} the bound follows from the estimate (4.57). Further, by (4.62)

$$\|F_{III}\|_W = \mathcal{R}(|z| + \|f_1\|_{L^\infty})(|z|^{10} + \|f_1\|_{L^\infty}^{10}) = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^5\log^{10}(2+\varepsilon t)(M_1^{10} + M_2^{10})$$

which together with (6.97) implies (6.98) for F_{III}. Finally, (4.58) implies

$$\|P^c\mathcal{N}_2[w,w]\|_{E_\sigma\cap W} = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\frac{\varepsilon}{1+\varepsilon t}\mathcal{M}_1^2$$

and then (6.98) follows. □

The term H_I is represented by (5.69) with C_m estimated in (5.70). For C_m we now obtain

Lemma 6.5. *For $m = 0, \ \pm 1$, the bounds hold*

$$\|C_m\|_{E_\sigma} = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\Big(\mathcal{M}_1^3 + \varepsilon^{1/2}(1+|\mathcal{M}|)^3\Big). \tag{6.99}$$

Proof. Estimate (5.70) implies

$$\|C_m\|_{E_\sigma} = \mathcal{R}(|z| + \|f\|_{E_{-\sigma}})|z|(|z| + \|g\|_{E_{-\sigma}} + \|h\|_{E_{-\sigma}})^2$$

$$= \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\mathcal{M}_1\Big(\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\mathcal{M}_1 + \frac{\varepsilon}{(1+t)^{3/2}} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(2+\varepsilon t)\mathcal{M}_2\Big)^2$$

which implies (6.99). □

6.4 Initial conditions

We suppose the smallness of initial condition:

$$|z(0)| \leq \varepsilon^{1/2} \tag{6.100}$$

$$\|f(0)\|_{E_\sigma} = \|h(0)\|_{E_\sigma} \leq \varepsilon^{3/2}h_0 \tag{6.101}$$

$$\|f(0)\|_{E_\sigma \cap W} \leq \varepsilon^{1/2}f_0 \tag{6.102}$$

where h_0, f_0 are some fixed constant, and $\varepsilon > 0$ is sufficiently small accordingly (2.21). Equation (5.90) implies $|z_1|^2 \leq |z|^2 + \mathcal{R}(|z|)|z|^3$. Therefore

$$y_0 = y(0) = |z_1(0)|^2 \leq \varepsilon + C(|z(0)|)\varepsilon^{3/2} \tag{6.103}$$

6.5 Estimates via majorants

This section is devoted to the study the system (5.86), (5.87), (5.91) under assumptions (6.101), (6.102), (6.103) on initial data and the estimates (6.96), (6.97), (6.98), (6.99) of the remainders.

First we consider equation (5.91) for y which is of Ricatti type.

Lemma 6.6. *([3], Proposition 5.6) The solution to (5.91) with an initial condition and a remainder satisfying (6.103) and (6.96) respectively admits the bound:*

$$|y - \frac{y_0}{1 + 2y_0 \operatorname{Im} Kt}| \leq \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\frac{\varepsilon^{5/2}}{(1+\varepsilon t)^2\sqrt{\varepsilon t}}\log(2+\varepsilon t)(1+|\mathcal{M}|)^5. \tag{6.104}$$

Corollary 6.7. *The majorant $\mathcal{M}_1$ satisfies*

$$\mathcal{M}_1^2 = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big(1 + \varepsilon^{1/2}(1+|\mathcal{M}|)^5\Big) \tag{6.105}$$

Proof. Bounds (6.103) and (6.104) imply

$$y \leq \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left[\frac{\varepsilon}{1+\varepsilon t} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(2+\varepsilon t)(1+|\mathcal{M}|)^5\right].$$

Using that $|z|^2 \leq y + \mathcal{R}(|z|)|z|^3$, we get

$$|z|^2 \leq \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\left[\frac{\varepsilon}{1+\varepsilon t} + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(2+\varepsilon t)(1+|\mathcal{M}|)^5 + \Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\mathcal{M}_1^3\right]$$

Hence, (6.105) follows. □

Second we consider equation (5.86) for f.

Lemma 6.8. *The solution to (5.86) admits the bound*

$$\|f_1\|_{L^\infty} \le C\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\log(2+\varepsilon t)\Big(f_0 + \mathcal{R}(\varepsilon^{1/2}\mathcal{M})(\mathcal{M}_1^2 + \varepsilon^{1/2}(1+|\mathcal{M}|)^{12}\Big). \tag{6.106}$$

Proof. The solution $f(x,t)$ to (5.86) is expressed as

$$f(t) = e^{At}f(0) + \int_0^t e^{A(t-\tau)}\tilde{F}_R(\tau)d\tau$$

Using the the bounds (3.36) and the estimates (6.98), (6.102), we obtain

$$\|f_1\|_{L^\infty} \le \frac{C}{(1+t)^{1/2}}\|f(0)\|_{E_\sigma\cap W} + \int_0^t \frac{C}{(1+(t-\tau))^{1/2}}\|\tilde{F}_R(\tau)\|_{E_\sigma\cap W}d\tau$$

$$\le C\Bigg[f_0\Big(\frac{\varepsilon}{1+t}\Big)^{1/2} + \mathcal{R}(\varepsilon^{1/2}\mathcal{M})(\mathcal{M}_1^2 + \varepsilon^{1/2}(1+|\mathcal{M}|)^{12}\int_0^t \frac{d\tau}{(t-\tau)^{1/2}}\frac{\varepsilon}{1+\varepsilon\tau}d\tau\Bigg]$$

$$\le C\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\log(2+\varepsilon t)\Big(f_0 + \mathcal{R}(\varepsilon^{1/2}\mathcal{M})(\mathcal{M}_1^2 + \varepsilon^{1/2}(1+|\mathcal{M}|)^{12}\Big)$$

□

Corollary 6.9.

$$\mathcal{M}_2 = \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big(M_1^2 + \varepsilon^{1/2}(1+|\mathcal{M}|)^{12}\Big). \tag{6.107}$$

Finally we consider equation (5.87) for h.

Lemma 6.10. *The solution to (5.87) admits the bound*

$$\|h\|_{E_{-\sigma}} \le C\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(2+\varepsilon t)\Bigg(h_0 + \mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big[(1+\mathcal{M}_1^2+\mathcal{M}_2)(1+\mathcal{M}_1^2)+\varepsilon^{1/2-\nu}(1+|\mathcal{M}|)^{12}\Big]\Bigg) \tag{6.108}$$

Proof. The solution $h(x,t)$ to (5.87) is expressed as

$$h = e^{At}h(0) + \int_0^t e^{A(t-\tau)}H_R(\tau)d\tau$$

Using the bounds (6.101), (6.97), (6.99), Proposition 3.1 and Corollary 3.3, we get

$$\|h\|_{E_{-\sigma}} \le \frac{C}{(1+t)^{3/2}}\|h(0)\|_{E_\sigma} + \int_0^t \frac{C}{(1+(t-\tau))^{3/2}}\Big(\|F_R(\tau)\|_{E_\sigma} + \sum_m \|C_m(\tau)\|_{E_\sigma}\Big)d\tau$$

$$\leq C\Big[h_0\Big(\frac{\varepsilon}{1+t}\Big)^{\frac{3}{2}}+\mathcal{R}(\varepsilon^{\frac{1}{2}}\mathcal{M})\Big[(\mathcal{M}_1+\mathcal{M}_2)(1+\mathcal{M}_1^2)+\varepsilon^{\frac{1}{2}-\nu}(1+|\mathcal{M}|)^{12}\Big]\int_0^t\frac{\log(2+\varepsilon t)\ d\tau}{(1+(t-\tau))^{\frac{3}{2}}}\Big(\frac{\varepsilon}{1+\varepsilon\tau}\Big)^{\frac{3}{2}}$$

$$+\sum_m\mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big(\mathcal{M}_1^3+\varepsilon^{1/2}(1+|\mathcal{M}|)^3\Big)\int_0^t\frac{d\tau}{(1+(t-\tau))^{3/2}}\Big(\frac{\varepsilon}{1+\varepsilon\tau}\Big)^{3/2}\Big]$$

which implies (6.108). □

Corollary 6.11.

$$\mathcal{M}_3=\mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big[(1+\mathcal{M}_1^2+\mathcal{M}_2)(1+\mathcal{M}_1^2)+\varepsilon^{1/2-\nu}(1+|\mathcal{M}|)^{12}\Big]. \tag{6.109}$$

6.6 Uniform bounds for majorants

The aim of this section is to prove that if ε is sufficiently small, all the $\mathcal{M}_i$ are bounded uniformly in T and ε.

Lemma 6.12. *For ε sufficiently small, there exists a constant M independent of T and ε, such that,*

$$|\mathcal{M}(T)|\leq M. \tag{6.110}$$

Proof. Combining the inequalities (6.105), (6.107), and (6.109) for the $\mathcal{M}_i$, we obtain the inequality

$$\mathcal{M}^2\leq\mathcal{R}(\varepsilon^{1/2}\mathcal{M})\Big[(1+\mathcal{M}_1^2+\mathcal{M}_2)^4+\varepsilon^{1/2-\nu}(1+|\mathcal{M}|)^{24}\Big]$$

Replacing $\mathcal{M}_1^2$ and $\mathcal{M}_2$ in the right-hand by its bound (6.105) and (6.107), we obtain

$$\mathcal{M}^2\leq\mathcal{R}(\varepsilon^{1/2}\mathcal{M})(1+\varepsilon^{1/2-\nu}F(\mathcal{M}))$$

where $F(\mathcal{M})$ is an appropriate function. This inequality implies that $\mathcal{M}$ is bounded uniformly in ε since $\mathcal{M}(0)$ is small and $\mathcal{M}(t)$ is continuous. □

Corollary 6.13. *The following estimates hold for all $t>0$, $\sigma>5/2$*

$$|z(t)|\leq M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}, \tag{6.111}$$

$$\|f_1\|_{L^\infty}\leq M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}\log(1+\varepsilon t), \tag{6.112}$$

$$\|h\|_{E_{-\sigma}}\leq M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(1+\varepsilon t). \tag{6.113}$$

$$\|f\|_{E_{-\sigma}}\leq M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big). \tag{6.114}$$

Thus we have proved the following result:

Theorem 6.14. *Let the conditions of Theorem 2.3 hold. Then*
i) for ε small enough, one can write the solution of (2.15) in the form

$$Y(x,t) = s(x) + (z(t) + \overline{z}(t))u + f(x,t), \tag{6.115}$$

ii) In addition, for all $t > 0$, there exists a constant $M > 0$ such that

$$|z(t)| \le M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{1/2}, \quad \|f\|_{E_{-\sigma}} \le M\Big(\frac{\varepsilon}{1+\varepsilon t}\Big), \quad \sigma > 5/2. \tag{6.116}$$

7 Soliton asymptotics

7.1 Long time behavior of $z(t)$

We start with equation (5.80) for z_1 that we rewrite

$$\dot z_1 = i\mu z_1 + iK|z_1|^2 z_1 + \widehat{Z}_R$$

By (5.79) $\widehat{Z}_R$ satisfies the estimate

$$\widehat{Z}_R = \mathcal{R}(|z| + \|f_1\|_{L^\infty})\Big[(|z|^2 + \|f\|_{E_{-\sigma}})^2 + |z|\|g\|_{E_{-\sigma}} + |z|\|h_1\|_{E_{-\sigma}}\Big]$$

$$= \mathcal{R}(\varepsilon^{1/2}M)\frac{\varepsilon^2\log(2+\varepsilon t)}{(1+\varepsilon t)^{3/2}\sqrt{\varepsilon t}}(1+M^4) \le \frac{C\varepsilon^2\log(2+\varepsilon t)}{(1+\varepsilon t)^{3/2}\sqrt{\varepsilon t}}$$

On the other hand, we have, from (6.103) and (6.104),

$$\Big|y - \frac{y_0}{1+2\,\mathrm{Im}\,Ky_0t}\Big| \le C\Big(\frac{\varepsilon}{1+\varepsilon t}\Big)^{3/2}\log(2+\varepsilon t)$$

with $|y_0 - \varepsilon| \le C\varepsilon^{3/2}$. With estimate (6.111) for $|z|$ and obviously the same one for $|z_1|$, we have

$$\dot z_1 = i\mu z_1 + iK\frac{y_0}{1+2\,\mathrm{Im}\,Ky_0t}z_1 + Z_1 \tag{7.117}$$

with

$$|Z_1| \le \frac{C\varepsilon^2\log(2+\varepsilon t)}{(1+\varepsilon t)^{3/2}\sqrt{\varepsilon t}}$$

Since $y_0 = \varepsilon + \mathcal{O}(\varepsilon^{3/2})$, we have that the coefficient $2\,\mathrm{Im}\,Ky_0 = k\varepsilon$. We also denote $\rho = \frac{\mathrm{Re}\,K}{\mathrm{Im}\,K}$. The solution z_1 of (7.117) is written in the form

$$z_1 = \frac{e^{i\mu t}}{(1+k\varepsilon t)^{1/2-i\rho}}\Big[z_1(0) + \int_0^t e^{-i\mu s}(1+k\varepsilon s)^{1/2-i\rho}Z_1(s)ds\Big] = z_{L^\infty}\frac{e^{i\mu t}}{(1+k\varepsilon t)^{1/2-i\rho}} + z_R$$

where

$$z_{L^\infty} = z_1(0) + \int_0^{L^\infty} e^{-\mu s}(1+k\varepsilon s)^{1/2-i\rho}Z_1(s)ds$$

and

$$z_R = -\int_t^{L^\infty} e^{i\mu t}\Big(\frac{1+k\varepsilon s}{1+k\varepsilon t}\Big)^{1/2-i\rho} Z_1(s)ds.$$

From the bound (7.117) on Z_1 it follows that

$$|z_R| \le \frac{C\varepsilon\log(2+\varepsilon t)}{(1+\varepsilon t)}.$$

Therefore $z_1(t)$ satisfies the estimate

$$z_1(t) = z_{L^\infty}\frac{e^{i\mu t}}{(1+k\varepsilon t)^{1/2-i\rho}} + \mathcal{O}\Big(\frac{\varepsilon}{1+\varepsilon t}\log(2+\varepsilon t)\Big) \tag{7.118}$$

Here $z_\infty = z_1(0)+\mathcal{O}(\varepsilon)$, $z = z_1+\mathcal{O}(\frac{\varepsilon}{1+\varepsilon t})$, and $|z(0)| = \varepsilon^{1/2}$. Thus $|z_\infty| = \varepsilon^{1/2}+\mathcal{O}(\varepsilon)$. Hence, the function $z(t)$ can be estimated as

$$z(t) = z_\infty\frac{e^{i\mu t}}{(1+k\varepsilon t)^{1/2-i\rho}} + \mathcal{O}\Big(\frac{\varepsilon}{1+\varepsilon t}\log(2+\varepsilon t)\Big) \tag{7.119}$$

7.2 Proof of soliton asymptotics

Here we prove our main Theorem 2.3. We have obtained the solution $Y(x,y)$ to (2.15) in the form

$$Y = S + w + f, \tag{7.120}$$

We include w into the remainder $r_\pm$ from (2.22) since $z(t)\sim t^{-1/2}$ by (7.119). It remains to extract the dispersive wave $W(t)\Phi_\pm$ from the term f.

7.2.1 The asymptotic completeness

Let us rewrite equation (4.54) as

$$\left\{\begin{array}{rcl} \dot f_1 &=& f_2 + Q_1 \\ \dot f_2 &=& f_1'' - m^2 f_1 + Q_2 \end{array}\right. \tag{7.121}$$

where

$$\begin{array}{rcl} Q_1 &=& (P^c\mathcal{N})_1 = -(P^d\mathcal{N})_1 = -\frac{1}{i\delta}\langle N,u_1\rangle u_1 + \frac{1}{i\delta}\langle N,u_1\rangle u_1 = 0 \\ Q_2 &=& (P^c\mathcal{N})_2 = (P^c\mathcal{N}_2[w,w])_2 + (F_R)_2 - Vf_1 \end{array}$$

by (3.33) and (5.74). The equations imply the asymptotics of type (2.23),

$$f(t) = W_0(t)f(0) + \int_0^t W_0(t-\tau)Q(\tau)d\tau = W_0(t)\Big(f(0) + \int_0^\infty W_0(-\tau)Q(\tau)d\tau\Big)$$

$$-\int_t^\infty W_0(t-\tau)Q(\tau)d\tau = W_0(t)\phi_+ + r_+(t), \tag{7.122}$$

if all the integrals converge. Here $W_0(t)$ is the dynamical group of the free Klein-Gordon equation, and $Q(t) := (0, Q_2(t))$. To complete the proof of (2.23), it remains to prove the following proposition

Proposition 7.1. *The bounds hold*

$$\|r_+(t)\|_E = \mathcal{O}(t^{-1/3}), \qquad t \to \infty \tag{7.123}$$

Proof. To check (7.123), we should obtain an appropriate decay for the function $Q_2(t)$.

Step i) First, according to (4.55), (4.57), (4.59), (6.111), (6.112), and (6.114), we have

$$\|(F_R)_2\|_{L^2} = \mathcal{O}(t^{-3/2}\log t) \tag{7.124}$$

By (3.33), (4.51), and (4.53)

$$(P^c\mathcal{N}_2[w,w])_2 = N_2[w,w] - (P^d\mathcal{N}_2[w,w])_2 = (z^2 + 2z\overline{z} + \overline{z}^2)\Big(N_2[u,u] - 2i\mu u_1 Z_2\Big)$$

Hence, (5.64) and (5.65) imply that

$$Q_2 = q_{20}z^2 + 2q_{11}z\overline{z} + q_{02}\overline{z}^2 + Q_{2R} \tag{7.125}$$

with

$$q_{ij} = N_2[u_1,u_1] - 2iZ_2\mu u_1 - Va_{ij,1}, \qquad Q_{2R} = (F_R)_2 - V(f_1 - k_1) \tag{7.126}$$

where $a_{ij,1}$ and k_1 are the first components of vector-functions a_{ij} and k from (5.65). By (1.10), (5.64), (6.95) and (6.113)

$$\|V(f_1 - k_1)\|_{L^2} = \mathcal{O}(t^{-3/2}\log t), \quad t \to \infty,$$

Hence, the last bound and (7.124) imply that

$$\|Q_{2R}\|_{L^2} = \mathcal{O}(t^{-3/2}\log t), \quad t \to \infty. \tag{7.127}$$

Therefore, the term Q_{2R} give the contribution of order $\mathcal{O}(t^{-1/2}\log t)$ to $r_+(t)$.

Step ii) It remains to estimate the contribution to $r_+(t)$ of the quadratic terms $q_{ij}z^i\overline{z}^j$ from (7.125). Let us note that the functions $q_{ij}(x)$ are smooth with exponential decay at infinity similarly to the functions $u_1(x)$ and $V(x)$ since $a_{ij} \in H^s_{-\sigma}$ with any $s > 0$ by Lemma 5.1.

On the other hand, the time decay of the functions $z^i(t)\overline{z}^j(t)$ is very slow like $\mathcal{O}(t^{-1})$. Therefore, the contribution of the term $q_{ij}z^i\overline{z}^j$ to $r_+(t)$ is the integral of type (7.122) which does not converge absolutely. Fortunately, we may define the integral as

$$\int_t^\infty W(t-\tau)q_{ij}(\tau)z^i\overline{z}^j d\tau := \lim_{T\to\infty}\int_t^T W(t-\tau)q_{ij}(\tau)z^i\overline{z}^j d\tau \tag{7.128}$$

We prove below the convergence of the integrals with the values in E and the decay rate $\mathcal{O}(t^{-1/3})$.

First we estimate the contribution of the term $q_{11}(x)z\overline{z}$. Note that (7.119) implies the asymptotics $z\overline{z} \sim (1 + k\varepsilon t)^{-1}$.

Lemma 7.2. *Let $q(x) \in L^2(\mathbb{R})$. Then*

$$I(t) := \Big\| \int\limits_t^\infty W_0(-\tau) \begin{pmatrix} 0 \\ q \end{pmatrix} \frac{d\tau}{1+\tau} \Big\|_E = \mathcal{O}(t^{-1}), \quad t \to \infty. \tag{7.129}$$

Proof. Denote $\omega = \omega(\xi) = \sqrt{\xi^2 + m^2}$. Then

$$I(t) = \Big\| \int\limits_t^\infty \begin{pmatrix} -\sin\omega\tau \ \hat{q}(\xi) \\ -\cos\omega\tau \ \hat{q}(\xi) \end{pmatrix} \frac{d\tau}{1+\tau} \Big\|_{L^2 \oplus L^2} \le \frac{C}{1+t} \|\hat{q}(\xi)/\omega(\xi)\|_{L^2} \tag{7.130}$$

since the partial integration implies that

$$\Big| \int\limits_t^\infty \frac{e^{i\omega\tau}}{1+\tau}\, d\tau \Big| = \Big| \int\limits_t^\infty \frac{de^{i\omega\tau}}{i\omega(1+\tau)}\, d\tau \Big| \le \Big| \frac{e^{i\omega\tau}}{\omega(1+t)} \Big| + \Big| \int\limits_t^\infty \frac{e^{i\omega\tau}}{\omega(1+\tau)^2}\, d\tau \Big| \le \frac{C}{\omega(1+t)} \tag{7.131}$$

□

Next we estimate the contribution from the terms with $q_{20}(x) z^2$ and $q_{02}(x) \overline{z}^2$ to (7.122) (cf. [3, Proposition 6.5]). Now (7.119) implies the asymptotics $z^2 \sim e^{2i\mu\tau}/(1+k\varepsilon t)^{1-2i\rho}$ and $\overline{z}^2 \sim e^{-2i\mu\tau}/(1+k\varepsilon t)^{1+2i\rho}$.

Lemma 7.3. *Let $q(x) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then*

$$\Big\| \int\limits_t^\infty W_0(-\tau) \begin{pmatrix} 0 \\ q \end{pmatrix} \frac{e^{\pm 2i\mu\tau} d\tau}{(1+\tau)^{1\mp 2i\rho}} \Big\|_E = \mathcal{O}(t^{-1/3}), \quad t \to \infty. \tag{7.132}$$

Proof. We consider for example the integral with $e^{-2i\mu\tau}$ and omit for simplicity the factor $(1+t)^{2i\rho}$ since with the factor the proof is similar. Let us represent $\sin\omega\tau$ and $\cos\omega\tau$ as linear combination of $e^{i\omega\tau}$ and $e^{-i\omega\tau}$. The contribution from the "nonresonant" terms with the $e^{-i\omega\tau}$ in (7.132) is $\mathcal{O}(t^{-1})$ similarly to (7.130)-(7.131). It remains to prove that

$$I(t) = \Big\| \int\limits_t^\infty \frac{e^{i(\omega-2\mu)\tau} \hat{q}(\xi)\ d\tau}{1+\tau} \Big\|_{L^2} = \mathcal{O}(t^{-1/3}) \tag{7.133}$$

For the fixed $\beta > 0$ let us denote

$$\chi_\tau(\xi) = \begin{cases} 1, & |\omega(\xi) - 2\mu| \le 1/\tau^\beta \\ 0, & |\omega(\xi) - 2\mu| > 1/\tau^\beta \end{cases}$$

Then

$$I(t) \le \Big\| \int\limits_t^\infty \frac{e^{i(2\omega-\mu)\tau} \chi_\tau(\xi) \hat{q}(\xi)\ d\tau}{1+\tau} \Big\|_{L^2} + \Big\| \int\limits_t^\infty \frac{e^{i(2\omega-\mu)\tau} (1-\chi_\tau(\xi)) \hat{q}(\xi)\ d\tau}{1+\tau} \Big\|_{L^2} = I_1(t) + I_2(t)$$

Since $\hat{q}(\xi)$ is bounded function, and $\|\chi_\tau\|^2 \le 1/\tau^\beta$, we have

$$I_1(t) \le \frac{C\|\hat{q}\|_{L^\infty}}{(1+t)^{\beta/2}}$$

On the other hand, the partial integration implies that

$$I_2(t) = \Big\|\int\limits_t^\infty \frac{(1-\chi_\tau(\xi))\hat{q}(\xi)\ de^{i(2\omega-\mu)\tau}}{(2\omega-\mu)(1+\tau)}\Big\|_{L^2} \le \frac{Ct^\beta}{1+t}\|\hat{q}\|_{L^2} + C\int\limits_t^\infty \frac{\tau^\beta d\tau}{(1+\tau)^2}\|\hat{q}\|_{L^2} \le \frac{C\|\hat{q}\|_{L^2}}{(1+t)^{1-\beta}}$$

Equating $\frac{\beta}{2} = 1-\beta$, we get $\beta = \frac{2}{3}$. □

Now Proposition 7.1 is proved. □

H Virial type estimates

We prove the weighted estimate (4.61). Let us recall that we split the solution $Y(t) = (\psi(\cdot,t),\pi(\cdot,t)) = S + X(t)$, and denote $X(t) = (\Psi(t),\Pi(t))$, $(\Psi_0,\Pi_0) := (\Psi(0),\Pi(0))$. Finally, our basic condition (2.21) implies that for some $\nu > 0$.

$$\|X_0\|_{E_{5/2+\nu}} \le d_0 < \infty \tag{H. 1}$$

Proposition H.1. *Let the potential U satisfy conditions* **U1***, and Ψ_0 satisfy (H. 1). Then the bounds hold*

$$\|\Psi(t)\|_{L^2_{5/2+\nu}} \le C(d_0)(1+t)^{4+\nu}, \qquad t > 0 \tag{H. 2}$$

We will deduce the proposition from the following two lemmas. The first lemma is well known. Denote

$$e(x,t) = \frac{|\pi(x,t)|^2}{2} + \frac{|\psi'(x,t)|^2}{2} + U(\psi(x,t)).$$

Lemma H.2. *For the solution $\psi(x,t)$ of Klein-Gordon equation (1.1) the local energy estimate holds*

$$\int\limits_a^b e(x,t)\ dx \le \int\limits_{a-t}^{b+t} e(x,0)\ dx, \quad a<b, \quad t>0. \tag{H. 3}$$

Proof. The estimate follows by standard arguments: multiplication of the equation (1.1) by $\dot{\psi}(x,t)$ and integration over the trapezium $ABCD$, where $A = (a-t,0)$, $B = (a,t)$, $C = (b,t)$, $D = (b+t,0)$. Then (H. 3) is obtained after partial integration using that $U(\psi) \ge 0$. □

Lemma H.3. *For any $\sigma \ge 0$*

$$\int (1+|x|^\sigma)e(x,t)dx \le C(\sigma)(1+t)^{\sigma+1}\int (1+|x|^\sigma)e(x,0)dx. \tag{H. 4}$$

Proof. By (H. 3)

$$\int (1+|x|^\sigma)\Big(\int\limits_{x-1}^{x} e(y,t)dy\Big)dx \le \int (1+|x|^\sigma)\Big(\int\limits_{x-1-t}^{x+t} e(y,0)dy\Big)dx.$$

Hence,

$$\int e(y,t)\Big(\int\limits_{y}^{y+1}(1+|x|^\sigma)dx\Big)dy \le \int e(y,0)\Big(\int\limits_{y-t}^{y+t+1}(1+|x|^\sigma)dx\Big)dy. \tag{H. 5}$$

Obviously,

$$\int\limits_{y}^{y+1}(1+|x|^\sigma)dx \ge c(\sigma)(1+|y|^\sigma) \tag{H. 6}$$

with some $c(\sigma)>0$. On the other hand,

$$\int\limits_{y-t}^{y+t+1}(1+|x|^\sigma)dx \le (2t+1)(1+t+|y|)^\sigma \le C(\sigma)(1+t)^{\sigma+1}(1+|y|^\sigma) \tag{H. 7}$$

since $\sigma\ge 0$. Now the bound (H. 4) follows from (H. 5)-(H. 7) . □

Proof of Proposition H.1 First we verify that

$$U_0 := \int (1+|x|^{5+2\nu})U(\psi_0(x))dx < \infty, \quad \psi_0(x)=\psi(x,0) \tag{H. 8}$$

Indeed, $\psi_0(x)=s(x)+\Psi_0(x)$ is bounded since $\Psi_0\in H^1(\mathbb{R})$. Hence **U1** implies for

$$|U(\psi_0(x))| \le C(d_0)(\psi_0(x)\pm a)^2 \le C(d_0)\Big((s(x)\pm a)^2+\Psi_0(x)^2\Big)$$

and then (H. 8) follows by (H. 1). Now (H. 4) with $\sigma=5+2\nu$ and (H. 1), (H. 8) imply that

$$\begin{aligned}
\|\Psi(t)\|^2_{L^2_{5/2+\nu}} &= \int (1+|x|^{5+2\nu})\Big(\int\limits_0^t \dot\Psi(x,s)ds-\Psi_0(x)\Big)^2dx \\
&\le 2\int (1+|x|^{5+2\nu})\Psi_0^2(x)dx + 2t\int (1+|x|^{5+2\nu})dx\int\limits_0^t \pi^2(x,s)ds \\
&\le 2d_0^2 + 2t\Big[\|X_0\|^2_{E_{5/2+\nu}}+U_0\Big]\int\limits_0^t (1+s)^{6+2\nu}ds \le C(d_0)(1+t)^{8+2\nu}
\end{aligned}$$

I Proof of Proposition 3.3

First we prove the following lemma. Let us denote by $\mathcal{L}(E_\sigma, E_{-\sigma})$ the Banach space of the linear bounded operators $E_\sigma \to E_{-\sigma}$.

Lemma I.1. *Let $L(\nu)$, $\nu \in \mathbb{R}$, be the operators $E_\sigma \to E_{-\sigma}$, and*

$$K(t) = \int \zeta(\nu) e^{i\nu t} Q(\nu)\, d\nu, \qquad Q(\nu) := \frac{L(\nu) - L(\nu_0)}{\nu - \nu_0} \tag{I. 1}$$

where $\zeta \in C_0^\infty(\mathbb{R})$, and for $k = 0, 1, 2$

$$M_k := \sup_{\nu \in \Sigma} \|\partial_\nu^k L(\nu)\|_{\mathcal{L}(E_\sigma, E_{-\sigma})} < \infty \tag{I. 2}$$

with $\sigma > 1/2 + k$ and $\Sigma := \text{supp}\ \zeta$. Then for $\sigma > 5/2$

$$\|K(t)\|_{\mathcal{L}(E_\sigma, E_{-\sigma})} = \mathcal{O}(t^{-3/2}), \quad t \to \infty, \tag{I. 3}$$

Proof. Let us take $\varphi \in C_0^\infty(\mathbb{R})$ and split $\zeta = \zeta_{1t} + \zeta_{2t}$, where

$$\zeta_{1t}(\nu) := \zeta(\nu)\varphi((\nu - \nu_0)\sqrt{t}), \quad \zeta_{2t}(\nu) := \zeta(\nu)[1 - \varphi((\nu - \nu_0)\sqrt{t})] \tag{I. 4}$$

Then

$$K(t) = \int \zeta_{1t}(\nu) e^{i\nu t} Q(\nu)\, d\nu + \int \zeta_{2t}(\nu) e^{i\nu t} Q(\nu)\, d\nu = K_1(t) + K_2(t)$$

Further we consider each term separately.
Step i) For the first term we obtain integrating twice by parts

$$\begin{aligned} K_1(t) &= -\frac{1}{it} \int\limits_{|\nu-\nu_0|<\frac{1}{\sqrt{t}}} \zeta_{1t} e^{i\nu t} Q'(\nu) d\nu - \frac{1}{t^2} \int\limits_{|\nu-\nu_0|<\frac{1}{\sqrt{t}}} \zeta_{1t}'' e^{i\nu t} Q(\nu) d\nu \\ &- \frac{1}{t^2} \int\limits_{|\nu-\nu_0|<\frac{1}{\sqrt{t}}} \zeta_{1t}' e^{i\nu t} Q'(\nu) d\nu \end{aligned} \tag{I. 5}$$

For the appropriate operator norms, we have the bounds

$$\begin{aligned} \|Q(\nu)\| &= \frac{1}{|\nu - \nu_0|} \| \int_{\nu_0}^{\nu} L'(r) dr \| \le M_1 \\ \|Q'(\nu)\| &= \frac{1}{|\nu - \nu_0|^2} \| - L'(\nu)(\nu_0 - \nu) - L(\nu) + L(\nu_0) \| \\ &= \frac{1}{|\nu - \nu_0|^2} \| L'(\nu) \int_{\nu_0}^{\nu} dr - \int_{\nu_0}^{\nu} L'(r) dr \| \\ &= \frac{1}{|\nu - \nu_0|^2} \| \int_{\nu_0}^{\nu} [L'(\nu) - L'(r)] dr \| \\ &= \frac{1}{|\nu - \nu_0|^2} \| \int_{\nu_0}^{\nu} [\int_r^{\nu} L''(s) ds] dr \| \le \frac{1}{2} M_2 \end{aligned} \tag{I. 6}$$

Hence, (I. 5) implies that

$$\|K_1(t)\|_{\mathcal{L}(E_\sigma,E_{-\sigma})} \le C_1 t^{-3/2}. \qquad \text{(I. 7)}$$

for $\sigma > 5/2$ since $|\partial_\nu^k \zeta_{1t}(\nu)| \le C(k)t^{k/2}$.

Step ii) For the second summand we obtain by triple partial integration

$$K_2(t) = -\frac{1}{t^2}\int e^{i\nu t}\zeta_{2t} Q''(\nu)d\nu - \frac{2}{t^2}\int e^{i\nu t}\zeta_{2t}' Q'(\nu)d\nu$$

$$+\frac{1}{it^3}\int e^{i\nu t}\zeta_{2t}''' Q(\nu)d\nu + \frac{1}{it^3}\int e^{i\nu t}\zeta_{2t}'' Q'(\nu)d\nu$$

$$= K_{21}(t) + K_{22}(t) + K_{23}(t) + K_{24}(t).$$

Using $|\partial_\nu^k \zeta_{1t}(\nu)| \le C(k)t^{k/2}$, we obtain

$$\|K_{2j}(t)\|_{\mathcal{L}(E_\sigma,E_{-\sigma})} \le C_2 t^{-3/2}, \quad j = 2,3,4.$$

Finally, to estimate $K_{21}(t)$, we use the identity

$$\begin{aligned} Q''(\nu) &= \frac{L''(\nu)(\nu-\nu_0)^2 - 2(L(\nu_0) - L(\nu) - L'(\nu)(\nu_0-\nu))}{(\nu-\nu_0)^3} \\ &= \frac{L''(\nu)(\nu-\nu_0)^2 - 2\displaystyle\int_{\nu_0}^{\nu}[\int_r^{\nu} L''(s)ds]dr}{(\nu-\nu_0)^3} \end{aligned} \qquad \text{(I. 8)}$$

which implies that $\|Q''(\nu)\| \le CM_2/|\nu-\nu_0|$. Therefore,

$$\|K_{21}(t)\|_{\mathcal{L}(E_\sigma,E_{-\sigma})} \le Ct^{-3/2}.$$

since $\zeta_{2t}(\nu) = 0$ for $|\nu - \nu_0| \le \frac{1}{2\sqrt{t}}$.

□

Proof of Proposition 3.3 The operator $e^{At}(A-2i\mu-0)^{-1}$ admits the Laplace representation

$$e^{At}(A-2i\mu-0)^{-1} = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\lambda t}R(\lambda+0)d\lambda\ R(2i\mu+0).$$

Let us apply the Hilbert identity for the resolvent:

$$R(\lambda_1)R(\lambda_2) = \frac{1}{\lambda_1-\lambda_2}[R(\lambda_1)-R(\lambda_2)],\ \operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2 > 0,$$

for $\lambda_1 = \lambda + 0$ and $\lambda_2 = 2i\mu + 0$. Then we obtain

$$e^{At}(A-2i\mu-0)^{-1} = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\lambda t}\frac{R(\lambda+0)-R(2i\mu+0)}{\lambda-2i\mu}\ d\lambda$$

$$= -\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} e^{\lambda t}\zeta(\lambda)\frac{R(\lambda+0)-R(2i\mu+0)}{\lambda-2i\mu}\,d\lambda - \frac{1}{2\pi i}\int\limits_{\mathcal{C}_+\cup\mathcal{C}_-} e^{\lambda t}(1-\zeta(\lambda))\frac{R(\lambda+0)-R(2i\mu+0)}{\lambda-2i\mu}\,d\lambda$$

$$-\frac{1}{2\pi i}\int\limits_{(-i\infty,i\infty)\setminus(\mathcal{C}_+\cup\mathcal{C}_-)} e^{\lambda t}(1-\zeta(\lambda))\frac{R(\lambda+0)-R(2i\mu+0)}{\lambda-2i\mu}\,d\lambda = K_1(t)+K_2(t)+K_3(t),$$

where $\zeta(\lambda)\in C_0^\infty(i\mathbb{R})$, $\zeta(\lambda)=1$ for $|\lambda-2i\mu|<\delta/2$ and $\zeta(\lambda)=0$ for $|\lambda-2i\mu|>\delta$, with $0<\delta<2\mu-\sqrt{2}$. By Lemma I.1 with $L(\nu)=R(i\nu+0)$,

$$\|K_1(t)\|_{\mathcal{L}(E_\sigma,E_{-\sigma})}=\mathcal{O}(t^{-3/2}),\quad t\to\infty$$

since $\sigma>5/2$. The bounds (I. 2) for $L(\nu)$ follow from Proposition 3.1. For the operator $K_2(t)$ we also apply Proposition 3.1 and obtain

$$\|K_2(t)\|_{\mathcal{L}(E_\sigma,E_{-\sigma})}=\mathcal{O}(t^{-3/2}),\quad t\to\infty$$

Here the choice of the sigh in $A-2i\mu-0$ plays the crucial role. Further, the integrand in $K_3(t)$ is an analytic function of $\lambda\neq 0,\pm i\mu$ with the values in $\mathcal{L}(E_\sigma,E_{-\sigma})$ for $\beta\geq 0$. At the points $\lambda=0$ and $\lambda=\pm i\mu$ the integrand has the poles of finite order. However, all the Laurent coefficients vanish when applied to P_ch. Hence for $K_3(t)$ we obtain, twice integrating by parts,

$$\|K_3(t)P^ch\|_{E_{-\sigma}}\leq c(1+t)^{-2}\|h\|_{E_\sigma},$$

completing the proof.

J Examples

We construct the examples of the potentials $U(\psi)$ satisfying all the conditions **U1** – **U3**. We will construct $U(\psi)$ by small perturbation of the cubic Ginzburg-Landau potential

$$U_0(\psi):=\frac{1}{4}(1-\psi^2)^2 \tag{J. 9}$$

For the potential $U_0(\psi)$ the kink is explicitly given by

$$s_0(x):=\tanh\frac{x}{\sqrt{2}} \tag{J. 10}$$

The potential V_0 of the linearized eduation reads

$$V_0(x)=U_0''(s_0(x))-2=-3\cosh^{-2}\frac{x}{\sqrt{2}}$$

Let us consider the corresponding Schrödinger operator

$$H_0=-\frac{d^2}{dx^2}+2+V_0(x)=-\frac{d^2}{dx^2}+2-\frac{3}{\cosh^2(x/\sqrt{2})}$$

restricted to the subspace of odd functions (2.16). The continuous spectrum of the operator H_0 coincides with the interval $[2, \infty)$. It is well known (see [11], pp. 64-65) that

i) The discrete spectrum of H_0 consists only of one point $\lambda_0 = 3/2$ with the corresponding eigenfunction $\varphi_0 = \sinh(x/\sqrt{2})/\cosh^2(x/\sqrt{2})$;

ii) The end point $\lambda = 2$ of the continuous spectrum is not eigenvalue nor resonance.

Hence, the Condition **U2** holds for the potential U_0. Further, the non-degeneracy condition **U3** reads

$$\int \phi_6(x) \frac{\sinh^3(x/\sqrt{2})}{\cosh^5(x/\sqrt{2})} dx \neq 0 \tag{J. 11}$$

where $\phi_6(x)$ is a nonzero odd solution to

$$H_0 \phi_6(x) = 6\psi_6(x)$$

Numerical calculation [16] demonstrate the validity of the condition (J. 11) and hence **U3** holds.

The potential $U_0(\psi)$ satisfies the conditions (1.2) with $a = 1$ and $m^2 = 2$. However, $U_0(\psi)$ does not satisfy the conditions (1.3) since $U_0'''(\pm 1) = \pm 6$, $U_0^{(4)}(\pm 1) = 6$.

Therefore we will construct a small perturbation of the potential U_0. Namely, for an appropriate fixed $C > 0$, and any sufficiently small $\delta > 0$, there exist the potentials $U(\psi)$ satisfying (1.3) such that

$$\left. \begin{array}{l} U(\psi) = U_0(\psi) \quad \text{for} \quad ||\psi| - 1| > \delta, \quad \sup\limits_{\psi \in \mathbb{R}} |U^{(k)}(\psi) - U_0^{(k)}(\psi)| \leq C\delta, \quad k = 0, 1, 2, \\ \sup\limits_{\psi \in \mathbb{R}} |U'''(\psi) - U_0'''(\psi)| \leq C \end{array} \right| \tag{J. 12}$$

For example, let us set

$$U(\psi) = U_0(\psi) - \Big[\frac{1}{4}(|\psi| - 1)^4 + (|\psi| - 1)^3\Big] \chi_\delta(|\psi| - 1)$$

where $\chi_\delta(z) = \chi(z/\delta)$, $\chi(z) \in C_0^\infty(\mathbb{R})$, $\chi(z) = 1$ for $|z| < 1/2$, and $\chi(z) = 0$ for $|z| > 1$. Then the conditions (J. 12) holds, and

$$U(\psi) = (|\psi| - 1)^2 \quad \text{for} \quad ||\psi| - 1| < \delta/2, \quad \text{and} \quad U(\psi) = U_0(\psi) \quad \text{for} \quad ||\psi| - 1| > \delta$$

Hence, $U(\psi)$ satisfies **U1**. It remains to prove that $U(\psi)$ satisfies **U2** and **U3**.

Denote $\mathcal{S} = \{x \in \mathbb{R} : ||s(x)| - 1|,\ ||s_0(x)| - 1| < \delta\}$. Then $s(x) = s_0(x)$ and $V(x) = V_0(x)$ for $x \in \mathbb{R} \setminus \mathcal{S}$. For $x \in \mathcal{S}$, using (J. 12), we obtain

$$\begin{aligned} \sup_{x \in \mathcal{S}} |V(x) - V_0(x)| &\leq \sup_{x \in \mathcal{S}} |U''(s(x)) - U''(s_0(x))| + \sup_{x \in \mathcal{S}} |U''(s_0(x)) - U_0''(s_0(x))| \\ &= \sup_{||\phi| - 1| < \delta} |U'''(\phi)| \sup_{x \in \mathcal{S}} |s_0(x) - s(x)| + \mathcal{O}(\delta) = \mathcal{O}(\delta) \end{aligned}$$

since $\sup\limits_{x \in \mathcal{S}} |s_0(x) - s(x)| \leq 2\delta$. Hence

$$\sup_{x \in \mathbb{R}} |V(x) - V_0(x)| = \mathcal{O}(\delta) \tag{J. 13}$$

Let us verify the uniform decay of $V(x)$ for small $\delta > 0$. We consider the case $x \geq 0$ (the case $x \leq 0$ can be consider similarly). Note that $U(\psi) \geq (\psi - 1)^2/4$ for $0 \leq \psi < 1$. Using the identity

$$\int_0^{s(x)} \frac{ds}{\sqrt{2U(s)}} = x$$

we obtain for $x > 0$ and $0 \leq s(x) < 1$

$$x \leq \int_0^{s(x)} \frac{\sqrt{2}\, ds}{\sqrt{(1-s)^2}} = \int_0^{s(x)} \frac{\sqrt{2}\, ds}{1-s} = -\sqrt{2}\ \ln(1 - s(x))$$

Hence, $1 - s(x) \leq e^{-x/\sqrt{2}}$ for $x \geq 0$, and then

$$|1 - |s(x)|| \leq e^{-|x|/\sqrt{2}}, \quad x \in \mathbb{R}$$

Therefore

$$|V(x)| \leq Ce^{-|x|/\sqrt{2}}, \quad x \in \mathbb{R} \tag{J. 14}$$

Finally, the unifirm bounds (J. 13) and (J. 14) imply that the conditions **U2** and **U3** hold for the potentials $U(\psi)$ for sufficiently small $\delta > 0$ since they hold for $U_0(\psi)$.

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