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Long time decay for 2D Klein–Gordon equation

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Abstract

We obtain a dispersive long-time decay in weighted energy norms for solutions of the 2D Klein–Gordon equations. The decay extends the results obtained by Jensen, Kato and Murata for the equations of Schrödinger's type by the spectral approach. For the proof we modify the approach to make it applicable to relativistic equations.

Keywords: Klein–Gordon equations; Resolvent; Cauchy problem; Long-time asymptotics

1. Introduction

In this paper, we establish an optimal long time decay for the solutions to 2D Klein–Gordon equation

$$\ddot{\psi}(x,t)=\Delta\psi(x,t)-m^2\psi(x,t)-V(x)\psi(x,t),\quad x\in\mathbb{R}^2,\ t\in\mathbb{R},\ m>0,\tag{1.1}$$

in weighted energy norms. In vectorial form, Eq. (1.1) reads

$$i\dot{\Psi}(t)=\mathcal{H}\Psi(t),\tag{1.2}$$

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where

$$\Psi(t)=\begin{pmatrix}\psi(t)\\ \dot\psi(t)\end{pmatrix},\qquad \mathcal{H}=\begin{pmatrix}0 & i\\ i(\Delta-m^2-V) & 0\end{pmatrix}. \tag{1.3}$$

For $s,\sigma\in\mathbb{R}$, let us denote by $H^s_\sigma=H^s_\sigma(\mathbb{R}^2)$ the weighted Sobolev spaces introduced by Agmon [1], with the finite norms

$$\|\psi\|_{H^s_\sigma}=\big\|\langle x\rangle^\sigma\langle\nabla\rangle^s\psi\big\|_{L^2}<\infty,\qquad \langle x\rangle=\big(1+|x|^2\big)^{1/2}.$$

We suppose that $V(x)\in C^1(\mathbb{R}^2)$ is a real function, and

$$\big|V(x)\big|+\big|\nabla V(x)\big|\leqslant C\langle x\rangle^{-\beta},\quad x\in\mathbb{R}^2 \tag{1.4}$$

with some $\beta>5$. Then the multiplication by $V(x)$ is bounded operator $H^1_s\to H^1_{s+\beta}$ for any $s\in\mathbb{R}$.

We restrict ourselves to the "nonsingular case", in the terminology of [20], where the truncated resolvent of the Schrödinger operator $H=-\Delta+V(x)$ is bounded at the end points of the continuous spectrum. In other words, the point $\lambda=0$ is neither eigenvalue nor resonance for the operator H.

Definition 1.1. $\mathcal{F}_\sigma$ is the Hilbert space $H^1_\sigma\oplus H^0_\sigma$ of vector-functions $\Psi=(\psi,\pi)$ with the norm

$$\|\Psi\|_{\mathcal{F}_\sigma}=\|\psi\|_{H^1_\sigma}+\|\pi\|_{H^0_\sigma}<\infty. \tag{1.5}$$

Our main result is the following long time decay of the solutions to (1.2): in the "nonsingular case", the asymptotics hold

$$\big\|\mathcal{P}_c\Psi(t)\big\|_{\mathcal{F}_{-\sigma}}=\mathcal{O}\big(|t|^{-1}\log^{-2}|t|\big),\quad t\to\pm\infty \tag{1.6}$$

for initial data $\Psi_0=\Psi(0)\in\mathcal{F}_\sigma$ with $\sigma>5/2$ where $\mathcal{P}_c$ is a Riesz projection onto the continuous spectrum of the operator $\mathcal{H}$. The decay is desirable for the study of asymptotic stability and scattering for the solutions to nonlinear hyperbolic equations. The study has been started in 90' for nonlinear Schrödinger equation [3,22–24], and continued last decade [4,5,13]. The study has been extended to the Klein–Gordon equation in [7,25]. Further extension need more information on the decay for the corresponding linearized equations that stipulated our investigation.

Let us comment on previous results in this direction. Local energy decay has been established first in the scattering theory for linear Schrödinger equation developed since 50' by Birman, Kato, Simon, and others.

For 3D Klein–Gordon equations with magnetic potential, the decay $\sim t^{-3/2}$ has been established primarily by Vainberg [28] in local energy norms for initial data with compact support. The results were extended to general hyperbolic partial differential equations by Vainberg in [29]. The decay in the L^p norms for wave and Klein–Gordon equations was obtained in [2,6,12,19, 31,32].

However, applications to asymptotic stability of solutions to the nonlinear equations also require an exact characterization of the decay for the corresponding linearized equations in weighted norms (see e.g. [3–5,25]).

The decay of type (1.6) in weighted norms has been established first by Jensen and Kato [10] for the Schrödinger equation in the dimension $n = 3$. The result has been extended to other dimensions by Jensen and Nenciu [8,9,11], and to more general PDEs of the Schrödinger type by Murata [20]. The survey of the results can be found in [27].

For discrete 1D, 2D and 3D Schrödinger and Klein–Gordon equations the decay of type (1.6) has been proved in [15–17] respectively.

For the continuous free 3D Klein–Gordon equation, the decay (1.6) in the weighted energy norms has been proved first in [7, Lemma 18.2]. However, for the perturbed relativistic equations, the decay was an open problem until our result [14]. The problem was that the Jensen–Kato approach is not applicable directly to the relativistic equations. The difference reflects distinct character of wave propagation in the relativistic and nonrelativistic equations (see the discussion in [14, Introduction]).

In [14] the decay of type (1.6) in the weighted energy norms has been proved for the first time for the Klein–Gordon equation in the dimension $n = 3$. The approach [14] develops the Jensen–Kato techniques to make it applicable to the relativistic equations. Namely, the decay of the low energy component of the solution follows by the Jensen–Kato techniques while the decay for the high energy component requires novel robust ideas. This problem has been resolved in [14] with a modified approach based on the Born series and convolution.

Here we extend our approach [14] to the dimension $n = 2$. The extension is not straightforward since the decay (1.6) violates for the free 2D Klein–Gordon equation corresponding to $V(x) = 0$ when the solutions decay slow, like $\sim t^{-1}$. Hence, the decay (1.6) cannot be deduced by perturbation arguments from the corresponding estimate for the free equation. The slow decay is caused by the "zero resonance function" $z(x) = \text{const}$ corresponding to the end point $\lambda = 0$ of the continuous spectrum of the 2D Schrödinger operator $-\Delta$.

Our approach to $n = 2$ relies on the following two main issues.

I. First is a spectral analysis of the "bad" term, with the slow decay $\sim t^{-1}$. Namely, we show that the bad term does not contribute to the high energy component. For example, this is obvious in the particular case of the free Green function

$$\begin{aligned} G(t,x,y) &= \frac{1}{2\pi}\theta\big(t-|x-y|\big)\frac{\cos m\sqrt{t^2-|x-y|^2}}{\sqrt{t^2-|x-y|^2}} \sim G_0(t) \\ &:= \frac{1}{2\pi}\theta(t)\frac{\cos mt}{t}, \quad t\to\infty. \end{aligned} \tag{1.7}$$

It is instructive to note that the asymptotics is proportional to the degenerate kernel $z(x)z(y)$, and its time spectrum is mainly concentrated at the frequencies $\pm m$. Hence, the slow decay $\sim t^{-1}$ should be entirely caused by the resonance at the end point $\lambda = 0$, and the decay $\sim t^{-3/2}$ for the high energy component follows by a development of our approach [14].

II. Second, we prove the decay $\sim t^{-1}\log^{-2} t$ for low energy component in the nonsingular case by an appropriate development of the methods [10,20]. Namely, we establish novel asymptotic expansions for the derivatives of the resolvent at the edge points of the continuous spectrum. The first expansion in (3.2) is proved in [20, (7.21)]. However, the expansions for the derivatives in (3.2) are new and necessary for the proof of the long time asymptotics.

Our paper is organized as follows. In Section 2 we obtain the time decay for the solution to the free Klein–Gordon equation and state the spectral properties of the free resolvent which follow from the corresponding known properties of the free Schrödinger resolvent. In Section 3 we obtain spectral properties of the perturbed resolvent and prove the decay (1.6). In Section 4 we apply the obtained decay to the asymptotic completeness.

2. Free Klein–Gordon equation

First, we consider the free Klein–Gordon equation:

$$\ddot{\psi}(x,t)=\Delta\psi(x,t)-m^2\psi(x,t),\quad x\in\mathbb{R}^2,\ t\in\mathbb{R}. \tag{2.1}$$

In vectorial form Eq. (2.1) reads

$$i\dot{\Psi}(t)=\mathcal{H}_0\Psi(t) \tag{2.2}$$

where

$$\Psi(t)=\begin{pmatrix}\psi(t)\\ \dot{\psi}(t)\end{pmatrix},\qquad \mathcal{H}_0=\begin{pmatrix}0 & i\\ i(\Delta-m^2) & 0\end{pmatrix}. \tag{2.3}$$

2.1. Spectral properties

We state spectral properties of the free Klein–Gordon dynamical group $\mathcal{G}(t)$ applying known results of [1,10,20] which concern the corresponding spectral properties of the free Schrödinger dynamical group. For $t>0$ and $\Psi_0=\Psi(0)\in\mathcal{F}_0$, the solution $\Psi(t)$ to the free Klein–Gordon equation (2.2) admits the spectral Fourier–Laplace representation

$$\theta(t)\Psi(t)=\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i(\omega+i\varepsilon)t}\mathcal{R}_0(\omega+i\varepsilon)\Psi_0\,d\omega,\quad t\in\mathbb{R} \tag{2.4}$$

with any $\varepsilon>0$ where $\theta(t)$ is the Heavyside function, $\mathcal{R}_0(\omega)=(\mathcal{H}_0-\omega)^{-1}$ for $\omega\in\mathbb{C}^+:=\{\operatorname{Im}\omega>0\}$ is the resolvent of the operator $\mathcal{H}_0$. The representation follows from the stationary equation $\omega\tilde{\Psi}^+(\omega)=\mathcal{H}_0\tilde{\Psi}^+(\omega)+i\Psi_0$ for the Fourier–Laplace transform $\tilde{\Psi}^+(\omega):=\int_{\mathbb{R}}\theta(t)e^{i\omega t}\Psi(t)\,dt$, $\omega\in\mathbb{C}^+$. The solution $\Psi(t)$ is continuous bounded function of $t\in\mathbb{R}$ with the values in $\mathcal{F}_0$ by the energy conservation for the free Klein–Gordon equation (2.2). Hence, $\tilde{\Psi}^+(\omega)=-i\mathcal{R}(\omega)\Psi_0$ is analytic function of $\omega\in\mathbb{C}^+$ with the values in $\mathcal{F}_0$, and bounded for $\omega\in\mathbb{R}+i\varepsilon$. Therefore, the integral (2.4) converges in the sense of distributions of $t\in\mathbb{R}$ with the values in $\mathcal{F}_0$. Similarly to (2.4),

$$\theta(-t)\Psi(t)=-\frac{1}{2\pi i}\int\limits_{\mathbb{R}} e^{-i(\omega-i\varepsilon)t}\mathcal{R}_0(\omega-i\varepsilon)\Psi_0\,d\omega,\quad t\in\mathbb{R}. \tag{2.5}$$

The resolvent $\mathcal{R}_0(\omega)$ can be expressed in terms of the resolvent $R_0(\zeta)=(-\Delta-\zeta)^{-1}$ of the free Schrödinger operator:

$$\mathcal{R}_0(\omega)=\begin{pmatrix}\omega R_0(\omega^2-m^2) & iR_0(\omega^2-m^2)\\ -i(1+\omega^2R_0(\omega^2-m^2)) & \omega R_0(\omega^2-m^2)\end{pmatrix}. \tag{2.6}$$

The free Schrödinger resolvent $R_0(\zeta)$ is an integral operator with the integral kernel

$$R_0(\zeta, x-y) = \frac{i}{4} H_0^{(1)}\big(\zeta^{1/2}|x-y|\big) = \frac{1}{2\pi} K_0\big(-i\zeta^{1/2}|x-y|\big), \quad \zeta \in \mathbb{C}^+,\ \operatorname{Im}\zeta^{1/2} > 0, \tag{2.7}$$

where $H_0^{(1)}$ is the modified Hankel function, and K_0 is the Macdonald's function.

Definition 2.1. Denote by $\mathcal{L}(B_1, B_2)$ the Banach space of bounded linear operators from a Banach space B_1 to a Banach space B_2.

Now we collect the properties of $R_0(\zeta)$ which are obtained in [1] and in [20]:

(i) $R_0(\zeta)$ is strongly analytic function of $\zeta \in \mathbb{C} \setminus [0, \infty)$ with the values in $\mathcal{L}(H_0^{-1}, H_0^1)$;
(ii) For $\zeta > 0$, the convergence holds

$$R_0(\zeta \pm i\varepsilon) \to R_0(\zeta \pm i0), \quad \varepsilon \to 0+$$

in $\mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)$ for $\sigma > 1/2$, uniformly in $\zeta \geqslant r$ for any $r > 0$.

Lemma 2.2. *(Cf. [20, formula (2.3)] and [11, formula (3.14)].) The asymptotic expansion holds*

$$R_0(\zeta) = A_0 \log\zeta + B_0 + \mathcal{O}\big(\zeta^{3/4}\big), \quad \zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0, \infty) \tag{2.8}$$

in the norm of $\mathcal{L}(H_\sigma^{-1}; H_{-\sigma}^1)$ with $\sigma > 5/2$. Here $A_0, B_0 \in \mathcal{L}(H_\sigma^{-1}; H_{-\sigma}^1)$, with $\sigma > 1$, are operators with the kernels $A_0(x-y)$, $B_0(x-y)$ respectively, and

$$A_0(x-y) = -\frac{1}{4\pi}, \quad x, y \in \mathbb{R}^2. \tag{2.9}$$

Furthermore,

$$R_0'(\zeta) = A_0\zeta^{-1} + \mathcal{O}\big(\zeta^{-1/4}\big),$$
$$R_0''(\zeta) = -A_0\zeta^{-2} + \mathcal{O}\big(\zeta^{-5/4}\big), \quad \zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0, \infty) \tag{2.10}$$

in the norm of $\mathcal{L}(H_\sigma^{-1}; H_{-\sigma}^1)$ with $\sigma > 5/2$.

Proof. The well-known asymptotics of Macdonald's functions [21] imply

$$K_0(z) = -\log\frac{z}{2} - \gamma + \mathcal{O}\big(z^{3/2}\big), \qquad K_1(z) = z^{-1} + \mathcal{O}\big(z^{1/2}\big),$$
$$K_2(z) = 2z^{-2} + \mathcal{O}\big(z^{-1/2}\big), \quad iz \in \mathbb{C}^+ \tag{2.11}$$

where γ is the Euler constant. Hence, (2.7) implies (2.8). Differentiating, we obtain that

$$R_0'(\zeta, x-y) = -\frac{i}{4\pi}\zeta^{-1/2}|x-y|K_0'\big(-i\zeta^{1/2}|x-y|\big)$$
$$= \frac{i}{4\pi}\zeta^{-1/2}|x-y|K_1\big(-i\zeta^{1/2}|x-y|\big),$$
$$R_0''(\zeta, x-y) = -\frac{i|x-y|}{8\pi\zeta^{3/2}}K_1\big(-i\zeta^{1/2}|x-y|\big)$$
$$-\frac{|x-y|^2}{16\pi\zeta}\big[K_0\big(-i\zeta^{1/2}|x-y|\big) + K_2\big(-i\zeta^{1/2}|x-y|\big)\big].$$

Hence, the asymptotics (2.10) follows. □

Let us denote $\Gamma := (-\infty, -m) \cup (m, \infty)$, and let $\mathcal{A}_0^{\pm}$ be the operator with the integral kernel

$$\mathcal{A}_0^{\pm}(x-y) = -\frac{1}{4\pi}\begin{pmatrix} \pm m & i \\ -im^2 & \pm m \end{pmatrix}. \tag{2.12}$$

Then the properties (i)–(ii), Lemma 2.2 and formula (2.6) imply the corresponding properties of $\mathcal{R}_0(\omega)$:

Lemma 2.3.

(i) *The resolvent $\mathcal{R}_0(\omega)$ is strongly analytic function of $\omega \in \mathbb{C} \setminus \overline{\Gamma}$ with the values in $\mathcal{L}(\mathcal{F}_0, \mathcal{F}_0)$.*
(ii) *For $\omega \in \Gamma$, the convergence holds*

$$\mathcal{R}_0(\omega \pm i\varepsilon) \to \mathcal{R}_0(\omega \pm i0), \quad \varepsilon \to 0+$$

in $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ with $\sigma > 1/2$, uniformly in $|\omega| \geqslant m + r$ for any $r > 0$.
(iii) *For $\omega \in \mathbb{C} \setminus \overline{\Gamma}$, the asymptotics hold as $\omega \to \pm m$*

$$\mathcal{R}_0(\omega) = \mathcal{A}_0^{\pm}\log(\omega \mp m) + \mathcal{B}_0^{\pm} + \mathcal{O}\big((\omega \mp m)^{3/4}\big),$$
$$\mathcal{R}_0'(\omega) = \mathcal{A}_0(\omega \mp m)^{-1} + \mathcal{O}\big((\omega \mp m)^{-1/4}\big),$$
$$\mathcal{R}_0''(\omega) = -\mathcal{A}_0(\omega \mp m)^{-2} + \mathcal{O}\big((\omega \mp m)^{-5/4}\big)$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$ with $\sigma > 5/2$.

Finally, we state the asymptotics of $\mathcal{R}_0(\omega)$ for large ω which follow from the corresponding asymptotics of R_0. In [14] we slightly strengthen known Agmon–Jensen–Kato decay of the resolvent [1, (A.2')], [10, (8.1)] for special case of free Schrödinger equation in arbitrary dimension $n \geqslant 1$.

Proposition 2.4. *The asymptotics hold for $s = 0, 1$ and $l = -1, 0, 1$*

$$\big\|R_0^{(k)}(\zeta)\big\|_{\mathcal{L}(H_\sigma^s, H_{-\sigma}^{s+l})} = \mathcal{O}\big(|\zeta|^{-\frac{1-l+k}{2}}\big), \quad |\zeta| \to \infty,\ \zeta \in \mathbb{C} \setminus (0, \infty), \tag{2.13}$$

with $\sigma > 1/2 + k$ for any $k = 0, 1, 2, \ldots$.

Then for $\mathcal{R}_0(\omega)$ we obtain

Lemma 2.5. *The asymptotics hold*

$$\left\|\mathcal{R}_0^{(k)}(\omega)\right\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}=\mathcal{O}(1),\quad |\omega|\to\infty,\ \omega\in\mathbb{C}\setminus\Gamma \tag{2.14}$$

with $\sigma>1/2+k$ *for* $k=0,1,2,\ldots.$

Proof. The asymptotics follow from representation (2.6) for $\mathcal{R}_0(\omega)$ and asymptotics (2.13) for $R_0(\zeta)$ with $\zeta=\omega^2-m^2$. □

Corollary 2.6. *For* $t\in\mathbb{R}$ *and* $\Psi_0\in\mathcal{F}_\sigma$ *with* $\sigma>1/2$, *the group* $\mathcal{G}(t)$ *admits the integral representation*

$$\mathcal{G}(t)\Psi_0=\frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t}\big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\big]\Psi_0\,d\omega \tag{2.15}$$

where the integral converges in the sense of distributions of $t\in\mathbb{R}$ *with the values in* $\mathcal{F}_{-\sigma}$.

Proof. Summing up the representations (2.4) and (2.5), and sending $\varepsilon\to 0+$, we obtain (2.15) by the Cauchy theorem and Lemmas 2.3 and 2.5. □

2.2. Time decay

The estimates (2.14) do not allow obtain the decay of $\mathcal{G}(t)$ by partial integration in (2.15). We deduce the decay from explicit formulas. The matrix kernel of the dynamical group $\mathcal{G}(t)$ for $t>0$ can be written as $\mathcal{G}(t,x-y)$, where

$$\mathcal{G}(t,z)=\begin{pmatrix}\dot G(t,z) & G(t,z)\\ \ddot G(t,z) & \dot G(t,z)\end{pmatrix},\quad z\in\mathbb{R}^2. \tag{2.16}$$

Here

$$G(t,z)=\frac{1}{2\pi}\theta\big(t-|z|\big)\frac{\cos m\sqrt{t^2-|z|^2}}{\sqrt{t^2-|z|^2}} \tag{2.17}$$

and θ is the Heavyside function. Therefore, the free Klein–Gordon group $\mathcal{G}(t)$ decays like t^{-1} that does not correspond to (1.6). We split $\mathcal{G}(t)$ as

$$\mathcal{G}(t)=\mathcal{G}_0(t)+\mathcal{G}_r(t)$$

where $\mathcal{G}_0(t)$ is the operator with the matrix kernel

$$\mathcal{G}_0(t,z):=\frac{1}{2\pi t}\begin{pmatrix}-m\sin mt & \cos mt\\ -m^2\cos mt & -m\sin mt\end{pmatrix},\quad z\in\mathbb{R}^2. \tag{2.18}$$

Below we show that $\mathcal{G}_0(t)$ is only term responsible for the slow decay. More exactly, in the next section we will prove the following basic proposition

Proposition 2.7. *Let $\sigma > 5/2$. Then the asymptotics hold*

$$\mathcal{G}_r(t) = \mathcal{O}\big(t^{-3/2}\big), \quad t \to \infty \tag{2.19}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$.

The following key observation is that the "bad term" $\mathcal{G}_0(t)$ does not contribute to the high energy component of the total group $\mathcal{G}(t)$ since (2.18) contains just two frequencies $\pm m$ which are the end points of the continuous spectrum. This suggests that the high energy component of the group $\mathcal{G}(t)$ decays faster then t^{-1}. More precisely, let us introduce the following *low energy* and *high energy* components of $\mathcal{G}(t)$:

$$\mathcal{G}_l(t) = \frac{1}{2\pi i}\int_\Gamma e^{-i\omega t} l(\omega)\big[\mathcal{R}_0(\omega + i0) - \mathcal{R}_0(\omega - i0)\big]\, d\omega, \tag{2.20}$$

$$\mathcal{G}_h(t) = \frac{1}{2\pi i}\int_\Gamma e^{-i\omega t} h(\omega)\big[\mathcal{R}_0(\omega + i0) - \mathcal{R}_0(\omega - i0)\big]\, d\omega \tag{2.21}$$

where $l(\omega) \in C_0^\infty(\mathbb{R})$ is an even function, $\operatorname{supp} l \in [-m - 2\varepsilon, m + 2\varepsilon]$, $l(\omega) = 1$ if $|\omega| \leqslant m + \varepsilon$ with an $\varepsilon > 0$, and $h(\omega) = 1 - l(\omega)$. In Appendix A we will prove the following lemma.

Lemma 2.8. *Let $\sigma > 5/2$. Then the asymptotics hold*

$$\mathcal{G}_l(t) = \mathcal{G}_0(t) + \mathcal{O}\big(t^{-7/4}\big), \quad t \to \infty \tag{2.22}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$.

Now we obtain the asymptotics of $\mathcal{G}_h(t)$.

Theorem 2.9. *Let $\sigma > 5/2$. Then the bound holds*

$$\big\|\mathcal{G}_h(t)\big\|_{\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})} \leqslant C\big(1 + |t|\big)^{-3/2}, \quad t \in \mathbb{R}. \tag{2.23}$$

Proof. First we consider a small $|t| \leqslant 1$. By energy conservation for the Klein–Gordon equation we obtain

$$\big\|\mathcal{G}(t)\Psi_0\big\|_{\mathcal{F}_{-\sigma}} \leqslant \big\|\mathcal{G}(t)\Psi_0\big\|_{\mathcal{F}_0} \leqslant C\|\Psi_0\|_{\mathcal{F}_0} \leqslant C\|\Psi_0\|_{\mathcal{F}\sigma}, \quad t \in \mathbb{R}. \tag{2.24}$$

The integrand in (2.20) has finite support and belongs to $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$ by Lemma 2.3(ii)–(iii). Hence

$$\big\|\mathcal{G}_l(t)\big\|_{\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})} \leqslant C, \quad t \in \mathbb{R}. \tag{2.25}$$

Then (2.23) for small $|t| \leqslant 1$ follows, since $\mathcal{G}_h(t) = \mathcal{G}(t) - \mathcal{G}_l(t)$.

For large $t \geqslant 1$ we deduce the bound (2.23) from Proposition 2.7 and Lemma 2.8. Using (2.22) we obtain

$$\mathcal{G}(t) = \mathcal{G}_l(t) + \mathcal{G}_h(t) = \mathcal{G}_0(t) + \mathcal{G}_h(t) + \mathcal{O}\big(t^{-7/4}\big), \quad t \to \infty \tag{2.26}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$. On the other hand, (2.19) implies that

$$\mathcal{G}(t) = \mathcal{G}_0(t) + \mathcal{G}_r(t) = \mathcal{G}_0(t) + \mathcal{O}\big(t^{-3/2}\big), \quad t \to \infty \tag{2.27}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$. Comparing the asymptotics (2.26) and (2.27) we obtain the bound (2.23) for large $t \geqslant 1$. For $t \leqslant -1$ the proof is similar. □

2.3. Proof of Proposition 2.7

We develop the method of [7, Lemma 18.2]. We split the initial function $\Psi_0 \in \mathcal{F}_\sigma$ in two terms, $\Psi_0 = \Psi'_{0,t} + \Psi''_{0,t}$ such that

$$\|\Psi'_{0,t}\|_{\mathcal{F}_\sigma} + \|\Psi''_{0,t}\|_{\mathcal{F}_\sigma} \leqslant C\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geqslant 1 \tag{2.28}$$

and

$$\Psi'_{0,t}(x) = 0 \quad \text{for } |x| > \frac{t}{3}, \quad \text{and} \quad \Psi''_{0,t}(x) = 0 \quad \text{for } |x| < \frac{t}{4}. \tag{2.29}$$

We estimate $\mathcal{G}_r(t)\Psi'_{0,t}$ and $\mathcal{G}_r(t)\Psi''_{0,t}$ separately.

Step (i). Let us consider $\mathcal{G}_r(t)\Psi''_{0,t} = \mathcal{G}(t)\Psi''_{0,t} - \mathcal{G}_0(t)\Psi''_{0,t}$. First we estimate $\mathcal{G}(t)\Psi''_{0,t}$ using energy conservation for the Klein–Gordon equation, and properties (2.28)–(2.29):

$$\begin{aligned}\|\mathcal{G}(t)\Psi''_{0,t}\|_{\mathcal{F}_{-\sigma}} &\leqslant \|\mathcal{G}(t)\Psi''_{0,t}\|_{\mathcal{F}_0} \leqslant C\|\Psi''_{0,t}\|_{\mathcal{F}_0} \leqslant C_1 t^{-\sigma}\|\Psi''_{0,t}\|_{\mathcal{F}_\sigma} \\ &\leqslant C_2 t^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geqslant 1\end{aligned} \tag{2.30}$$

since $\sigma > 5/2$. Second we estimate $\mathcal{G}_0(t)\Psi''_{0,t}$ which is constant matrix. Let $\Psi''_{0,t} = (\psi''_{0,t}, \pi''_{0,t})$. Formula (2.18) and Cauchy inequality imply

$$\begin{aligned}\big|\mathcal{G}_0^{i1}(t)\psi''_{0,t}\big| &\leqslant \frac{C}{t}\int \big|\psi''_{0,t}(x)\big|\,dx \\ &\leqslant \frac{C}{t}\bigg(\int \big|\pi''_{0,t}(x)\big|^2\big(1+|x|^2\big)^\sigma\,dx\bigg)^{1/2}\bigg(\int\limits_{|x|>t/4}\frac{dx}{(1+|x|^2)^\sigma}\bigg)^{1/2} \\ &\leqslant C_1 t^{-\sigma}\|\psi''_{0,t}\|_{H^0_\sigma} \leqslant C_1 t^{-5/2}\|\psi''_{0,t}\|_{H^0_\sigma}, \quad i = 1,2\end{aligned}$$

since $\sigma > 5/2$. Similarly, $|\mathcal{G}_0^{i2}(t)\pi''_{0,t}| \leqslant Ct^{-5/2}\|\pi''_{0,t}\|_{H^0_\sigma}$, $i = 1,2$. Therefore,

$$\|\mathcal{G}_0(t)\Psi''_{0,t}\|_{\mathcal{F}_{-\sigma}} \leqslant Ct^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geqslant 1 \tag{2.31}$$

and (2.30)–(2.31) imply that

$$\|\mathcal{G}_r(t)\Psi''_{0,t}\|_{\mathcal{F}_{-\sigma}} \leqslant Ct^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geqslant 1. \tag{2.32}$$

Step (ii). Now we consider $\mathcal{G}_r(t)\Psi'_{0,t}$. We split the operator $\mathcal{G}_r(t)$, for $t \geqslant 1$ in two terms:

$$\mathcal{G}_r(t) = (1-\zeta)\mathcal{G}_r(t) + \zeta\mathcal{G}_r(t)$$

where ζ is the operator of multiplication by the function $\zeta(|x|/t)$ such that $\zeta = \zeta(s) \in C_0^\infty(\mathbb{R})$, $\zeta(s) = 1$ for $|s| < 1/4$, $\zeta(s) = 0$ for $|s| > 1/3$. Obviously, for $|\alpha| \leqslant 1$, we have

$$\left|\partial_x^\alpha \zeta\big(|x|/t\big)\right| \leqslant C < \infty, \quad t \geqslant 1.$$

Furthermore, $1 - \zeta(|x|/t) = 0$ for $|x| < t/4$, then by the energy conservation and (2.28), we obtain

$$\begin{aligned}\left\|(1-\zeta)\mathcal{G}(t)\Psi'_{0,t}\right\|_{\mathcal{F}_{-\sigma}} &\leqslant Ct^{-\sigma}\left\|(1-\zeta)\mathcal{G}(t)\Psi'_{0,t}\right\|_{\mathcal{F}_0} \leqslant C_1 t^{-\sigma}\left\|\mathcal{G}(t)\Psi'_{0,t}\right\|_{\mathcal{F}_0} \\ &\leqslant C_2 t^{-\sigma}\left\|\Psi'_{0,t}\right\|_{\mathcal{F}_0} \leqslant C_3 t^{-\sigma}\left\|\Psi'_{0,t}\right\|_{\mathcal{F}_\sigma} \\ &\leqslant C_4 t^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geqslant 1 \end{aligned} \tag{2.33}$$

since $\sigma > 5/2$.

Now we estimate $(1-\zeta)\mathcal{G}(t)\Psi'_{0,t}$. Let $\Psi'_{0,t} = (\psi'_{0,t}, \pi'_{0,t})$. We have

$$\left|\mathcal{G}_0^{i1}(t)\psi'_{0,t}\right| \leqslant \frac{C}{t}\left\|\psi'_{0,t}\right\|_{H^0_\sigma}, \qquad \left|\mathcal{G}_0^{i2}(t)\pi'_{0,t}\right| \leqslant \frac{C}{t}\left\|\pi'_{0,t}\right\|_{H^0_\sigma}, \quad i = 1, 2.$$

Hence,

$$\begin{aligned}\left\|(1-\zeta)\mathcal{G}_0^{i2}(t)\psi'_{0,t}\right\|_{H^1_{-\sigma}} &\leqslant \frac{C}{t}\left\|\psi'_{0,t}\right\|_{H^0_\sigma}\Bigg(\int\limits_{|x|>t/4} \frac{dx}{(1+|x|^2)^\sigma}\Bigg)^{\frac{1}{2}} \\ &\leqslant C_1 t^{-\sigma}\left\|\psi'_{0,t}\right\|_{H^0_\sigma}, \quad i = 1, 2.\end{aligned}$$

Similarly, $\|(1-\zeta)\mathcal{G}_0^{i2}(t)\pi'_{0,t}\|_{H^1_{-\sigma}} \leqslant Ct^{-\sigma}\|\pi'_{0,t}\|_{H^0_\sigma}$, $i = 1, 2$. Therefore,

$$\left\|(1-\zeta)\mathcal{G}_0(t)\Psi'_{0,t}\right\|_{\mathcal{F}_{-\sigma}} \leqslant Ct^{-\sigma}\|\Psi_0\|_{\mathcal{F}_\sigma} \leqslant Ct^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geqslant 1 \tag{2.34}$$

and (2.33)–(2.34) imply

$$\left\|(1-\zeta)\mathcal{G}_r(t)\Psi'_{0,t}\right\|_{\mathcal{F}_{-\sigma}} \leqslant Ct^{-5/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geqslant 1. \tag{2.35}$$

Step (iii). Finally, we estimate $\zeta\mathcal{G}_r(t)\Psi'_{0,t}$. Let χ_t be the characteristic function of the ball $|x| \leqslant t/3$. We will use the same notation for the operator of multiplication by this characteristic function. By (2.29), we have

$$\zeta\mathcal{G}_r(t)\Psi'_{0,t} = \zeta\mathcal{G}_r(t)\chi_t\Psi'_{0,t}. \tag{2.36}$$

The matrix kernel of the operator $\zeta\mathcal{G}_r(t)\chi_t$ is equal to

$$\mathcal{G}'_r(x-y,t)=\zeta\big(|x|/t\big)\mathcal{G}_r(x-y,t)\chi_t(y).$$

Lemma 2.10. *The bound holds*

$$\big|\partial_z^\alpha \mathcal{G}_r(t,z)\big|\leqslant Ct^{-3/2}\big(1+|z|\big)^{3/2},\quad |z|\leqslant 2t/3,\ t\geqslant 1,\ |\alpha|\leqslant 1. \tag{2.37}$$

We prove the lemma in Appendix B.

Since $\zeta(|x|/t)=0$ for $|x|>t/3$ and $\chi_t(y)=0$ for $|y|>t/3$, the estimate (2.37) implies that

$$\big|\partial_x^\alpha \mathcal{G}'_r(x-y,t)\big|\leqslant Ct^{-3/2}\big(1+|z|\big)^{3/2},\quad |\alpha|\leqslant 1,\ t\geqslant 1. \tag{2.38}$$

The norm of the operator $\zeta\mathcal{G}_r(t)\chi_t:\mathcal{F}_\sigma\to\mathcal{F}_{-\sigma}$ is equivalent to the norm of the operator

$$\langle x\rangle^{-\sigma}\zeta\mathcal{G}_r(t)\chi_t(y)\langle y\rangle^{-\sigma}:\mathcal{F}_0\to\mathcal{F}_0.$$

The norm of the later operator does not exceed the sum in α, $|\alpha|\leqslant 1$ of the norms of operators

$$\partial_x^\alpha\big[\langle x\rangle^{-\sigma}\zeta\mathcal{G}_r(t)\chi_t(y)\langle y\rangle^{-\sigma}\big]:L^2\big(\mathbb{R}^2\big)\oplus L^2\big(\mathbb{R}^2\big)\to L^2\big(\mathbb{R}^2\big)\oplus L^2\big(\mathbb{R}^2\big). \tag{2.39}$$

The estimates (2.38) imply that operators (2.39) are Hilbert–Schmidt operators since $\sigma>5/2$, and their Hilbert–Schmidt norms do not exceed $Ct^{-3/2}$. Hence, (2.28) and (2.36) imply that

$$\big\|\zeta\mathcal{G}_r(t)\Psi'_{0,t}\big\|_{\mathcal{F}_{-\sigma}}\leqslant Ct^{-3/2}\big\|\Psi'_{0,t}\big\|_{\mathcal{F}_\sigma}\leqslant Ct^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\geqslant 1. \tag{2.40}$$

Finally, the estimates (2.40), (2.35) imply

$$\big\|\mathcal{G}_r(t)\Psi'_{0,t}\big\|_{\mathcal{F}_{-\sigma}}\leqslant Ct^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\geqslant 1. \tag{2.41}$$

Proposition 2.7 is proved.

3. Perturbed Klein–Gordon equation

To prove the long time decay for the perturbed Klein–Gordon equation, we first establish the spectral properties of the generator.

3.1. Spectral properties

According [20, formula (3.1)], let us introduce a generalized eigenspace $\mathbf{M}$ for the perturbed Schrödinger operator $H=-\Delta+V$:

$$\mathbf{M}=\big\{\psi\in H^1_{-1/2-0}\colon\ (1+B_0V)\psi\in\Re(A_0),\ A_0V\psi=0\big\}.$$

Where A_0 and B_0 are defined in (2.8), and $\Re(A_0)$ is the range of A_0.

Below we assume that

$$\mathbf{M} = 0. \tag{3.1}$$

Remark 3.1. $N(H) \subset \mathbf{M}$ where $N(H)$ is the zero eigenspace of the operator H. This embedding is obtained in [20, Lemma 3.2]. The functions from $\mathbf{M} \setminus N(H)$ are called *zero resonance functions*. Hence, the condition (3.1) means that $\lambda = 0$ is neither eigenvalue nor resonance for the operator H.

The condition (3.1) corresponds to the “nonsingular case” in [20, Section 7].

Denote by $R(\zeta) = (H - \zeta)^{-1}$, $\zeta \in \mathbb{C} \setminus [0, \infty)$, the resolvent of the Schrödinger operator H. Let us collect the properties of $R(\zeta)$ which are obtained in [1,10,20] under conditions (1.4) and (3.1). Note, that in [10] is considered 3D case, but corresponding properties can be proved in 2D case similarly.

R1. $R(\zeta)$ is strongly meromorphic function of $\zeta \in \mathbb{C} \setminus [0, \infty)$ with the values in $\mathcal{L}(H_0^{-1}, H_0^1)$; the poles of $R(\zeta)$ are located at a finite set of eigenvalues $\zeta_j < 0$, $j = 1, \ldots, N$, of the operator H with the corresponding eigenfunctions $\psi_j^1, \ldots, \psi_j^{k_j} \in H_s^2$ with any $s \in \mathbb{R}$ where k_j is the multiplicity of ζ_j.

R2. For $\zeta > 0$, the convergence holds $R(\zeta \pm i\varepsilon) \to R(\zeta \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)$ with $\sigma > 1/2$, uniformly in $\zeta \geqslant \rho$ for any $\rho > 0$ (cf. [10, Lemma 9.1]).

Now we obtain the asymptotics for $R(\zeta)$, $R'(\zeta)$ and $R''(\zeta)$ at $\zeta = 0$.

Proposition 3.2. *Under the conditions* (1.4) *and* (3.1) *the asymptotics hold*

$$\left.\begin{aligned} R(\zeta) &= A_1 + A_2 \log^{-1}\zeta + \mathcal{O}\big(\log^{-2}\zeta\big) \\ R'(\zeta) &= -A_2\zeta^{-1}\log^{-2}\zeta + \mathcal{O}\big(\zeta^{-1}\log^{-3}\zeta\big) \\ R''(\zeta) &= \mathcal{O}\big(\zeta^{-2}\log^{-2}\zeta\big) \end{aligned}\right| \quad \zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0, \infty) \tag{3.2}$$

in the norms of $\mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)$ *with* $\sigma > 5/2$.

We deduce Proposition 3.2 from the following three lemmas. The first lemma is proved in [20].

Lemma 3.3. *(See [20, Theorem 7.2].) The family* $\{R(\zeta),\ |\zeta| < \varepsilon,\ \zeta \in \mathbb{C} \setminus [0, \infty)\}$ *is bounded in the operator norm of* $\mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)$ *for any* $\sigma > 1$ *and sufficiently small* $\varepsilon > 0$.

Corollary 3.4. *For any* $1 < \sigma < \beta/2$, *the operators* $(1 + R_0(\zeta)V)^{-1} = 1 - R(\zeta)V$ *and* $(1 + VR_0(\zeta))^{-1} = 1 - VR(\zeta)$ *are bounded respectively in* $\mathcal{L}(H_{-\sigma}^1, H_{-\sigma}^1)$ *and in* $\mathcal{L}(H_\sigma^{-1}, H_\sigma^{-1})$ *for* $|\zeta| < \varepsilon$, $\zeta \in \mathbb{C} \setminus [0, \infty)$ *and sufficiently small* $\varepsilon > 0$.

Lemma 3.5.

(i) *The bound holds*

$$\big\| \big(1 + R_0(\lambda)V\big)^{-1}[1] \big\|_{H_{-\sigma}^1} = \mathcal{O}\big(\big|\log^{-1}\zeta\big|\big), \quad \zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0, \infty),\ \sigma > 5/2 \tag{3.3}$$

where 1 *stands for the constant function* $f(x) \equiv 1$.

(ii) *For any* $f \in H^{-1}_\sigma$ *with* $\sigma > 5/2$

$$\int \big[\big(1 + VR_0(\zeta)\big)^{-1} f\big](y)\,dy = \mathcal{O}\big(\log^{-1}\zeta\big), \quad \zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0,\infty). \tag{3.4}$$

Proof. The asymptotics (2.8) implies

$$R(\zeta) = \big(1 + R_0(\zeta)V\big)^{-1}R_0(\zeta) = \big(1 + R_0(\zeta)V\big)^{-1}\big[A_0\log\zeta + B_0 + \mathcal{O}\big(\zeta^{3/4}\big)\big],$$
$$R(\zeta) = R_0(\zeta)\big(1 + VR_0(\zeta)\big)^{-1} = \big[A_0\log\zeta + B_0 + \mathcal{O}\big(\zeta^{3/4}\big)\big]\big(1 + VR_0(\zeta)\big)^{-1}. \tag{3.5}$$

Hence, the boundedness $R(\zeta)$, $(1 + R_0(\zeta)V)^{-1}$ and $(1 + VR_0(\zeta))^{-1}$ at $\zeta = 0$ in the corresponding norms imply the bounds

$$\big(1 + R_0(\zeta)V\big)^{-1}A_0 = \mathcal{O}\big(\log^{-1}\zeta\big),$$
$$A_0\big(1 + VR_0(\zeta)\big)^{-1} = \mathcal{O}\big(\log^{-1}\zeta\big), \quad \zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0,\infty)$$

in $\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})$ with $\sigma > 5/2$. Then (3.3) and (3.4) follow by (2.9). □

Now we obtain the bounds for the first and second derivatives of $R(\zeta)$ at $\zeta = 0$.

Lemma 3.6. *The bounds hold*

$$R'(\zeta) = \mathcal{O}\big(\zeta^{-1}\log^{-2}\zeta\big), \quad \zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0,\infty), \tag{3.6}$$
$$R''(\zeta) = \mathcal{O}\big(\zeta^{-2}\log^{-2}\zeta\big), \quad \zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0,\infty) \tag{3.7}$$

in the norm $\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})$ *with* $\sigma > 5/2$.

Proof. The statement follow from the bounds (2.10), (3.3)–(3.4) and the identities

$$R' = (1 + R_0V)^{-1}R_0'(1 + VR_0)^{-1},$$
$$R'' = \big[(1 + R_0V)^{-1}R_0'' - 2R'VR_0'\big](1 + VR_0)^{-1}. \qquad \square \tag{3.8}$$

Proof of Proposition 3.2. Integrating (3.6), we obtain

$$R(\zeta) = A_1 + \mathcal{O}\big(\log^{-1}\zeta\big), \quad \zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0,\infty) \tag{3.9}$$

in the norm $\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})$ with $\sigma > 5/2$. Therefore we can refine the bounds (3.3) and (3.4). Namely, formulas (3.5) and asymptotics (3.9) imply

$$\big(1 + R_0(\lambda)V\big)^{-1}A_0 = D_1\log^{-1}\zeta + \mathcal{O}\big(\log^{-2}\zeta\big),$$
$$A_0\big(1 + VR_0(\zeta)\big)^{-1} = D_2\log^{-1}\zeta + \mathcal{O}\big(\log^{-2}\zeta\big), \tag{3.10}$$

as $\zeta \to 0$, $\zeta \in \mathbb{C} \setminus [0,\infty)$ in the norm $\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})$ with $\sigma > 5/2$. Applying (3.10) to (3.8), we obtain by (2.8)

$$R'(\zeta) = -A_2\zeta^{-1}\log^{-2}\zeta + \mathcal{O}\big(\zeta^{-1}\log^{-3}\zeta\big), \quad \zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0,\infty) \tag{3.11}$$

in the norm $\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})$ with $\sigma > 5/2$. Finally, integrating (3.11), we obtain

$$R(\zeta) = A_1 + A_2\log^{-1}\zeta + \mathcal{O}\big(\log^{-2}\zeta\big), \quad \zeta \to 0,\ \zeta \in \mathbb{C} \setminus [0,\infty) \tag{3.12}$$

in the norm $\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})$ with $\sigma > 5/2$. Proposition 3.2 is proved. □

Further, the resolvent $\mathcal{R}(\omega) = (\mathcal{H} - \omega)^{-1}$, can be expressed similarly to (2.6):

$$\mathcal{R}(\omega) = \begin{pmatrix} \omega R(\omega^2 - m^2) & iR(\omega^2 - m^2) \\ -i(1 + \omega^2 R(\omega^2 - m^2)) & \omega R(\omega^2 - m^2) \end{pmatrix}. \tag{3.13}$$

Hence, the properties R1–R2 and Proposition 3.2 imply the corresponding properties of $\mathcal{R}(\omega)$:

Lemma 3.7. *Let the potential V satisfy* (1.4) *and* (3.1). *Then*

(i) *$\mathcal{R}(\omega)$ is strongly meromorphic function of $\omega \in \mathbb{C} \setminus \overline{\Gamma}$ with the values in $\mathcal{L}(\mathcal{F}_0, \mathcal{F}_0)$;*
(ii) *The poles of $\mathcal{R}(\omega)$ are located at a finite set*

$$\Sigma = \big\{\omega_j^{\pm} = \pm\sqrt{m^2 + \zeta_j},\ j = 1, \ldots, N\big\}$$

of eigenvalues of the operator $\mathcal{H}$ with the corresponding eigenfunctions $\begin{pmatrix} \psi_j^k(x) \\ -i\omega_j^{\pm}\psi_j^k(x) \end{pmatrix}$, $k = 1, \ldots, k_j$;
(iii) *For $\omega \in \Gamma$, the convergence holds $\mathcal{R}(\omega \pm i\varepsilon) \to \mathcal{R}(\omega \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ for $\sigma > 1/2$, uniformly in $|\omega| \geqslant m + r$ for any $r > 0$;*
(iv) *The asymptotics hold in the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$ with $\sigma > 5/2$*

$$\begin{aligned} \mathcal{R}(\omega) &= \mathcal{A}_1^{\pm} + \mathcal{A}_2^{\pm}\log^{-1}(\omega \mp m) + \mathcal{O}\big(\log^{-2}(\omega \mp m)\big), \\ \mathcal{R}'(\omega) &= -\mathcal{A}_2^{\pm}(\omega \mp m)^{-1}\log^{-2}(\omega \mp m) + \mathcal{O}\big((\omega \mp m)^{-1}\log^{-3}(\omega \mp m)\big), \\ \mathcal{R}''(\omega) &= \mathcal{O}\big((\omega \mp m)^{-2}\log^{-2}(\omega \mp m)\big) \end{aligned} \tag{3.14}$$

as $\omega \to \pm m$, $\omega \in \mathbb{C} \setminus \overline{\Gamma}$.

Finally, we obtain the asymptotics of $\mathcal{R}(\omega)$ for large ω.

Lemma 3.8. *Let the potential V satisfy* (1.4). *Then for $s = 0, 1$ and $l = -1, 0, 1$ with $s + l \in \{0, 1\}$ we have*

$$\big\|R^{(k)}(\zeta)\big\|_{\mathcal{L}(H^s_\sigma, H^{s+l}_{-\sigma})} = \mathcal{O}\big(|\zeta|^{-\frac{1-l+k}{2}}\big), \quad |\zeta| \to \infty,\ \zeta \in \mathbb{C} \setminus [0,\infty) \tag{3.15}$$

with $\sigma > 1/2 + k$ for $k = 0, 1, 2$.

Proof. The lemma follows from [14, Proposition A1] by the arguments from the proof of Theorem 9.2 in [10], where the bounds are proved for $s = 0$ and $l = 0, 1$. $\square$

Hence (3.13) implies

Corollary 3.9. *Let the potential V satisfy* (1.4). *Then the bounds hold*

$$\big\|\mathcal{R}^{(k)}(\omega)\big\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(1), \quad |\omega| \to \infty,\ \omega \in \mathbb{C} \setminus \Gamma \tag{3.16}$$

with $\sigma > 1/2 + k$ for $k = 0, 1, 2$.

Further, let us denote by $\mathcal{V}$ the matrix

$$\mathcal{V} = \begin{pmatrix} 0 & 0 \\ -iV & 0 \end{pmatrix}. \tag{3.17}$$

Then the vectorial equation (1.2) reads

$$i\dot{\Psi}(t) = (\mathcal{H}_0 + \mathcal{V})\Psi(t). \tag{3.18}$$

The resolvents $\mathcal{R}(\omega)$, $\mathcal{R}_0(\omega)$ are related by the Born perturbation series

$$\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega) + \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega), \quad \omega \in \mathbb{C} \setminus [\overline{\Gamma} \cup \Sigma] \tag{3.19}$$

which follows by iteration of $\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)$. An important role in (3.19) plays the product $\mathcal{W}(\omega) := \mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}$. We obtain the asymptotics of $\mathcal{W}(\omega)$ for large ω.

Lemma 3.10. *Let $k = 0, 1, 2$, and the potential V satisfy* (1.4) *with $\beta > 1/2 + k + \sigma$ where $\sigma > 0$. Then the asymptotics hold*

$$\big\|\mathcal{W}^{(k)}(\omega)\big\|_{\mathcal{L}(\mathcal{F}_{-\sigma},\mathcal{F}_{\sigma})} = \mathcal{O}\big(|\omega|^{-2}\big), \quad |\omega| \to \infty,\ \omega \in \mathbb{C} \setminus \Gamma. \tag{3.20}$$

Proof. Bounds (3.20) follow from the algebraic structure of the matrix

$$\mathcal{W}^{(k)}(\omega) = \mathcal{V}\mathcal{R}_0^{(k)}(\omega)\mathcal{V} = \begin{pmatrix} 0 & 0 \\ -iV\partial_\omega^k R_0(\omega^2 - m^2)V & 0 \end{pmatrix} \tag{3.21}$$

since (2.13) with $s = 1$ and $l = -1$ implies that

$$\begin{aligned} \big\|VR_0^{(k)}(\zeta)Vf\big\|_{H^0_\sigma} &\leqslant C\big\|R_0^{(k)}(\zeta)Vf\big\|_{H^0_{\sigma-\beta}} = \mathcal{O}\big(|\zeta|^{-1-\frac{k}{2}}\big)\|Vf\|_{H^1_{\beta-\sigma}} \\ &= \mathcal{O}\big(|\zeta|^{-1-\frac{k}{2}}\big)\|f\|_{H^1_{-\sigma}} \end{aligned}$$

for $1/2 + k < \beta - \sigma$. $\square$

3.2. Time decay

In this section we combine the spectral properties of the perturbed resolvent and time decay for the unperturbed dynamics using the (finite) Born perturbation series. Our main result is the following.

Theorem 3.11. *Let the potential V satisfy* (1.4) *and* (3.1). *Then*

$$\Big\| e^{-it\mathcal{H}} - \sum_{\omega_J\in\Sigma} e^{-i\omega_J t} P_j \Big\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}\big(|t|^{-1}\log^{-2}|t|\big), \quad t\to\pm\infty \tag{3.22}$$

with $\sigma > 5/2$, where P_j are the Riesz projectors onto the corresponding eigenspaces.

Proof. *Step* (i). Lemma 3.7(iii) and asymptotics (3.14) and (3.16) with $k=0$ imply similarly to (2.15), that

$$\begin{aligned}\Psi(t) - \sum_{\omega_J\in\Sigma} e^{-i\omega_j t} P_J\Psi_0 &= \frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}\big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\big]\Psi_0\,d\omega \\ &= \Psi_l(t)+\Psi_h(t)\end{aligned} \tag{3.23}$$

where P_J stands for the corresponding Riesz projection

$$P_j\Psi_0 := -\frac{1}{2\pi i}\int_{|\omega-\omega_J|=\delta}\mathcal{R}(\omega)\Psi_0\,d\omega$$

with a small $\delta>0$, and

$$\Psi_l(t) = \frac{1}{2\pi i}\int_\Gamma l(\omega)e^{-i\omega t}\big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\big]\Psi_0\,d\omega,$$

$$\Psi_h(t) = \frac{1}{2\pi i}\int_\Gamma h(\omega)e^{-i\omega t}\big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\big]\Psi_0\,d\omega$$

where $l(\omega)$ and $h(\omega)$ are defined in Section 2.2. Further we analyze $\Psi_l(t)$ and $\Psi_h(t)$ separately.

3.2.1. Low energy component

We consider only the integral over $(m, m+2\varepsilon)$. The integral over $(-m-2\varepsilon, -m)$ deal with the same way. We prove the desired decay of $\Psi_l(t)$ using a special case of Lemma 10.2 from [10].

Lemma 3.12. *Assume* **B** *be a Banach space, and $F\in C(a,b;\mathbf{B})$ satisfies $F(a)=0$ and $F(\omega)=0$ for $\omega>b>a$, $F'\in L^1(a+\delta,b;\mathbf{B})$ for any $\delta>0$. Moreover, $F'(\omega)=\mathcal{O}((\omega-a)^{-1}\ln^{-3}(\omega-a))$ as well as $F''(\omega)=\mathcal{O}((\omega-a)^{-2}\log^{-2}(\omega-a))$ as $\omega\to a+0$. Then*

$$\int_a^\infty e^{-it\omega}F(\omega)\,d\omega = \mathcal{O}\big(t^{-1}\ln^{-2}t\big), \quad t\to\infty$$

in the norm of **B**.

Proof. Extending F by $F(\omega)=0$ for $\omega<a$, we obtain a function F on $(-\infty,\infty)$ with $F'\in L^1(-\infty,\infty;\mathbf{B})$. For $t>0$ we have

$$\int_{-\infty}^{\infty} F'(\omega)e^{-it\omega}\,d\omega=-\frac{1}{2}\int_{-\infty}^{\infty}\biggl(F'\biggl(\omega+\frac{\pi}{t}\biggr)-F'(\omega)\biggr)e^{-it\omega}\,d\omega. \tag{3.24}$$

Finally,

$$\begin{aligned}\int_{-\infty}^{\infty}\biggl\|F'\biggl(\omega+\frac{\pi}{t}\biggr)-F'(\omega)\biggr\|\,d\omega&=\int_{-\infty}^{a+\pi/t}+\cdots+\int_{a+\pi/t}^{\infty}\cdots\\&\leqslant 2\int_{a}^{a+2\pi/t}\bigl\|F'(\omega)\bigr\|\,d\omega+\int_{a+\pi/t}^{\infty}d\omega\int_{\omega}^{\omega+\pi/t}\bigl\|F''(\mu)\bigr\|\,d\mu\\&=\mathcal{O}\bigl(\ln^{-2}t\bigr)+\frac{\pi}{t}\int_{a+\pi/t}^{\infty}\bigl\|F''(\mu)\bigr\|\,d\mu=\mathcal{O}\bigl(\ln^{-2}t\bigr).\end{aligned} \tag{3.25}$$

Therefore, (3.24) implies that the Fourier transform of F' is $\mathcal{O}(\ln^{-2}t)$, and hence the Fourier transform of F is $\mathcal{O}(t^{-1}\ln^{-2}t)$ as $t\to\infty$. □

Due to (3.14), we can apply Lemma 3.12 with $F=l(\omega)(\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0))$, $\mathbf{B}=\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$, $a=m$, $b=m+2\varepsilon$ with a small $\varepsilon>0$ and $\sigma>5/2$, to get

$$\bigl\|\Psi_l(t)\bigr\|_{\mathcal{F}_{-\sigma}}\leqslant C\|\Psi_0\|_{\mathcal{F}_\sigma}\bigl(1+|t|\bigr)^{-1}\log^{-2}\bigl(1+|t|\bigr),\quad t\in\mathbb{R},\ \sigma>5/2. \tag{3.26}$$

3.2.2. High energy component

Let us substitute the series (3.19) into the spectral representation for $\Psi_h(t)$:

$$\begin{aligned}\Psi_h(t)&=\frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}h(\omega)\bigl[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\bigr]\Psi_0\,d\omega\\&\quad+\frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}h(\omega)\bigl[\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\bigr]\Psi_0\,d\omega\\&\quad+\frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}h(\omega)\bigl[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega+i0)-\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega-i0)\bigr]\Psi_0\,d\omega\\&=\Psi_{h1}(t)+\Psi_{h2}(t)+\Psi_{h3}(t),\quad t\in\mathbb{R}.\end{aligned} \tag{3.27}$$

Further we analyze each term Ψ_{hk} separately.

Step (i). The first term $\Psi_{h1}(t)=\mathcal{G}_h(t)\Psi_0$ by (2.21). Hence, Theorem 2.9 implies that

$$\left\|\Psi_{h1}(t)\right\|_{\mathcal{F}_{-\sigma}}\leqslant\frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}},\quad t\in\mathbb{R},\ \sigma>5/2.\tag{3.28}$$

Step (ii). Let us consider the second term $\Psi_{h2}(t)$. Denote $h_1(\omega)=\sqrt{h(\omega)}$ (we can assume that $h(\omega)\geqslant 0$ and $h_1\in\mathbb{C}_0^\infty(\mathbb{R})$). Let us set

$$\Phi_{h1}=\frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t}h_1(\omega)\big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\big]\Psi_0\,d\omega.$$

It is obvious that for Φ_{h1} the inequality (3.28) also holds. Namely,

$$\left\|\Phi_{h1}(t)\right\|_{\mathcal{F}_{-\sigma}}\leqslant\frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t|)^{3/2}},\quad t\in\mathbb{R},\ \sigma>5/2.\tag{3.29}$$

Now the second term $\Psi_{h2}(t)$ can be rewritten as a convolution.

Lemma 3.13. *The convolution representation holds*

$$\Psi_{h2}(t)=i\int\limits_0^t\mathcal{G}_h(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\,d\tau,\quad t\in\mathbb{R}\tag{3.30}$$

where the integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma>5/2$.

Proof. Then the term $\Psi_{h2}(t)$ can be rewritten as

$$\begin{aligned}\Psi_{h2}(t)=\frac{1}{2\pi i}\int\limits_{\mathbb{R}}e^{-i\omega t}h_1^2(\omega)\big[&\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0)\\&-\mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\big]\Psi_0\,d\omega.\end{aligned}\tag{3.31}$$

Let us integrate the first term in the right-hand side of (3.31), denoting

$$\mathcal{G}_h^\pm(t):=\theta(\pm t)\mathcal{G}_h(t),\qquad\Phi_{h1}^\pm(t):=\theta(\pm t)\Phi_{h1}(t),\quad t\in\mathbb{R}.$$

We know that $h_1(\omega)\mathcal{R}_0(\omega+i0)\Psi_0=i\tilde{\Phi}_{h1}^+(\omega)$, hence integrating the first term in the right-hand side of (3.31), we obtain that

$$\begin{aligned}\Psi_{h2}^+(t)&=\frac{1}{2\pi}\int\limits_{\mathbb{R}}e^{-i\omega t}h_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\tilde{\Phi}_{h1}^+(\omega)\,d\omega\\&=\frac{1}{2\pi}\int\limits_{\mathbb{R}}e^{-i\omega t}h_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int\limits_{\mathbb{R}}e^{i\omega\tau}\Phi_{h1}^+(\tau)\,d\tau\Big]\,d\omega\end{aligned}$$

$$= \frac{1}{2\pi}(i\partial_t + i)^2 \int_{\mathbb{R}} \frac{e^{-i\omega t}}{(\omega+i)^2} h_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int_{\mathbb{R}} e^{i\omega\tau}\Phi_{h1}^+(\tau)\,d\tau\Big]\,d\omega. \quad (3.32)$$

The last double integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma > 5/2$ by (3.29), Lemma 2.3(ii), and (2.14) with $k = 0$. Hence, we can change the order of integration by the Fubini theorem. Then we obtain that

$$\Psi_{h2}^+(t) = i\int_{\mathbb{R}} \mathcal{G}_h^+(t-\tau)\mathcal{V}\Phi_{h1}^+(\tau)\,d\tau = \begin{cases} i\int_0^t \mathcal{G}_h(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\,d\tau, & t>0, \\ 0, & t<0 \end{cases} \quad (3.33)$$

since

$$\mathcal{G}_h^+(t-\tau) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega(t-\tau)} h_1(\omega)\mathcal{R}_0(\omega+i0)\ d\omega$$
$$= \frac{1}{2\pi i}(i\partial_t + i)^2 \int_{\mathbb{R}} \frac{e^{-i\omega(t-\tau)}}{(\omega+i)^2} h_1(\omega)\mathcal{R}_0(\omega+i0)\,d\omega \quad (3.34)$$

by (2.4). Similarly, integrating the second term in the right-hand side of (3.31), we obtain

$$\Psi_{h2}^-(t) = i\int_{\mathbb{R}} \mathcal{G}_h^-(t-\tau)\mathcal{V}\Phi_{h1}^-(\tau)\,d\tau$$
$$= \begin{cases} 0, & t>0, \\ i\int_0^t \mathcal{G}_h(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\,d\tau, & t<0. \end{cases} \quad (3.35)$$

Now (3.30) follows since $\Psi_{h2}(t)$ is the sum of two expressions (3.33) and (3.35). □

Applying Theorem 2.9 with h_1 instead of h to the integrand in (3.30), we obtain that

$$\big\|\mathcal{G}_h(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\big\|_{\mathcal{F}_{-\sigma}} \leqslant \frac{C\|\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}'_\sigma}}{(1+|t-\tau|)^{3/2}} \leqslant \frac{C\|\Phi_{h1}(\tau)\|_{\mathcal{F}_{\sigma'-\beta}}}{(1+|t-\tau|)^{3/2}}$$
$$\leqslant \frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{3/2}(1+|\tau|)^{3/2}}$$

where $\sigma' \in (5/2, \beta - 5/2)$. Therefore integrating here in τ, we obtain by (3.30) that

$$\big\|\Psi_{h2}(t)\big\|_{\mathcal{F}_{-\sigma}} \leqslant C\big(1+|t|\big)^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t\in\mathbb{R},\ \sigma>5/2. \quad (3.36)$$

Step (iv). Finally, let us rewrite the last term as

$$\Psi_{h3}(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{-i\omega t} h(\omega)\mathcal{N}(\omega)\Psi_0\,d\omega,$$

where $\mathcal{N} := \mathcal{M}(\omega + i0) - \mathcal{M}(\omega - i0)$ for $\omega \in \Gamma$, and

$$\mathcal{M}(\omega) = \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega) = \mathcal{R}_0(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega), \quad \omega \in \mathbb{C} \setminus [\overline{\Gamma} \cup \Sigma]. \tag{3.37}$$

Now we obtain the asymptotics of $\mathcal{M}(\omega)$ and its derivatives for large ω.

Lemma 3.14. *For $k = 0, 1, 2$ the asymptotics hold*

$$\big\|\mathcal{M}^{(k)}(\omega)\big\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} = \mathcal{O}\big(|\omega|^{-2}\big), \quad |\omega| \to \infty,\ \omega \in \mathbb{C} \setminus \Gamma,\ \sigma > 1/2 + k. \tag{3.38}$$

Proof. The asymptotics (3.38) follow from the asymptotics (2.14), (3.16) and (3.20) for $\mathcal{R}_0^{(k)}(\omega)$, $\mathcal{R}^{(k)}(\omega)$ and $\mathcal{W}^{(k)}(\omega)$. For example, let us consider the case $k = 2$. Differentiating (3.37), we obtain

$$\mathcal{M}'' = \mathcal{R}_0''\mathcal{W}\mathcal{R} + \mathcal{R}_0\mathcal{W}''\mathcal{R} + \mathcal{R}_0\mathcal{W}\mathcal{R}'' + 2\mathcal{R}_0'\mathcal{W}'\mathcal{R} + 2\mathcal{R}_0'\mathcal{W}\mathcal{R}' + 2\mathcal{R}_0\mathcal{W}'\mathcal{R}'. \tag{3.39}$$

For a fixed $\sigma > 5/2$, let us choose $\sigma' \in (5/2, \beta - 1/2)$. Then for the first term in (3.39) we obtain by (3.16) and (3.20)

$$\begin{aligned}\big\|\mathcal{R}_0''(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega)f\big\|_{\mathcal{F}_{-\sigma}} &\leqslant \big\|\mathcal{R}_0''(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega)f\big\|_{\mathcal{F}_{-\sigma'}} \leqslant C\big\|\mathcal{W}(\omega)\mathcal{R}(\omega)f\big\|_{\mathcal{F}_{\sigma'}}\\ &= \mathcal{O}\big(|\omega|^{-2}\big)\big\|\mathcal{R}(\omega)f\big\|_{\mathcal{F}_{-\sigma'}} = \mathcal{O}\big(|\omega|^{-2}\big)\|f\|_{\mathcal{F}_{\sigma'}}\\ &= \mathcal{O}\big(|\omega|^{-2}\big)\|f\|_{\mathcal{F}_\sigma}, \quad |\omega| \to \infty,\ \omega \in \mathbb{C} \setminus \Gamma.\end{aligned}$$

Other terms can be estimated similarly choosing an appropriate value of σ'. Namely, $\sigma' \in (1/2, \beta - 5/2)$ for the second term, $\sigma' \in (5/2, \beta - 1/2)$ for the third, $\sigma' \in (3/2, \beta - 3/2)$ for the forth and sixth terms, and $\sigma' \in (3/2, \beta - 1/2)$ for the fifth term. □

Now we prove the decay of $\Psi_{h3}(t)$. By Lemma 3.14

$$(h\mathcal{N})'' \in L^1\big((-\infty, -m - \varepsilon) \cup (m + \varepsilon, \infty); \mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})\big)$$

with $\sigma > 5/2$. Hence, two times partial integration implies that

$$\big\|\Psi_{h3}(t)\big\|_{\mathcal{F}_{-\sigma}} \leqslant C\big(1 + |t|\big)^{-2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}.$$

This completes the proof of Theorem 3.11. □

Corollary 3.15. *The asymptotics* (3.22) *imply* (1.6) *with the projection*

$$\mathcal{P}_c = 1 - \mathcal{P}_d, \qquad \mathcal{P}_d = \sum_{\omega_J \in \Sigma} P_J. \tag{3.40}$$

4. Application to the asymptotic completeness

We apply the obtained results to prove the asymptotic completeness which follows by standard Cook's argument. Let us note that the asymptotic completeness is proved in [18,26,30] by another methods for more general Klein–Gordon equations with an external Maxwell field. Our results give some refinement to the estimate of the remainder term.

Theorem 4.1. *Let conditions* (1.4) *and* (3.1) *hold. Then*

(i) *For solution to* (1.2) *with any initial function* $\Psi(0)\in\mathcal{F}_0$*, the following long time asymptotics hold,*

$$\Psi(t)=\sum_{\omega_J\in\Sigma}e^{-i\omega_J t}\Psi_J+\mathcal{U}_0(t)\Phi_\pm+r_\pm(t) \tag{4.1}$$

where Ψ_j *are the corresponding eigenfunctions,* $\Phi_\pm\in\mathcal{F}_0$ *are the scattering states, and*

$$\|r_\pm(t)\|_{\mathcal{F}_0}\to 0,\quad t\to\pm\infty. \tag{4.2}$$

(ii) *Furthermore,*

$$\|r_\pm(t)\|_{\mathcal{F}_0}=\mathcal{O}\big(\log^{-1}|t|\big) \tag{4.3}$$

if $\Psi(0)\in\mathcal{F}_\sigma$ *with* $\sigma>5/2$.

Proof. First let us denote $\mathcal{X}_d:=\mathcal{P}_d\mathcal{F}_0$ and $\mathcal{X}_c:=\mathcal{P}_c\mathcal{F}_0$. For $\Psi(0)\in\mathcal{X}_d$ the asymptotics (4.1) obviously hold with $\Phi_\pm=0$ and $r_\pm(t)=0$. Hence, it remains to prove for $\Psi(0)\in\mathcal{X}_c$ the asymptotics

$$\Psi(t)=\mathcal{U}_0(t)\Phi_\pm+r_\pm(t) \tag{4.4}$$

with the remainder satisfying (4.2). Moreover, it suffices to prove the asymptotics (4.4), (4.3) for $\Psi(0)\in\mathcal{X}_c\cap\mathcal{F}_\sigma$ where $\sigma>5/2$ since the space $\mathcal{F}_\sigma$ is dense in $\mathcal{X}_c$, while the group $\mathcal{U}_0(t)$ is unitary in $\mathcal{F}_0$ after a suitable modification of the norm. In this case Theorem 3.11 implies the decay

$$\|\Psi(t)\|_{\mathcal{F}_{-\sigma}}\leqslant C\big(1+|t|\big)^{-1}\log^{-2}\big(1+|t|\big)\|\Psi(0)\|_{\mathcal{F}_\sigma},\quad t\to\pm\infty. \tag{4.5}$$

The function $\Psi(t)$ satisfies Eq. (3.18),

$$i\dot\Psi(t)=(\mathcal{H}_0+\mathcal{V})\Psi(t).$$

Hence, the corresponding Duhamel equation reads

$$\Psi(t)=\mathcal{U}_0(t)\Psi(0)+\int_0^t\mathcal{U}_0(t-\tau)\mathcal{V}\Psi(\tau)\,d\tau,\quad t\in\mathbb{R}. \tag{4.6}$$

Finally, let us rewrite (4.6) as

$$\Psi(t)=\mathcal{U}_0(t)\left[\Psi(0)+\int_0^{\pm\infty}\mathcal{U}_0(-\tau)\mathcal{V}\Psi(\tau)\,d\tau\right]$$
$$-\int_t^{\pm\infty}\mathcal{U}_0(t-\tau)\mathcal{V}\Psi(\tau)\,d\tau=\mathcal{U}_0(t)\Phi_\pm+r_\pm(t). \qquad (4.7)$$

It remains to prove that $\Phi_\pm\in\mathcal{F}_0$ and (4.3) holds. Let us consider the sign "+" for the concreteness. The "unitarity" of $\mathcal{U}_0(t)$ in $\mathcal{F}_0$, the condition (1.4) and the decay (3.22) imply that for $\sigma'\in(5/2,\beta]$

$$\int_0^\infty\big\|\mathcal{U}_0(-\tau)\mathcal{V}\Psi(\tau)\big\|_{\mathcal{F}_0}\,d\tau\leqslant C\int_0^\infty\big\|\mathcal{V}\Psi(\tau)\big\|_{\mathcal{F}_0}\,d\tau\leqslant C_1\int_0^\infty\big\|\Psi(\tau)\big\|_{\mathcal{F}_{-\sigma'}}\,d\tau$$
$$\leqslant C_2\int_0^\infty(1+\tau)^{-1}\log^{-2}(2+\tau)\big\|\Psi(0)\big\|_{\mathcal{F}_\sigma}\,d\tau<\infty \qquad (4.8)$$

since $|V(x)|\leqslant C\langle x\rangle^{-\beta}\leqslant C\langle x\rangle^{-\sigma'}$. Hence, $\Phi_+\in\mathcal{F}_0$. The estimate (4.3) follows similarly. □

Appendix A. Proof of Lemma 2.8

For any operator $A\in\mathcal{L}(H^{-1}_\sigma;H^1_{-\sigma})$, denote $\operatorname{Re}A:=(A+A^*)/2$ and $\operatorname{Im}A:=(A-A^*)/2i$.

Step (i). First, we obtain a convenient representation for $\mathcal{G}_l(t)$. Formula (2.20) implies that

$$\mathcal{G}_l(t)=\frac{1}{2\pi i}\int_\Gamma l(\omega)\begin{pmatrix}\omega & i\\ -i\omega^2 & \omega\end{pmatrix}e^{-i\omega t}\big(P_0(\omega+i0)-P_0(\omega-i0)\big)\,d\omega$$

where $P_0(\omega)=R_0(\omega^2-m^2)$. Using the identity

$$R_0(\zeta-i0)=R_0^*(\zeta+i0),\quad \text{for }\zeta\in\mathbb{R}, \qquad (A.1)$$

we obtain that $P_0(\omega-i0)=P_0^*(\omega+i0)$, and then

$$\mathcal{G}_l(t)=\frac{1}{\pi}\int_\Gamma l(\omega)\begin{pmatrix}\omega & i\\ -i\omega^2 & \omega\end{pmatrix}e^{-i\omega t}\operatorname{Im}P_0(\omega+i0)\,d\omega$$
$$=\frac{1}{\pi}\int_m^\infty l(\omega)\left[\begin{pmatrix}\omega & i\\ -i\omega^2 & \omega\end{pmatrix}e^{-i\omega t}\operatorname{Im}P_0(\omega+i0)\right.$$
$$\left.+\begin{pmatrix}-\omega & i\\ -i\omega^2 & -\omega\end{pmatrix}e^{i\omega t}\operatorname{Im}P_0(-\omega+i0)\right]d\omega.$$

Applying (A.1) again, we have $P_0(-\omega+i0)=P_0^*(\omega+i0)$. Hence,

$$\mathcal{G}_l(t)=\frac{2}{\pi}\operatorname{Re}\int_m^\infty l(\omega)\begin{pmatrix}\omega & i\\ -i\omega^2 & \omega\end{pmatrix}e^{-i\omega t}\operatorname{Im}P_0(\omega+i0)\,d\omega$$
$$=\frac{2}{\pi}\operatorname{Re}\int_m^\infty l(\omega)e^{-i\omega t}\mathcal{P}(\omega). \tag{A.2}$$

Step (ii). Second, we obtain the asymptotics for the matrix operator

$$\mathcal{P}(\omega)=\begin{pmatrix}\omega & i\\ -i\omega^2 & \omega\end{pmatrix}\operatorname{Im}P(\omega+i0).$$

Using (2.8) and (2.9), we have

$$\mathcal{P}(\omega)=\mathcal{P}_0+\mathcal{P}_r(\omega),\quad \omega\to m+0 \tag{A.3}$$

where $\mathcal{P}_0$ is the operator with the matrix integral kernel

$$\mathcal{P}_0=\frac{1}{4}\begin{pmatrix}m & i\\ -im^2 & m\end{pmatrix}$$

and for the remainder $\mathcal{P}_r(\omega)$ we have

$$\mathcal{P}_r(\omega)=\mathcal{O}\big(|\omega-m|^{3/4}\big),\qquad \mathcal{P}_r'(\omega)=\mathcal{O}\big(|\omega-m|^{-1/4}\big),$$
$$\mathcal{P}_r''(\omega)=\mathcal{O}\big(|\omega-m|^{-5/4}\big),\quad \omega\to m+0 \tag{A.4}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})$ with $\sigma>5/2$.

Step (iii). Further, let us consider the contribution of the first term from (A.3) into the RHS of (A.2). Using integration by parts N times, $N=1,2,3,\dots,$ we obtain that

$$\int_m^\infty e^{-i\omega t}l(\omega)\,d\omega=\frac{e^{-imt}}{it}+\frac{1}{it}\int_m^\infty e^{-i\omega t}l'(\omega)\,d\omega=\cdots$$
$$=\frac{e^{-imt}}{it}+\mathcal{O}\big(t^{-N}\big),\quad t\to\infty \tag{A.5}$$

since $l(m)=1$, $l^{(k)}(m)=0$, $k=1,2,\dots$. Hence,

$$\frac{2}{\pi}\operatorname{Re}\int_m^\infty l(\omega)e^{-i\omega t}\mathcal{P}_0\,d\omega=\mathcal{G}_0(t)+\mathcal{O}\big(t^{-N}\big),\quad t\to\infty$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})$ with $\sigma>5/2$.

Step (iv). Finally, the contributions of the remainder $\mathcal{P}_r(\omega)$ into the RHS of (A.2) is $\mathcal{O}(t^{-7/4})$ in the norm of $\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})$. It follows from (A.4) and from the next lemma (cf. [10, Lemma 10.2]).

Lemma A.1. *Assume* $\mathbf{B}$ *be a Banach space, and* $F\in C(a,b;\mathbf{B})$ *satisfies* $F(a)=0$ *and* $F(\omega)=0$ *for* $\omega\geqslant b>a$, $F'\in L^1(a+\delta,b;\mathbf{B})$ *for any* $\delta>0$. *Moreover,* $F'(\omega)=\mathcal{O}((\omega-a)^{-1/4})$ *as well as* $F''(\omega)=\mathcal{O}((\omega-a)^{-5/4})$ *as* $\omega\to a+0$. *Then*

$$\int\limits_a^\infty e^{-it\omega}F(\omega)\,d\omega=\mathcal{O}\big(t^{-7/4}\big),\quad t\to\infty$$

in the norm of $\mathbf{B}$.

Appendix B. Proof of Lemma 2.10

Differentiating $\mathcal{G}(t,z)$, we obtain for $|z|<t$

$$\dot G(t,z)=-\frac{mt}{2\pi}\frac{\sin m\sqrt{t^2-|z|^2}}{t^2-|z|^2}-\frac{t}{2\pi}\frac{\cos m\sqrt{t^2-|z|^2}}{\sqrt{(t^2-|z|^2)^3}},$$

$$\ddot G(t,z)=-\frac{m}{2\pi}\frac{\sin m\sqrt{t^2-|z|^2}}{t^2-|z|^2}-\frac{m^2t^2}{2\pi}\frac{\cos m\sqrt{t^2-|z|^2}}{\sqrt{(t^2-|z|^2)^3}}+\frac{3mt^2}{2\pi}\frac{\sin m\sqrt{t^2-|z|^2}}{(t^2-|z|^2)^2}$$
$$-\frac{1}{2\pi}\frac{\cos m\sqrt{t^2-|z|^2}}{\sqrt{(t^2-|z|^2)^3}}+\frac{3t^2}{2\pi}\frac{\cos m\sqrt{t^2-|z|^2}}{\sqrt{(t^2-|z|^2)^5}}.$$

Hence, (2.16) implies

$$\mathcal{G}(t,z)=\tilde{\mathcal{G}}_0(t,z)+\tilde{\mathcal{G}}_r(t,z)$$

where

$$\tilde{\mathcal{G}}_0(t,z):=\frac{\theta(t-|z|)}{2\pi}\begin{pmatrix}-\frac{mt\sin m\sqrt{t^2-|z|^2}}{t^2-|z|^2} & \frac{\cos m\sqrt{t^2-|z|^2}}{\sqrt{t^2-|z|^2}}\\ -\frac{m^2t^2\cos m\sqrt{t^2-|z|^2}}{\sqrt{(t^2-|z|^2)^3}} & -\frac{mt\sin m\sqrt{t^2-|z|^2}}{t^2-|z|^2}\end{pmatrix}$$

and for remainder $\tilde{\mathcal{G}}_r(t,z)$ the bound holds

$$\big|\partial_z^\alpha\tilde{\mathcal{G}}_r(t,z)\big|\leqslant Ct^{-2},\quad |z|\leqslant 2t/3,\ |\alpha|\leqslant 1.$$

It remains to prove the bounds of type (2.37) for the difference $Q(t,z)=\tilde{\mathcal{G}}_0(t,z)-\mathcal{G}_0(t,z)$. Let us consider the entry $Q^{12}(t,z)$. Applying the Lagrange formula, we obtain

$$\begin{aligned}\left|Q^{12}(t,z)\right| &= \frac{1}{2\pi}\left|\frac{\cos m\sqrt{t^2-|z|^2}}{\sqrt{t^2-|z|^2}} - \frac{\cos mt}{t}\right| \\ &\leqslant C|z|^2 t^{-2} \leqslant C_1|z|^{3/2}t^{-3/2}, \quad |z| \leqslant 2t/3. \end{aligned} \tag{B.6}$$

Differentiating $Q^{12}(t,z)$ we obtain for $|z| < t$

$$\partial_{z_j} Q^{12}(t,z) = \frac{z_j}{2\pi}\left[\frac{1}{2\sqrt{(t^2-|z|^2)^3}}\cos m\sqrt{t^2-z^2} + \frac{m}{t^2-|z|^2}\sin m\sqrt{t^2-|z|^2}\right], \quad j=1,2.$$

Hence,

$$\left|\partial_{z_j} Q^{12}(t,z)\right| \leqslant C|z|t^{-2}, \quad |z| \leqslant 2t/3,\ j=1,2. \tag{B.7}$$

Other entries $Q^{ij}(t,z)$ also admit the estimates of type (B.6)–(B.7). Hence, Lemma 2.10 follows since $\mathcal{G}_r(t,z) = \tilde{\mathcal{G}}_r(t,z) + Q(t,z)$.

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