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Dispersive Estimates for the 2D Wave Equation

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Abstract. We obtain a dispersive long-time decay with respect to weighted energy norms for solutions of the 2D wave equation with generic potential. The decay extends results obtained by Murata for the 2D Schrödinger equation.

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1. INTRODUCTION

We consider the 2D wave equation

$$\ddot\psi(x,t)=\Delta\psi(x,t)-V(x)\psi(x,t),\qquad x\in\mathbb{R}^2, \tag{1.1}$$

In vector form, equation (1.1) reads

$$i\dot\Psi(t)=\mathcal{H}\Psi(t), \tag{1.2}$$

where

$$\lambda H\Psi(t)=\begin{pmatrix}\psi(t)\\ \dot\psi(t)\end{pmatrix},\qquad \mathcal{H}=\begin{pmatrix}0 & i\\ i(\Delta-V) & 0\end{pmatrix}. \tag{1.3}$$

For $s,\sigma\in\mathbb{R}$, write $H^s_\sigma=H^s_\sigma(\mathbb{R}^2)$ for the weighted Sobolev spaces with finite norms

$$\|\psi\|_{H^s_\sigma}=\|(1+|x|^2)^{\sigma/2}(1+|\nabla|^2)^{s/2}\psi\|_{L^2}<\infty.$$

Assume that $V(x)$ is a real function and

$$|V(x)|+|\nabla V(x)|\leqslant C(1+|x|)^{-\beta},\qquad x\in\mathbb{R}^2, \tag{1.4}$$

for some $\beta>5$. Then the multiplication by $V(x)$ is a bounded operator taking H^1_s to $H^1_{s+\beta}$ for any $s\in\mathbb{R}$. We restrict ourselves to the "nonsingular case," in the terminology of [13], where the truncated resolvent of the Schrödinger operator $H=-\Delta+V(x)$ is bounded at the endpoints of the continuous spectrum. In other words, the point $\lambda=0$ is neither an eigenvalue nor a resonance for the operator H.

Definition 1.1. (i) Let $\mathcal{F}$ be the Hilbert space $\dot H^1(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2)$ of vector functions $\Psi=(\psi,\pi)$ with the norm

$$\|\,\Psi\|_{\mathcal{F}}=\|\nabla\psi\|_{L^2}+\|\pi\|_{L^2}<\infty.$$

(ii) Let $\mathcal{F}_\sigma$ be the Hilbert space $H^1_\sigma\oplus H^0_\sigma$ of vector functions $\Psi=(\psi,\pi)$ with the norm

$$\|\,\Psi\|_{\mathcal{F}_\sigma}=\|\psi\|_{H^1_\sigma}+\|\pi\|_{H^0_\sigma}<\infty.$$

Our main result is the following long-time decay of the solutions to (1.1): in the "nonsingular case,"

$$\|\mathcal{P}_c\Psi(t)\|_{\mathcal{F}_{-\sigma}}=\mathcal{O}(|t|^{-1}\log^{-2}|t|),\qquad t\to\pm\infty, \tag{1.5}$$

for any initial data $\Psi_0=\Psi(0)\in\mathcal{F}_\sigma$ with $\sigma>5/2$. Here $\mathcal{P}_c$ stands for the Riesz projection to the continuous spectrum of the operator $\mathcal{H}$.

Let us comment on previous results in this direction. The local energy decay has been established for the first time in scattering theory for the linear Schrödinger equation in the fifties by Birman, Kato, Simon, and others. For wave equations with compactly supported potentials and for similar hyperbolic PDEs, Vainberg [16] established the decay in local energy norms for solutions with compactly supported initial data. However, applications to asymptotic stability of solutions to nonlinear equations require an exact characterization of the decay for the corresponding linearized equations with respect to weighted norms (see, e.g., [2, 3, 4, 15]).

The decay in weighted norms has been established by Jensen and Kato [7] for the Schrödinger equation in the dimension $n = 3$. The result has been extended to other dimensions by Jensen and Nenciu [5, 6, 8] and, for more general PDEs of Schrödinger type, by Murata [13]. In our papers [9]-[12], the decay in the weighted energy norms has been proved for 1D and 3D Klein–Gordon equations and for the wave equation. Note that the decay rate for wave equations depends of the spatial decay of the initial function $\Psi(0)$ and of the potential $V(x)$, in contrast to the case of Klein–Gordon equation, where the decay rate is $t^{-3/2}$ for sufficiently large β and σ. This difference is related to the presence of an interior lacuna for the wave equations. Our approach develops the Jensen–Kato technique to relativistic equations. Namely, the decay of the low-energy component of the solution follows by using the Jensen–Kato technique, whereas the decay for the high-energy component requires novel robust ideas. This problem has been resolved with a modified approach based on the Born series and the convolution.

Here we extend our approach to 2D wave equation. In this case, the decay rate in (1.5) does not depend of $\beta > 5$ and $\sigma > 5/2$, as in the case of 1D and 3D wave equation, due to the absence of lacunae.

Our paper is organized as follows. In Section 2, we obtain the time decay for the solution to the free wave equation and state the spectral properties of the free resolvent. In Section 3, we obtain spectral properties of the perturbed resolvent and prove the decay relation (1.5).

2. FREE WAVE EQUATION

First, consider the free wave equation

$$\dot{\psi}(x,t) = \pi(x,t), \qquad \dot{\pi}(x,t) = \Delta\psi(x,t), \qquad x \in \mathbb{R}^2. \tag{2.1}$$

Write system (2.1) in the form

$$i\dot{\Psi}(t) = \mathcal{H}_0\Psi(t), \tag{2.2}$$

where

$$\Psi(t) = \begin{pmatrix} \psi(t) \\ \pi(t) \end{pmatrix}, \quad \mathcal{H}_0 = \begin{pmatrix} 0 & i \\ i\Delta & 0 \end{pmatrix}. \tag{2.3}$$

2.1. Spectral properties

We state spectral properties of the free wave dynamical group $\mathcal{G}(t)$. For $t > 0$ and $\Psi_0 = \Psi(0) \in \mathcal{F}$, there exists a unique solution $\Psi(t) \in C_b(\mathbb{R}, \mathcal{F})$ to the free wave equation (2.2). Hence, $\Psi(t)$ admits the spectral Fourier–Laplace representation

$$\theta(t)\Psi(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i(\omega+i\varepsilon)t}\mathcal{R}_0(\omega+i\varepsilon)\Psi_0\, d\omega, \qquad t \in \mathbb{R} \tag{2.4}$$

with any $\varepsilon > 0$ where $\theta(t)$ stands for the Heaviside function and $\mathcal{R}_0(\omega) = (\mathcal{H}_0 - \omega)^{-1}$ provided that $\omega \in \mathbb{C}^+ := \{\operatorname{Im}\omega > 0\}$ is the resolvent of the operator $\mathcal{H}_0$. The representation follows from the stationary equation $\omega\tilde{\Psi}^+(\omega) = \mathcal{H}_0\tilde{\Psi}^+(\omega) + i\Psi_0$ for the Fourier–Laplace transform

$$\tilde{\Psi}^+(\omega) := \int_{\mathbb{R}} \theta(t)e^{i\omega t}\Psi(t)dt, \qquad \omega \in \mathbb{C}^+.$$

The solution $\Psi(t)$ is a continuous bounded function of $t \in \mathbb{R}$ with values in $\mathcal{F}$ by the energy conservation for the free wave equation (2.2). Hence, $\tilde{\Psi}^+(\omega) = -i\mathcal{R}_0(\omega)\Psi_0$ is an analytic function of $\omega \in \mathbb{C}^+$ with values in $\mathcal{F}$ which is bounded for $\omega \in \mathbb{R} + i\varepsilon$. Therefore, the integral (2.4) converges in the sense of distributions of $t \in \mathbb{R}$ with values in $\mathcal{F}$. Similarly to (2.4),

$$\theta(-t)\Psi(t) = -\frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i(\omega - i\varepsilon)t}\mathcal{R}_0(\omega - i\varepsilon)\Psi_0\, d\omega, \qquad t \in \mathbb{R}. \tag{2.5}$$

The resolvent $\mathcal{R}_0(\omega)$ admits the matrix representation

$$\mathcal{R}_0(\omega) = \begin{pmatrix} \omega R_0(\omega^2) & iR_0(\omega^2) \\ -i(1 + \omega^2 R_0(\omega^2)) & \omega R_0(\omega^2) \end{pmatrix}, \tag{2.6}$$

where $R_0(\zeta) = (-\Delta - \zeta)^{-1}$ is the free Schrödinger resolvent with the integral kernel

$$R_0(\zeta, x - y) = \frac{1}{2\pi}K_0(-i\zeta^{1/2}|x - y|), \qquad \zeta \in \mathbb{C}^+, \qquad \operatorname{Im}\zeta^{1/2} > 0, \tag{2.7}$$

where K_0 stands for the Macdonald function.

Definition 2.1. Denote by $\mathcal{L}(B_1, B_2)$ the Banach space of bounded linear operators from a Banach space B_1 to a Banach space B_2.

Now we collect some properties of $R_0(\zeta)$ (see [1, 7, 13, 9, 14]).

Lemma 2.2. (i) *$R_0(\zeta)$ is a strongly analytic function of the variable $\zeta \in \mathbb{C} \setminus [0, \infty)$ with values in $\mathcal{L}(H_0^{-1}, H_0^1)$.*

(ii) *For $\zeta > 0$ and $\sigma > 1/2$, the following convergence holds:*

$$R_0(\zeta \pm i\varepsilon) \to R_0(\zeta \pm i0), \qquad \varepsilon \to 0+,$$

in $\mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)$ uniformly with respect to $\zeta \geqslant r$ for any $r > 0$.

(iii) *The following asymptotic expansions hold:*

$$R_0(\zeta) = A_0 \log\zeta + B_0 + \mathcal{O}(\zeta^{3/4}), \qquad \zeta \to 0, \qquad \zeta \in \mathbb{C} \setminus [0, \infty) \tag{2.8}$$

with respect to the norm of $\mathcal{L}(H_\sigma^{-1}; H_{-\sigma}^1)$ with $\sigma > 5/2$. Here $A_0, B_0 \in \mathcal{L}(H_\sigma^{-1}; H_{-\sigma}^1)$. $\sigma > 1$, are operators with the kernels $A_0(x - y), B_0(x - y)$, respectively, and

$$A_0(x - y) = -\frac{1}{4\pi}, \qquad x, y \in \mathbb{R}^2 \tag{2.9}$$

The asymptotics (2.8) can be differentiated twice with respect to the norm of $\mathcal{L}(H_\sigma^{-1}; H_{-\sigma}^1)$ with $\sigma > 5/2$.

(iv) *For $m = 0, 1$, $l = -1, 0, 1, \dots$, $k = 0, 1, 2, \dots$ and any $\sigma > 1/2 + k$, the following asymptotic relations hold:*

$$\|R_0^{(k)}(\zeta)\|_{\mathcal{L}(H_\sigma^m, H_{-\sigma}^{m+l})} = \mathcal{O}(|\zeta|^{-\frac{1-l+k}{2}}), \qquad \zeta \to \infty, \quad \zeta \in \mathbb{C} \setminus (0, \infty) \tag{2.10}$$

Lemma 2.2 and formula (2.6) imply the corresponding properties of $\mathcal{R}_0(\omega)$.

Lemma 2.3. (i) *The resolvent $\mathcal{R}_0(\omega)$ is a strongly analytic function of $\omega \in \mathbb{C} \setminus \mathbb{R}$ with values in $\mathcal{L}(\mathcal{F}_0, \mathcal{F}_0)$.*

(ii) *For $\omega \neq 0$, the convergence $\mathcal{R}_0(\omega \pm i\varepsilon) \to \mathcal{R}_0(\omega \pm i0)$ as $\varepsilon \to 0+$ holds in $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ with $\sigma > 1/2$, uniformly with respect to $|\omega| \geqslant r$ for any $r > 0$.*

(iii) *The following asymptotic expansion holds:*

$$\mathcal{R}_0(\omega) = \mathcal{A}_0 \log\omega + \mathcal{B}_0 + \mathcal{O}(\omega\log\omega), \qquad \omega \to 0, \qquad \omega \in \mathbb{C} \setminus \mathbb{R},$$

with respect to the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$ with $\sigma > 5/2$.

(iv) *For $k = 0, 1, 2, \dots$ and any $\sigma > 1/2 + k$, the following asymptotic relation holds:*

$$\|\mathcal{R}_0^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} = \mathcal{O}(1), \qquad \omega \to \infty, \qquad \omega \in \mathbb{C} \setminus \mathbb{R}. \tag{2.11}$$

Corollary 2.4. *For $t \in \mathbb{R}$ and $\Psi_0 \in \mathcal{F}_\sigma$ with $\sigma > 1/2$, the group $\mathcal{G}(t)$ admits the integral representation*

$$\mathcal{G}(t)\Psi_0 = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}\Big[\mathcal{R}_0(\omega+i0) - \mathcal{R}_0(\omega-i0)\Big]\Psi_0\, d\omega, \tag{2.12}$$

where the integral converges in the sense of distributions of $t \in \mathbb{R}$ with values in $\mathcal{F}_{-\sigma}$.

Proof. Summing up the representations (2.4) and (2.5), in the limit as $\varepsilon \to 0+$, we obtain (2.12) (by the Cauchy theorem) and prove Lemma 2.3.

2.2. Time decay

Estimates (2.11) give no possibility to prove the decay of $\mathcal{G}(t)$ by partial integration in (2.12). Let us derive the decay by using explicit formulas. To be definite, consider the case of $t > 0$. The matrix kernel of the dynamical group $\mathcal{G}(t)$ reads

$$\mathcal{G}(t, x-y) = \begin{pmatrix} \dot G(t, x-y) & G(t, x-y) \\ \ddot G(t, x-y) & \dot G(t, x-y) \end{pmatrix}, \quad x, y \in \mathbb{R}^2 \tag{2.13}$$

Here

$$G(t, z) = \frac{1}{2\pi}\frac{\theta(t-|z|)}{\sqrt{t^2-|z|^2}} \tag{2.14}$$

The group $\mathcal{G}(t)$ decays like t^{-1}, which does not correspond to (1.5). Split $\mathcal{G}(t)$ as

$$\mathcal{G}(t) = \mathcal{G}_0(t) + \mathcal{G}_r(t),$$

where $G_0(t)$ is the operator with the kernel

$$\mathcal{G}_0(t, x-y) = \frac{1}{2\pi t}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad x, y \in \mathbb{R}^2. \tag{2.15}$$

Below we show that $\mathcal{G}_0(t)$ is the only term responsible for the slow decay. More exactly, in the next section we prove the following basic proposition

Proposition 2.5. *Let $\sigma > 5/2$. Then the following asymptotic relation holds*:

$$\mathcal{G}_r(t) = \mathcal{O}(|t|^{-3/2}), \qquad t \to \infty, \tag{2.16}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$.

Further, introduce the following *low-energy* and *high-energy* components of $\mathcal{G}(t)$:

$$\mathcal{G}_l(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t} l(\omega)\Big[\mathcal{R}_0(\omega+i0) - \mathcal{R}_0(\omega-i0)\Big]\, d\omega, \tag{2.17}$$

$$\mathcal{G}_h(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t} h(\omega)\Big[\mathcal{R}_0(\omega+i0) - \mathcal{R}_0(\omega-i0)\Big]\, d\omega, \tag{2.18}$$

where $l(\omega) \in C_0^\infty(\mathbb{R})$ is an even function, supp $l \in [-2\varepsilon, 2\varepsilon]$, $l(\omega) = 1$ if $|\omega| \leqslant \varepsilon$ with an $\varepsilon > 0$, and $h(\omega) = 1 - l(\omega)$. The following key observation is that the term $\mathcal{G}_0(t)$ does not contribute to the high-energy component.

Lemma 2.6. *Let $\sigma > 5/2$. Then the following asymptotic relation holds*:

$$\mathcal{G}_l(t) = \mathcal{G}_0(t) + \mathcal{O}(t^{-7/4}), \quad t \to \infty, \tag{2.19}$$

with respect to the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$.

We prove this lemma in Appendix A. Let us now find the asymptotics of $\mathcal{G}_h(t)$.

Theorem 2.7. *Let $\sigma > 5/2$. Then the following asymptotic relation holds*:

$$\mathcal{G}_h(t) = \mathcal{O}(|t|^{-3/2}), \qquad t \to \infty, \tag{2.20}$$

with respect to the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$.

Proof. First, consider a small $t \leqslant 1$. By the energy conservation for the wave equation,

$$\|\psi'(t)\|_{L^2} + \|\dot\psi(t)\|_{L^2} = \|\mathcal{G}(t)\Psi(0)\|_{\mathcal{F}} = \|\Psi(0)\|_{\mathcal{F}} \leqslant \|\Psi(0)\|_{\mathcal{F}_0} \tag{2.21}$$

where $(\psi(t), \dot\psi(t)) = \mathcal{G}(t)\Psi(0)$. Further, the Hölder inequality and (2.21) imply

$$\|\psi(t)\|^2_{L^2} = \int \Big(\int_0^t \dot\psi(x,s)ds - \psi(x,0)\Big)^2 dx \leqslant 2\|\psi(0)\|^2_{L^2} + 2t\int_0^t \|\dot\psi(s)\|^2_{L^2} ds \leqslant 2\|\Psi(0)\|^2_{\mathcal{F}_0}.$$

Hence,

$$\|\mathcal{G}(t)\Psi(0)\|_{\mathcal{F}_0} \leqslant C\|\Psi(0)\|_{\mathcal{F}_0}, \qquad t \leqslant 1. \tag{2.22}$$

The integrand in (2.17) is finitely supported, and it belongs to $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$ for $\sigma > 1/2$. Hence, for $\sigma > 1/2$,

$$\|\mathcal{G}_l(t)\|_{\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})} \leqslant C, \qquad t > 0 \tag{2.23}$$

This implies (2.20) for small $t \leqslant 1$, since $\mathcal{G}_h(t) = \mathcal{G}(t) - \mathcal{G}_l(t)$.

For large $t \geqslant 1$, we derive the asymptotic relation (2.20) from Proposition 2.5 and Lemma 2.6. Using (2.19), we obtain

$$\mathcal{G}(t) = \mathcal{G}_l(t) + \mathcal{G}_h(t) = \mathcal{G}_0(t) + \mathcal{G}_h(t) + \mathcal{O}(t^{-7/4}), \qquad t \to \infty \tag{2.24}$$

with respect to the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$. On the other hand, (2.16) implies

$$\mathcal{G}(t) = \mathcal{G}_0(t) + \mathcal{G}_r(t) = \mathcal{G}_0(t) + \mathcal{O}(t^{-3/2}), \qquad t \to \infty, \tag{2.25}$$

in the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$. Comparing (2.24) and (2.25), we obtain the asymptotic relation (2.20).

2.3. Proof of Proposition 2.5

Consider an arbitrary $t \geqslant 1$. Split the initial function Ψ into two terms,

$$\Psi_0 = \Psi_{1,t} + \Psi_{2,t},$$

such that

$$\Psi_{1,t}(x) = 0 \quad \text{for} \quad |x| > t/3, \qquad \Psi_{2,t}(x) = 0 \quad \text{for} \quad |x| < t/4, \tag{2.26}$$

$$\|\Psi_{1,t}\|_{\mathcal{F}_\sigma} + \|\Psi_{2,t}\|_{\mathcal{F}_\sigma} \leqslant C\|\Psi_0\|_{\mathcal{F}_\sigma}, \qquad t \geqslant 1, \tag{2.27}$$

and estimate $\mathcal{G}_r(t)\Psi_{1,t}$ and $\mathcal{G}_r(t)\Psi_{2,t}$ separately.

Step (i). Let us first estimate

$$\mathcal{G}_r(t)\Psi_{2,t} = \mathcal{G}(t)\Psi_{2,t} - \mathcal{G}_0(t)\Psi_{2,t}.$$

Let

$$\mathcal{G}(s)\Psi_{2,t} = (\psi_{2,t}(s), \dot\psi_{2,t}(s)).$$

Using energy conservation for the wave equation and properties (2.26)–(2.27), we obtain

$$\|\nabla\psi_{2,t}(t)\|_{H^0_{-\sigma}} + \|\dot\psi_{2,t}(t)\|_{H^0_{-\sigma}} \leqslant \|\mathcal{G}(t)\Psi_{2,t}\|_{\mathcal{F}} = \|\Psi_{2,t}\|_{\mathcal{F}} \leqslant Ct^{-\sigma}\|\Psi_{2,t}\|_{\mathcal{F}_\sigma} \leqslant Ct^{-\sigma}\|\Psi_0\|_{\mathcal{F}_\sigma}. \tag{2.28}$$

The Hölder inequality, energy conservation, and properties (2.26)–(2.27) imply

$$\|\psi_{2,t}(t)\|^2_{H^0_{-\sigma}} \leqslant \int \Big(\int_0^t \dot\psi_{2,t}(x,s)ds - \psi_{2,t}(x,0)\Big)^2 dx \leqslant 2\|\psi_{2,t}(0)\|^2_{L^2} + 2t\int_0^t \|\dot\psi_{2,t}(s)\|^2_{L^2} ds$$
$$\leqslant 2\Big(\|\Psi_{2,t}\|^2_{\mathcal{F}_0} + t\int_0^t \|\Psi_{2,t}\|^2_{\mathcal{F}} ds\Big) \leqslant C\Big(t^{-2\sigma}\|\Psi_0\|^2_{\mathcal{F}_\sigma} + t^{2-2\sigma}\|\Psi_0\|^2_{\mathcal{F}_\sigma}\Big) \leqslant Ct^{2-2\sigma}\|\Psi_0\|^2_{\mathcal{F}_\sigma}. \tag{2.29}$$

Hence, (2.28) and (2.29) yield

$$\|\mathcal{G}(t)\Psi_{2,t}\|_{\mathcal{F}_{-\sigma}} \leqslant Ct^{-\sigma+1}\|\Psi_0\|_{\mathcal{F}_\sigma}, \qquad t \geqslant 1. \tag{2.30}$$

Let us now estimate $\mathcal{G}_0(t)\Psi_{2,t} = (\varphi_2(t), 0)$. By the Cauchy inequality,

$$|\varphi_2(t)| = \Big|\frac{1}{2\pi t}\int \pi_{2,t}(x)dx\Big| \leqslant \frac{C}{t}\Big(\int |\pi_{2,t}(x)|^2(1+|x|^2)^\sigma dx\Big)^{1/2}\Big(\int_{|x|>t/3}\frac{dx}{(1+|x|^2)^\sigma}\Big)^{1/2}$$
$$\leqslant Ct^{-\sigma}\|\pi_{2,t}\|_{H^0_\sigma} \leqslant Ct^{-\sigma}\|\Psi_0\|_{\mathcal{F}_\sigma}, \tag{2.31}$$

where $\pi_{2,t}$ is the second component of $\Psi_{2,t}$. Therefore,

$$\|\mathcal{G}_0(t)\Psi_{2,t}\|_{\mathcal{F}_{-\sigma}} = |\varphi_2(t)|\Big(\int \frac{dx}{(1+|x|^2)^\sigma}\Big)^{1/2} \leqslant Ct^{-\sigma}\|\Psi_0\|_{\mathcal{F}_\sigma}, \qquad t \geqslant 1. \tag{2.32}$$

Finally, (2.30)–(2.32) imply

$$\|\mathcal{G}_r(t)\Psi_{2,t}\|_{\mathcal{F}_{-\sigma}} \leqslant Ct^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \qquad t \geqslant 1 \tag{2.33}$$

since $\sigma > 5/2$.

Step (ii). Now we consider $\mathcal{G}_r(t)\Psi_{1,t}$. We split the operator $\mathcal{G}_r(t)$, for $t \geqslant 1$, in two terms:

$$\mathcal{G}_r(t) = (1-\zeta)\mathcal{G}_r(t) + \zeta\mathcal{G}_r(t)$$

where ζ is the operator of multiplication by the function $\zeta(|x|/t)$ such that $\zeta = \zeta(s) \in C_0^\infty(\mathbb{R})$, $\zeta(s) = 1$ for $|s| < 1/4$, and $\zeta(s) = 0$ for $|s| > 1/3$. Obviously, for $|\alpha| \leqslant 1$, we have

$$|\partial_x^\alpha \zeta(|x|/t)| \leqslant C < \infty, \qquad t \geqslant 1. \tag{2.34}$$

Furthermore, $1 - \zeta(|x|/t) = 0$ for $|x| < t/4$, and therefore

$$||(1-\zeta)\mathcal{G}(t)\Psi_{1,t}||_{\mathcal{F}_{-\sigma}} \leqslant Ct^{-\sigma}\|(1-\zeta)\mathcal{G}(t)\Psi_{1,t}\|_{\mathcal{F}_0} \leqslant C_1t^{-\sigma}||\mathcal{G}(t)\Psi_{1,t}||_{\mathcal{F}_0}. \tag{2.35}$$

Let $\mathcal{G}(s)\Psi_{1,t} = (\psi_{1,t}(s), \dot\psi_{1,t}(s))$. By the energy conservation and by (2.9), we obtain

$$\|\nabla\psi_{1,t}(t)\|_{L^2} + \|\dot\psi_{1,t}(t)\|_{L^2} = \|\mathcal{G}(t)\Psi_{1,t}\|_{\mathcal{F}} = \|\Psi_{1,t}\|_{\mathcal{F}} \leqslant \|\Psi_{1,t}\|_{\mathcal{F}_\sigma} \leqslant C\|\Psi_0\|_{\mathcal{F}_\sigma} \tag{2.36}$$

Further, similarly to (2.29), the energy conservation implies

$$\|\psi_{1,t}(t)\|^2_{L^2} \leqslant 2\|\psi_{1,t}(0)\|^2_{L^2} + 2t\int_0^t \|\dot\psi_{1,t}(s)\|^2_{L^2}\, ds \leqslant 2\Big(\|\Psi_{1,t}\|^2_{\mathcal{F}_0} + t\int_0^t \|\mathcal{G}(s)\Psi_{1,t}\|^2_{\mathcal{F}}\, ds\Big)$$
$$\leqslant C\Big(\|\Psi_0\|^2_{\mathcal{F}_\sigma} + t^2\|\Psi_{1,t}\|^2_{\mathcal{F}}\Big) \leqslant Ct^2\|\Psi_0\|^2_{\mathcal{F}_\sigma}. \tag{2.37}$$

Hence, (2.35)–(2.37) yield

$$\|(1-\zeta)\mathcal{G}(t)\Psi_{1,t}\|_{\mathcal{F}_{-\sigma}} \leqslant Ct^{-\sigma+1}\|\Psi_0\|_{\mathcal{F}_\sigma} \tag{2.38}$$

Let us now estimate $(1-\zeta)\mathcal{G}_0(t)\Psi_{1,t}$. Similarly to (2.31), we obtain (by the Cauchy inequality)

$$|\varphi_1(t)| = \Big|\frac{1}{2\pi t}\int \pi_{1,t}(x)dx\Big| \leqslant Ct^{-1}\|\Psi_0\|_{\mathcal{F}_\sigma}, \quad t \geqslant 1$$

where $\pi_{1,t}$ is the second component of $\Psi_{1,t}$. Hence, (2.34) yields

$$\|(1-\zeta)\mathcal{G}_0(t)\Psi_{1,t}\|_{\mathcal{F}_{-\sigma}} = |\varphi_1(t)|\|(1-\zeta)\|_{H^1_{-\sigma}} \leqslant \frac{C}{t}\|\Psi_0\|_{\mathcal{F}_\sigma}\Big(\int_{|x|>t/4}\frac{dx}{(1+|x|^2)^\sigma}\Big)^{\frac{1}{2}} \leqslant \frac{C(\varepsilon)}{t^\sigma}\|\Psi_0\|_{\mathcal{F}_\sigma}$$

The last inequality and (2.38) imply

$$\|(1-\zeta)\mathcal{G}_r(t)\Psi_{1,t}\|_{\mathcal{F}_{-\sigma}} \leqslant \|(1-\zeta)\mathcal{G}(t)\Psi_{1,t}\|_{\mathcal{F}_{-\sigma}} + \|(1-\zeta)\mathcal{G}_0(t)\Psi_{1,t}\|_{\mathcal{F}_{-\sigma}} \leqslant C(\varepsilon)t^{-\sigma+1}\|\Psi_0\|_{\mathcal{F}_\sigma} \tag{2.39}$$

Step (iii). Finally, let us estimate $\zeta\mathcal{G}_r(t)\Psi_{1,t}$. Let χ_t be the characteristic function of the ball $|x| \leqslant t/4$. Use the same notation for the operator of multiplication by this characteristic function. By (2.26),

$$\zeta\mathcal{G}_r(t)\Psi_{1,t} = \zeta\mathcal{G}_r(t)\chi_t\Psi_{1,t} \tag{2.40}$$

The matrix kernel of the operator $\zeta\mathcal{G}_r(t)\chi_t$ is equal to

$$\mathcal{G}_r'(x-y,t) = \zeta(|x|/t)\mathcal{G}_r(x-y,t)\chi_t(y)$$

In Appendix B, we prove the following lemma.

Lemma 2.8. *The following bound holds*:

$$|\partial_z^\alpha \mathcal{G}_r(t,z)| \leqslant Ct^{-2}(1+|z|), \quad |z| \leqslant t/2, \qquad t \geqslant 1, \qquad |\alpha| \leqslant 1. \tag{2.41}$$

Since $\zeta(|x|/t) = 0$ for $|x| > t/4$ and $\chi_t(y) = 0$ for $|y| > t/4$, the estimate (2.41) implies that

$$|\partial_x^\alpha \mathcal{G}_r'(x-y,t)| \leqslant Ct^{-2}(1+|x-y|), \qquad |\alpha| \leqslant 1, \quad t \geqslant 1. \tag{2.10}$$

The norm of the operator $\zeta\mathcal{G}_r(t)\chi_t : \mathcal{F}_\sigma \to \mathcal{F}_{-\sigma}$ is equivalent to the norm of the operator

$$\langle x\rangle^{-\sigma}\zeta\mathcal{G}_r(t)\chi_t(y)\langle y\rangle^{-\sigma} : \mathcal{F}_0 \to \mathcal{F}_0.$$

The norm of this operator does not exceed the sum over α, $|\alpha| \leqslant 1$, of the norms of the operators

$$\partial_x^\alpha[\langle x\rangle^{-\sigma}\zeta\mathcal{G}_r(t)\chi_t(y)\langle y\rangle^{-\sigma}] : L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2). \tag{2.43}$$

Estimates (2.42) imply that the operators (2.43) are of Hilbert–Schmidt class, since $\sigma > 5/2 > 2$, and their Hilbert–Schmidt norms do not exceed Ct^{-2}. Hence, (2.27) and (2.40) yield

$$||\zeta\mathcal{G}_r(t)\Psi_{1,t}||_{\mathcal{F}_{-\sigma}} \leqslant Ct^{-2}||\Psi_{1,t}||_{\mathcal{F}_\sigma} \leqslant Ct^{-2}||\Psi_0||_{\mathcal{F}_\sigma}, \quad t \geqslant 1. \tag{2.44}$$

Finally, estimates (2.33), (2.39), and (2.12) imply

$$||\mathcal{G}_r(t)\Psi_0||_{\mathcal{F}_{-\sigma}} \leqslant Ct^{-3/2}||\Psi_0||_{\mathcal{F}_\sigma}, \quad t \geqslant 1. \quad \Box(2.45)$$

3. PERTURBED WAVE EQUATION

To prove the long-time decay for the perturbed wave equation, we first establish the spectral properties of the generator.

3.1. Spectral properties

According to [13, formula (3.1)], introduce a generalized eigenspace $\mathbf{M}$ for the perturbed Schrödinger operator $H = -\Delta + V$ as follows:

$$\mathbf{M} = \{\psi \in H^1_{-1/2-0}: \ (1 + B_0 V)\psi \in \Re(A_0),\ A_0 V\psi = 0\}.$$

Where A_0 and B_0 are defined in (2.8), and $\Re(A_0)$ is the range of A_0. Below we assume that

$$\mathbf{M} = 0 \tag{3.1}$$

Remark 3.1. $N(H) \subset \mathbf{M}$ where $N(H)$ is the zero eigenspace of the operator H. This embedding is obtained in [13, Lemma 3.2]. The functions in $\mathbf{M} \setminus N(H)$ are referred to as *zero-resonance functions*. Hence, the condition (3.1) means that $\lambda = 0$ is neither eigenvalue nor resonance for the operator H.

Let us collect the properties of the perturbed Schrödinger resolvent $R(\zeta) = (H - \zeta)^{-1}$ obtained in [1, 7, 13, 9] under assumptions (1.4) and (3.1).

R1. $R(\zeta)$ is a strongly meromorphic function of $\zeta \in \mathbb{C} \setminus [0, \infty)$ with values in $\mathcal{L}(H_0^{-1}, H_0^1)$; the poles of $R(\zeta)$ are placed at a finite set of eigenvalues $\zeta_j < 0$, $j = 1, \ldots, N$, of the operator H with the corresponding eigenfunctions

$$\psi_j^1(x), .., \psi_j^{k_j}(x) \in H_s^2$$

for any $s \in \mathbb{R}$, where k_j is the multiplicity of ζ_j.

R2. For $\zeta > 0$, the convergence $R(\zeta \pm i\varepsilon) \to R(\zeta \pm i0)$ holds as $\varepsilon \to 0+$ in $\mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)$ with $\sigma > 1/2$, uniformly in $\zeta \geqslant r$ for any $r > 0$.

R3. The following expansion holds:

$$\left.\begin{array}{r} R(\zeta) = A_1 + A_2 \log^{-1}\zeta + \mathcal{O}(\log^{-2}\zeta) \\ R'(\zeta) = -A_2\zeta^{-1}\log^{-2}\zeta + \mathcal{O}(\zeta^{-1}\log^{-3}\zeta) \\ R''(\zeta) = \mathcal{O}(\zeta^{-2}\log^{-2}\zeta) \end{array}\right| \quad \zeta \to 0, \quad \zeta \in \mathbb{C} \setminus [0, \infty) \tag{3.2}$$

with respect to the norms of $\mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)$ with $\sigma > 5/2$. Here $A_1, A_2 \in \mathcal{L}(H_\sigma^{-1}, H_{-\sigma}^1)$ with $\sigma > 5/2$.

R4. For $k = 0, 1, 2$, $s = 0, 1$, and $l = -1, 0, 1$ with $s + l \in \{0, 1\}$, the following asymptotic relation holds:

$$\|R^{(k)}(\zeta)\|_{\mathcal{L}(H_\sigma^s, H_{-\sigma}^{s+l})} = \mathcal{O}(|\zeta|^{-\frac{1-l+k}{2}}),\ |\zeta| \to \infty, \quad \zeta \in \mathbb{C} \setminus [0, \infty) \tag{3.3}$$

with $\sigma > 1/2 + k$.

The resolvent $\mathcal{R}(\omega) = (\mathcal{H} - \omega)^{-1}$, can be expressed similarly to (2.6),

$$\mathcal{R}(\omega) = \begin{pmatrix} \omega R(\omega^2) & iR(\omega^2) \\ -i(1 + \omega^2 R(\omega^2)) & \omega R(\omega^2) \end{pmatrix} \tag{3.4}$$

Hence, properties **R1-R3** imply the corresponding properties of $\mathcal{R}(\omega)$. The corresponding equation is as follows.

Lemma 3.2. *Let the potential V satisfy* (1.4) *and* (3.1). *The following assertions hold.*

(i) $\mathcal{R}(\omega)$ *is a strongly meromorphic function of* $\omega \in \mathbb{C} \setminus \mathbb{R}$ *with values in* $\mathcal{L}(\mathcal{F}_0, \mathcal{F}_0)$. *The poles of* $\mathcal{R}(\omega)$ *are placed at a finite set*

$$\Sigma = \{\omega_j^{\pm} = \pm\sqrt{\zeta_j},\ j = 1, \ldots, N\}$$

of eigenvalues of the operator $\mathcal{H}$ *with the corresponding eigenfunctions*

$$\begin{pmatrix} c\psi_j^k(x) \\ i\omega_j^{\pm}\psi_j^k(x) \end{pmatrix}, \qquad k = 1, \dots k_j.$$

(ii) *For* $\omega \neq 0$, *the convergence* $\mathcal{R}(\omega \pm i\varepsilon) \to \mathcal{R}(\omega \pm i0)$ *holds as* $\varepsilon \to 0+$ *in* $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ *for* $\sigma > 1/2$, *uniformly in* $|\omega| \geqslant r$ *for any* $r > 0$.

(iii) *The following asymptotic relation holds*:

$$\left.\begin{array}{r} \mathcal{R}(\omega) = \mathcal{A}_1 + \mathcal{A}_2 \log^{-1}\omega + \mathcal{O}(\log^{-2}\omega) \\ \mathcal{R}'(\omega) = -\mathcal{A}_2\omega^{-1}\log^{-2}\omega + \mathcal{O}(\omega^{-1}\log^{-3}\omega) \\ \mathcal{R}''(\omega) = \mathcal{O}(\omega^{-2}\log^{-2}\omega) \end{array}\right| \qquad \omega \to 0, \quad \omega \in \mathbb{C} \setminus \mathbb{R}, \tag{3.5}$$

with respect to the norm of $\mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$ *with* $\sigma > 5/2$. *Here* $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{L}(\mathcal{F}_\sigma; \mathcal{F}_{-\sigma})$ *with* $\sigma > 5/2$.

(iv) *For* $k = 0, 1, 2$,

$$\|\mathcal{R}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})} = \mathcal{O}(1), \qquad |\omega| \to \infty, \quad \omega \in \mathbb{C} \setminus \mathbb{R} \tag{3.6}$$

with $\sigma > 1/2 + k$.

Finally, denote by $\mathcal{V}$ the matrix

$$\mathcal{V} = \begin{pmatrix} 0 & 0 \\ -iV & 0 \end{pmatrix}. \tag{3.7}$$

Then the vector equation (1.2) reads

$$i\dot{\Psi}(t) = (\mathcal{H}_0 + \mathcal{V})\Psi(t)$$

The resolvents $\mathcal{R}(\omega)$ and $\mathcal{R}_0(\omega)$ are related by the Born perturbation series

$$\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega) + \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega), \qquad \omega \in \mathbb{C} \setminus [\mathbb{R} \cup \Sigma], \tag{3.8}$$

which follows by iterating the relation

$$\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega).$$

An important role in (3.8) plays the product $\mathcal{W}(\omega) := \mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}$. We obtain the asymptotics of $\mathcal{W}(\omega)$ for large ω.

Lemma 3.3. *Let the potential* V *satisfy* (1.4) *with* $\beta > 1/2 + k + \sigma$, *for* $k = 0, 1, 2, \dots,$ *with some* $\sigma > 0$. *Then the following asymptotic relation holds*:

$$\|\mathcal{W}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_{-\sigma}, \mathcal{F}_\sigma)} = \mathcal{O}(|\omega|^{-2}), \quad |\omega| \to \infty, \quad \omega \in \mathbb{C} \setminus \mathbb{R}. \tag{3.9}$$

Proof. The bounds (3.9) follow from the algebraic structure of the matrix

$$\mathcal{W}^{(k)}(\omega) = \mathcal{V}\mathcal{R}_0^{(k)}(\omega)\mathcal{V} = \begin{pmatrix} 0 & 0 \\ -iVR_0^{(k)}(\omega^2)V & 0 \end{pmatrix}, \tag{3.10}$$

since (2.10) for $s = 1$ and $l = -1$ implies that, for large $\zeta \in \mathbb{C} \setminus [0, \infty)$,

$$\|VR_0^{(k)}(\zeta)Vf\|_{H^0_\sigma} \leqslant C\|R_0^{(k)}(\zeta)Vf\|_{H^0_{\sigma-\beta}} = \mathcal{O}(|\zeta|^{-1-\frac{k}{2}})\|Vf\|_{H^1_{\beta-\sigma}} = \mathcal{O}(|\zeta|^{-1-\frac{k}{2}})\|f\|_{H^1_{-\sigma}} \tag{3.611}$$

with $1/2 + k < \beta - \sigma$ for $k = 0, 1, 2, \dots$.

3.2. Time decay

Our main result is the following.

Theorem 3.4. *Let conditions* (1.4) *and* (3.1) *hold. Then for* $\sigma > 5/2$

$$\|e^{-it\mathcal{H}} - \sum_{\omega_J\in\Sigma} e^{-i\omega_J t}P_J\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(|t|^{-1}\log^{-2}|t|),\ t\to\pm\infty \tag{3.12}$$

Here P_J *are the Riesz projections onto the corresponding eigenspaces.*

Proof. Lemma 3.2 and the bounds (3.6) with $k=0$ implym similarly to (2.12), that

$$\Psi(t) - \sum_{\omega_J\in\Sigma} e^{-i\omega_J t}P_J\Psi_0 = \frac{1}{2\pi i}\int e^{-i\omega t}\Big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\Big]\Psi_0\ d\omega = \Psi_l(t)+\Psi_h(t) \tag{3.13}$$

where P_J stands for the corresponding Riesz projection

$$P_j\Psi_0 := -\frac{1}{2\pi i}\int_{|\omega-\omega_J|=\delta}\mathcal{R}(\omega)\Psi_0 d\omega$$

with a small $\delta>0$, and

$$\Psi_l(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} l(\omega)e^{-i\omega t}\Big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\Big]\Psi_0\ d\omega \tag{3.14}$$

$$\Psi_h(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} h(\omega)e^{-i\omega t}\Big[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\Big]\Psi_0\ d\omega, \tag{3.15}$$

where $l(\omega)$ and $h(\omega)$ are defined in Section 2.2. Further, let us study $\Psi_l(t)$ and $\Psi_h(t)$ separately.

3.2.1. Low energy component. We consider only the integral over $(0,2\varepsilon)$. The integral over $(-2\varepsilon,0)$ deal with the same way. We prove the desired decay of $\Psi_l(t)$ using a special case of Lemma 10.2 in [7].

Lemma 3.5. *Assume* $\mathbf{B}$ *be a Banach space, and* $F\in C(0,b;\mathbf{B})$ *satisfies* $F(0)=0$ *and* $F(\omega)=0$ *for* $\omega>b$,

$$F'\in L^1(\delta,b;\mathbf{B})$$

for any $\delta>0$. *Moreover,*

$$F'(\omega)=\mathcal{O}(\omega^{-1}\log^{-3}\omega)$$

as well as

$$F''(\omega)=\mathcal{O}(\omega^{-2}\log^{-2}\omega)$$

as $\omega\to 0+$. *Then*

$$\int_0^\infty e^{-it\omega}F(\omega)d\omega = \mathcal{O}(|t|^{-1}\log^{-2}|t|),\quad t\to\infty.$$

Due to (3.8), we can apply Lemma 3.5 for

$$F = l(\omega)\big(\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)\big),$$

for $\mathbf{B}=\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ with $\sigma>5/2$, and for $b=2\varepsilon$ to obtain

$$\|\Psi_l(t)\|_{\mathcal{F}_{-\sigma}} \leqslant C(1+|t|)^{-1}\log^{-2}(1+|t|)\|\Psi_0\|_{\mathcal{F}_\sigma},\qquad t\in\mathbb{R}.$$

3.2.2. High energy component. Substitute the series (3.8) into the spectral representation (3.15) for $\Psi_h(t)$:

$$\begin{aligned}\Psi_h(t) &= \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}h(\omega)\Big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\Big]\Psi_0\ d\omega\\ &\quad+\frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}h(\omega)\Big[\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\ d\omega\\ &\quad+\frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}h(\omega)\Big[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega+i0)-\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega-i0)\Big]\Psi_0\ d\omega\\ &=\Psi_{h1}(t)+\Psi_{h2}(t)+\Psi_{h3}(t),\qquad t\in\mathbb{R}.\end{aligned}\tag{3.16}$$

Below we study each of the terms Ψ_{hk}, $k=1,2,3$, separately.

Step (i). The first term is $\Psi_{h1}(t)=\mathcal{G}_h(t)\Psi_0$ by (2.18). Hence, Theorem 2.7 implies

$$\|\Psi_{h1}(t)\|_{\mathcal{F}_{-\sigma}}\leqslant C(1+|t|)^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma},\qquad t\in\mathbb{R}\tag{3.17}$$

Step (ii). Consider the second term $\Psi_{h2}(t)$. Write $h_1(\omega)=\sqrt{h(\omega)}$ (we can assume that $h(\omega)\geqslant 0$ and $h_1\in\mathbb{C}_0^\infty(\mathbb{R})$). Set

$$\Phi_{h1}=\frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}h_1(\omega)\Big[\mathcal{R}_0(\omega+i0)-\mathcal{R}_0(\omega-i0)\Big]\Psi_0\ d\omega$$

It is obvious that, for Φ_{h1}, the inequality (3.17) also holds. Namely,

$$\|\Phi_{h1}(t)\|_{\mathcal{F}_{-\sigma}}\leqslant C(1+|t|)^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma},\qquad t\in\mathbb{R}.\tag{3.18}$$

Now the second term $\Psi_{h2}(t)$ can be represented as a convolution.

Lemma 3.6 (cf. [10, Lemma 3.11])**.** *The following convolution representation holds*:

$$\Psi_{h2}(t)=i\int_0^t\mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\ d\tau,\qquad t\in\mathbb{R},\tag{3.19}$$

where the integral converges in $\mathcal{F}_{-\sigma}$ *with* $\sigma>5/2$.

Let us now apply Theorem 2.7, with h_1 instead of h, to the integrand in (3.19). For an arbitrary $\sigma'\in(5/2,\beta-5/2)$, we obtain

$$\|\mathcal{G}_{h1}(t-\tau)\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_{-\sigma}}\leqslant\frac{C\|\mathcal{V}\Phi_{h1}(\tau)\|_{\mathcal{F}_{\sigma'}}}{(1+|t-\tau|)^{3/2}}\leqslant\frac{C\|\Phi_{h1}(\tau)\|_{\mathcal{F}_{\sigma'-\beta}}}{(1+|t-\tau|)^{3/2}}\leqslant\frac{C\|\Psi_0\|_{\mathcal{F}_\sigma}}{(1+|t-\tau|)^{3/2}(1+|\tau|)^{3/2}}$$

Hence, the convolution representation (3.19) gives

$$\|\Psi_{h2}(t)\|_{\mathcal{F}_{-\sigma}}\leqslant C(1+|t|)^{-3/2}\|\Psi_0\|_{\mathcal{F}_\sigma},\quad t\in\mathbb{R}.\tag{3.20}$$

Step (iii). Finally, let us rewrite the last term Ψ_{h3} as

$$\Psi_{h3}(t)=\frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}h(\omega)\mathcal{N}(\omega)\Psi_0\ d\omega,\tag{3.21}$$

where

$$\mathcal{N}(\omega):=\mathcal{M}(\omega+i0)-\mathcal{M}(\omega-i0)$$

for $\omega\in\mathbb{R}$, and

$$\mathcal{M}(\omega):=\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)=\mathcal{R}_0(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega),\qquad\omega\in\mathbb{C}\setminus[\mathbb{R}\cup\Sigma].\tag{3.22}$$

Let us now obtain the asymptotics for $\mathcal{M}$ and its derivatives for large ω.

Lemma 3.7. *For $k = 0, 1, 2$, the following asymptotic relation holds*:

$$\|\mathcal{M}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(|\omega|^{-2}), \quad |\omega| \to \infty, \quad \omega \in \mathbb{C} \setminus \mathbb{R}, \quad \sigma > 1/2 + k \tag{3.23}$$

Proof. The asymptotic relation (3.23) follows from the asymptotic relations (2.11), (3.6), and (3.9) for $\mathcal{R}_0^{(k)}(\omega)$, $\mathcal{R}^{(k)}(\omega)$, and $\mathcal{W}^{(k)}(\omega)$. For example, consider the case $k = 2$. Differentiating (3.22), we obtain

$$\mathcal{M}'' = \mathcal{R}_0''\mathcal{W}\mathcal{R} + \mathcal{R}_0\mathcal{W}''\mathcal{R} + \mathcal{R}_0\mathcal{W}\mathcal{R}'' + 2\mathcal{R}_0'\mathcal{W}'\mathcal{R} + 2\mathcal{R}_0'\mathcal{W}\mathcal{R}' + 2\mathcal{R}_0\mathcal{W}'\mathcal{R}' \tag{3.24}$$

For a fixed $\sigma > 5/2$, choose $\sigma' \in (5/2, \beta - 1/2\})$. Then, for the first term in (3.24), for large $\omega \in \mathbb{C} \setminus \mathbb{R}$, we obtain (by (3.6) and (3.9))

$$\|\mathcal{R}_0''(\omega)\mathcal{W}(\omega)\mathcal{R}(\omega)f\|_{\mathcal{F}_{-\sigma}} \leqslant C\|\mathcal{W}(\omega)\mathcal{R}(\omega)f\|_{\mathcal{F}_{\sigma'}} \leqslant C_1|\omega|^{-2}\|\mathcal{R}(\omega)f\|_{\mathcal{F}_{-\sigma'}} \leqslant C_2|\omega|^{-2}\|f\|_{\mathcal{F}_\sigma}$$

Other terms can be estimated similarly, by choosing an appropriate value of σ'. Namely, we can take $\sigma' \in (1/2, \beta - 5/2)$ for the second term, $\sigma' \in (5/2, \beta - 1/2)$ for the third one, $\sigma' \in (3/2, \beta - 3/2)$ for the forth and sixth terms, and $\sigma' \in (3/2, \beta - 1/2)$ for the fifth term. □

Let us now prove the desired decay for $\Psi_{h3}(t)$. By Lemma 3.7,

$$(h\mathcal{N})'' \in L^1((-\varepsilon, \varepsilon); \mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma}))$$

with $\sigma > 5/2$. Then we can integrate by parts twice in (3.21) to obtain

$$\|\Psi_{h3}(t)\|_{\mathcal{F}_{-\sigma}} \leqslant C(1 + |t|)^{-2}\|\Psi_0\|_{\mathcal{F}_\sigma}, \qquad t \in \mathbb{R}.$$

This completes the proof of Theorem 3.4.

Corollary 3.8. *The asymptotic relation* (3.12) *implies* (1.5) *with the projection*

$$\mathcal{P}_c = 1 - \sum_{\omega_J \in \Sigma} P_J. \tag{3.25}$$

APPENDIX A. PROOF OF LEMMA 2.6

For an operator $A \in \mathcal{L}(H_\sigma^{-1}; H_{-\sigma}^1)$, write $\operatorname{Re} A := (A + A^*)/2$ and $\operatorname{Im} A := (A - A^*)/2i$.

Step i) First, we obtain a convenient representation for $\mathcal{G}_l(t)$. Formula (2.17) yields

$$\mathcal{G}_l(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} l(\omega) \begin{pmatrix} \omega & i \\ -i\omega^2 & \omega \end{pmatrix} e^{-i\omega t}\Big(P(\omega + i0) - P(\omega - i0)\Big)d\omega,$$

where $P(\omega) = R_0(\omega^2)$. Using the identity

$$R_0(\zeta - i0) = R_0^*(\zeta + i0), \qquad \zeta \in \mathbb{R}, \tag{A.1}$$

we obtain $P(\omega - i0) = P^*(\omega + i0)$. Hence,

$$\begin{aligned}\mathcal{G}_l(t) &= \frac{1}{\pi}\int_{\mathbb{R}} l(\omega) \begin{pmatrix} \omega & i \\ -i\omega^2 & \omega \end{pmatrix} e^{-i\omega t} \operatorname{Im} P(\omega + i0)d\omega \\ &= \frac{1}{\pi}\int_0^\infty l(\omega)\Big[\begin{pmatrix} \omega & i \\ -i\omega^2 & \omega \end{pmatrix} e^{-i\omega t} \operatorname{Im} P(\omega + i0) + \begin{pmatrix} -\omega & i \\ -i\omega^2 & -\omega \end{pmatrix} e^{i\omega t} \operatorname{Im} P(-\omega + i0)\Big]d\omega\end{aligned}$$

Applying (A.1) again, we see that $P(-\omega+i0)=P^*(\omega+i0)$. Hence,

$$\mathcal{G}_l(t)=\frac{2}{\pi}\,\mathrm{Re}\int_0^\infty l(\omega)\begin{pmatrix}\omega & i\\ -i\omega^2 & \omega\end{pmatrix}e^{-i\omega t}\,\mathrm{Im}\,P(\omega+i0)d\omega=\frac{2}{\pi}\,\mathrm{Re}\int_0^\infty l(\omega)e^{-i\omega t}\mathcal{P}(\omega). \tag{A.2}$$

Step (ii). Second, we obtain the asymptotics for the matrix operator

$$\mathcal{P}(\omega)=\begin{pmatrix}\omega & i\\ -i\omega^2 & \omega\end{pmatrix}\mathrm{Im}\,P(\omega+i0)$$

Using (2.8) and (2.9), we see that

$$\mathcal{P}(\omega)=\mathcal{P}_0(\omega)+\mathcal{P}_r(\omega),\qquad \omega\to+0 \tag{A.3}$$

where $\mathcal{P}_0(\omega)$ is the operator with the matrix integral kernel

$$\mathcal{P}_0(\omega,x,y)=\begin{pmatrix}0 & i/4\\ 0 & 0\end{pmatrix},$$

and, for the remainder $\mathcal{P}_r(\omega)$, we obtain

$$\mathcal{P}_r(\omega)=\mathcal{O}(\omega^{3/4}),\qquad \mathcal{P}_r'(\omega)=\mathcal{O}(\omega^{-1/4}),\qquad \mathcal{P}_r''(\omega)=\mathcal{O}(\omega^{-5/4}),\qquad \omega\to+0 \tag{A.4}$$

with respect to the norm of $\mathcal{L}(\mathcal{F}_\sigma;\mathcal{F}_{-\sigma})$ with $\sigma>5/2$.

Step (iii). Further, consider the contribution of the first term in (A.3) into the right-hand side of (A.2). Integrating by parts N times, $N=1,2,3,\dots$, we obtain

$$\int_0^\infty e^{-i\omega t}l(\omega)d\omega=\frac{1}{it}+\frac{1}{it}\int_0^\infty e^{-i\omega t}l'(\omega)d\omega=\dots=\frac{1}{it}+\mathcal{O}(t^{-N}),\ t\to\infty, \tag{A.5}$$

since $l(0)=1$ and $l^{(k)}(0)=0$, $k=1,2,\dots$. Hence,

$$\frac{2}{\pi}\,\mathrm{Re}\int_0^\infty l(\omega)e^{-i\omega t}\mathcal{P}_0d\omega=\mathcal{G}_0(t)+\mathcal{O}(t^{-N}),\quad t\to\infty.$$

Step (iv). Finally, Let us prove that the contribution of the remainder $\mathcal{P}_r(\omega)$ into the right-hand side of (A.2) is $\mathcal{O}(t^{-7/4})$. This results from the following lemma (cf. [7, Lemma 10.2]).

Lemma A.1. *Let $\mathcal{B}$ be a Banach space, let $F\in C(0,b;\mathbf{B})$ satisfy $F(0)=0$ and $F(\omega)=0$ for $\omega\geqslant b>0$, and let $F'\in L^1(\delta,b;\mathbf{B})$ for any $\delta>0$. Moreover, let $F'(\omega)=\mathcal{O}(\omega^{-1/4})$ and $F''(\omega)=\mathcal{O}(\omega^{-5/4})$ as $\omega\to 0+$. Then*

$$\int_0^\infty e^{-it\omega}F(\omega)d\omega=\mathcal{O}(t^{-7/4}),\qquad t\to\infty.$$

APPENDIX B. PROOF OF LEMMA 2.8

Differentiating $\mathcal{G}(t,z)$ for $|z|<t$, we obtain

$$\mathcal{G}^{11}(t,z)=\mathcal{G}^{22}(t,z)=-\frac{t}{2\pi\sqrt{(t^2-|z|^2)^3}},\quad \mathcal{G}^{21}(t,z)=-\frac{1}{2\pi\sqrt{(t^2-|z|^2)^3}}+\frac{3t^2}{2\pi\sqrt{(t^2-|z|^2)^5}}.$$

Hence, (2.13)–(2.15) imply that, for $|\alpha| \leqslant 1$,

$$|\partial_z^\alpha \mathcal{G}_r^{ij}(t,z)| = |\partial_z^\alpha \mathcal{G}^{ij}(t,z)| \leqslant Ct^{-2}, \quad |z| \leqslant t/2, \quad t \geqslant 1, \quad (i,j) \neq (1,2)$$

Further,

$$\mathcal{G}_r^{12}(t,z) = \frac{1}{2\pi}\Big(\frac{1}{\sqrt{t^2-|z|^2}} - \frac{1}{t}\Big), \quad |z| < t$$

Then the Lagrange formula implies

$$|\mathcal{G}_r^{12}(t,z)| \leqslant C|z|^2 t^{-3} \leqslant C|z| t^{-2}, \quad |z| \leqslant t/2$$

Differentiating $\mathcal{G}_r^{12}(t,z)$, we obtain

$$|\partial_{z_j} \mathcal{G}_r^{12}(t,z)| = \frac{|z_j|}{2\pi\sqrt{(t^2-|z|^2)^3}} \leqslant Ct^{-2},\ j=1,2,\ |z| \leqslant t/2$$

Hence, the bound (2.41) follows.

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