On decay of the Schrödinger resolvent

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Abstract

We strengthen known Agmon-Jensen-Kato decay of the resolvent [1, 4] for special case of Schrödinger equation in arbitrary dimension $n \geq 1$. The decay is of crucial importance in application to linear and nonlinear hyperbolic PDEs.

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1 Introduction

In this paper, we establish a decay for the resolvent of Schrödinger equation

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R} \tag{1.1}$$

in weighted energy norms.

Definition 1.1. *For $s, \sigma \in \mathbb{R}$, let us denote by $H^s_\sigma = H^s_\sigma(\mathbb{R}^n)$ the weighted Sobolev spaces introduced by Agmon, [1], with the finite norms*

$$\|\psi\|_{H^s_\sigma} = \|\langle x\rangle^\sigma \langle\nabla\rangle^s \psi\|_{L^2} < \infty, \qquad \langle x\rangle = (1+|x|^2)^{1/2}.$$

Definition 1.2. *Denote by $\mathcal{L}(B_1, B_2)$ the Banach space of bounded linear operators from a Banach space B_1 to a Banach space B_2.*

Denote by $R(\zeta) = (-\Delta - \zeta)^{-1}$, $\operatorname{Im}\zeta > 0$ the resolvent of the operator $-\Delta$. The following properties of $R(\zeta)$ are well known (see [1], and [4] in 3D case):

i) $R(\zeta)$ is strongly analytic function of $\zeta \in \mathbb{C} \setminus [0,\infty)$ with the values in $\mathcal{L}(H^{-1}_0, H^1_0)$;
ii) For $\zeta > 0$, the convergence holds $R(\zeta \pm i\varepsilon) \to R(\zeta \pm i0)$ as $\varepsilon \to 0+$ in $\mathcal{L}(H^{-1}_\sigma, H^1_{-\sigma})$ with $\sigma > 1/2$, uniformly in $\zeta \geq r$ for any $r > 0$.

For $t \in \mathbb{R}$ and $\Psi_0 \in H^{-1}_\sigma$ with $\sigma > 1$, the solution $\Psi(t)$ to the equation (1.1) admits the spectral Fourier-Laplace representation

$$\Psi(t) = \frac{1}{2\pi i}\int_0^\infty e^{-i\zeta t}\Big[R(\zeta+i0) - R(\zeta-i0)\Big]\Psi_0 \, d\zeta \tag{1.2}$$

where the integral converges in the sense of distributions of $t \in \mathbb{R}$ with the values in $H^1_{-\sigma}$.

In present paper we study the asymptotics of $R(\zeta)$ as $\zeta \to \infty$. We strengthen known Agmon-Jensen-Kato decay of the resolvent [1, (A.2')], [4, (8.1)] and proof the following statements

Theorem 1.3. *For $k = 0, 1, 2, ...,$ $m \in \mathbb{R}$, and $\sigma > 1/2 + k$, the asymptotics hold*

$$\|R^{(k)}(\zeta)\|_{\mathcal{L}(H^m_\sigma, H^{m+l}_{-\sigma})} = \mathcal{O}(|\zeta|^{-\frac{1-l+k}{2}}), \quad |\zeta| \to \infty, \quad \zeta \in \mathbb{C} \setminus [0,\infty) \tag{1.3}$$

with $l = -1, 0, 1, 2$ for $k = 0$, and $l = -1, 0, 1$ for $k = 1, 2, ...$

We give the proof with details
i) since the bound (A.2') is stated in [1] without a formal proof,
ii) to consider the simple special case and to avoid any ambiguity since the basic Lemma A.3 in [1] is stated for $\zeta \in \mathcal{K}$ where $\mathcal{K} \subset \mathbb{C}$ is a compact set (the lemma includes some additional bounds which are valid only for the compact set and are nonrelevant to our goals),

We also include in the statement a rapid decay with the loss of the smoothness corresponding to $l = -1$ in the proposition below. The rapid decay is of crucial importance in the different application. We deduce the rapid decay using some Vainberg's idea from the proof of [7, formula (17), p. 348].

Remark 1.4. *Note that asymptotics* (1.3) *does not follow from the explicit formulas for the resolvent. For example, in 3D case the resolvent of the Schrödinger equation is the integral operator with the kernel*

$$R(\zeta, x-y) = \frac{e^{i\sqrt{\zeta}|x-y|}}{4\pi|x-y|},$$

and the decay (1.3) *is not obvious since the kernel does not decay for* $|\zeta| \to \infty$.

2 The decay of the resolvent

Here we consider the case $k = 0$. We reduce the proof of Theorem 1.3 to the proof of certain lemmas. The first two lemmas are well known (see [1, Lemma A.1 and A.2], and [6, Lemma 3 and 4, p.442]).

Lemma 2.1. *(n=1) For* $s > 1/2$ *the following inequality holds:*

$$\|v\|_{0,-s} \le C_s \|[\frac{d}{dx} - \lambda]v\|_{0,s}, \qquad \lambda \in \mathbb{C}, \quad v \in C_0^\infty(\mathbb{R}). \tag{2.1}$$

Proof. We set $f(x) = [\frac{d}{dx} - \lambda]v(x)$. Then we have

$$v(x) = \int\limits_{-\infty}^{x} f(y) e^{\lambda(x-y)} dy.$$

It suffices to consider $\mathrm{Re}\,\lambda \le 0$. Then using the Cauchy inequality, we obtain

$$|v(x)|^2 \le \Big(\int\limits_{-\infty}^{x} |f(y)|dy\Big)^2 \le C_s \int\limits_{-\infty}^{\infty} (1+y^2)^s |f(y)|^2 dy \tag{2.2}$$

Multiplying (2.2) by $(1+x^2)^{-s}$ and integrating over $\mathbb{R}$, we obtain

$$\int\limits_{-\infty}^{\infty} |v(x)|^2 (1+x^2)^{-s} dx \le C_s^2 \int\limits_{-\infty}^{\infty} (1+y^2)^s |f(y)|^2 dy$$

This yields the lemma. □

Let us denote $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1, ..., n$.

Lemma 2.2. *For* $s > 1/2$, *the following inequality holds:*

$$\int\limits_{\mathbb{R}^n} (1+x_j^2)^{-s} |\partial_j u|^2 dx \le C_s^2 \int\limits_{\mathbb{R}^n} (1+x_j^2)^s |\Delta u - z|^2 dx, \quad z \in \mathbb{C}\ , \quad u \in C_0^\infty(\mathbb{R}^n). \tag{2.3}$$

Proof. It suffices to consider $j = 1$. We have

$$-\xi_1^2 - \xi_2^2 - ... - \xi_n^2 - z = [i\xi_1 - \lambda_1(\xi')][i\xi_1 - \lambda_2(\xi')], \tag{2.4}$$

where $\xi' = (\xi_2, ..., \xi_n)$. Differentiating in ξ_1, we obtain

$$2i\xi_1 = [i\xi_1 - \lambda_1(\xi')] + [i\xi_1 - \lambda_2(\xi')] \tag{2.5}$$

Let us write $u(x) = u(x_1, x')$ where $x' = (x_2, ..., x_n)$. We denote by $\tilde{u}(x_1, \xi')$ the Fourier transform of u with respect to the variable x'. Then (2.5) implies that

$$2\frac{\partial}{\partial x_1}\tilde{u}(x_1, \xi') = [\partial_1 - \lambda_1(\xi'))\tilde{u}(x_1, \xi'] + [\partial_1 - \lambda_2(\xi'))\tilde{u}(x_1, \xi'] \tag{2.6}$$

Let us apply Lemma 2.1 to the functions $v(x_1) = [\partial_1 - \lambda_j(\xi')]\tilde{u}(x_1, \xi')$ for $j = 1, 2$, taking $\lambda = \lambda_k(\xi')$ with $k \neq j$. Then (2.1) and (2.4) imply that

$$\begin{aligned} &\int_{-\infty}^{\infty} (1 + x_1^2)^{-s} |[\partial_1 - \lambda_j(\xi')]\tilde{u}(x_1, \xi')|^2 dx_1 \qquad (2.7) \\ \leq\ & C_s^2 \int_{-\infty}^{\infty} (1 + x_1^2)^{s} |[\partial_1 - \lambda_k(\xi')][\partial_1 - \lambda_j(\xi')]\tilde{u}(x_1, \xi')|^2 dx_1 \\ =\ & C_s^2 \int_{-\infty}^{\infty} (1 + x_1^2)^{s} |[\partial_1^2 - \xi_2^2 - ... - \xi_n^2 - z]\tilde{u}(x_1, \xi')|^2 dx_1 \end{aligned}$$

Combining (2.6) and (2.11), we get

$$\int_{-\infty}^{\infty} (1 + x_1^2)^{-s} |\partial_1 \tilde{u}(x_1, \xi')|^2 dx_1 \leq C_s^2 \int_{-\infty}^{\infty} (1 + x_1^2)^{s} |[\partial_1^2 - \xi_2^2 - ... - \xi_n^2 - z]\tilde{u}(x_1, \xi')|^2 dx_1 \tag{2.8}$$

Integrating (2.8) with respect to ξ', and using Parseval's formula, we find that

$$\begin{aligned} &\int_{\mathbb{R}^n} (1 + x_1^2)^{-s} |\partial_1 u(x)|^2 dx = \int_{\mathbb{R}^{n-1}} d\xi' \int_{-\infty}^{\infty} (1 + x_1^2)^{-s} |\partial_1 \tilde{u}(x_1, \xi')|^2 dx_1 \qquad (2.9) \\ \leq\ & \int_{\mathbb{R}^{n-1}} d\xi' \int_{-\infty}^{\infty} (1 + x_1^2)^{-s} |[\partial_1^2 - \xi_2^2 - ... - \xi_n^2 - z]\tilde{u}(x_1, \xi')|^2 dx_1 \\ =\ & C_s^2 \int_{\mathbb{R}^n} (1 + x_1^2)^{s} |(\Delta - z)u(x)|^2 dx. \end{aligned}$$

This establishes the lemma. $\square$

The next lemma (and its proof) is a streamlined version of corresponding [1, Lemma A.3].

Lemma 2.3. *Let us fix any $\rho > 0$ and $s \in \mathbb{R}$. Then the following estimate holds for $\psi \in C_0^\infty(\mathbb{R}^n)$*

$$\|\psi\|_{H_s^l} \le C(s,\rho)|\zeta|^{-\frac{1-l}{2}}\Big(\|(-\Delta-\zeta)\psi\|_{H_s^0} + \sum_{j=1}^n \|\partial_j\psi(x)\|_{H_s^0}\Big), \quad \zeta \in \mathbb{C}, \quad |\zeta| \ge \rho, \quad l = 0,1. \tag{2.10}$$

Proof. Step i) First we prove (2.10) for $s = 0$. For the proof we use the bound

$$(1+|\xi|^l)^2 \le C(\rho)|\zeta|^{-(1-l)}\Big(||\xi|^2-\zeta|^2+|\xi|^2\Big), \quad \xi \in \mathbb{R}^n, \quad |\zeta| \ge \rho, \quad l = 0,1. \tag{2.11}$$

For $l = 1$ the bound is obvious. For $l = 0$ we assume for the simplicity that $\rho > 1/2$. Then (2.11) reduces to a quadratic inequality for $y = |\xi|^2 - |\zeta| \ge -|\zeta|$ since then

$$\begin{aligned} ||\xi|^2-\zeta|^2+|\xi|^2 &\ge ||\xi|^2-|\zeta||^2+|\xi|^2 = y^2+y+|\zeta| \\ &\ge \min_{y\ge-|\zeta|}(y^2+y)+|\zeta| \ge \frac{|\zeta|}{2}, \qquad |\zeta| \ge 1/2. \end{aligned} \tag{2.12}$$

Further, let us multiply both sides of (2.11) by $|\hat\psi(\xi)|^2$ and integrate over $\mathbb{R}^n$. Then using Parseval's formula, we find for $|\zeta| \ge \rho$ that

$$\sum_{|\alpha|\le l}\|D^\alpha\psi\|^2 \le C\int_{\mathbb{R}^n}(1+|\xi|^l)^2|\hat\psi(\xi)|^2d\xi \le C_0(\rho)|\zeta|^{-(1-l)}\Big(\|(-\Delta-\zeta)\psi\|^2+\sum_{j=1}^n\|\partial_j\psi(x)\|^2\Big) \tag{2.13}$$

Step ii) To prove (2.10) for arbitrary $s \in \mathbb{R}$, let us introduce the family of weight functions $\rho_\varepsilon(x) = (1+|\varepsilon x|^2)^{s/2}$ with $0 < \varepsilon \le 1$. Observe that the weight with any fixed $\varepsilon > 0$ is equivalent to $\rho_1(x)$ defining the Agmon spaces, and

$$|\partial_j\rho_\varepsilon(x)| = \Big|\frac{s}{2}(1+|\varepsilon x|^2)^{s/2-1}2\varepsilon^2x_j\Big| \le \frac{|s|}{2}(1+|\varepsilon x|^2)^{s/2-1}\varepsilon(1+\varepsilon^2x_j^2) \le \varepsilon C\rho_\varepsilon(x)$$

where $C = C(s)$. Similarly, we have

$$|\partial^\alpha\rho_\varepsilon(x)| \le \varepsilon^{|\alpha|}C\rho_\varepsilon(x), \quad x \in \mathbb{R}^n, \quad 0 < \varepsilon \le 1, \quad 0 \le |\alpha| \le 2. \tag{2.14}$$

Further, we obtain

$$\partial^\alpha(\rho_\varepsilon\psi) - \rho_\varepsilon\partial^\alpha\psi = \sum_{0\le\beta_j\le\alpha_j,\ |\beta|\ \ge1} C_{\alpha,\beta}\partial^\beta\rho_\varepsilon\cdot\partial^{\alpha-\beta}\psi,$$

Hence, (2.14) implies that

$$\|\partial^\alpha(\rho_\varepsilon\psi) - \rho_\varepsilon\partial^\alpha\psi\| \le \varepsilon C_1\sum_{|\gamma|\le|\alpha|-1}\|\rho_\varepsilon\partial^\gamma\psi\|. \tag{2.15}$$

Therefore,

$$\|(-\Delta-\zeta)(\rho_\varepsilon\psi) - \rho_\varepsilon(-\Delta-\zeta)\psi\| \le \varepsilon C_2\sum_{|\alpha|\le1}\|\rho_\varepsilon\partial^\alpha\psi\|. \tag{2.16}$$

Now let us apply the estimate (2.10) for $s = 0$ proved above to the function $\rho_\varepsilon\psi$:

$$\sum_{|\alpha|\le l} \|\partial^\alpha(\rho_\varepsilon\psi)\| \le C_0(\rho)|\zeta|^{-\frac{1-l}{2}}\Big(\|(-\Delta-\zeta)(\rho_\varepsilon\psi)\| + \sum_{j=1}^n \|\partial_j(\rho_\varepsilon\psi)\|\Big), \qquad |\zeta| \ge \rho. \tag{2.17}$$

Step iii) Let us prove (2.10) for $l = 0$. Applying (2.17) with $l = 0$, we obtain by (2.16)

$$\begin{aligned}\|\rho_\varepsilon\psi\| &\le C_0(\rho)\frac{1}{\sqrt{|\zeta|}}\Big(\|(-\Delta-\zeta)(\rho_\varepsilon\psi)\| + \sum_{j=1}^n \|\partial_j(\rho_\varepsilon\psi)\|\Big)\\ &\le C_0(\rho)\frac{1}{\sqrt{|\zeta|}}\Big(\|\rho_\varepsilon(-\Delta-\zeta)\psi\| + \varepsilon C_2(\|\rho_\varepsilon\psi\| + \sum_{j=1}^n \|\rho_\varepsilon\partial_j\psi\|) + \sum_{j=1}^n \|\rho_\varepsilon\partial_j\psi\| + \varepsilon C_1\|\rho_\varepsilon\psi\|\Big)\\ &\le C_0(\rho)\frac{1}{\sqrt{|\zeta|}}\Big(\|\rho_\varepsilon(-\Delta-\zeta)\psi\| + \sum_{j=1}^n \|\rho_\varepsilon\partial_j\psi\|\Big) + \frac{1}{\sqrt{\zeta}}\varepsilon C_3(\rho)\|\rho_\varepsilon\psi\|\end{aligned}$$

Choosing $\varepsilon > 0$ small enough such that $\frac{1}{\sqrt{|\zeta|}}\varepsilon C_3(\rho) < 1$ we obtain

$$\|\rho_\varepsilon\psi\| \le C_4(\rho)\frac{1}{\sqrt{|\zeta|}}\Big(\|\rho_\varepsilon(-\Delta-\zeta)\psi\| + \sum_{j=1}^n \|\rho_\varepsilon\frac{\partial}{\partial x_j}\psi\|\Big)$$

Hence, (2.10) with $l = 0$ follows since the weight function $\rho_\varepsilon(x)$, with any fixed $\varepsilon > 0$, is equivalent to $\rho_1(x)$ defining the Agmon spaces.
Step iv) Finally let us prove (2.10) for $l = 1$. Applying (2.15) with $|\alpha| = 1$ and (2.17) with $l = 1$, we obtain

$$\begin{aligned}\sum_{|\alpha|\le 1} \|\rho_\varepsilon\partial^\alpha\psi\| &\le \sum_{|\alpha|\le 1} \|\partial^\alpha(\rho_\varepsilon\psi)\| + \varepsilon C_5\|\rho_\varepsilon\psi\|\\ &\le C_0(\rho)\Big(\|(-\Delta-\zeta)(\rho_\varepsilon\psi)\| + \sum_{j=1}^n \|\partial_j(\rho_\varepsilon\psi)\|\Big) + \varepsilon C_5\|\rho_\varepsilon\psi\|\\ &\le C_0(\rho)\Big(\|\rho_\varepsilon(-\Delta-\zeta)\psi\| + \sum_{j=1}^n \|\rho_\varepsilon\partial_j\psi\|\Big) + \varepsilon C_6(\rho)\|\rho_\varepsilon\psi\|\end{aligned}$$

using (2.16) as above. Choosing $\varepsilon > 0$ small enough, we obtain

$$\sum_{|\alpha|\le 1} \|\rho_\varepsilon\partial^\alpha\psi\| \le C_7(\rho)\Big(\|\rho_\varepsilon(-\Delta-\zeta)\psi\| + \sum_{j=1}^n \|\rho_\varepsilon\frac{\partial}{\partial x_j}\psi\|\Big)$$

that implies (2.10) with $l = 1$. □

Proof of Theorem 1.3 for $k = 0$**.** It suffices to verify the case $m = 0$ since $R(\zeta)$ commutes with the operator $\langle\nabla\rangle^m$. We must prove that

$$\|R(\zeta)\|_{\mathcal{L}(H^0_\sigma, H^l_{-\sigma})} = \mathcal{O}(|\zeta|^{-\frac{1-l}{2}}), \quad |\zeta| \to \infty, \quad \zeta \in \mathbb{C}\setminus[0,\infty), \quad l = -1, 0, 1, 2 \tag{2.18}$$

for $\sigma > 1/2$. By Lemma 2.3 with $s = -\sigma$,

$$\|\psi\|_{H^l_{-\sigma}} \le C(\sigma,\rho)|\zeta|^{-\frac{1-l}{2}}\Big(\|(-\Delta-\zeta)\psi\|_{H^0_{-\sigma}} + \sum_{j=1}^n \|\partial_j\psi\|_{H^0_{-\sigma}}\Big), \quad |\zeta| \ge \rho > 0, \quad l = 0,1 \tag{2.19}$$

for all $\psi \in H^2_\sigma(\mathbb{R}^n)$. By Lemma 2.2, we have

$$\sum_{j=1}^n \|\partial_j\psi\|_{H^0_{-\sigma}} \le C(\sigma)\|(\Delta-\zeta)\psi\|_{H^0_\sigma}, \quad j = 1,...,n. \tag{2.20}$$

Combining (2.19) and (2.20), we obtain

$$\|\psi\|_{H^l_{-\sigma}} \le C(\sigma,\rho)|\zeta|^{-\frac{1-l}{2}}\Big(\|(\Delta-\zeta)\psi\|_{H^0_{-\sigma}} + C(\sigma)\|(-\Delta-\zeta)\psi\|_{H^0_\sigma}\Big) \le C_1(\sigma,\rho)|\zeta|^{-\frac{1-l}{2}}\|(\Delta-\zeta)\psi\|_{H^0_\sigma}$$

and then (2.18) with $l = 0,1$ is proved. It remains to prove (2.18) for $l = -1$ and $l = 2$.

The bound with $l = 1$ implies that $R(\zeta) = \mathcal{O}(1)$ in $\mathcal{L}(H^0_\sigma, H^1_{-\sigma})$, hence $\Delta R(\zeta) = \mathcal{O}(1)$ in $\mathcal{L}(H^0_\sigma, H^{-1}_{-\sigma})$. Therefore

$$R(\zeta) = -\frac{1}{\zeta} - \frac{\Delta R(\zeta)}{\zeta} = \mathcal{O}(|\zeta|^{-1}) \quad \text{in} \quad \mathcal{L}(H^0_\sigma, H^{-1}_{-\sigma}).$$

Hence, the bound (2.18) follow with $l = -1$.

Using the identity $(1-\Delta)R(\zeta) = 1 + (1+\zeta)R(\zeta)$, we obtain

$$\begin{aligned} \|R(\zeta)\|_{\mathcal{L}(H^0_\sigma, H^2_{-\sigma})} &= \|(1-\Delta)R(\zeta)\|_{\mathcal{L}(H^0_\sigma, H^0_{-\sigma})} = \|1 + (1+\zeta)R(\zeta)\|_{\mathcal{L}(H^0_\sigma, H^0_{-\sigma})} \\ &= 1 + \mathcal{O}(|\zeta|)\|R(\zeta)\|_{\mathcal{L}(H^0_\sigma, H^0_{-\sigma})} = \mathcal{O}(|\zeta|^{1/2}) \end{aligned} \tag{2.21}$$

then the bound (2.18) follow with $l = 2$. □

3 The decay of the derivatives

First we prepare two lemma concerning the relation between derivatives of $R(\zeta)$ stated in [4, (8.2)].

Lemma 3.1. *The following identity (Lavine-type) holds*

$$\zeta R_0'(\zeta) = -R(\zeta) + \frac{1}{2}[x\cdot\nabla, R(\zeta)], \quad \zeta \in \mathbb{C}\setminus[0,\infty) \tag{3.1}$$

where $[\cdot,\cdot]$ *stands for the commutator.*

Proof. Applying Fourier transform to RHS of (3.1), we obtain

$$F_{x-y\to\xi}\{-R(\zeta) + \frac{1}{2}[x\cdot\nabla, R(\zeta)]\} = -\frac{1}{|\xi|^2-\zeta} + \frac{1}{2}i\nabla\Big(\frac{1}{|\xi|^2-\zeta}\Big)\cdot(-i\xi) \tag{3.2}$$

since

$$[x\cdot\nabla, R(\zeta)] = x\cdot\nabla R(\zeta) - R(\zeta)x\cdot\nabla = xR(\zeta)\cdot\nabla - R(\zeta)x\cdot\nabla = [x, R(\zeta)]\cdot\nabla$$

Simplifying (3.2), we have

$$F_{x-y\to\xi}\{-R(\zeta)+\frac{1}{2}[x\cdot\nabla,R(\zeta)]\}=\frac{-1}{|\xi|^2-\zeta}+\frac{|\xi|^2}{(|\xi|^2-\zeta)^2}=\frac{-1}{|\xi|^2-\zeta}+\frac{|\xi|^2-\zeta+\zeta}{(|\xi|^2-\zeta)^2}=\frac{\zeta}{(|\xi|^2-\zeta)^2}$$

which coincides with the Fourier transform of LHS (3.1) □

Lemma 3.2. *The following identity holds*

$$2\zeta R^{(k)}(\zeta)=-(2k-n)R^{(k-1)}(\zeta)-\frac{1}{2}[x,[x,R^{(k-2)}(\zeta)]],\quad k\geq 2 \tag{3.3}$$

Proof. Applying Fourier transform to RHS of (3.3), we obtain

$$-(2k-n)\frac{(k-1)!}{(|\xi|^2-\zeta)^k}-\frac{1}{2}(i\nabla)^2\frac{(k-2)!}{(|\xi|^2-\zeta)^{k-1}}=-(2k-n)\frac{(k-1)!}{(|\xi|^2-\zeta)^k}-\nabla\cdot\frac{(k-1)!\ \xi}{(|\xi|^2-\zeta)^k}$$

$$=-(2k-n)\frac{(k-1)!}{(|\xi|^2-\zeta)^k}-n\frac{(k-1)!}{(|\xi|^2-\zeta)^k}+\frac{2k!|\xi|^2}{(|\xi|^2-\zeta)^{k+1}}=-\frac{2k!}{(|\xi|^2-\zeta)^k}+\frac{2k!(|\xi|^2-\zeta+\zeta)}{(|\xi|^2-\zeta)^{k+1}}$$

$$=2\zeta\frac{k!}{(|\xi|^2-\zeta)^{k+1}}$$

which coincides with the Fourier transform of LHS (3.3). □

Proof of Theorem 1.3

For $k=1$ the asymptotics (1.3) follows from (2.18) and (3.1) since

$$x\in\mathcal{L}(H^m_\sigma,H^m_{\sigma-1}),\quad \nabla\in\mathcal{L}(H^m_\sigma,H^{m-1}_\sigma)$$

Namely, (2.18) for $\sigma>3/2$ implies

$$\|R(\zeta)\|_{\mathcal{L}(H^0_{\sigma-1},H^{1+l}_{-\sigma+1})}=\mathcal{O}(|\zeta|^{-\frac{1-(1+l)}{2}}),\quad 0\leq 1+l\leq 2$$

Hence,

$$\|x\cdot\nabla R(\zeta)\|_{\mathcal{L}(H^0_{\sigma-1},H^l_{-\sigma})}=\mathcal{O}(|\zeta|^{\frac{l}{2}}),\quad \|R(\zeta)x\cdot\nabla\|_{\mathcal{L}(H^0_\sigma,H^l_{-\sigma+1})}=\mathcal{O}(|\zeta|^{\frac{l}{2}})$$

Therefore (2.18) and (3.1) imply that

$$\|R'(\zeta)\|_{\mathcal{L}(H^0_\sigma,H^l_{-\sigma})}=\mathcal{O}(|\zeta|^{-\frac{2-l}{2}})$$

Finally, we obtain (1.3) for all $k\geq 2$ by induction using the identity (3.3).

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