Weighted energy decay for magnetic Klein-Gordon equation

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Abstract

We obtain a dispersive long-time decay in weighted energy norms for solutions of 3D Klein-Gordon equation with magnetic and scalar potentials. The decay extends the results obtained by Jensen and Kato for the Schrödinger equation with scalar potential. For the proof we develop the spectral theory of Agmon, Jensen and Kato and minimal escape velocities estimates of Hunziker, Sigal and Soffer.

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1 Introduction

We establish a dispersive long time decay for solutions to 3D magnetic Klein-Gordon equation

$$\ddot{\psi}(x,t) = \big(\nabla - iA(x)\big)^2\psi(x,t) - m^2\psi(x,t) - V(x)\psi(x,t), \quad m > 0. \tag{1.1}$$

For $s, \sigma \in \mathbb{R}$, denote by $\mathcal{H}^s_\sigma = \mathcal{H}^s_\sigma(\mathbb{R}^3)$ the weighted Sobolev spaces introduced by Agmon, [1], with the finite norms

$$\|\psi\|_{\mathcal{H}^s_\sigma} = \|\langle x\rangle^\sigma \langle\nabla\rangle^s \psi\|_{L^2(\mathbb{R}^3)} < \infty, \qquad \langle x\rangle = (1+|x|^2)^{1/2}. \tag{1.2}$$

We assume that $V(x) \in C^1(\mathbb{R}^3)$, $A_j \in C^4(\mathbb{R}^3)$ are real functions, and for some $\beta > 3$ the bounds hold

$$|V(x)| + |\nabla V(x)| + \sum_{|\alpha|\le 4}\sum_{j=1}^{3} |D^\alpha A_j(x)| \le C\langle x\rangle^{-\beta}, \quad x \in \mathbb{R}^3. \tag{1.3}$$

We restrict ourselves to the "regular case" in the terminology of [10] where the truncated resolvent of the corresponding magnetic Schrödinger operator

$$H = (i\nabla + A)^2 + V = -\Delta + 2iA\cdot\nabla + i\nabla\cdot A + A^2 + V$$

is bounded at the edge point 0 of the continuous spectrum. In other words, the point 0 is neither an eigenvalue nor a resonance for the operator H; this holds for *generic potentials*.

In vector form, equation (1.1) reads

$$i\dot{\Psi}(t) = \mathcal{K}\Psi(t), \tag{1.4}$$

where

$$\Psi(t) = \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix}, \qquad \mathcal{K} = \begin{pmatrix} 0 & i \\ i\big((\nabla - iA(x))^2 - m^2 - V(x)\big) & 0 \end{pmatrix}.$$

Denote $\mathcal{F}_\sigma = \mathcal{H}^1_\sigma \oplus \mathcal{H}^0_\sigma$, and let $U(t) : \mathcal{F}_0 \to \mathcal{F}_0$ be the dynamical group of equation (1.4).

Our main result is the following long time decay: in the regular case for any $\sigma > 5/2$

$$\|U(t)\Psi(0)\|_{\mathcal{F}_{-\sigma}} \le C(1+t)^{-3/2}\|\Psi(0)\|_{\mathcal{F}_\sigma}, \quad t > 0 \tag{1.5}$$

for solutions to (1.4) with initial data $\Psi(0)$ from the space of continuous spectrum of $\mathcal{K}$.

Let us comment on previous results in this direction. The decay of type (1.5) in weighted norms has been established first by Jensen and Kato [10] for 3D Schrödinger equation with scalar potential. The result has been extended to more general PDEs of the Schrödinger type by Murata [14]. The survey of the results can be found in [16]. For the Klein-Gordon and wave equations with scalar potential, the weighed energy decay has been established in [11] and [13], and for the Dirac equation in [3]. The Strichartz estimates for magnetic Schrödinger, wave, Klein-Gordon and Dirac equations with smallness conditions on the potentials were obtained in [5, 6, 7] and for magnetic Schrödinger equations with large potentials in [4].

The decay in weighted norms for magnetic Schrödinger equation has been established in [12]. For the Klein-Gordon equations with magnetic potential, the decay $\sim t^{-3/2}$ was obtained by Vainberg [22, 23] in local energy norms for initial data with a compact support. However, the decay in weighed energy norms for magnetic Klein-Gordon equation was not obtain up to now.

Let us comment on our approach. We extend the method of Jensen and Kato [10] to the Klein-Gordon equation with magnetic potential. The main problem consists in the presence of the first order derivatives in the perturbation. These derivatives cannot be handled with the perturbation theory like [11] since the corresponding terms do not decay in suitable norms.

Our main novelties are Propositions 3.2 and 3.3 on decay of propagators far from thresholds. First, we prove the decay for magnetic Klein-Gordon equation with $V = 0$. The proof rely on the Mourre estimates for the operator $B_A = ((i\nabla + A)^2 + m^2)^{1/2}$ and the minimal escape velocity estimates of Hunziker, Sigal and Soffer [9] and their development by Boussaid [3]. Finally, we obtain the decay for the Klein-Gordon equation with $V \neq 0$ using the Born perturbation series and our recent results on the decay of the magnetic Schrödinger resolvent [12].

2 Spectral properties

Denote $L^2 = L^2(\mathbb{R}^3)$. Similarly to [10, p. 589], [14, formula (3.1)] and [12, §3.2], let us introduce a generalized eigenspace $\mathcal{M}$ for the Schrödinger operator H:

$$\mathcal{M} = \{\psi \in \mathcal{H}^0_{-1/2-0} : \ (1 + A_0 W)\psi = 0\},$$

where A_0 is the operator with the integral kernel $1/4\pi|x-y|$ and $W = 2iA\cdot\nabla + i\nabla\cdot A + A^2 + V$. Functions $\psi \in \mathcal{M} \cap L^2$ are the zero eigenfunctions of H and functions $\psi \in \mathcal{M} \setminus L^2$ are the **zero resonances** of H.

Our key assumption is the following spectral condition (cf. Condition (i) in [14, Theorem 7.2]):

$$\mathcal{M} = 0 \tag{2.1}$$

In other words, the point zero is neither an eigenvalue nor a resonance for the operator H. Condition (2.1) holds for *a generic* W. Denote by $\mathcal{L}(B_1, B_2)$ the Banach space of bounded linear operators from a Banach space B_1 to a Banach space B_2. Denote by $R(\omega) = (H - \omega)^{-1}$ the resolvent of the operator H. Let us collect the properties of $R(\omega)$ obtained in [12] under conditions (1.3) and (2.1):

Lemma 2.1. *Let condition (*1.3*) holds. Then*
i) For $\omega > 0$, the limiting absorption principle holds

$$R(\omega \pm i\varepsilon) \to R(\omega \pm i0), \quad \varepsilon \to 0+ \tag{2.2}$$

in $\mathcal{L}(\mathcal{H}^0_\sigma, \mathcal{H}^2_{-\sigma})$ with $\sigma > 1/2$.

ii) For $k = 0, 1, 2$, $\sigma > 1/2 + k$, and $s = 0, 1$, the asymptotics hold

$$\|R^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{H}^s_\sigma;\mathcal{H}^{s+l}_{-\sigma})} = \mathcal{O}(|\omega|^{-\frac{1-l+k}{2}}), \quad |\omega| \to \infty, \quad \omega \in \mathbb{C} \setminus [0, \infty). \tag{2.3}$$

where $l = 0, 1$ for $s = 0$, and $l = 0, -1$ for $s = 1$.

Note that asymptotics (2.3) have been proved in [12] for $s = 0$ only. In the case $s = 1$ the proof is given in Appendix.

Lemma 2.2. *Let conditions* (1.3) *and* (2.1) *hold. Then*
i) For $\sigma > 1$ the asymptotics hold

$$\|R(\omega) - R(0)\|_{\mathcal{L}(\mathcal{H}^0_\sigma,\mathcal{H}^2_{-\sigma})} \to 0, \quad \omega \to 0, \quad \omega \in \mathbb{C} \setminus [0,\infty) \tag{2.4}$$

where $R(0) : \mathcal{H}^0_\sigma \to \mathcal{H}^2_{-\sigma'}$ is a continuous operator.
ii) For $k = 1, 2$ and $\sigma > 1/2 + k$ the asymptotics hold

$$\|R^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{H}^0_\sigma,\mathcal{H}^2_{-\sigma})} = \mathcal{O}(|\omega|^{1/2-k}), \quad \omega \to 0, \quad \omega \in \mathbb{C} \setminus [0,\infty). \tag{2.5}$$

Denote $\Gamma := (-\infty, -m) \cup (m, \infty)$ and let $\mathcal{R}(\omega) = (\mathcal{K} - \omega)^{-1}$ be the resolvent of the operator $\mathcal{K}$. The resolvent $\mathcal{R}$ can be expressed in terms of the resolvent R:

$$\mathcal{R}(\omega) = \begin{pmatrix} \omega R(\omega^2 - m^2) & iR(\omega^2 - m^2) \\ -i(1 + \omega^2 R(\omega^2 - m^2)) & \omega R(\omega^2 - m^2) \end{pmatrix} \tag{2.6}$$

Hence, the properties of R imply the corresponding properties of $\mathcal{R}$:

Lemma 2.3. *Let conditions* (1.3) *and* (2.1) *hold. Then*
i) The limiting absorption principle holds:

$$\mathcal{R}(\omega \pm i\varepsilon) \to \mathcal{R}(\omega \pm i0), \quad \omega \in \Gamma \quad \varepsilon \to 0+ \tag{2.7}$$

in $\mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma})$ with $\sigma > 1/2$.
ii) For $\omega \in \mathbb{C} \setminus \overline{\Gamma}$ the asymptotics hold

$$\|\mathcal{R}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(1), \quad \omega \pm m \to 0, \quad \sigma > 1 \tag{2.8}$$

$$\|\mathcal{R}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(|\omega \pm m|^{1/2-k}), \quad \omega \pm m \to 0, \quad \sigma > 1/2 + k, \quad k = 1,\ 2, \ldots \tag{2.9}$$

iii) For $k = 0, 1, 2, \ldots$ and $\sigma > 1/2 + k$ the asymptotics hold

$$\|\mathcal{R}^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(1) \quad \omega \to \infty, \quad \omega \in \mathbb{C} \setminus \Gamma \tag{2.10}$$

Under conditions (1.3) and (2.1) the representation holds

$$U(t)P_c(\mathcal{K})\Psi(0) = \frac{1}{2\pi i} \int_\Gamma e^{-i\omega t}[\mathcal{R}(\omega + i0) - \mathcal{R}(\omega - i0)]\Psi(0)\, d\omega, \quad t \in \mathbb{R} \tag{2.11}$$

for initial state $\Psi(0) \in \mathcal{F}_\sigma$ with $\sigma > 1$. Here

$$P_c(\mathcal{K}) = \mathbf{I}_{\overline{\Gamma}}(\mathcal{K})$$

is the projector associated with the continuous spectrum of $\mathcal{K}$. The representation (2.11) follows from the Cauchy residue theorem, and Lemma 2.3 (cf. [11, §2.2]).

3 Time decay

We are now able to state our main result

Theorem 3.1. *(Weighed energy decay) Let assumptions (1.3) and (2.1) hold. Then for $\sigma > 5/2$ the time decay holds*

$$\|U(t)P_c(\mathcal{K})\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} \le C\langle t\rangle^{-3/2}, \quad t\in\mathbb{R}. \tag{3.1}$$

We prove the decay separately for the components of the solution near thresholds and far from thresholds. More precisely, we choose a function $\chi_m \in C_0^\infty(\mathbb{R})$ supported in a sufficiently small neighborhood of $[-m,m]$. Then

$$\|U(t)P_c(\mathcal{K})\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} \le \|U(t)\chi_m(\mathcal{K})P_c(K)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} + \|U(t)(1-\chi_m)(\mathcal{K})\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})}. \tag{3.2}$$

The decay of the first *low energy component* can be treated by the method of Jensen and Kato [10]. Namely, using the spectral representation (cf. (2.11))

$$U(t)\chi_m(\mathcal{K})P_c(\mathcal{K})\Psi(0) = \frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}\chi_m(\omega)[\mathcal{R}(\omega+i0)-\mathcal{R}(\omega-i0)]\Psi(0)\,d\omega, \quad t\in\mathbb{R}, \tag{3.3}$$

and asymptotics (2.8) - (2.9), we obtain for $\sigma > 5/2$

$$\|U(t)\chi_m(\mathcal{K})P_c(\mathcal{K})\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} \le C(\sigma)\langle t\rangle^{-3/2}, \quad t\in\mathbb{R} \tag{3.4}$$

by [10, Lemma 10.2]. To treat the decay of the second *high energy component* we cannot use the spectral representation since the resolvent $\mathcal{R}(\omega)$ does not decay in $\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})$ as $\omega\to\infty$ (see (2.10)). We obtain the required decay in the following way. First, we consider the Klein-Gordon equation without a scalar potential, i.e with $V=0$:

$$i\dot\Psi(t) = \mathcal{K}_0\Psi(t), \qquad \mathcal{K}_0 = \begin{pmatrix} 0 & i \\ i\big((\nabla - iA(x))^2 - m^2\big) & 0 \end{pmatrix}. \tag{3.5}$$

Denote $U_0(t):\mathcal{F}_0\to\mathcal{F}_0$ the dynamical group of equation (3.5). Applying the minimal escape velocity estimates of Hunziker, Sigal and Soffer [9, Theorem 1.1.] and their modification [3, Proposition 2.2] we will prove

Proposition 3.2. *(The case $V=0$) Let assumption (1.3) hold. Then for any bounded $\chi \in C^\infty(\mathbb{R})$ supported in Γ, any $\sigma\ge 2$ and any $\varepsilon>0$ the decay holds*

$$\|U_0(t)\chi(\mathcal{K}_0)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} \le C(\varepsilon)\langle t\rangle^{-2+\varepsilon}, \quad t\in\mathbb{R}. \tag{3.6}$$

Finally, we will prove the decay for the Klein-Gordon equation with $V\ne 0$ using the Born perturbation series (5.3).

Proposition 3.3. *(The case $V\ne 0$) Let assumption (1.3) hold. Then for any bounded $\chi\in C^\infty(\mathbb{R})$ supported in Γ, any $\sigma > 5/2$ and any $\varepsilon > 0$ the decay holds*

$$\|U(t)\chi(\mathcal{K})\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} \le C(\sigma,\varepsilon)\langle t\rangle^{-2+\varepsilon}, \quad t\in\mathbb{R}. \tag{3.7}$$

Theorem 3.1 follows from (3.2), (3.4), and Propositions 3.2 and 3.3. We prove Propositions 3.2 and 3.3 in the remaining part of the paper.

4 The case $V = 0$

First, we prove Proposition 3.2. Denote

$$B = \Big[\big(i\nabla - A(x)\big)^2 + m^2\Big]^{1/2}$$

which is positive and self-adjoint in L^2. Then

$$U_0(t) = \left(\begin{array}{cc} \cos Bt & B^{-1}\sin Bt \\ \\ -B\sin Bt & \cos Bt \end{array}\right). \tag{4.1}$$

Hence, for the proof of (3.6) it suffices to check that

$$\|e^{-itB}\chi(B)\|_{\mathcal{L}(H^0_\sigma, H^0_{-\sigma})} \le C(\varepsilon)\langle t\rangle^{-2+\varepsilon}, \quad t \in \mathbb{R} \tag{4.2}$$

for $\sigma \ge 2$, $\varepsilon > 0$, and any bounded $\chi \in C^\infty(\mathbb{R})$ with support in (m, ∞). We will deduce (4.2) from the minimal escape velocity estimates [9] which rely on the Mourre estimates for the operator B.

4.1 Mourre estimates

Denote

$$P = \frac{i}{2}(x\cdot\nabla + \nabla\cdot x), \quad P_B = PB^{-1} + B^{-1}P. \tag{4.3}$$

and let $\mathbf{I}_M$ be the characteristic function of a set M.

Lemma 4.1. *Suppose that assumption (*1.3*) holds. Then*
i) For any $\theta \in (0,1)$ there exists $\nu \ge 0$ such that

$$\mathbf{I}_{|B|\ge m+\nu}\, i[B, P_B]\,\mathbf{I}_{|B|\ge m+\nu} \ge \theta\,\mathbf{I}_{|B|\ge m+\nu}. \tag{4.4}$$

ii) For any $\lambda \in \Gamma$ and any $\delta > 0$, there exists $\mu > 0$ such that

$$\mathbf{I}_{|B-\lambda|\le\mu}\, i[B, P_B]\,\mathbf{I}_{|B-\lambda|\le\mu} \ge \Big(\frac{\lambda^2 - m^2}{\lambda^2} - \delta\Big)\,\mathbf{I}_{|B-\lambda|\le\mu}. \tag{4.5}$$

Proof. Step i) Let us obtain a suitable formula for commutator $[B, P_B]$. First,

$$[B, P_B] = [B, P]B^{-1} + B^{-1}[B, P].$$

Further, we express $[B, P]$ via $[B^2, P]$ following [20]. Namely, using the Kato square root formula [15, page 317], we obtain for any $\psi \in L^2$

$$B\psi = \frac{1}{\pi}\int_0^\infty \omega^{-1/2}B^2(B^2+\omega)^{-1}\psi\, d\omega. \tag{4.6}$$

Hence, one has

$$[B, P] = \frac{1}{\pi}\int_0^\infty \omega^{-1/2}[B^2(B^2+\omega)^{-1}, P]\, d\omega. \tag{4.7}$$

Further,

$$(B^2+\omega)[B^2(B^2+\omega)^{-1},P](B^2+\omega)=B^2P(B^2+\omega)-(B^2+\omega)PB^2=\omega[B^2,P].$$

Then (4.6) becomes

$$[B,P]=\frac{1}{\pi}\int_0^\infty \omega^{1/2}(B^2+\omega)^{-1}[B^2,P](B^2+\omega)^{-1}\,d\omega. \tag{4.8}$$

It is easy to calculate

$$i[B^2,P]=B^2-m^2+Q, \tag{4.9}$$

where

$$Q=-A^2+2i(\nabla\cdot A)-x\cdot(\nabla A^2)+2ix\cdot(\nabla(\nabla\cdot A))+i\sum_{j,k}x_j(\nabla_jA_k)\nabla_k.$$

Substituting into (4.8), we get

$$\left.\begin{aligned}
i[B,P]B^{-1} &= \frac{1}{\pi}\int_0^\infty \omega^{1/2}(B^2-m^2)B^{-1}(B^2+\omega)^{-2}\,d\omega\\
&+\frac{1}{\pi}\int_0^\infty \omega^{1/2}(B^2+\omega)^{-1}QB^{-1}(B^2+\omega)^{-1}\,d\omega = J_1+J_2\\
iB^{-1}[B,P] &= \frac{1}{\pi}\int_0^\infty \omega^{1/2}(B^2-m^2)B^{-1}(B^2+\omega)^{-2}\,d\omega\\
&+\frac{1}{\pi}\int_0^\infty \omega^{1/2}(B^2+\omega)^{-1}B^{-1}Q(B^2+\omega)^{-1}\,d\omega = J_1+J_3
\end{aligned}\right| \tag{4.10}$$

Applying the integration by parts, we rewrite J_1 as

$$\begin{aligned}
J_1 &= -\frac{1}{\pi}\int_0^\infty \omega^{1/2}\frac{d}{d\omega}(B^2+\omega)^{-1}(B^2-m^2)B^{-1}\,d\omega\\
&= \frac{1}{2}\int_0^\infty \omega^{-1/2}(B^2+\omega)^{-1}(B^2-m^2)B^{-1}\,d\omega=\frac{1}{2}(B^2-m^2)B^{-2},
\end{aligned} \tag{4.11}$$

which follows from (4.6) and the bound

$$\|(B^2+\omega)^{-1}\psi\|_{L^2}\le(m^2+\omega)^{-1}\|\psi\|_{L^2}. \tag{4.12}$$

Finally,

$$i[B,P_B]=\frac{B^2-m^2}{B^2}+J,\quad J=J_2+J_3. \tag{4.13}$$

Step ii) Let us prove that $J = J_2 + J_3 : L^2 \to L^2$ is a compact operator. First, bounds (1.3) and (4.12) imply for any $0 < \sigma < \beta$ and $0 < \alpha < 1$

$$\|(B^2+\omega)^{-1+\alpha} Q B^{-1}(B^2+\omega)^{-1}\psi\|_{\mathcal{H}^0_\sigma} \;\le\; C(1+\omega)^{-2+\alpha}\|\psi\|_{L^2},$$

$$\|(B^2+\omega)^{-1+\alpha} B^{-1} Q(B^2+\omega)^{-1}\psi\|_{\mathcal{H}^0_\sigma} \;\le\; C(1+\omega)^{-2+\alpha}\|\psi\|_{L^2}$$

by the technique of [17] and the standard technique of PDOs [2, 19, 21]. Second, for any $0 < \alpha < 1$ and $\phi \in \mathcal{H}^0_\sigma$ the bound holds

$$\|(B^2+\omega)^{-\alpha}\phi\|_{\mathcal{H}^{2\alpha}_\sigma} \le C\|\phi\|_{\mathcal{H}^0_\sigma}. \tag{4.14}$$

Indeed, using the technique [17], we get

$$\|(B^2+\omega)^{-\alpha}\phi\|_{\mathcal{H}^{2\alpha}_\sigma} \le C\|B^{2\alpha}(B^2+\omega)^{-\alpha}\phi\|_{\mathcal{H}^0_\sigma} \le C_1\|\phi\|_{\mathcal{H}^0_\sigma}$$

since B is a positive elliptic first order PDO. Finally, choosing $0 < \alpha < 1/2$, we obtain

$$\|J_2\psi\|_{\mathcal{H}^{2\alpha}_\sigma} + \|J_3\psi\|_{\mathcal{H}^{2\alpha}_\sigma} \le C\|\psi\|_{L^2}. \tag{4.15}$$

Thetefore, $J_2,\ J_3 : L^2 \to L^2$ are compact operators since the embedding $\mathcal{H}^{2\alpha}_\sigma \subset L^2$ is compact by Sobolev's Embedding Theorem.

Step iii) In the case $A = 0$ (and then $J = 0$) bounds (4.4) and (4.5) follow from (4.13). For $A \neq 0$ and any $\varkappa > 0$ we split the compact operator J as

$$J = J_\varkappa + \sum_1^N |f_j\rangle\langle g_j|,$$

where $\|J_\varkappa\| \le \varkappa$, and $f_j, g_j \in L^2$. Then

$$\|\,\mathbf{I}_{|B-\lambda|\le\mu}|\,f_j\rangle\langle g_j\,|\mathbf{I}_{|B-\lambda|\le\mu}\|_{L^2\to L^2} \le \|\,\mathbf{I}_{|B-\lambda|\le\mu}\hat f_j\|_{L^2}\cdot\|\,\mathbf{I}_{|B-\lambda|\le\mu}\hat g_j\|_{L^2} \to 0, \quad \mu\to 0$$

due to absolute continuity of spectral representatives $\hat f_j, \hat g_j \in L^2([0,\infty), X)$ of $f_j,\ g_j$ in the spectral resolution of B, where X is an appropriate Hilbert space. Hence, for sufficiently small $\varkappa$ and μ bound (4.5) follows. Similarly, bound (4.4) follows for sufficiently small $\varkappa$ and sufficiently large ν. □

4.2 Minimal escape velocity

Here we adapt the methods of [9, Theorem 1.1] to our case (see also [3, Theorem 2.1])

Lemma 4.2. *Let assumption (*1.3*) hold. Then for any bounded $\chi \in C^\infty$ with support in Γ, there exists $\theta > 0$ such that for any $v \in (0,\theta)$, any $a \in \mathbb{R}$, and any $\varepsilon > 0$ the bound holds*

$$\|\mathbf{I}_{P_B\le a+v|t|}\, e^{-itB}\chi(B)\mathbf{I}_{P_B\ge a}\| \le C(v,\varepsilon)\langle t\rangle^{-2+\varepsilon}, \quad t\in\mathbb{R}, \tag{4.16}$$

where C does not depend on a and t.

Proof. According to [9, Theorem 1.1] and [3, Theorem 2.1] bound (4.16) follows from the Mourre estimates (4.4) - (4.5) and the boundedness of commutators $ad^k_{P_B}(B): L^2 \to L^2$ for $1 \le k \le 3$, where

$$ad^1_{P_B}(B) = [B, P_B], \quad \text{and} \quad ad^k_{P_B}(B) = [ad^{k-1}_{P_B}(B), P_B].$$

The boundedness of $ad^1_{P_B}(B) = [B, P_B]$ follows from (4.13) and (4.15).
For $k = 2, 3$ we have

$$ad^k_{P_B}(B) = -i\big[ad^{k-1}_{P_B}(2J_1) + ad^{k-1}_{P_B}(J_2) + ad^{k-1}_{P_B}(J_3)\big]$$

by (4.13). The boundedness of $ad^{k-1}_{P_B}(J_2)$ and $ad^{k-1}_{P_B}(J_3)$ is obvious due to (1.3) and definition (4.10) of J_2 and J_3. Hence, it remains to prove that $[P_B, 2J_1]$ and $[P_B, [P_B, 2J_1]]$ are bounded in L^2. The boundedness of $[B, P_B]$ imply the boundedness of

$$[P_B, B^{-1}] = B^{-1}[B, P_B]B^{-1}.$$

Then by (4.11) the operator

$$[P_B, 2J_1] = [P_B, (B^2 - m^2)B^{-2}] = -m^2[P_B, B^{-2}] = -m^2\big([P_B, B^{-1}]B^{-1} + B^{-1}[P_B, B^{-1}]\big)$$

is also bounded in L^2. Further, (4.3) and (4.9) imply

$$\begin{aligned}[P_B, 2J_1] &= m^2B^{-2}[P_B, B^2]B^{-2} = m^2B^{-3}[P, B^2]B^{-2} + m^2B^{-2}[P, B^2]B^{-3} \\ &= m^2 i\Big(2(B^2 - m^2)B^{-5} + B^{-3}QB^{-2} + B^{-2}QB^{-3}\Big).\end{aligned}$$

Hence, the boundedness of $[P_B, [P_B, 2J_1]]$ in L^2 follows from (1.3) and the boundedness of $[P_B, B^{-l}]$ for any $l \in \mathbb{N}$. $\square$

Proof of Proposition 3.2 For any $c \ge 0$ and any $\sigma > 0$ one has

$$\langle P_B\rangle^{-\sigma} = \langle P_B\rangle^{-\sigma}\mathbf{I}_{\pm P_B \le c|t|} + \mathcal{O}(|t|^{-\sigma}), \quad |t| > 1$$

in $\mathcal{L}(L^2, L^2)$. Hence,

$$\langle P_B\rangle^{-\sigma}e^{-itB}\chi(B)\langle P_B\rangle^{-\sigma} = \langle P_B\rangle^{-\sigma}\mathbf{I}_{P_B\le(\theta-\gamma)|t|/2}e^{-itB}\chi(B)\mathbf{I}_{P_B\ge-\theta|t|/2}\langle P_B\rangle^{-\sigma} + \mathcal{O}(|t|^{-\sigma}), \quad \gamma < \theta.$$

Choosing $a = -\frac{\theta|t|}{2}$ and $v = \theta - \frac{\gamma}{2}$ in Lemma 4.2, we obtain for $\sigma = 2$ and $\varepsilon > 0$

$$\|\langle P_B\rangle^{-\sigma}\langle x\rangle^{\sigma}\langle x\rangle^{-\sigma}e^{-itB}\chi(B)\langle x\rangle^{-\sigma}\langle x\rangle^{\sigma}\langle P_B\rangle^{-\sigma}\|_{\mathcal{L}(L^2,L^2)} \le C(\varepsilon)\langle t\rangle^{-2+\varepsilon}, \quad t \in \mathbb{R}.$$

Now (4.2) follows since $\langle P_B\rangle^{-\sigma}\langle x\rangle^{\sigma}$ and $\langle x\rangle^{\sigma}\langle P_B\rangle^{-\sigma}$ are bounded in $\mathcal{L}(L^2, L^2)$. This follows by the arguments from the proof of Proposition 2.2 in [3] (page 770), relying on the multi-commutator expansion [8, Identity (B.24)] and the identity [18, (1.2)].

5 The case $V \neq 0$

Here we prove Proposition 3.3. Denote

$$X(t) := U(t)\chi(\mathcal{K})\Psi(0) = \frac{1}{2\pi i}\int\limits_{\Gamma} \chi(\omega)e^{-i\omega t}\Big[\mathcal{R}(\omega+i0) - \mathcal{R}(\omega-i0)\Big]\Psi_0\, d\omega. \tag{5.1}$$

Our final goal is the bound

$$\|X(t)\|_{\mathcal{F}_{-\sigma}} \le C(\sigma,\varepsilon)\|\Psi_0\|_{\mathcal{F}_\sigma}\langle t\rangle^{-2+\varepsilon}, \quad t\in\mathbb{R}, \quad \sigma > 5/2. \tag{5.2}$$

Let us apply the Born perturbation series

$$\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega) + \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega), \tag{5.3}$$

which follows by iteration of $\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)$. Here $\mathcal{R}_0(\omega) = (\mathcal{K}_0 - \omega)^{-1}$ is the resolvent of the operator $\mathcal{K}_0$ and

$$\mathcal{V} = \begin{pmatrix} 0 & 0 \\ -iV & 0 \end{pmatrix}. \tag{5.4}$$

Substituting (5.3) into (5.1) we obtain

$$\begin{aligned} X(t) &= \frac{1}{2\pi i}\int\limits_{\Gamma} \chi(\omega)e^{-i\omega t}\Big[\mathcal{R}_0(\omega+i0) - \mathcal{R}_0(\omega-i0)\Big]\Psi_0\, d\omega \\ &+ \frac{1}{2\pi i}\int\limits_{\Gamma} \chi(\omega)e^{-i\omega t}\Big[\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0) - \mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\, d\omega \\ &+ \frac{1}{2\pi i}\int\limits_{\Gamma} \chi(\omega)e^{-i\omega t}\Big[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega+i0) - \mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega-i0)\Big]\Psi_0\, d\omega \\ &= X_1(t) + X_2(t) + X_3(t), \qquad t\in\mathbb{R} \end{aligned} \tag{5.5}$$

We analyze each term X_k separately.

Step i) For $X_1(t) = U_0(t)\chi(\mathcal{K}_0)\Psi(0)$ Proposition 3.2 implies that for any $\sigma \ge 2$ and any $\varepsilon > 0$

$$\|X_1(t)\|_{\mathcal{F}_{-\sigma}} \le C(\varepsilon)\|\Psi_0\|_{\mathcal{F}_\sigma}\langle t\rangle^{-2+\varepsilon}, \quad t\in\mathbb{R}. \tag{5.6}$$

Step ii) Consider the second term $X_2(t)$. We can choose the function $\chi(\omega)$ such that $\chi(\omega) = \chi_1^2(\omega)$. Denote

$$Y_1(t) = \frac{1}{2\pi i}\int\limits_{\Gamma} \chi_1(\omega)e^{-i\omega t}\Big[\mathcal{R}_0(\omega+i0) - \mathcal{R}_0(\omega-i0)\Big]\Psi_0\, d\omega$$

It is obvious that for $Y_1(t)$ the inequality (5.6) also holds. Namely,

$$\|Y_1(t)\|_{\mathcal{F}_{-\sigma}} \le C(\varepsilon)\|\Psi_0\|_{\mathcal{F}_\sigma}\langle t\rangle^{-2+\varepsilon}, \quad t\in\mathbb{R}, \quad \sigma\ge 2, \quad \varepsilon>0. \tag{5.7}$$

Now the second term $X_2(t)$ can be rewritten as a convolution.

Lemma 5.1. *The convolution representation holds*

$$X_2(t) = i\int_0^t U(t-\tau)\mathcal{V}Y_1(\tau)\, d\tau, \qquad t\in\mathbb{R} \tag{5.8}$$

where the integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma \geq 2$.

Proof. We have

$$\begin{aligned} X_2(t) &= \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}\chi_1(\omega)^2\mathcal{R}_0(\omega+i0)\mathcal{V}\mathcal{R}_0(\omega+i0)\Psi_0\, d\omega \\ &- \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega t}\chi_1(\omega)^2\mathcal{R}_0(\omega-i0)\mathcal{V}\mathcal{R}_0(\omega-i0)\Big]\Psi_0\, d\omega = X_2^+(t)+X_2^-(t) \end{aligned} \tag{5.9}$$

Denote $Y_1^+(t) := \theta(t)Y_1(t)$. Then $\chi_1(\omega)\mathcal{R}_0(\omega+i0)\Psi_0 = i\tilde{Y}_1^+(\omega)$ and we obtain that

$$\begin{aligned} X_2^+(t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\chi_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\tilde{Y}_1^+(\omega)\, d\omega \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\chi_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int_{\mathbb{R}} e^{i\omega\tau}Y_1^+(\tau)d\tau\Big]d\omega \\ &= \frac{1}{2\pi}(i\partial_t+i)^2\int_{\mathbb{R}}\frac{e^{-i\omega t}}{(\omega+i)^2}\chi_1(\omega)\mathcal{R}_0(\omega+i0)\mathcal{V}\Big[\int_{\mathbb{R}} e^{i\omega\tau}Y_1^+(\tau)d\tau\Big]d\omega. \end{aligned}$$

The last double integral converges in $\mathcal{F}_{-\sigma}$ with $\sigma \geq 2$ by (5.7) with $0<\varepsilon<1$, Lemma 2.3 i), and (2.10) with $k=0$. Hence, we can change the order of integration by the Fubini theorem and we obtain that

$$X_2^+(t) = \begin{cases} i\displaystyle\int_0^t U_0(t-\tau)\chi_1(\mathcal{K}_0)\mathcal{V}Y_1(\tau)d\tau &, \quad t>0 \\ 0 &, \quad t<0 \end{cases} \tag{5.10}$$

since

$$\begin{aligned} \frac{1}{2\pi i}(i\partial_t+i)^2\int_{\mathbb{R}}\frac{e^{-i\omega(t-\tau)}}{(\omega+i)^2}\chi_1(\omega)\mathcal{R}_0(\omega+i0)\, d\omega &= \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i\omega(t-\tau)}\chi_1(\omega)\mathcal{R}_0(\omega+i0)\, d\omega \\ &= \theta(t-\tau)U_0(t-\tau)\chi_1(\mathcal{K}_0) \end{aligned}$$

Similarly, we obtain

$$X_2^-(t) = \begin{cases} 0 &, \quad t>0 \\ i\displaystyle\int_0^t U_0(t-\tau)\chi_1(\mathcal{K}_0)\mathcal{V}Y_1(\tau)d\tau &, \quad t<0 \end{cases} \tag{5.11}$$

Now (5.8) follows since $X_2(t)$ is the sum of two expressions (5.10) and (5.11). $\square$

Now we choose an arbitrary $\sigma \geq 2$, $0 < \varepsilon < 1$ and $\sigma_1 \in [2, \min\{\sigma, \beta/2\})$. Applying Proposition 3.2 with χ_1 instead χ to the integrand in (5.8), we obtain that

$$\|U_0(t-\tau)\chi_1(\mathcal{K}_0)\mathcal{V}Y_1(\tau)\|_{\mathcal{F}_{-\sigma}} \leq \|U_0(t-\tau)\chi_1(\mathcal{K}_0)\mathcal{V}Y_1(\tau)\|_{\mathcal{F}_{-\sigma_1}} \leq \frac{C\|\mathcal{V}Y_1(\tau)\|_{\mathcal{F}_{\sigma_1}}}{(1+|t-\tau|)^{2-\varepsilon}}$$

$$\leq \frac{C\|Y_1(\tau)\|_{\mathcal{F}_{-\sigma_1}}}{(1+|t-\tau|)^{2-\varepsilon}} \leq \frac{C\|\Psi_0\|_{\mathcal{F}_{\sigma_1}}}{(1+|t-\tau|)^{2-\varepsilon}(1+|\tau|)^{2-\varepsilon}} \leq \frac{C\|\Psi_0\|_{\mathcal{F}_{\sigma}}}{(1+|t-\tau|)^{2-\varepsilon}(1+|\tau|)^{2-\varepsilon}}$$

Integrating, we obtain by (5.8) that

$$\|X_2(t)\|_{\mathcal{F}_{-\sigma}} \leq C(\varepsilon)\|\Psi_0\|_{\mathcal{F}_\sigma}\langle t\rangle^{-2+\varepsilon}, \quad t \in \mathbb{R}, \quad \sigma \geq 2. \tag{5.12}$$

Step iii) Finally, we rewrite the last term in (5.5) as

$$X_3(t) = \frac{1}{2\pi i}\int_\Gamma e^{-i\omega t}\chi(\omega)N(\omega)\Psi_0\, d\omega, \tag{5.13}$$

where $N(\omega) := M(\omega + i0) - M(\omega - i0)$ and

$$M(\omega) := \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega) = \mathcal{R}_0 L(\omega)\mathcal{R}(\omega).$$

First, we obtain the asymptotics of $L(\omega) := \mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}$ for large ω.

Lemma 5.2. *Let $\sigma > 0$, $k = 0, 1, 2$, and V satisfy (1.3) with $\beta > 1/2 + k + \sigma$. Then the asymptotics hold*

$$\|L^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_{-\sigma},\mathcal{F}_\sigma)} = \mathcal{O}(|\omega|^{-2}), \quad |\omega| \to \infty, \quad \omega \in \mathbb{C}\setminus\Gamma. \tag{5.14}$$

Proof. Denote $R_0(\omega) = (H_0 - \omega)^{-1}$, where H_0 corresponds to H with $V = 0$, i.e., $H_0 = (i\nabla + A)^2$. Bounds (5.14) follow from the algebraic structure of the matrix

$$L^{(k)}(\omega) = \mathcal{V}\mathcal{R}_0^{(k)}(\omega)\mathcal{V} = \begin{pmatrix} 0 & 0 \\ -iVR_0^{(k)}(\omega^2 - m^2)V & 0 \end{pmatrix} \tag{5.15}$$

For $\sigma > 1/2 + k$ asymptotics (2.3) with $s = 1$ and $l = -1$ implies that

$$\|R_0^{(k)}(\omega^2 - m^2)\|_{\mathcal{L}(\mathcal{H}^1_\sigma;\mathcal{H}^0_{-\sigma})} = \mathcal{O}(|\omega|^{-2}), \quad |\omega| \to \infty, \quad \omega \in \mathbb{C}\setminus\Gamma, \quad k = 0, 1, 2.$$

Therefore, for $1/2 + k < \beta - \sigma$ the asymptotics hold

$$\|VR_0^{(k)}(\omega^2 - m^2)Vf\|_{\mathcal{H}^0_\sigma} \leq C\|R_0^{(k)}(\omega^2 - m^2)Vf\|_{\mathcal{H}^0_{\sigma-\beta}} = \mathcal{O}(|\omega|^{-2})\|Vf\|_{\mathcal{H}^1_{\beta-\sigma}} = \mathcal{O}(|\omega|^{-2})\|f\|_{\mathcal{H}^1_{-\sigma}}.$$

□

Further, we obtain the asymptotics of $M(\omega)$ and its derivatives for large ω.

Lemma 5.3. *Let V satisfy (1.3) with $\beta > 3$. Then for $k = 0, 1, 2$, the asymptotics hold*

$$\|M^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{F}_\sigma,\mathcal{F}_{-\sigma})} = \mathcal{O}(|\omega|^{-2}), \quad |\omega| \to \infty, \ \ \omega \in \mathbb{C} \setminus \Gamma, \quad \sigma > 1/2 + k. \tag{5.16}$$

Proof. The asymptotics (5.16) follow from asymptotics (2.10) for $\mathcal{R}_0^{(k)}$ and $\mathcal{R}^{(k)}$, and asymptotics (5.14) for $L^{(k)}$. For example, consider the case $k = 2$. We have

$$M'' = \mathcal{R}_0'' L\mathcal{R} + \mathcal{R}_0 L''\mathcal{R} + \mathcal{R}_0 L\mathcal{R}'' + 2\mathcal{R}_0' L'\mathcal{R} + 2\mathcal{R}_0' L\mathcal{R}' + 2\mathcal{R}_0 L'\mathcal{R}'. \tag{5.17}$$

For a fixed $\sigma > 5/2$, let us choose $\sigma' \in (5/2, \min\{\sigma, \beta - 1/2\})$. Then for the first term in (5.17) we obtain by (2.10) and (5.14)

$$\begin{aligned}\|\mathcal{R}_0''(\omega)L(\omega)\mathcal{R}(\omega)f\|_{\mathcal{F}_{-\sigma}} &\le \|\mathcal{R}_0''(\omega)L(\omega)\mathcal{R}(\omega)f\|_{\mathcal{F}_{-\sigma'}} \le \|L(\omega)\mathcal{R}(\omega)f\|_{\mathcal{F}_{\sigma'}} \\ \le \frac{C}{|\omega|^2}\|\mathcal{R}(\omega)f\|_{\mathcal{F}_{-\sigma'}} &\le \frac{C_1}{|\omega|^2}\|f\|_{\mathcal{F}_{\sigma'}} \le \frac{C_2}{|\omega|^2}\|f\|_{\mathcal{F}_\sigma}, \quad \omega \to \infty, \quad \omega \in \mathbb{C} \setminus \Gamma.\end{aligned}$$

Other terms can be estimated similarly choosing an appropriate values of σ'. □

Now we prove the decay of $X_3(t)$. By Lemma 5.3

$$(\chi N)'' \in L^1(\Gamma; \mathcal{L}(\mathcal{F}_\sigma, \mathcal{F}_{-\sigma}))$$

with $\sigma > 5/2$. Hence, two times partial integration in (5.13) implies that

$$\|X_3(t)\|_{\mathcal{F}_{-\sigma}} \le C(\sigma)\|\Psi_0\|_{\mathcal{F}_\sigma}\langle t\rangle^{-2}, \quad t \in \mathbb{R}$$

Together with (5.6) and (5.12) this completes the proof of Proposition 3.3.

A Decay of magnetic Schrödinger resolvent

Here we prove Lemma 2.1 ii) for $s = 1$. First, we consider the case $V = 0$. Recall that

$$H_0 = (i\nabla + A)^2, \quad \text{and} \quad R_0(\omega) = (H_0 - \omega)^{-1}.$$

Lemma A.1. *Let $A(x) \in C^2(\mathbb{R}^3)$ be a real function, and for some $\beta > 2$ the bound holds*

$$|A(x)| + |\nabla A(x)| + |\nabla\nabla A(x)| \le C\langle x\rangle^{-\beta}. \tag{A.1}$$

Then for $l = -1, 0, 1$ and $\sigma > 1/2$, the asymptotics hold

$$\|R_0(\omega)\|_{\mathcal{L}(\mathcal{H}^1_\sigma;\mathcal{H}^{1+l}_{-\sigma})} = \mathcal{O}(|\omega|^{-\frac{1-l}{2}}), \quad |\omega| \to \infty, \quad \omega \in \mathbb{C} \setminus [0, \infty). \tag{A.2}$$

Proof. Step i) Consider $l = 0$. Applying the technique of PDO [19, 21] we obtain for large $\omega \in \mathbb{C} \setminus [0, \infty)$

$$\begin{aligned}\|R_0(\omega)\psi\|_{\mathcal{H}^1_{-\sigma}} &\le \|\nabla R_0(\omega)\psi\|_{\mathcal{H}^0_{-\sigma}} + \|R_0(\omega)\psi\|_{\mathcal{H}^0_{-\sigma}} \le C\|\sqrt{H_0+1}R_0(\omega)\psi\|_{\mathcal{H}^0_{-\sigma}} \\ &= C_1\|R_0(\omega)\sqrt{H_0+1}\,\psi\|_{\mathcal{H}^0_{-\sigma}} \le C_2|\omega|^{-1/2}\|\sqrt{H_0+1}\,\psi\|_{\mathcal{H}^0_\sigma} \\ &\le C_3|\omega|^{-1/2}\|\psi\|_{\mathcal{H}^1_\sigma}\end{aligned}$$

by (2.3) with $k = s = l = 0$ and $V = 0$.
Step ii) Similarly, (2.3) with $k = s = 0$ and $l = 1$, implies for large $\omega \in \mathbb{C} \setminus [0, \infty)$

$$\begin{aligned}
\|R_0(\omega)\psi\|_{\mathcal{H}^2_{-\sigma}} &= \|(H_0+1)R_0(\omega)\psi\|_{\mathcal{H}^0_{-\sigma}} \le C\|(\sqrt{-\Delta+1}\sqrt{H_0+1}\,R_0(\omega)\psi\|_{\mathcal{H}^0_{-\sigma}} \\
&= C\|\sqrt{-\Delta+1}\,R_0(\omega)\sqrt{H_0+1}\,\psi\|_{\mathcal{H}^0_{-\sigma}} \le C_1\|R_0(\omega)\sqrt{H_0+1}\,\psi\|_{\mathcal{H}^1_{-\sigma}} \\
&\le C_2\|\sqrt{H_0+1}\,\psi\|_{\mathcal{H}^0_{\sigma}} \le C\|\psi\|_{\mathcal{H}^1_{\sigma}}
\end{aligned}$$

Then (A.2) with $l = 1$ follows.
Step iii) It remains to consider the case $l = -1$. We have by (A.2) with $l = 1$

$$\begin{aligned}
\|R_0(\omega)\psi\|_{\mathcal{H}^0_{-\sigma}} &= \|\omega^{-1}(-1 + H_0R_0(\omega))\psi\|_{\mathcal{H}^0_{-\sigma}} \le C|\omega|^{-1}\Big[\|\psi\|_{\mathcal{H}^0_{-\sigma}} + \|R_0(\omega)\psi\|_{\mathcal{H}^2_{-\sigma}}\Big] \\
&\le C_1|\omega|^{-1}\|\psi\|_{\mathcal{H}^1_{-\sigma}}
\end{aligned}$$

□

Now we consider $V \ne 0$.

Lemma A.2. *Let for some $\beta > 3$*

$$|V(x)| + |A(x)| + |\nabla A(x)| + |\nabla\nabla A(x)| \le C\langle x\rangle^{-\beta}.$$

Then for $k = 0, 1, 2$, $\sigma > 1/2 + k$, and $l = -1, 0$, the asymptotics hold

$$\|R^{(k)}(\omega)\|_{\mathcal{L}(\mathcal{H}^1_\sigma;\mathcal{H}^{1+l}_{-\sigma})} = \mathcal{O}(|\omega|^{-\frac{1-l+k}{2}}), \quad |\omega| \to \infty, \quad \omega \in \mathbb{C} \setminus [0, \infty). \tag{A.3}$$

Proof. For $k = 0$ asymptotics (2.3) follow from the Born splitting

$$R(\omega) = R_0(\omega)[1 + VR_0(\omega)]^{-1}$$

and (A.2), since the norm of the operator $[1 + VR_0(\omega)]^{-1} : \mathcal{H}^1_\sigma \to \mathcal{H}^1_\sigma$ is bounded for large $\omega \in \mathbb{C} \setminus [0, \infty)$ and $\sigma \in (1/2, \beta/2]$.
For $k = 1$ and $k = 2$ we use the identities

$$R' = (1 - RW)R'_\Delta(1 - WR) = R'_\Delta - RWR'_\Delta - R'_\Delta WR + RWR'_\Delta WR. \tag{A.4}$$

$$\begin{aligned}
R'' &= (1 - RW)R''_\Delta(1 - WR) - 2R'WR'_\Delta(1 - WR) \qquad\qquad (A.5)\\
&= R''_\Delta - RWR''_\Delta - R''_\Delta WR + RWR''_\Delta WR - 2R'WR'_\Delta + 2R'WR'_\Delta WR.
\end{aligned}$$

and well-known asymptotics for $R_\Delta(\omega) = (-\Delta - \omega)^{-1}$ (see [10, 11]):

$$\|R^{(k)}_\Delta(\omega)\|_{\mathcal{L}(\mathcal{H}^s_\sigma,\mathcal{H}^{s+l}_{-\sigma})} = \mathcal{O}(|\omega|^{-\frac{1-l+k}{2}}), \quad \omega \to \infty, \quad \omega \in \mathbb{C} \setminus [0, \infty) \tag{A.6}$$

for $s \in \mathbb{R}$, $l = -1, 0, 1$, $k = 0, 1, 2, \ldots$ and $\sigma > k + 1/2$. Identities (A.4)-(A.5) and asymptotics (A.6) imply (A.3) (cf. [12, Theorem 3.8]). □

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