ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 15, No. 4, 2008, pp. 487–492. © Pleiades Publishing, Ltd., 2008.

Existence of Solitary Waves for the Discrete Schrödinger Equation Coupled to a Nonlinear Oscillator

E. A. Kopylova

Institute for Information Transmission Problems RAS
B. Karetnyi per. 19, Moscow, 101447 Russia
E-mail: ek@vpti.vladimir.ru

Received April 28, 2008; in final form: May 2, 2008

Abstract. The discrete Schrödinger equation with a nonlinearity concentrated at a single point is an interesting and important model to study the long-time behavior of solutions, including the asymptotic stability of solitary waves and properties of global attractors. In this note, the global well-posedness of this equation and the existence of solitary waves is proved and the properties of these waves are studied.

DOI: 10.1134/S1061920808040067

1. INTRODUCTION

Consider a model $U(1)$-invariant discrete Schrödinger equation with a nonlinearity concentrated at a single point,

$$i\dot{\psi}(x,t) = -\Delta_L\,\psi(x,t) - \delta(x)F(\psi(0,t)), \qquad x \in \mathbb{Z}. \tag{1.1}$$

Here $\delta(x) = \delta_{0x}$, the symbol Δ_L stands for the difference Laplacian on $\mathbb{Z}$ defined by $\Delta_L\psi(x) = \psi(x+1) - 2\psi(x) + \psi(x-1)$, $x \in \mathbb{Z}$, for functions $\psi\colon \mathbb{Z} \to \mathbb{C}$, and F for a continuous function. Physically, equation (1.1) describes the system of the free Schrödinger equation coupled to an oscillator attached at the point $x = 0$, and F is a nonlinear "oscillator force."

Assume that $F(\psi) = -\nabla U(\psi)$, where $U(\psi) = u(|\psi|^2)$. In this case, (1.1) defines a $U(1)$-invariant Hamiltonian system. The value of the Hamiltonian functional is preserved on solutions to (1.1) belonging to $l^2 = l^2(\mathbb{Z})$. We claim the well-posedness in l^2 for equation (1.1) and the existence of special solutions of type $\psi_\omega(x)e^{i\omega t}$, the so-called *solitary waves* or *nonlinear eigenfunctions*. The solitary waves form a two-dimensional *solitary manifold* in the Hilbert phase space of finite-energy states of the system.

Our ultimate goal here is to prove the asymptotic stability of solitary waves for equation (1.1) and to establish the global attraction to the solitary manifold. The asymptotic stability of solitary waves for a continuous Schrödinger equation coupled to a nonlinear oscillator was considered recently in [1]. The asymptotic stability of solitary waves in continuous nonlinear Schrödinger equations was treated in [2–4, 9–11]. The global attractor for the continuous Klein–Gordon equation coupled to a nonlinear oscillator was found in [7]. However, no work has been reported towards the proof of the asymptotic stability of solitary waves and global attraction to a solitary manifold for discrete nonlinear equations. This paper is a preparatory step in this direction.

The paper is organized as follows. In Section 2, the notation and definitions are given. The global well-posedness is proved in Section 3. In Section 4, we describe the nonzero solitary waves and analyze their properties.

2. NOTATION AND DEFINITIONS

We identify a complex number $\psi = \psi_1 + i\psi_2$ with the real two-dimensional vector $\Psi = (\psi_1, \psi_2) \in \mathbb{R}^2$ and assume that the vector version $\mathbf{F}$ of the oscillator force F admits a real-valued potential,

$$\mathbf{F}(\Psi) = -\nabla U(\Psi), \quad \Psi \in \mathbb{R}^2, \quad U \in C^2(\mathbb{R}^2). \tag{2.1}$$

Supported partly by FWF grant P19138-N13, DFG grant 436 RUS 113/929/0-1 and RFBR grant 07-01-00018a.

Then (1.1) becomes a Hamiltonian system with the Hamiltonian

$$\mathcal{H}(\Psi) = \frac{1}{2}\langle -\Delta_L \Psi, \Psi\rangle + U(\Psi(0)) = \frac{1}{2}\langle \nabla_L \Psi, \nabla_L \Psi\rangle + U(\Psi(0)), \tag{2.2}$$

where $\langle\cdot\rangle$ stands for the inner product in l^2 and $\nabla_L \psi(x) = \psi(x+1) - \psi(x)$. The Hamiltonian form of (1.1) is

$$\dot{\Psi} = JD\mathcal{H}(\Psi), \tag{2.3}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.4}$$

and $D\mathcal{H}$ is the Fréchet derivative in the Hilbert space l^2. Our key assumption concerns the $U(1)$-invariance of the oscillator, where $U(1)$ stands for the group $e^{i\theta}$, $\theta \in [0, 2\pi]$, acting by phase rotations $\psi \mapsto e^{i\theta}\psi$. Namely, assume that $U(\psi) = u(|\psi|^2)$ with $u \in C^2(\mathbb{R})$. Then, by (2.1),

$$F(\psi) = a(|\psi|^2)\psi, \quad \psi \in \mathbb{C}, \quad a \in C^1(\mathbb{R}), \tag{2.5}$$

where $a(|\psi|^2)$ is real. In this case, $F(e^{i\theta}\psi) = e^{i\theta}F(\psi)$, $\theta \in [0, 2\pi]$, and $F(0) = 0$ for continuous F. Hence, $e^{i\theta}\psi(x,t)$ is a solution to (1.1) if $\psi(x,t)$ is. Therefore, equation (1.1) is $U(1)$-invariant in the sense of [5], and the Noether theorem implies the conservation of l^2 norm, $\|\psi(t)\| = \|\psi(0)\|$. Here and further, $\|\cdot\| = \|\cdot\|_{l^2}$.

The main objective of this paper is to study the special role played by "quantum stationary states," or *solitary waves*, in the sense of [5], which are finite-energy solutions of the form

$$\psi(x,t) = \psi_\omega(x)e^{i\omega t}, \quad \omega \in \mathbb{R}. \tag{2.6}$$

The frequency ω and the amplitude $\psi_\omega(x)$ solve the following *nonlinear eigenvalue problem*:

$$-\omega\psi_\omega(x) = -\Delta_L\psi_\omega(x) - \delta(x)F(\psi_\omega(0)), \quad x \in \mathbb{Z}, \tag{2.7}$$

which follows directly from (1.1) and (2.5) since $\omega \in \mathbb{R}$.

Definition 2.1. The symbol $\mathcal{S}$ denotes the set of all solutions $\psi_\omega(x) \in l^2(\mathbb{Z})$ to (2.7) with all possible $\omega \in \mathbb{R}$.

Below, in Section 4, we present a complete analysis of the set $\mathcal{S}$ of all nonzero solitary waves $\psi_\omega(x)$ by an explicit calculation. For $\omega \in (-\infty, 0)$ and $\omega \in (4, \infty)$, the set $\mathcal{S}$ consists of the functions $C(\omega)e^{-k(\omega)|x|+i\theta}$ and $C(\omega)(-1)^{|x|}e^{-k(\omega)|x|+i\theta}$, respectively, for $C > 0$ and any $\theta \in [0, 2\pi]$, where C must belong to a set which is the finite union of one-dimensional intervals if F is a polynomial. Note that $C = 0$ corresponds to the zero function $\psi(x) = 0$, which is always a solitary wave because $F(0) = 0$, and, for $\omega \in [0, 4]$, the only solitary wave is zero.

3. GLOBAL WELL-POSEDNESS

The existence of a global solution to (1.1) is guaranteed by the following theorem whose proof is similar to that of Theorem 1.1 in [7].

Theorem 3.1. i) *Let conditions* (2.1) *and* (2.5) *hold. Then, for any* $\psi_0 \in l^2$*, there exists a unique solution* $\psi \in C_b(\mathbb{R}, l^2)$ *to equation* (1.1) *with initial condition* $\psi(x,0) = \psi_0(x)$.

ii) *The value of energy functional is preserved,*

$$\mathcal{H}(\psi(t)) = \mathcal{H}(\psi_0), \qquad t \in \mathbb{R}. \tag{3.1}$$

iii) *The norm of the solution is preserved,*

$$\|\psi(t)\| = \|\psi_0\|. \tag{3.2}$$

iv) *The mapping* $\mathcal{U}\colon \psi_0 \mapsto \psi$ *is continuous from* l^2 *to* $C_b([0,T], l^2)$ *for any* $T > 0$.

Let us outline the proof. We first show that, without loss of generality, it suffices to prove the theorem assuming that U is uniformly bounded together with its derivatives. Indeed, the a priori bounds for the l^2-norm of ψ imply that the nonlinearity $F(z)$ can be modified for large values of $|z|$. After this, we establish the existence and uniqueness of a solution $\psi \in C_b([0,\tau], l^2)$ for some $\tau > 0$ by using the contraction mapping theorem. Further, we show that ψ can be extended to all $t \geqslant 0$, and the energy and the norm are preserved. We then use this preservation to extend the solution $\psi(x,t)$ for $t \in \mathbb{R}$ and prove that $\psi \in C_b(\mathbb{R}, l^2)$.

Lemma 3.2 (see [7, Lemma 2.2]). *Suppose that Theorem* 3.1 *holds for the nonlinearities U that satisfy the following additional condition*:

$$\text{For}\quad k = 0,1,2, \quad \text{there exist some}\quad U_k < \infty \quad \text{such that}\quad \sup_{z\in\mathbb{C}} |\nabla^k U(z)| \leqslant U_k. \tag{3.3}$$

Then Theorem 3.1 *remains valid without this additional condition.*

Proof. Suppose U does not satisfy (3.3). For any initial data $\psi_0 \in l^2$, choose $\tilde U(z) \in C^2(\mathbb{C})$ in such a way that $\tilde U(z) = \tilde U(|z|)$ for $z \in \mathbb{C}$ and $\tilde U(z) = U(z)$ for $|z| \leqslant \|\psi_0\|$. Here $\tilde U$ can be chosen to satisfy the uniform bounds $\sup_{z\in\mathbb{C}} |\nabla^k \tilde U(z)| \leqslant \infty$, $k = 0,1,2$. By the assumption of the lemma, Theorem 3.1 holds for the nonlinearity $\tilde F = -\nabla \tilde U$ instead of $F = -\nabla U$. Hence, there is a unique solution $\psi(x,t) \in C_b(\mathbb{R}, l^2)$ to the equation $i\dot\psi(x,t) = -\Delta_L \psi(x,t) - \delta(x)\tilde F(\psi(0,t))$ with $\psi(x,0) = \psi_0$. Identity (3.2) implies that $|\psi(0,t)| \leqslant \|\psi_0\|$ for $t \in \mathbb{R}$. Therefore, $\tilde F(\psi(0,t)) = F(\psi(0,t))$ for $t \in \mathbb{R}$, and $\psi(x,t)$ is also a solution to (1.1) with the nonlinearity $F = -\nabla U$.

From now on, in the proof of Theorem 3.1, we assume that the bounds (3.3) hold.

Lemma 3.3. i) *Let $\psi_0 \in l^2$. There is a $\tau > 0$ depending only on U_2 in* (3.3) *for which there is a unique solution $\psi \in C_b([0,\tau], l^2)$ to equation* (1.1) *with initial data ψ_0.*

ii) *The mapping $\mathcal{U}\colon \psi_0 \mapsto \psi$ is continuous from l^2 to $C_b([0,\tau], l^2)$.*

iii) *The values of the functionals $\mathcal{H}$ and Q are preserved in time.*

Proof. Denote the dynamical group for the free Scfrödinger equation by $W(t)$. The discrete Fourier transform of $u\colon \mathbb{Z} \to \mathbb{C}$ is defined by the formula $\hat u(\theta) = \sum_{x\in\mathbb{Z}} u(x) e^{i\theta x}$, $\quad \theta \in T := \mathbb{R}/2\pi\mathbb{Z}$. After taking the Fourier transform, the operator $-\Delta_L$ becomes the operator of multiplication by $2 - 2\cos\theta$, $-\widehat{\Delta_L u}(\theta) = \phi(\theta)\widehat u(\theta)$, and, in the Fourier transform,

$$F_{x\to\theta}[W(t)u(x)](\theta) = e^{-i(2-2\cos\theta)t}\widehat u(\theta), \quad \theta \in T. \tag{3.4}$$

The solution ψ to (1.1) with the initial data $\psi(x,0) = \psi_0(x)$ admits the Duhamel representation

$$\psi(x,t) = W(t)\psi_0(x) + Z\psi(x,t), \tag{3.5}$$

where

$$Z\psi(x,t) = -\int_0^t W(s)\delta F(\psi(0,t-s))ds, \quad \delta := \delta(x). \tag{3.6}$$

By the Parseval identity, $\|W(s)\delta\| = \|F_{x\to\theta}[W(t)\delta](\theta)\|_{L^2(T)} = \|e^{-i(2-2\cos\theta)t}\|_{L^2(T)} = C < \infty$. Hence, if ψ_1, $\psi_2 \in C_b([0,\tau], l^2)$, then

$$\begin{aligned}
\|Z\psi_2(\cdot,t) - Z\psi_1(\cdot,t)\| &= \Big\| \int_0^t W(s)\delta\Big(F(\psi_2(0,t-s)) - F(\psi_2(0,t-s))\Big)ds\Big\| \\
&\leqslant \int_0^t \|W(s)\delta\| |F(\psi_2(0,t-s)) - F(\psi_2(0,t-s))| ds \\
&= C\int_0^t |F(\psi_2(0,t-s)) - F(\psi_2(0,t-s))| ds \leqslant U_2 t \sup_{0\leqslant s\leqslant t} |\psi_2(s) - \psi_1(s)|,
\end{aligned} \tag{3.7}$$

where we have used (3.3) with $k = 2$. In this case, the mapping $\psi \mapsto W(t)\psi_0 + Z\psi$ is contracting in the space $C_b([0,\tau], l^2)$ for $\tau = 1/(2U_2)$. This proves part i) of the lemma. Part ii) also follows by contraction. The energy and the charge preservation follows from the Hamiltonian structure (2.3). Namely, the differentiation of the Hamiltonian functional gives (by the chain rule) $(d/dt)\mathcal{H}(\Psi(t)) = \langle D\mathcal{H}(\Psi(t)), \dot\Psi(t)\rangle = \langle D\mathcal{H}(\Psi(t)), JD\mathcal{H}(\Psi(t))\rangle = 0$, $t \in [0,\tau]$. Similarly, the norm preservation follows by differentiation,

$$\begin{aligned}\frac{d}{dt}\|\Psi(t)\|^2 &= 2\langle \Psi(t), \dot\Psi(t)\rangle = 2\langle \Psi(t), JD\mathcal{H}(\Psi(t))\rangle \\ &= 2\langle \Psi(x,t), J\Delta_L\Psi(t)\rangle - 2\langle \Psi(x,t), J\delta(x)F(\Psi(0,t))\rangle = 0, \quad t \in [0.\tau],\end{aligned}$$

since $\langle \Psi(x,t), J\Delta_L\Psi(t)\rangle = \langle \nabla_L\Psi(x,t), J\nabla_L\Psi(t)\rangle = 0$ and $\langle \Psi(x,t), J\delta(x)F(\Psi(0,t))\rangle = \Psi(0,t)\cdot JF(\Psi(0,t)) = a(|\Psi(0,t)|)\Psi(0,t)\cdot J\Psi(0,t) = 0$, where "$\cdot$" stands for the real inner product on $\mathbb{R}^2$.

Proof of Theorem 3.1. The solution $\psi \in C_b([0,\tau], l^2)$ constructed in the last lemma exists for $0 \leqslant t \leqslant \tau$, where the τ depends only on U_2. Hence, the bound (3.2) at $t = \tau$ allows us to extend the solution ψ to the time interval $[\tau, 2\tau]$. We proceed by induction. This completes the proof of Theorem 3.1.

4. SOLITARY WAVES

There are two different sets of solitary waves. The first set $\mathcal{S}_1$ corresponds to $\omega \in (0,\infty)$, and the other set $\mathcal{S}_2$ corresponds to $\omega \in (-\infty,-4)$. Denote by $k(\omega)$ the positive solution of the equation $\cosh k(\omega) = |\omega - 2|/2$.

Lemma 4.1. *The sets of all nonzero solitary waves are given by*

$$\mathcal{S}_+ = \big\{\psi_\omega e^{i\theta} = Ce^{i\theta - k(\omega)|x|} : \ \omega \in (0,\infty),\ C > 0,\ \sinh k(\omega) = a(C^2)/2 > 0,\ \theta \in [0,2\pi]\big\},$$

$$\mathcal{S}_- = \big\{\psi_\omega e^{i\theta} = C(-1)^{|x|}e^{i\theta - k(\omega)|x|} : \ \omega \in (-\infty,-4),\ C > 0,\ \sinh k(\omega) = -a(C^2)/2 > 0,\ \theta \in [0,2\pi]\big\}.$$

Proof. Let us calculate all solitary waves (2.6). After taking the Fourier transform, equation (2.7) becomes

$$(2 - 2\cos\theta + \omega)\widehat{\psi}_\omega = F(C), \tag{4.1}$$

where $C = \psi_\omega(0)$. Therefore, $\psi_\omega(x) = (2\pi)^{-1}\int_T \big(e^{-i\theta x}F(C)/(\phi(\theta) + \omega)\big)\, d\theta$, $\omega \in \mathbb{C} \setminus [-4,0]$. Using the results of [8] (see Lemma 2.1), we see that, for $\omega \in \mathbb{C} \setminus [-4,0]$,

$$\psi_\omega(x) = -i\,\frac{F(C)e^{-i\theta(\omega)|x|}}{2\sin\theta(\omega)}, \quad x \in \mathbb{Z}, \tag{4.2}$$

where $\theta(\omega)$ is the unique root of the equation

$$2\cos\theta - 2 = \omega \tag{4.3}$$

in the domain $D := \{-\pi \leqslant \operatorname{Re}\theta \leqslant \pi,\ \operatorname{Im}\theta < 0\}$. Below we consider two cases separately.

First, let us consider the case $\omega \in (0,\infty)$. Then $\theta(\omega) = -ik(\omega)$, $k(\omega) > 0$, and $\sin\theta(\omega) = -\sin ik(\omega) = -i\sinh k(\omega)$. Therefore, (4.2) yields $\psi_\omega(x) = F(C)e^{-k(\omega)|x|}/(2\sinh k(\omega))$, $x \in \mathbb{Z}$. Hence, $C = \psi_\omega(0) = F(C)/(2\sinh k(\omega))$. This implies that

$$\sinh k(\omega) = \frac{F(C)}{2C} = \frac{a(C^2)}{2}. \tag{4.4}$$

Now we consider the case of $\omega \in (-\infty,-4)$. Then $\theta(\omega) = \pi - ik(\omega)$, $k(\omega) > 0$, and $\sin\theta(\omega) = \sin ik(\omega) = i\sinh k(\omega)$. Therefore, (4.2) yields $\psi_\omega(x) = -F(C)(-1)^{|x|}e^{-k(\omega)|x|}/(2\sinh k(\omega))$, $x \in \mathbb{Z}$. Hence, $C = \psi_\omega(0) = -F(C)/(2\sinh k(\omega))$, and

$$\sinh k(\omega) = -\frac{F(C)}{2C} = -\frac{a(C^2)}{2} > 0. \tag{4.5}$$

Corollary 4.2. *The set $\mathcal{S}_+$ ($\mathcal{S}_-$) is a smooth manifold with the coordinates $\theta \in \mathbb{R} \bmod 2\pi$ and $C > 0$ such that $a(C^2) > 0$ ($a(C^2) < 0$, respectively).*

Remark 4.3. We shall analyze only the solitary waves with $a'(C^2) \neq 0$. On the manifolds $\mathcal{S}_\pm$, we have $\omega = \pm 2\cosh k - 2$ with $\sinh k = \pm a(C^2)/2$, according to (4.4) and (4.5). Hence, the parameters θ, ω are locally smooth coordinates on $\mathcal{S}_\pm$ at the points with $a'(C^2) \neq 0$, because $\omega' = \pm 2k' \sinh k = \dfrac{2a'(C^2)C\sinh k}{\cosh k} \neq 0$ in this case.

Example 4.4. Consider the function $F(C) = (-C^2+1)C = -C^3+C$, $C > 0$. The interval $(0,1)$ corresponds to $a(C^2) = -C^2+1 > 0$, and the interval $(1,\infty)$ corresponds to $a(C^2) < 0$ (see Figure).

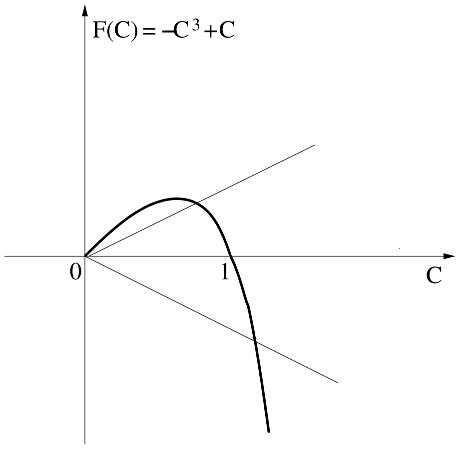

Figure.

For $C \in (0,1)$, the equation $2\sinh k(\omega)C = F(C)$ has a unique solution if $\sinh k \in (0, 1/2)$, $\cosh k \in (1, \sqrt{5}/2)$ and $\omega = 2\cosh k - 2 \in (0, \sqrt{5} - 2)$.

For $C \in (1,\infty)$, the equation $2\sinh k(\omega)C = -F(C)$ has a unique solution for $\sinh k \in (0,\infty)$, $\cosh k \in (1,\infty)$ and $\omega = -2\cosh k - 2 \in (-\infty, -4)$. Therefore, nonzero solitary waves exist for $\omega \in (0, \sqrt{5}-2) \cup (-\infty, -4)$.

A soliton solution is a trajectory $\psi_{\omega(t)}(x)e^{i\theta(t)}$ in which the parameters satisfy the equation $\dot\theta = \omega$, $\dot\omega = 0$. In time, the solitary waves $e^{i\theta}\psi_\omega(x)$ form an orbit of the $U(1)$ symmetry group. This group acts on the phase space $l^2(\mathbb{Z})$, preserving the Hamiltonian $\mathcal{H}$. The orbital stability of solitary waves in Hamiltonian $U(1)$-invariant systems is a well-studied topic (see [5] for very general theorems in this area and [12] for an approach closer to that used here). The standard condition for orbital stability in the continuous case reads as $\partial_\omega \int |\psi_\omega(x)|^2 dx > 0$; this is expected to be a necessary and sufficient condition for orbital stability when the Hessian of the augmented Hamiltonian (see [12]) has a single negative eigenvalue. A similar condition for the discrete equation is $\partial_\omega \sum_{x\in\mathbb{Z}} |\psi_\omega(x)|^2 > 0$.

Write $N(C) = \sum_{x\in\mathbb{Z}} |\psi_\omega(x)|^2$ with $\omega = \pm 2\cosh k - 2$ and $\sinh k = \pm a(C^2)/2$, according to (4.4)–(4.5). Then

$$N(C) = C^2\Big(\sum_{x=0}^{\infty} e^{-2kx} + \sum_{x=1}^{\infty} e^{-2kx}\Big) = C^2\Big(\frac{1}{1-e^{-2k}} + \frac{e^{-2k}}{1-e^{-2k}}\Big) = C^2\frac{e^k+e^{-k}}{e^k-e^{-k}} = C^2\frac{\cosh k}{\sinh k}.$$

Differentiating gives $N'(C) = 2C\cosh k/\sinh k - C^2\, k'/\sinh^2 k$. Differentiating (4.4)–(4.5), we obtain $k'\cosh k = \pm a'C$, $a' = a'(C^2)$. Thus, again by (4.4)–(4.5), we have

$$\begin{aligned} N'(C) &= 2C\frac{\cosh k}{\sinh k} \mp C^2\frac{a'C}{\cosh k\sinh^2 k} = \pm\frac{4C}{a^2\cosh k}(a\cosh^2 k - a'C^2) \\ &= \pm\frac{4C}{a^2\cosh k}(a + a\sinh^2 k - a'C^2) = \pm\frac{4C}{a^2\cosh k}\Big(a + \frac{a^3}{4} - a'C^2\Big) \neq 0 \end{aligned} \tag{4.6}$$

if $C > 0$ and $a' \neq (4a + a^3)/(4C^2)$. Therefore, noticing that

$$N'(C) = \omega'(C)\partial_\omega \sum_{x\in\mathbb{Z}} |\psi_\omega(x)|^2 \tag{4.7}$$

with $\omega'(C) = \pm 2k' \sinh k = 2a'C \sinh k/\cosh k = \pm\, a'aC/\cosh k$, we obtain the following result.

Lemma 4.5. *For $C > 0$, the inequality $\partial_\omega \sum_{x\in\mathbb{Z}} |\psi_\omega(x)|^2 > 0$ holds if*

i) $a' \in (0, (a^3 + 4a)/(4C^2))$ *for* $\psi_\omega \in \mathcal{S}_+$,

ii) $a' \in (-\infty, (a^3 + 4a)/(4C^2)) \cup (0, +\infty)$ *for* $\psi_\omega \in \mathcal{S}_-$.

Proof. Equations (4.6) and (4.7) yield

$$\partial_\omega \sum_{x\in\mathbb{Z}} |\psi_\omega(x)|^2 = \frac{N'(C)}{\omega'(C)} = \frac{4}{a^3 a'}\Big(a + \frac{a^3}{4} - a'C^2\Big).$$

Taking into account that $a > 0$ for $\psi_\omega \in \mathcal{S}_+$ and $a < 0$ for $\psi_\omega \in \mathcal{S}_-$, we obtain the conclusion of the lemma.

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