Number theory, dynamical systems and statistical mechanics

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Abstract

We shortly review recent work interpreting the quotient $\zeta(s-1)/\zeta(s)$ of Riemann zeta functions as a dynamical zeta function. The corresponding interaction function (Fourier transform of the energy) has been shown to be ferromagnetic, i.e. positive.

On the additive group

$$\mathbf{G}_k := (\mathbb{Z}/2\mathbb{Z})^k, \quad \text{with} \quad \mathbb{Z}/2\mathbb{Z} = (\{0,1\}, +).$$

we set inductively

$$\mathbf{h}_0 := 1, \quad \mathbf{h}_{k+1}(\sigma, 0) := \mathbf{h}_k(\sigma) \quad \text{and} \quad \mathbf{h}_{k+1}(\sigma, 1) := \mathbf{h}_k(\sigma) + \mathbf{h}_k(1-\sigma), \quad (1)$$

where $\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbf{G}_k$ and $1 - \sigma := (1-\sigma_1, \ldots, 1-\sigma_k)$ is the inverted configuration. The sequences $\mathbf{h}_k(\sigma)$ of integers, written in lexicographic order, coincide with the denominators of the modified Farey sequence.

We now formally interpret $\sigma \in \mathbf{G}_k$ as a *configuration* of a spin chain with k spins and *energy function*

$$\mathbf{H}_k := \ln(\mathbf{h}_k).$$

Thus we may interpret

$$Z_k(s) := \sum_{\sigma \in \mathbf{G}_k} \exp(-s \cdot \mathbf{H}_k(\sigma))$$

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as the *partition function* of that finite spin chain for *inverse temperature* s. The quotient

$$Z(s) := \frac{\zeta(s-1)}{\zeta(s)} \equiv \sum_{n=1}^{\infty} \varphi(n) n^{-s} \tag{2}$$

is the *thermodynamic limit*

$$\lim_{k \to \infty} Z_k(s) = Z(s) \qquad (\Re(s) > 2) \tag{3}$$

of partition functions $Z_k(s) = \sum_{n=1}^{\infty} \varphi_k(n) n^{-s}$. That *number-theoretical spin chain* was introduced in [5], see also Cvitanović [3]. The *Gibbs measure* for inverse temperature $\beta \in \mathbb{R}$ assigns probabilities

$$\sigma \mapsto \frac{\exp(-\beta \mathbf{H}_k(\sigma))}{Z_k(\beta)} \qquad (\sigma \in \mathbf{G}_k) \tag{4}$$

to the configurations of the spin chain. We denote the expectation of a random variable by

$$\langle f \rangle_k (\beta) := \sum_{\sigma \in \mathbf{G}_k} f(\sigma) \frac{\exp(-\beta \mathbf{H}_k(\sigma))}{Z_k(\beta)} \qquad (f : \mathbf{G}_k \to \mathbb{R}).$$

To show that the analogy with statistical mechanics is not only formal, the *Fourier coefficients* of $\mathbf{H}_k$ were estimated in [5]. One notes that the dual group $\mathbf{G}_k^*$ of $\mathbf{G}_k$ is naturally isomorphic to $\mathbf{G}_k$, since the characters on $\mathbf{G}_k$ can be written in the form

$$\chi_t : \mathbf{G}_k \to \{-1, 1\} \quad , \quad \chi_t(\sigma) := (-1)^{\sum_{l=1}^{k} \sigma_i t_i} \qquad (t \in \mathbf{G}_k^*).$$

The Fourier coefficients

$$j_k(t) := -2^{-k} \sum_{\sigma \in \mathbf{G}_k} \mathbf{H}_k(\sigma) \cdot \chi_t(\sigma) \qquad (t \in \mathbf{G}_k^*)$$

of $-\mathbf{H}_k$ are called *interaction coefficients* in the statistical mechanics terminology, and

$$\mathbf{H}_k(\sigma) = -\sum_{t \in \mathbf{G}_k^*} j_k(t) \cdot \chi_\sigma(t).$$

The negative mean $j_k(0)$ of $\mathbf{H}_k$ has special properties. In the thermodynamic limit it is asymptotic to $j_k(0) \sim -c \cdot k$ for some $c > 0$ [6], but it is the only coefficient whose value does not affect the Gibbs probability measure (4).

When we write $t \equiv (t_1, \ldots, t_k) \in \mathbf{G}_k^* \setminus \{0\}$ in the form

$$t = (0, \ldots, 0, 1, t_{l+1}, \ldots, t_{r-1}, 1, 0, \ldots, 0),$$

$s := r - l$ will be called the *size of* t, and $d := \min(l, k+1-r)$ its *distance* from the ends of the chain at 1 and k. Finally we say that t is *even* (*odd*) if $\sum_{i=1}^{k} t_i$ is *even* (*odd*). With these notations the following estimates were shown.

Theorem [5].

- The even interactions decay exponentially in the size:

$$j_k(t) < 2^{-s} \qquad (t \in \mathbf{G}_k^* \setminus \{0\} \text{ even}), \tag{5}$$

  whereas in the odd case one even has

$$j_k(t) < 2^{-(k-l)} \qquad (t \in \mathbf{G}_k^* \text{ odd}). \tag{6}$$

  So odd interactions are small in comparison to the even ones except near the right end of the chain.

- The interaction is *asymptotically translation invariant* in the sense that, up to a relative error which is exponentially small in the distance from the ends, the interactions only depend on the relative positions of the spins involved:

$$0 \le (j_{k+1}(0,t) - j_{k+1}(t,0)) \cdot 2^s < C \cdot 2^{-d} \qquad (t \in \mathbf{G}_k^*).$$

- The interaction has a *thermodynamic limit* in the sense

$$0 \le (j_{k+1}(0,t) - j_k(t)) \cdot 2^s < C \cdot 2^{-d} \qquad (t \in \mathbf{G}_k^* \setminus \{0\}).$$

  For $\beta > 0$ the thermodynamic limit

$$F(\beta) := \lim_{k\to\infty} F_k(\beta) \quad \text{with} \quad F_k(\beta) := -\frac{1}{\beta \cdot k} \ln\left(Z_k(\beta)\right) \tag{7}$$

  of the *free energy* per spin exists [7].

- The interaction is *ferromagnetic*, that is,

$$j_k(t) \ge 0 \qquad (t \in \mathbf{G}_k^* \setminus \{0\}). \tag{8}$$

  This is in accordance with earlier speculations (see Ruelle [14]). Note, however, that the system is not of Ising type, since multi-body interactions are present.

- The *effective interaction*

$$A_k(l,r) := \sum_{t' \in \mathbf{G}^*_{r-l-1}} j_k(0,\ldots,0,1,t_{l+1},\ldots,t_{r-1},1,0,\ldots,0)$$

  between spins at positions l and r decays quadratically with their distance $s = r - l$ in the bulk:

$$A_k(l,r) \le \frac{1}{s^2} + \frac{2^{-(k-r)}}{s}. \tag{9}$$

  This is just the borderline decay rate for a phase transition

A proof of the ferromagnetic property can be based on abstract polymer models, see [4], and this kind of combinatorics appears naturally in the context of number-theoretical zeta functions [2].

Switching to a multiplicative representation $s_i(\sigma) := (-1)^{\sigma_i}$ of the ith spin, an important variable is the *mean magnetization* per spin

$$M_k(\beta) := \frac{1}{k} \sum_{i=1}^{k} \langle s_i \rangle_k (\beta)$$

and its thermodynamic limit $M(\beta)$. By analyzing a Perron-Frobenius operator with *PF* eigenvalue $\exp(-\beta \cdot F(\beta))$, the following statements were proved.

Theorem [1]. The only phase transition of the number-theoretical spin chain occurs for inverse temperature $\beta_{\mathrm{cr}} := 2$. For lower temperatures

$$F(\beta) = U(\beta) = 0 \quad \text{and} \quad M(\beta) = 1 \qquad (\beta > \beta_{\mathrm{cr}}),$$

whereas for high temperatures

$$U(\beta) \ge \tfrac{1}{4}(\beta - \beta_{\mathrm{cr}}) \quad (1 < \beta < \beta_{\mathrm{cr}}) \quad \text{and} \quad M(\beta) = 0 \quad (0 \le \beta < \beta_{\mathrm{cr}}).$$

The energy function can be interpreted as the time delay of scattering geodesics in the modular domain [8]. There exist direct relations with the works [12], [13] by Mayer, and [11] by Lanford and Ruedin, and the Riemann Hypothesis can be related to a problem concerning the spectral radius of a related Markov chain [9]. More details can be found in the lecture notes [10].

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