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The Construction of Non-Commutative Manifolds Using Coherent States

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Abstract

We describe the construction of non-commutative manifolds, which are the non-commutative analogs of homogeneous spaces using coherent states. In the commutative limit we obtain standard manifolds. Applications to the Fuzzy sphere and to the Fuzzy hyperboloid are discussed in more detail.

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1 Introduction

In recent years the method of non-commutative geometry was developed [1] and applied to various physical problems [2]. The main idea consists in reformulating the geometry of a manifold in terms of an algebra of smooth functions defined on it, and then to generalize the differential calculus to a non-commutative algebra. If one takes for this algebra the set of $n \times n$ complex matrices M_n, one calls the resulting formalism a matrix geometry [3,4]. A sequence of matrix geometries on M_n has been given which allows to recover the geometry of the smooth 2-sphere in the limit $n \to \infty$ [5]. It was possible to formulate the Schwinger model on this Fuzzy sphere and to discuss the continuum limit [6].

Here we construct non-commutative manifolds which are the non-commutative analogs of homogeneous spaces. We extend some of the matrix models given in [5–7].

A non-commutative geometry is based on

i) a suitable non-commutative algebra $\mathcal{A}$ (replacing the abelian algebra of smooth functions on a manifold) together with a representation in a Hilbert space $\mathcal{H}$,

ii) an exterior differential algebra $\Omega(\mathcal{A})$ over $\mathcal{A}$ together with an operator d mapping from n-forms to $(n+1)$-forms,

iii) eventually some additional structures like p-summable Fredholm modules connected to Dirac operators.

Our construction of the algebra $\mathcal{A}$ and the exterior differential algebra $\Omega(\mathcal{A})$ is based on coherent states defined on homogeneous spaces [8–10]. Mathematical details, relevant for our purposes, can be found in [11].

We describe the construction of the algebras $\mathcal{A}$ and $\Omega(\mathcal{A})$ for the non-commutative analogs of homogeneous spaces in Section 2. More specific non-commutative manifolds (non-commutative co-adjoint orbits) leading to standard manifolds in the commutative limit are investigated in Section 3. The next Section 4 contains applications to the non-commutative sphere and to the non-commutative hyperboloid. The last Section 5 is devoted to concluding remarks.

2 Coherent States and the Algebras $\mathcal{A}$ and $\Omega(\mathcal{A})$

We briefly describe the construction of coherent states on homogeneous spaces following Refs. [10] and [12]. Let $T(g)$ be an unitary irreducible representation of an unimodular Lie group G in a Hilbert space $\mathcal{H}$. We assume that the normalized state $|x_0\rangle \in \mathcal{H}$ is from the Gårding space of the representation $T(g)$. The function

$$\omega(g, x_0) = \langle x_0 | T(g) | x_0 \rangle \tag{1}$$

is then a smooth (analytic) function of $g \in G$. Let H be the stability group of $|x_0\rangle$ for which

$$T(h)|x_0\rangle = d(h)|x_0\rangle, \qquad h \in H, \tag{2}$$

where $d(h)$, a phase factor, is a unitary character of the subgroup $H \subset G$.

To any point $x = g_x x_0 \in M = G/H$ we assign a coherent state

$$|x\rangle = d(g_x^{-1}) T(g_x) |x_0\rangle, \tag{3}$$

where $d(g) = \omega(g, x_0)/|\omega(g, x_0)|$ denotes a phase factor. To $x_0 \in M$ we assign $|x_0\rangle$ by definition. The kernel function

$$\omega(g, x) = \langle x|T(g)|x\rangle = \omega(g_x^{-1} g g_x, x_0) \tag{4}$$

is a smooth function in both variables.

To any finite distribution (with compact support) $\widetilde{f}(g)$ over smooth functions on G we assign the operator

$$\widehat{f} = \int dg \widetilde{f}(g) T(g), \tag{5}$$

where dg denotes the both-sided invariant Haar measure on G. The operator $\widehat{f}$ has well-defined smooth coherent states matrix elements. In this way we assign to any $\widetilde{f}(g)$ the smooth function on M:

$$f(x) = \langle x|\widehat{f}|x\rangle = \int dg \widetilde{f}(g)\omega(g, x). \tag{6}$$

Due to (4) $f(x)$ may be reinterpreted as a smooth function on G

$$f(g) = \int dg' \widetilde{f}(g')\omega(g^{-1}g'g, x_0) = \int dg' \widetilde{f}(gg'g^{-1})\omega(g', x_0), \tag{7}$$

which is right H-invariant: $f(gh) = f(g)$ for $h \in H$; we use the same symbol for $f(x)$ and $f(g)$.

The non-commutative algebra $\mathcal{A}$ is defined as the algebra of functions (6). The *-product of two functions $f_1(x) = \langle x|\widehat{f}_1|x\rangle$ and $f_2(x) = \langle x|\widehat{f}_2|x\rangle$ is given by

$$(f_1 * f_2)(x) = \langle x|\widehat{f}_1\widehat{f}_2|x\rangle = \int dg (\widetilde{f}_1 * \widetilde{f}_2)(g)\omega(g, x). \tag{8}$$

Here $\widetilde{f}_1 * \widetilde{f}_2$ denotes the convolution of two finite distributions defined by

$$\int dg (\widetilde{f}_1 * \widetilde{f}_2)(g)\omega(g, x) = \int dg_1 \int dg_2 \widetilde{f}_1(g_1)\widetilde{f}_2(g_2)\omega(g_1 g_2, x). \tag{9}$$

Let x_i be a basis of the Lie algebra $\mathcal{G}$ of the group G satisfying the relations

$$[x_i, x_j] = f^k{}_{ij} x_k \tag{10}$$

where $[\cdot,\cdot]$ denotes the Lie algebra structure in $\mathcal{G}$. To any x_i we assign differential operators $\widehat{x}_i$ acting on smooth functions on G as

$$(\widehat{x}_i \varphi)(g) = \lim_{t \searrow 0} \frac{1}{t}[\varphi(g\, e^{tAd_g x_i}) - \varphi(g)]. \tag{11}$$

The operators $\widehat{x}_i$ are the left-invariant vector fields on G and satisfy the same commutation relations as (10)

$$[\widehat{x}_i, \widehat{x}_j] = f^k{}_{ij} \widehat{x}_k, \tag{12}$$

where $[\cdot,\cdot]$ now denotes the commutator.

The canonical 1-forms Θ^i form a dual basis to x_i and satisfy the Maurer-Cartan equations

$$d\Theta^i = -\frac{1}{2} f^i{}_{jk} \Theta^j \wedge \Theta^k. \tag{13}$$

The elements of the exterior differential algebra $\Omega(G)$ over G have the form

$$A = A_{i_1 \ldots i_p} \Theta^{i_1} \wedge \ldots \wedge \Theta^{i_p}. \tag{14}$$

The differential of a function $f(g)$ is given by

$$df = (\widehat{x}_i f)\Theta^i. \tag{15}$$

Eqs. (13) and (15) together with the Leibniz rule

$$d(A \wedge B) = dA \wedge B + (-1)^{\deg A} A \wedge dB \tag{16}$$

define the exterior differentiation of p-forms on G (14) algebraically.

As a next step we choose the coefficient functions $A_{i_1 \ldots i_p}$ so that (14) represents only differential forms from $\Omega(M)$, $M = G/H$. This is guaranteed if all $A_{i_1 \ldots i_p}$ are represented by differentials (15) of right-invariant functions on G, i.e. functions on M such that

$$A = \sum_\alpha a^0_\alpha da^1_\alpha \wedge \ldots \wedge da^p_\alpha, \tag{17}$$

where a^k_α are smooth functions on M, and da^k_α are given by (15). Obviously,

$$dA = \sum_\alpha da^0_\alpha \wedge \ldots \wedge da^p_\alpha. \tag{18}$$

In the non-commutative case the elements of $\Omega(A)$ and their differentials will be given by the same expressions, but with the *-product replacing the usual product among different factors:

$$A = \sum_\alpha a^0_\alpha * da^1_\alpha \overset{*}{\wedge} \ldots \overset{*}{\wedge} da^p_\alpha, \tag{19}$$

and

$$dA = \sum_\alpha da^0_\alpha \overset{*}{\wedge} \ldots \overset{*}{\wedge} da^p_\alpha. \tag{20}$$

Here

$$da^\ell_\alpha \overset{*}{\wedge} \ldots \overset{*}{\wedge} da^p_\alpha = (\widehat{x}_{i_\ell} a^\ell_\alpha) \; * \; \ldots \; * \; (\widehat{x}_{i_p} a^p_\alpha) \Theta^{i_\ell} \wedge \ldots \wedge \Theta^{i_p}, \tag{21}$$

where Θ^i will be treated as Grassmann variables satisfying (13) and commuting with the elements of $\mathcal{A}$.

It remains to be shown that $\widehat{x}_i f \in \mathcal{A}$ if $f \in \mathcal{A}$. This can be proven easily: we rewrite (7) as

$$f(g) = \int dg' \widetilde{f}_{g'}(g) \omega(g', x_0) \in \mathcal{A}, \tag{22}$$

where we have put $\widetilde{f}_{g'}(g) = \widetilde{f}(gg'g^{-1})$ in order to indicate explicitly the variable g on which $\widehat{x}_i$ acts. Then $(\widehat{x}_i f)(g) = \int dg' (\widehat{x}_i \widetilde{f}_{g'})(g) \omega(g', x_0)$. Performing the shift of the integration variable $g' \to g^{-1} g' g$, we obtain

$$(\widehat{x}_i f)(g) = \int dg' (\widehat{x}_i \widetilde{f}_{g^{-1}g'g})(g) \omega(g', g x_0) \in \mathcal{A}. \tag{23}$$

This expression is simple but non-trivial, since it might contain only a rather limited set of functions (confirm Sec. 4).

3 Non-Commutative Algebras $\mathcal{A}_0$ and $\Omega(\mathcal{A}_0)$

To any element $x_i \in \mathcal{G}$ we can assign the distribution $\widetilde{x}_i(g)$ (with support at the one point $e \in G$) defined by

$$\int dg \widetilde{x}_i(g)\omega(g,x) = (\widehat{x}_i\omega)(e,x) =: (-i)\varphi_i(x). \tag{24}$$

The *-product of the functions φ_i is given by (8)

$$(\varphi_{i_1} \ * \ \dots \ * \ \varphi_{i_k})(x) = (-i)^k(\widehat{x}_{i_1}\dots\widehat{x}_{i_k}\omega)(e,x). \tag{25}$$

¿From (12) it follows that the *-commutator

$$[\varphi_i, \varphi_j]_* =: \varphi_i \ * \ \varphi_j - \varphi_j \ * \ \varphi_i \tag{26}$$

satisfies the relations

$$[\varphi_i, \varphi_j]_* = i f^k{}_{ij}\varphi_k. \tag{27}$$

We denote by $\mathcal{A}_0$ the algebra generated by *-polynomials in φ_i's.

Using (10), (11) and (24), (27) we can show that

$$i\widehat{x}_i\varphi_j = i f^k{}_{ij}\varphi_k = [\varphi_i, \varphi_j]_*. \tag{28}$$

By the Leibniz rule this extends to any polynomial $f \in \mathcal{A}_0$

$$i\widehat{x}_i f = [\varphi_i, f]_*, \qquad f \in \mathcal{A}_0. \tag{29}$$

The elements of $\Omega(\mathcal{A}_0)$ and their exterior differentials are given by eqs. (19) and (20), but the differential of any $f \in \mathcal{A}_0$ is given by the following algebraic rule

$$df = [D, f]_* = [\varphi_i, f]_*\Theta^i, \tag{30}$$

where $D = \varphi_i\Theta^i$ is the *-analog of a Dirac operator.

It is interesting to analyze the relation between the algebra $\mathcal{A}$ and its subalgebra $\mathcal{A}_0$. The latter is determined by the behaviour of $\omega(g,x)$ in the neighbourhood of the point $g = e \in G$. We put $g = e^X$, $X \in \mathcal{G}$, and obtain

$$\omega(e^X, x_0) = \exp[i(\langle F_0, X\rangle + r_0(X))], \tag{31}$$

where F_0 is a fixed element of the Lie co-algebra $\mathcal{G}^*$ and $r_0(X)$ contains quadratic and higher terms. If $x = g_x x_0$, then

$$\omega(e^X, x) = \omega(\exp[Ad_{g_x}X], x_0) = \exp[i(\langle F, X\rangle + r(X,x))], \tag{32}$$

where $F = Ad^*_{g_x}F_0$ and $r(X,x) = r_0(Ad_{g_x}X)$. Since (32) should be H-invariant, we obtain

$$Ad^*_h F_0 = F_0, \qquad r_0(Ad_h X) = r_0(X), \qquad h \in H. \tag{33}$$

The first condition tells us that H is a subgroup of the stability group K of F_0:

$$Ad^*_k F_0 = F_0, \qquad k \in K. \tag{34}$$

In general, $H \subset K$ and therefore the co-orbit $\Gamma = Ad^*_G F_0 \simeq G/K$ is a submanifold of $M = G/H$. Using (32) we obtain for φ_i, defined in (24), the expression

$$\varphi_i = i\langle F, x_i\rangle. \tag{35}$$

We see that φ_i is a linear function on Γ. Thus, the algebra $\mathcal{A}_0$ is a non-commutative algebra of *-polynomials on Γ with generators φ_i satisfying the *-commutator relations (26).

Finally, we shall describe the commutative limit of $\mathcal{A}_0$ within the tower of unitary irreducible representations $T_\lambda(g)$ leading, instead of (31), to

$$\omega_\lambda(e^X, x) = \langle x|T_\lambda(e^X)|x\rangle = \exp(i\lambda\langle F, X\rangle + r_\lambda(X, x)), \tag{36}$$

where $F = Ad^*_{g_x} F_0 \in \Gamma$, and we shall suppose that

$$r_\lambda\left(\frac{1}{\lambda}X, x\right) \to 0 \qquad \text{for } \lambda \to \infty. \tag{37}$$

We note that for a compact group G only discrete values for λ are admissible, but in any case arbitrary large values of λ are allowed, and this limit is assumed in what follows.

We stress the explicit presence of λ in the first term in the exponent in (36), which guarantees that the size of Γ remains unchanged. In order to eliminate here the λ-dependence we put $X = x^i X_i^\lambda$ with $X_i^\lambda = \frac{1}{\lambda}X_i$ (leading to $\lambda\langle F, X\rangle = x^i\varphi_i$). The X_i^λ satisfy relations

$$[X_i^\lambda, X_j^\lambda] = \frac{1}{\lambda} f^k{}_{ij} X_k^\lambda. \tag{38}$$

Repeating the construction given above one obtains that (26) is changed to

$$[\varphi_i, \varphi_j]_* = \frac{i}{\lambda} f^k{}_{ij}\varphi_k. \tag{39}$$

There are two direct consequences:

(i) The algebra $\mathcal{A}_0$ becomes commutative in the limit $\lambda \to \infty$,

(ii) the rescaled *-commutator

$$\frac{\lambda}{i}[\varphi_i, \varphi_j]_* = f^k{}_{ij}\varphi_k =: \{\varphi_i, \varphi_j\} \tag{40}$$

is λ-independent and generates the Lie-Poisson bracket $\{\cdot,\cdot\}$ on Γ (corresponding to the standard Lie-Kirillov symplectic structure on Γ).

Note: Eq. (40) plays the same role as $\frac{1}{i\hbar}[x, p] = 1 = \{x, p\}$ in quantum and classical mechanics.

4 Applications and Concluding Remarks

We interpret S^2 as an orbit of $G = SU(2)$, the group of matrices

$$g = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}, \qquad u = \cos\frac{\vartheta}{2}\exp\left[-\frac{i}{2}(\varphi+\psi)\right], \qquad v = \sin\frac{\vartheta}{2}\exp\left[-\frac{i}{2}(\varphi-\psi)\right], \tag{41}$$

where $0 \leq \vartheta \leq \pi$, $|\varphi| \leq \pi$, $|\psi| \leq \pi$.

Let $T(g)$ be the standard $(2\ell + 1)$-dimensional unitary representation of G. In this representation we can choose

$$\omega(g, x_0) = (u^*)^n, \tag{42}$$

where $n = 2\ell$ is an integer. The subgroup $H = U(1)$ of diagonal matrices

$$h = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \qquad u = \exp\left(-\frac{i}{2}\psi\right), \tag{43}$$

is a stability group, since

$$\omega(h, x_0) = e^{in\psi}. \tag{44}$$

The points of the factor space $M = G/H$ we parametrize by $x \in \mathbf{C} \cup \{\infty\} \simeq S^2$. We identify $|x_0\rangle$ with $|0\rangle$, and we define the coherent state $|x\rangle$ by

$$|x\rangle = T(g_x)|0\rangle, \qquad g_x = (1 + |x|^2)^{-1/2} \begin{pmatrix} 1 & x \\ -x^* & 1 \end{pmatrix}. \tag{45}$$

Then

$$\omega(g, x) = \langle x|T(g_x)|x\rangle = \left(\frac{u^* + vx^* - v^*x + u|x|^2}{1 + |x|^2}\right)^n. \tag{46}$$

Choosing the basis in the algebra $\mathcal{G} = su(2)$ as

$$X_j = \frac{1}{in}\sigma_j, \qquad \sigma_j\text{– Pauli matrices}, \qquad j = 1, 2, 3, \tag{47}$$

we obtain the left-invariant vector fields

$$\begin{aligned} \widehat{X}_1 &= \frac{2}{n}\left[\cos\varphi \cot\vartheta \partial_\varphi + \sin\varphi \partial_\vartheta - \frac{\cos\varphi}{\sin\vartheta}\partial_\psi\right], \\ \widehat{X}_2 &= \frac{2}{n}\left[\sin\varphi \cot\vartheta \partial_\varphi - \cos\varphi \partial_\vartheta - \frac{\sin\varphi}{\sin\vartheta}\partial_\psi\right], \\ \widehat{X}_3 &= -\frac{2}{n}\partial_\varphi, \end{aligned} \tag{48}$$

satisfying the commutation relations

$$[\widehat{X}_i, \widehat{X}_j] = f_{ij}^k \widehat{X}_k, \tag{49}$$

where $f_{ij}^k = \frac{2}{n}\varepsilon_{ijk}$, ε_{ijk} – totally antisymmetric and $\varepsilon_{123} = 1$. For completeness we give the dual 1-forms

$$\begin{aligned} \Theta^1 &= \frac{n}{2}[\sin\varphi d\vartheta - \cos\varphi \sin\vartheta d\psi], \\ \Theta^2 &= \frac{n}{2}[-\cos\varphi d\vartheta - \sin\varphi \sin\vartheta d\psi], \\ \Theta^3 &= \frac{n}{2}[-d\varphi - \cos\varphi d\psi], \end{aligned} \tag{50}$$

satisfying the Maurer-Cartan relations

$$d\Theta^i = -\frac{1}{2} f_{jk}^i \Theta^j \wedge \Theta^k. \tag{51}$$

Putting $X = x^j X_j$ from (46) we obtain

$$\omega(e^X, x) = \exp(in\langle F, X\rangle + o(X^2)), \tag{52}$$

where $F = Ad^*_{g_x} F_0 \in \Gamma$ and $F_0 = \frac{1}{2i}\sigma_3$. The stability group of F_0 is H, and thus $\Gamma = M$ in this case.

According to (46) the quantities $\varphi_i = i^{-1}(\widehat{X}_i\omega)(e,x)$ are

$$\varphi_1 = \frac{2\ \mathrm{Re}\ x}{|x|^2+1}, \qquad \varphi_2 = \frac{2\ \mathrm{Im}\ x}{|x|^2+1}, \qquad \varphi_3 = \frac{|x|^2-1}{|x|^2+1}. \tag{53}$$

The direct calculation gives

$$\varphi_i \ * \ \varphi_j = -(\widehat{X}_i\widehat{X}_j\omega)(e,x) = \left(1 - \frac{1}{n}\right)\varphi_i\varphi_j + if^k_{ij}\varphi_k + \frac{1}{n}\delta_{ij}. \tag{54}$$

Similarly one obtains

$$\varphi_{i_1} \ * \ \ldots \ * \ \varphi_{i_k} = \left(1 - \frac{1}{n}\right)\ldots\left(1 - \frac{k-1}{n}\right)\varphi_{i_1}\ldots\varphi_{i_k} + o(1/n), \tag{55}$$

where the $o(1/n)$ term contains only polynomials of degree $< k$.

The specific factor in the first term on rhs of (55) guarantees that the *-product generates a finite dimensional algebra $\mathcal{A}_0 = \mathcal{A}$ of polynomials of the degree $\leq n$. These polynomials can be taken as *-polynomials (or as usual polynomials, the relation among them is given by (55), or its inversion). The algebra of polynomials is factorized by the relation $\sum\varphi_i \ * \ \varphi_i = 1 + \frac{2}{n}$ (or equivalently by $\sum\varphi_i^2 = 1$).

The exterior algebra $\Omega(\mathcal{A}_0)$ is generated by (abstract) Grassmann variables Θ^i, $i = 1,2,3$, satisfying Maurer-Cartan relations (51), with *-polynomial coefficients. Besides 0-forms the algebra $\Omega(\mathcal{A}_0)$ contains 1- and 2-forms

$$A = \sum_\alpha a^0_\alpha \ * \ da^1_\alpha, \qquad F = \sum_\alpha a^0_\alpha \ * \ da^1_\alpha \overset{*}{\wedge} da^2_\alpha, \tag{56}$$

where $a^k_\alpha \in \mathcal{A}_0$. The exterior differential of any element $f \in \mathcal{A}_0$ is given by

$$df = \{D, f\}_*, \qquad D = \varphi_i\widetilde{\Theta}^i, \tag{57}$$

where $\{\cdot,\cdot\}_* = \frac{n}{i}[\cdot,\cdot]_*$ and the rescaled Grassmann variables $\widetilde{\Theta}^i = \frac{1}{n}\Theta^i$ satisfy Maurer-Cartan relations

$$d\widetilde{\Theta}^i = -\frac{1}{2}\varepsilon_{ijk}\widetilde{\Theta}^j \wedge \widetilde{\Theta}^k \tag{58}$$

not depending on n.

Our construction for the Fuzzy sphere S^2 is in fact identical to the construction of matrix models for S^2 presented in [6] and [7]. The next example is connected with the non-compact group $SU(1,1)$ and goes beyond matrix models.

The group $G = SU(1,1)$ consists of complex matrices

$$g = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}, \qquad u = \cosh\frac{\vartheta}{2}\exp\left[-\frac{i}{2}(\varphi+\psi)\right], \qquad v = \sinh\frac{\vartheta}{2}\exp\left[-\frac{i}{2}(\varphi-\psi)\right], \tag{59}$$

where $\vartheta \in \mathbf{R}$, $|\varphi| \leq \pi$, $|\psi| \leq \pi$. We briefly sketch the construction of coherent states in the unitary representation $T(g)$ in the discrete series (see [9], [10] and [12]), in which

$$\omega(g, x_0) = (y^*)^{-n}. \tag{60}$$

The Bargmann index $n/2$ takes the values $1, 3/2, 2, \ldots$. The stability subgroup of (60) is the compact group $H = U(1)$ of diagonal matrices (43).

The factor space $M = SU(1,1)/U(1)$ is one sheet of a two-sided hyperboloid and its points are parametrized by the complex parameter x, $|x| < 1$. The coherent state is given as

$$|x\rangle = T(g_x)|0\rangle, \qquad g_x = (1 - |x|^2)^{-1/2} \begin{pmatrix} 1 & x \\ x^* & 1 \end{pmatrix}, \tag{61}$$

(we identified $|x_0\rangle$ with $|0\rangle$). Then

$$\omega(g, x) = \langle x|T(g_x)|x\rangle = \left(\frac{u^* - vx^* + v^*x - u|x|^2}{1 - |x|^2} \right)^{-n}. \tag{62}$$

The left-invariant vector fields on $G = su(1,1)$ are

$$\begin{aligned} \widehat{X}_1 &= \frac{2}{n} \left[\coth\vartheta \cos\varphi \partial_\varphi + \sin\varphi \partial_\vartheta - \frac{\cos\varphi}{\sinh\tau} \partial_\psi \right], \\ \widehat{X}_2 &= \frac{2}{n} \left[\coth\vartheta \sin\varphi \partial_\varphi - \cos\varphi \partial_\vartheta - \frac{\sin\varphi}{\sinh\tau} \partial_\psi \right], \\ \widehat{X}_3 &= -\frac{2}{n} \partial_\varphi. \end{aligned} \tag{63}$$

The quantities $\varphi_i = i^{-1}(\widehat{X}_i \omega)(e, x)$ now are

$$\varphi_1 = \frac{2 \text{ Re } x}{1 - |x|^2}, \qquad \varphi_2 = \frac{2 \text{ Im } x}{1 - |x|^2}, \qquad \varphi_3 = \frac{1 + |x|^2}{1 - |x|^2}. \tag{64}$$

The *-product is now given as

$$\varphi_i \, * \, \varphi_j = -(\widehat{X}_i \widehat{X}_j \omega)(e, x) = \left(1 + \frac{1}{n} \right) \varphi_i \varphi_j + i f^k{}_{ij} \varphi_k + \frac{1}{n} \delta_{ij} \tag{65}$$

where $f^k{}_{ij} = g^{k\ell} \varepsilon_{ij\ell}$, $g_{i\ell} = \text{diag } (1, -1, -1)$. Similarly

$$\varphi_{i_1} \, * \, \ldots \, * \, \varphi_{i_k} = \left(1 + \frac{1}{n} \right) \ldots \left(1 + \frac{k-1}{n} \right) \varphi_{i_1} \ldots \varphi_{i_k} + o(1/n), \tag{66}$$

where the $o(1/n)$ term contains polynomials of degree $< k$. We stress the different form of the factor in the first term on the rhs of (66). Consequently the algebra A_0 is infinite dimensional (and A is a closure of it in a suitable topology).

The formulas (56) – (58) defining the exterior algebras remain unchanged (except the proper definition of $f^i{}_{jk}$ has to be taken into account).

Note: The group $su(1,1)$ has more series of unitary representations. For a specific purpose, a difference to the discussed one could be more appropriate.

5 Concluding Remarks

Our approach is close to the geometrical quantization (see [11], [14] and [15]) but the interpretation is different. We intensively use the notions of coherent spaces (see [8] and [9]), and of the *-product (introduced in [16]) and specify them for particular Lie orbits. For recent results see [17] and [18].

The supercoherent states were introduced recently, see e.g. [12] and [13]. Obviously the non-commutative homogeneous super-spaces can be introduced too following the scheme given above. We shall treat them separately.

Our construction is simple, transparent and it allows to investigate the commutative limit, but many important questions remain open. For physical application more realistic models (higher dimensional and/or relativistic) are required. Moreover, important structures (like spin structure, monopoles, ...) should be well understood within non-commutative geometry. These problems are under current study.

After finishing this letter, Prof. A. Perelomov informed us that the procedure how we introduced the limit $n \to \infty$ is similar to that one used in the Berezin quantization scheme (for details see [10]).

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References

[1] A. Connes, Publ. IHES *62* (1986) 257.

[2] A. Connes, Geometrie Noncommutative (Inter Editions, Paris 1990).

[3] M. Dubois-Violette, C. R. Acad. Sci. Paris *307*, Ser. I (1988) 403.

[4] J. Madore, Lectures on Noncommutative Geometry, Orsay preprint, January 1993.

[5] J. Madore, Journ. Math. Phys. *32* (1991) 332.

[6] H. Grosse and J. Madore, Phys. Lett. *B283* (1992) 218.

[7] J. Madore, Class. Qu. Grav. *9* (1992) 69.

[8] J.R. Klauder and B.-S. Skagerstam, Coherent States (World Sci., Singapore 1985).

[9] A.M. Perelomov, Commun. Math. Phys. *26* (1972) 222.

[10] A.M. Perelomov, Generalized Coherent States and their Applications (Springer Verlag, Berlin 1986).

[11] A.A. Kirillov, Elements of Theory of Representations (Springer Verlag, Berlin).

[12] M. Chaichian, D. Ellinas and P. Prešnajder, Journ. Math. Phys. *32* (1992) 3381.

[13] B.W. Fatyga, V.A. Kostelecky, M.M. Nieto and D.R. Truax, Phys. Rev. *D43* (1991) 1403.

[14] B. Konstant, Lect. Notes Math. *170*, Springer Verlag, Berlin 1970.

[15] J.-M. Souriau, Structures des systémes dynamiques, Maîtrises de mathem., Dunod, Paris 1970.

[16] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. Phys. *111* (1978) 61 and 111.

[17] A. Odziewicz, Commun. Math. Phys. *150* (1992) 385.

[18] J. Grabowski, Jour. Geo and Phys. *9* (1992) 45.