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INVERTIBLE LINEAR DIFFERENTIAL OPERATORS ON TWO-DIMENSIONAL MANIFOLDS

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Abstract. In case of two independent variables invertible linear differential operators structure is described. It is proved that a two-sided invertible operator can be written as a composition of triangular invertible operators in the stable sense. The form to which a left-invertible operator can be reduced by composing it with triangular operators is given.

1.Introduction.

There are many various applications of invertible differential operators. An algebraic method for solving underdetermined systems of partial differential equations, and diverse classifying problems, and many other less obvious applications from [G] are among them. M.Gromov' results (see [G]) show that left-invertible differential operators occur quite frequently, and so the problem of finding invertible differential operators is of a considerable interest.

The most interesting is the invertibility problem for nonlinear operators, but since the linearization of a nonlinear invertible operator is a linear invertible operator (see, for example, [G]), investigation of linear invertible operators structure can be considered as the first step towards a description of nonlinear invertible operators.

In what follows we present a proof of a theorem which gives a necessary and sufficient condition for two-sided invertibility and a necessary condition for left-invertibility of differential operators on two-dimensional manifolds.

2.Preliminaries.

In this section we recall some definitions and results from [G, KLV].

Let K be a commutative ring, A be a commutative unitary algebra over K, P and Q be modules over A. Define the mapping

$$\delta_a : \mathrm{Hom}_K(Q, P) \longrightarrow \mathrm{Hom}_K(Q, P)$$

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by the formula

$$\delta_a(f)(q) = f(aq) - af(q),$$

where $a \in A, q \in Q, f \in \mathrm{Hom}_K(Q, P)$.

Definition. *A K-homomorphism $\nabla \in Hom_K(Q, P)$ is called a linear differential operator of the order $\leq k$, if*

$$(\delta_{a_0} \circ \ldots \circ \delta_{a_k})(\nabla) = 0$$

for any $a_0, \ldots, a_k \in A$.

The set of all linear differential operators of the order $\leq k$ acting from Q into P is denoted by $Diff_k(Q, P)$. In the case of $Q = A$ we shall also use the notation $Diff_k P$ instead of $Diff_k(A, P)$.

Remark. It follows from the definition that a differential operator of the order zero is an A-homomorphism of modules. And the mapping $\mathcal{D}_0 : Diff_0 P \longrightarrow P : \Delta \longmapsto \Delta(1)$ determines an one-to-one correspondence for any P. In the case of $P = A$ we shall not distinguish between the elements of A and operators of the order zero (A-homomorphisms) from A into A.

By $\mathrm{ord}\Delta$ we shall denote the order of a differential operator Δ, i.e. $k = \mathrm{ord}\Delta$ iff Δ is an operator of the order $\leq k$ but not $\leq k - 1$. Besides introduce the following

Definition. *Let $a \in A$, Δ be a differential operator. The minimal integer s for which*

$$\delta_a^{s+1}(\Delta) = 0$$

is called the order with respect to a of the operator Δ and is denoted by $ord_a\Delta$.

Note that if P, Q, R are A-modules, $\Delta : P \longrightarrow Q$ and $\nabla : Q \longrightarrow R$ are differential operators of the order k and l respectively, then $\nabla \circ \Delta : P \longrightarrow R$ is an operator of the order $\leq k + l$. This statement follows from the formula

$$\delta_a(\nabla \circ \Delta) = \delta_a(\nabla) \circ \Delta + \nabla \circ \delta_a(\Delta)$$

and the fact, that the mapping δ_a decreases the order of an operator by 1, i.e.

$$\delta_a : Diff_s(*, *) \longrightarrow Diff_{s-1}(*, *)$$

for $s > 0$ and any $a \in A$.

Definitions. *A differential operator $\nabla : Q \longrightarrow P$ is called left-invertible if there exists a differential operator $\Delta : P \longrightarrow Q$, such that the composition $\nabla \circ \Delta$ is the identity mapping id_P of P. In this case the operator Δ is called the inversion of ∇. If the operators ∇ and Δ, in addition, satisfy the condition $\Delta \circ \nabla = id_Q$ then they are called two-sided invertible.*

In what follows we consider the geometric case only, which means that $K = \mathbb{R}$ is the field of real numbers, $A = C^\infty(M)$ is the algebra of smooth functions on a manifold M, $P = C^\infty(\pi)$ and $Q = C^\infty(\tilde{\pi})$ are modules of sections of some vector bundles π and $\tilde{\pi}$ over M. In this case the problem of finding invertible operators is a local one. In fact, if an operator ∇ has an inversion Δ_α in a neighborhood U_α of any point $x \in M$, $\bigcup_\alpha U_\alpha = M$ is a covering of M, and $\sum_\alpha f_\alpha = 1$ is a partition of unity, then setting

$$\Delta(p) = \sum_\alpha \Delta_\alpha(f_\alpha p)$$

we obtain

$$(\nabla \circ \Delta)(p) = \sum_\alpha (\nabla \circ \Delta_\alpha)(f_\alpha p) = \sum_\alpha f_\alpha p = p$$

for any section p of the bundle π.

Example 1. Let $M = \mathbb{R}^2$ be a plane, $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $\tilde{\pi} : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be trivial bundles, x_1, x_2 be coordinates on M, $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, 2$, be the derivative operators. Consider operators

$$\nabla = (\, \partial_2, \quad \partial_1 + f \,)\,, \ \Box = \begin{pmatrix} \partial_1 + f \\ -\partial_2 \end{pmatrix}.$$

We have

$$\nabla \circ \Box = [\partial_2, \partial_1 + f] = \partial_2(f).$$

Hence, the operator ∇ is invertible in a neighborhood of any point $x \in M$ where $\partial_2(f)(x) \neq 0$ and

$$\nabla = \Box \circ \frac{1}{\partial_2(f)}$$

is its inversion. But there exist many inversions for ∇ and a difference of two inversions lies in the kernel of ∇. It is easy to show that

$$\ker \nabla = (1 - \Delta \circ \nabla) \circ L,$$

where L is an arbitrary operator from Q into P. Thus any inversion of the operator ∇ has the form

$$\Delta + (1 - \Delta \circ \nabla) \circ L. \blacktriangleright$$

A. Vinogradov [V] had found a relation between two-sided invertible operators and $\mathcal{C}$-transformations of jet spaces. By $\mathcal{C}$-transformations he called diffeomorphisms of infinite jet spaces which preserve the Cartan distribution (see [KLV]). It is easy to see that any nonlinear two-sided invertible differential operator is a $\mathcal{C}$-transformation and that $\mathcal{C}$-transformations should be considered as isomorphisms in the category of nonlinear differential equations. The linearization of a $\mathcal{C}$-transformation is a linear two-sided invertible operator. A. Vinogradov considered the scalar case, that is, when the bundles π and $\tilde{\pi}$ have one-dimensional fibers. He proved that in this case any $\mathcal{C}$-transformation is a contact transformation. It means that a two-sided invertible operator in the scalar case must have the order zero. In addition, he noted that it is not true in general, since there exist triangular invertible operators of nonzero order (see Example 2 below).

Note that if $P = C^\infty(\pi), Q = C^\infty(\tilde{\pi})$, then one has

$$\delta_a = \sum_{i=1}^n a_{x_i} \delta_{x_i},$$

where $a \in A$ is a function on M, $x_1, \dots, x_n$ are coordinate functions on M, and $a_{x_i} = \frac{\partial a}{\partial x_i}$, $i = 1, \dots, n$, are partial derivatives.

3.Triangular operators.

In this section we consider some examples of invertible operators.

Example 2. Let $M=\mathbb{R}^2, \pi=\tilde{\pi}:\mathbb{R}^4\longrightarrow\mathbb{R}^2, \Phi$ be an arbitrary linear differential operator from $C^\infty(M)$ into $C^\infty(M)$. Then

$$\nabla=\begin{pmatrix}1 & \Phi\\ 0 & 1\end{pmatrix}$$

is a two-sided invertible operator and its inversion is

$$\Delta=\begin{pmatrix}1 & -\Phi\\ 0 & 1\end{pmatrix}. \blacktriangleright$$

This example can be generalized in the following way.

Definition. *In the geometric case a linear differential operator $\nabla:Q\longrightarrow P$ is called a triangular invertible operator if in a neighborhood of any point $x\in M$ there exist basises in fibers of the bundles π and $\tilde{\pi}$ in which the matrix of the operator ∇ has an upper-triangular form, that is, identities are on the diagonal, zeroes are under it and any operators are over it.*

It is obvious that any triangular invertible operator is two-sided invertible and its inversion is also triangular at the same basises.

Example 3. If the operators $\beta\in Diff_0(Q,A), \xi\in Diff_k Q$ are such that $\beta\circ\xi=0$, then the operator

$$\Theta=\mathrm{id}_Q+\xi\circ\beta\ \in Diff_k(Q,Q)$$

is triangular and its inversion is $\Theta^{-1}=\mathrm{id}_Q-\xi\circ\beta$. It follows from the fact that in the decomposition $Q=Q_1\oplus A$, where $Q_1=\ker\beta,\ \beta\big|_A=\beta_0\in Diff_0(A,A)$, the operator Θ has the form

$$\Theta=\begin{pmatrix}\mathrm{id}_{Q_1} & \xi\circ\beta_0\\ 0 & \mathrm{id}_A,\end{pmatrix}$$

and hence in the basis of Q which is an union of basises of Q_1 and A the matrix of Θ has an upper-triangular form. $\blacktriangleright$

A composition of triangular invertible operators is evidently an invertible operator. Are there other two-sided invertible differential operators besides the compositions of triangular operators? The answer is positive and is given by the following

Example 4. Let $M=\mathbb{R}^2, x_1, x_2$ be coordinates on M, $\pi=\tilde{\pi}:\mathbb{R}^4\longrightarrow\mathbb{R}^2$ be a trivial bundle. Then

$$\nabla=\begin{pmatrix}1-\partial_{12} & \partial_{22}\\ -\partial_{11} & 1+\partial_{12}\end{pmatrix},$$

where $\partial_{12}=\frac{\partial^2}{\partial x_1\partial x_2}$, etc., is an invertible operator with the inversion

$$\Delta=\begin{pmatrix}1+\partial_{12} & -\partial_{22}\\ \partial_{11} & 1-\partial_{12}\end{pmatrix}.$$

Using the symbols it is possible to prove that these operators are not compositions of triangular operators. But the operators $\nabla \oplus \mathrm{id}_A$ and $\Delta \oplus \mathrm{id}_A$ are represented as such compositions:

$$\nabla \oplus \mathrm{id}_A = \begin{pmatrix} 1-\partial_{12} & \partial_{22} & 0 \\ -\partial_{11} & 1+\partial_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} 1 & 0 & \partial_2 \\ 0 & 1 & \partial_1 \\ 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\partial_1 & \partial_2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & -\partial_2 \\ 0 & 1 & -\partial_1 \\ 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_1 & -\partial_2 & 1 \end{pmatrix}. \blacktriangleright$$

4.Main result.

Definition. *An ST-transformation for a differential operator $\nabla : Q \longrightarrow P$ is a change of the operator ∇ to an operator $\tilde{\nabla} : \tilde{Q} \longrightarrow \tilde{P}$ for which there exist A-modules R and $\tilde{R}$, such that $Q \oplus R \simeq \tilde{Q} \oplus \tilde{R}, P \oplus R \simeq \tilde{P} \oplus \tilde{R}$, and*

$$\tilde{\nabla} \oplus id_{\tilde{R}} = \Theta_1 \circ (\nabla \oplus id_R) \circ \Theta_2,$$

where Θ_1 and Θ_2 are compositions of triangular operators.

Remarks. 1. An ST-transformation preserves invertibility of operators, since compositions of triangular operators are invertible operators, and the operator $\nabla \oplus \mathrm{id}_R$ is invertible iff ∇ is invertible.
2. A composition of ST-transformations is a ST-transformation.

Theorem. *Let M be a two-dimensional manifold, x_1, x_2 be coordinates on M, $\pi : E \longrightarrow M$ be a vector bundle, $P = Q = C^\infty(\pi)$. Then*
1) for any two-sided invertible linear differential operator $\nabla : Q \longrightarrow P$ of nonzero order there exists a trivial bundle $\pi_R : \mathbb{R}^m \times U \longrightarrow U$ over a neighborhood U of an arbitrary generic point $x \in M$, such that the operator

$$\nabla\big|_U \oplus id_R : Q\big|_U \oplus R \longrightarrow P\big|_U \oplus R,$$

where $R = C^\infty(\pi_R)$, and $\nabla\big|_U, Q\big|_U, P\big|_U$ are restrictions of ∇, Q, P onto the neighborhood U, is a composition of triangular operators.
2) Any left-invertible linear differential operator ∇ of nonzero order in a neighborhood of a generic point can be reduced by an ST-transformation to an operator $\tilde{\nabla} : \tilde{Q} \longrightarrow \tilde{P}$ satisfying the following condition: It has the 1st order, and there exist decompositions

$$\tilde{Q} = Q_1 \oplus \ldots \oplus Q_r \oplus Q_{r+1} \quad and \quad \tilde{P} = P_1 \oplus \ldots \oplus P_r,$$

such that $\tilde{\nabla}$ has the form

$$\begin{pmatrix} \Theta_1 & \Phi_{1,2} & \cdots & \Phi_{1,r} & \Phi_{1,r+1} \\ 0 & \Theta_2 & \cdots & \Phi_{2,r} & \Phi_{2,r+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \Theta_r & \Phi_{r,r+1} \end{pmatrix}, \tag{1}$$

where

$$\delta_{x_2}(\Theta_i) = 0,\ 1 \le i \le r,$$
$$\delta_{x_1}(\Phi_{i,j}) = 0,\ 1 \le i < j \le r+1,$$

and $\delta_{x_1}(\Theta_i) : Q_i \longrightarrow P_i, i = 1, \dots, r,$ *are isomorphisms.*
3) The generic points of an invertible operator constitute a residual subset of M.

The concept of a generic point will be clarified during the proof of Theorem.

Proof. We shall consider only such points of M which have neighborhoods satisfying the following condition of nonsingularity: Any module of sections $\{p\}$ of a vector bundle over M which may occur below at every point x of the neighborhood defines a linear space of vectors $\{p(x)\}$ whose dimension is independent of the point x. These points will be called generic points. Since only a finite number of the modules of sections will be dealt with in the proof, the generic points form a residual set.

In addition, since the problem of finding invertible operators is a local one, we shall consider a coordinate neighborhood of a generic point. Fix coordinates x_1 and x_2 in this neighborhood and let $\partial_i = \partial/\partial_{x_i}, i = 1, 2$, be the corresponding derivation operators. By Q and P we shall denote the modules of sections of bundles $\tilde{\pi}$ and π (or of their extensions) over this neighborhood.

The proof consists of four parts:
1. The problem is reduced to the case $\mathrm{ord}\nabla = 1$.
2. The possibility of decreasing $\mathrm{ord}_{x_2}\Delta$ with conserving $\mathrm{ord}\nabla$ is considered.
3. The case of $\mathrm{ord}\nabla = 1$, $\mathrm{ord}_{x_2}\Delta = 0$ is studied.
4. The operators ∇ for which $\mathrm{ord}_{x_2}\Delta$ cannot be decreased with preserving $\mathrm{ord}\nabla = 1$ are investigated and it is proved that in this case there are no that two-sided invertible operators.

Part 1. Any linear differential operator of order > 0 can be represented as $\nabla = \nabla_1 \circ \partial_1 + \nabla_2 \circ \partial_2 + \nabla_0$, where $\nabla_0, \nabla_1, \nabla_2$ are operators of less order than the order of ∇. Hence, extending the modules Q and P to $Q \oplus Q \oplus Q$ and $P \oplus Q \oplus Q$ we obtain the representation

$$\nabla \oplus \mathrm{id}_Q \oplus \mathrm{id}_Q = \begin{pmatrix} \mathrm{id}_P & \nabla_1 & \nabla_2 \\ 0 & \mathrm{id}_Q & 0 \\ 0 & 0 & \mathrm{id}_Q \end{pmatrix} \circ \begin{pmatrix} \nabla_0 & -\nabla_1 & -\nabla_2 \\ \partial_1 & \mathrm{id}_Q & 0 \\ \partial_2 & 0 & \mathrm{id}_Q \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_Q & 0 & 0 \\ -\partial_1 & \mathrm{id}_Q & 0 \\ -\partial_2 & 0 & \mathrm{id}_Q \end{pmatrix},$$

where the first and the third operators of the composition are triangular ones. They act from $P \oplus Q \oplus Q$ into $P \oplus Q \oplus Q$ and from $Q \oplus Q \oplus Q$ into $Q \oplus Q \oplus Q$ respectively. Now, if ∇ is an invertible operator of the order > 1, then the operator in the middle is invertible, being a composition of invertible operators, and its order is less than the one of ∇. We denote the middle operator by ∇ again and repeat this construction till $\mathrm{ord}\nabla > 1$.

Consider the case of $\mathrm{ord}\nabla = 1$. Let $\delta_i = \delta_{x_i}(\nabla), i = 1, 2$. Since ∇ is the operator of the first order, δ_1 and δ_2 are module homomorphisms from Q into P. If

$$Q_0 = \ker \delta_1 \cap \ker \delta_2 \neq \{0\},$$

then $P_0 = \nabla(Q_0)$ is a module and $\nabla\big|_{Q_0} : Q_0 \longrightarrow P_0$ is an isomorphism. In decompositions $Q = Q_0 \oplus Q_1$ and $P = P_0 \oplus P_1$ the operator ∇ has the form

$$\nabla = \begin{pmatrix} \nabla_{11} & \nabla_{12} \\ 0 & \nabla_{22} \end{pmatrix},$$

where $\nabla_{11} = \nabla\big|_{Q_0}$ is an isomorphism. Therefore

$$\Theta = \begin{pmatrix} \nabla_{11}^{-1} & -\nabla_{11}^{-1} \circ \nabla_{12} \\ 0 & \mathrm{id}_{Q_1} \end{pmatrix}$$

is a triangular operator from $P_0 \oplus Q_1$ to $Q_0 \oplus Q_1$ and

$$\nabla \circ \Theta = \begin{pmatrix} \mathrm{id}_{Q_0} & 0 \\ 0 & \nabla_{22} \end{pmatrix}.$$

In this case the composition

$$\begin{aligned} \Theta^{-1} \circ \Delta &= \begin{pmatrix} \nabla_{11} & \nabla_{12} \\ 0 & \mathrm{id}_{Q_1} \end{pmatrix} \circ \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} = \\ &= \begin{pmatrix} \nabla_{11} \circ \Delta_{11} + \nabla_{12} \circ \Delta_{21} & \nabla_{11} \circ \Delta_{12} + \nabla_{12} \circ \Delta_{22} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \end{aligned} \tag{2}$$

is to be equal

$$\begin{pmatrix} \mathrm{id}_{Q_0} & 0 \\ 0 & \Delta_{22} \end{pmatrix} \tag{3}$$

otherwise $(\nabla \circ \Theta) \circ (\Theta^{-1} \circ \Delta) \neq \mathrm{id}_P$. That is, by ST-transformations we can reduce the considered operator to a new operator ∇ which either satisfies the conditions

$$\mathrm{ord}\nabla = 1, \ \ker \delta_1 \cap \ker \delta_2 = \{0\} \tag{4}$$

or is an invertible operator of zero order. In the last case the conclusions of Theorem hold, since the composition of a triangular invertible operator and an invertible operator of zero order is a triangular operator.

Part 2. Let conditions (4) hold and

$$\begin{aligned} &\mathrm{ord}_{x_2}\Delta = k \geq 1, \\ &\Box = \delta_{x_2}^k(\Delta), \end{aligned}$$

T be the submodule of Q generated by $\Box(P)$. Then

$$\delta_2 \circ \Box = 0$$

and

$$\delta_2(S) = 0,$$

since $0 = \delta_{x_2}^{k+1}(\nabla \circ \Delta) = \delta_2 \circ \Box$ and δ_2 is a module homomorphism. Introduce differential operators

$$\nu : Diff_0 Q \longrightarrow Diff_0 P$$

and

$$\mu : Diff_0(P, A) \longrightarrow Diff_0(Q, A)$$

by the equalities

$$\begin{aligned} \nu(\xi) &= \nabla \circ \xi - \delta_1 \circ \xi \circ \partial_1 - \delta_2 \circ \xi \circ \partial_2, \\ \mu(\alpha) &= \alpha \circ \nabla - \partial_1 \circ \alpha \circ \delta_1 - \partial_2 \circ \alpha \circ \delta_2 \end{aligned}$$

and prove two lemmas.

Lemma 1. *Let the hypotheses of Theorem and the conditions (4) hold and $\nabla(S) \not\subset \delta_1(S)$ for any nonzero submodule $S \subset T$. Then there exist an integer $s \geq 0$ and a set of operators $\xi_i, \eta_j \in Diff_0 Q, i = 0, \dots, s, j = 0, \dots, s+1$, which satisfy the conditions*

$$\xi_0(1) \in \Box(P),\ \xi_i(1) \in T, i = 0, 1, \dots, s,\ \nu(\xi_0)(1) \notin \delta_1(T)$$

and the system

$$\begin{aligned}
&\delta_2 \circ \xi_i = 0,\ i = 0, 1, \dots, s, \\
&\delta_1 \circ \xi_s + \delta_2 \circ \eta_{s+1} = 0, \\
&\nu(\xi_i) + \delta_1 \circ \xi_{i-1} + \delta_2 \circ \eta_i = 0,\ i = 1, \dots, s, \\
&\nu(\xi_0) + \delta_2 \circ \eta_0 = 0.
\end{aligned} \tag{5}$$

Proof. If $(\nabla \circ \Box)(P) \subset \delta_1(T)$, then $\nabla(T) \subset \delta_1(T)$, since for any $r \in T$ we have $r = \sum_{i=1}^{l} a_i \Box(p_i)$, where $a_i \in A, p_i \in P, i = 1, \dots, l$, and

$$\begin{aligned}
\nabla(r) &= \sum_{i=1}^{l} (\delta_{a_i}(\nabla) + a_i \nabla)(\Box(p_i)) \\
&= \sum_{i=1}^{l} ((\partial_1(a_i)\delta_1 + \partial_2(a_i)\delta_2 + a_i\nabla) \circ \Box)(p_i) \\
&= \sum_{i=1}^{l} (\partial_1(a_i)(\delta_1 \circ \Box)(p_i) + 0 + \sum_{j=1}^{l_i} b_{ij}(\delta_1 \circ \Box)(\tilde{p}_j)) \\
&= \delta_1(\sum_{i=1}^{l} (\partial_1(a_i)\Box(p_i) + \sum_{j=1}^{l_i} b_{ij}\Box(\tilde{p}_j))) \in \delta_1(T),
\end{aligned}$$

where $(\nabla \circ \Box)(p_i) = \sum_{j=1}^{l_i} b_{ij}(\delta_1 \circ \Box)(\tilde{p}_j), i = 1, \dots, l, b_{ij} \in A$. But it disagrees with the hypotheses of the lemma. Therefore $(\nabla \circ \Box)(P) \not\subset \delta_1(T)$ and there exists an element $p \in P$ such that $q = \Box(p), \nabla(q) \notin \delta_1(S)$.

Let θ be an operator from $Diff_0 P$ such that $\theta(1) = p$. Then $\Box \circ \theta(1) = \Box(p) = q \neq 0$. Since $\mathrm{ord}_{x_2}\Box = 0$ and $\mathrm{ord}\theta = 0$ then $\delta_{x_2}(\Box \circ \theta) = 0$ and hence

$$\Box \circ \theta = \sum_{i=0}^{s} \xi_i \circ \partial_1^i$$

for some $s \geq 0, \xi_i \in Diff_0 Q, i = 0, \dots, s$. Obviously, from the definitions of $\xi_i, i = 0, \dots, s$, and T it follows that $\xi_i(a) = a\xi_i(1) \in T$ for any $a \in A, i = 0, 1, \dots, s$. Moreover

$$\begin{aligned}
&\xi_0(1) = \sum_{i=0}^{s} (\xi_i \circ \partial_1^i)(1) = (\Box \circ \theta)(1) = \Box(p), \\
&\nu(\xi_0)(1) = (\nabla \circ \xi_0)(1) = \nabla(\Box(p)) = \nabla(q) \notin \delta_1(T),
\end{aligned}$$

and it follows from the equality $\delta_2 \circ \Box = 0$ that $\delta_2 \circ (\sum_{i=0}^{s} \xi_i \circ \partial_1^i) = \delta_2 \circ \Box \circ \theta = 0$. Thus we obtain the first of equalities of (5).

Since

$$\delta_{x_2}(\delta_{x_2}^{k-1}(\Delta)\circ\theta)=\Box\circ\theta=\sum_{i=0}^{s}\xi_i\circ\partial_1^i,$$

one has

$$\delta_{x_2}^{k-1}(\Delta)\circ\theta=\sum_{i=0}^{s}\xi_i\circ\partial_1^i\circ\partial_2+\frac{1}{k}\sum_{i=0}^{s_1}\eta_i\circ\partial_1^i$$

for some $s_1\geq 0, \eta_i\in Diff_0Q, i=0,\dots,s_1$. From this and from the first of equalities of (5) it follows that

$$\begin{aligned}\delta_{x_2}^{k}(\nabla\circ\Delta)\circ\theta&=\nabla\circ\delta_{x_2}^{k}(\Delta)\circ\theta+k\delta_2\circ\delta_{x_2}^{k-1}(\Delta)\circ\theta\\&=\nabla\circ(\sum_{i=0}^{s}\xi_i\circ\partial_1^i)+k\delta_2\circ(\sum_{i=0}^{s}\xi_i\circ\partial_1^i\circ\partial_2+\frac{1}{k}\sum_{i=0}^{s_1}\eta_i\circ\partial_1^i)\\&=\sum_{i=0}^{s}(\nu(\xi_i)+\delta_1\circ\xi_i\circ\partial_1)\circ\partial_1^i+\sum_{i=0}^{s_1}\delta_2\circ\eta_i\circ\partial_1^i,\end{aligned}$$

from where we obtain the rest equalities of (5). ►

Lemma 2. *Let the hypotheses of Lemma 1 holds. Then by ST-transformations we can make $s=0$ in the conclusions of Lemma 1 so that its hypotheses remain valid, and the dimension of the module T does not change.*

Proof. If $\nu(\xi_s)=\delta_2\circ\varepsilon$ for some $\varepsilon\in Diff_0Q$, then we can decrease s in the system (5), since from the third of equalities of (5) it follows

$$\delta_2\circ\varepsilon+\delta_1\circ\xi_{s-1}+\delta_2\circ\eta_s=\delta_1\circ\xi_{s-1}+\delta_2\circ\tilde{\eta}_s=0,$$

where $\tilde{\eta}_s=\eta_s+\varepsilon$. In this case the operators $\xi_i, i=0,\dots,s-1$, do not change, hence the conclusions of Lemma 1 remain valid.

If $\nu(\xi_s)\notin\delta_2\circ Diff_0Q$ then we can reduce the system (5) by transformations which we shall construct now. Denote

$$\begin{aligned}\phi&=\nu(\eta_{s+1})+\delta_1\circ\eta_{s+1}\circ\partial_1,\\\theta&=\xi_s\circ\partial_2+\eta_{s+1}\circ\partial_1.\end{aligned}$$

We have

$$\begin{aligned}&\delta_{x_2}(\phi)=0,\\&\delta_{x_1}(\phi)=\delta_1\circ\eta_{s+1},\\&\nabla\circ\eta_{s+1}-\phi=\delta_2\circ\eta_{s+1}\circ\partial_2.\end{aligned}\tag{6}$$

Substitute $Q\oplus A$ for Q, $P\oplus A$ for P and the following composition of invertible operators for ∇:

$$\begin{pmatrix}\mathrm{id}_P & -\phi\circ\partial_1\\0&1\end{pmatrix}\circ\begin{pmatrix}\nabla&0\\0&1\end{pmatrix}\circ\begin{pmatrix}\mathrm{id}_Q&\theta\\0&1\end{pmatrix}==\begin{pmatrix}\nabla&\nabla\circ\theta-\phi\circ\partial_1\\0&1\end{pmatrix}=\tilde{\nabla}.$$

From (5) and (6) it follows that

$$\begin{aligned}\nabla\circ\theta-\phi\circ\partial_1&=\nabla\circ\xi_s\circ\partial_2+\nabla\circ\eta_{s+1}\circ\partial_1-\phi\circ\partial_1\\&=(\nu(\xi_s)+\delta_1\circ\xi_s\circ\partial_1)\circ\partial_2+\delta_2\circ\eta_{s+1}\circ\partial_2\circ\partial_1\\&=\nu(\xi_s)\circ\partial_2.\end{aligned}$$

Therefore

$$\tilde{\nabla} = \begin{pmatrix} \nabla & \nu(\xi_s)\circ\partial_2 \\ 0 & 1 \end{pmatrix}$$

and

$$\delta_{x_2}(\tilde{\nabla}) = \begin{pmatrix} \delta_2 & \nu(\xi_s) \\ 0 & 0 \end{pmatrix}.$$

Hence, identifying $Diff_0 P$ with $Diff_0 P \oplus \{0\} \subset Diff_0(P\oplus A)$ we obtain

$$\nu(\xi_s) \in \delta_{x_2}(\tilde{\nabla}) \circ Diff_0(Q\oplus A)$$

and just as above we can decrease s in (5). Conditions (4) for the operator $\tilde{\nabla}$ are also realized, since

$$\ker\delta_{x_1}(\tilde{\nabla}) = \ker\delta_1 \oplus A,$$
$$\nu(\xi_s) \notin \delta_2 \circ Diff_0 Q$$

and hence

$$\ker\delta_{x_2}(\tilde{\nabla}) = \ker\delta_2 \oplus \{0\} \subset Q\oplus A,$$
$$\ker\delta_{x_1}(\tilde{\nabla}) \cap \ker\delta_{x_2}(\tilde{\nabla}) = \ker\delta_1 \cap \ker\delta_2 \oplus \{0\} = \{0\}.$$

We shall now prove that the dimension of the module T does not change when passing from ∇ to $\tilde{\nabla}$ (respectively from Δ to $\tilde{\Delta}$). We have

$$\tilde{\Delta} = \begin{pmatrix} \mathrm{id}_Q & -\theta \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \Delta & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_P & \phi\circ\partial_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Delta & \Delta\circ\phi\circ\partial_1 - \theta \\ 0 & 1 \end{pmatrix}.$$

Since $\delta_{x_2}(\phi) = 0, \delta_{x_2}(\theta) = \xi_s$, one has

$$\delta^k_{x_2}(\tilde{\Delta}) = \begin{pmatrix} \Box & \Box\circ\phi\circ\partial_1 - \xi_s\rho \\ 0 & 0 \end{pmatrix}, \tag{7}$$

where $\rho = 1$, if $k = 1$, and $\rho = 0$, if $k > 1$. From this and the inclusion $\xi_s \in T$ it follows that the linear span of $\delta^k_{x_2}(\tilde{\Delta})(P\oplus A\oplus A)$ which plays the role of the new T coincides with the module $T\oplus\{0\}$. It means that under such transformations the dimension of the module T does not change. ►

Let

$$\begin{aligned} &\delta_2\circ\xi_0 = 0, \\ &\delta_1\circ\xi_0 + \delta_2\circ\eta_1 = 0, \\ &\nu(\xi_0) + \delta_2\circ\eta_0 = 0, \end{aligned} \tag{8}$$

where

$$\xi_0, \eta_1, \eta_0 \in Diff_0 Q,$$
$$\xi_0(1) \in \Box(P),$$
$$\nu(\xi_0)(1) \notin \delta_1(T)$$

(see Lemmas 1 and 2). We shall now decrease the dimension of the module T still staying in the situation (4). To do this we choose an operator $\beta \in Diff_0(P, A)$, such that $\beta(\delta_1(T)) = 0$ and $\beta(\nu(\xi_0)(1)) = (\beta\circ\nu(\xi_0))(1) = 1$. This is possible because $\nu(\xi_0)(1) \notin \delta_1(T)$. Since T is the module generated by elements from $\square(P)$, and $\beta,\ \nu(\xi_0)$ are operators of order zero, then

$$\begin{aligned} &\beta \circ \delta_1 \circ \square = 0, \\ &\beta \circ \nu(\xi_0) = 1. \end{aligned} \tag{9}$$

The operator

$$\Theta = \mathrm{id}_Q - (\xi_0 \circ \partial_2 + \eta_1 \circ \partial_1) \circ \beta \circ \delta_2 \in \mathit{Diff}_1(Q, Q)$$

is the triangular operator from Example 3. In fact, $\beta \circ \delta_2 \in \mathit{Diff}_0(Q, A)$. In addition, from the equalities (8), (9) and the inclusion $\xi_0(1) \in \square(P)$ it follows that $\beta \circ \delta_2 \circ \eta_1 = -\beta \circ \delta_1 \circ \xi_0 = 0$ and

$$(\beta \circ \delta_2) \circ (\xi_0 \circ \partial_2 + \eta_1 \circ \partial_1) = 0.$$

Let $\alpha \in \mathit{Diff}_0(P, A)$ be an operator, such that

$$\alpha \circ \delta_2 \circ \eta_1 = 1.$$

Denote

$$\gamma = \nu(\eta_1) + \delta_1 \circ \eta_1 \circ \partial_1.$$

Then

$$\begin{aligned} &\delta_{x_2}(\gamma) = 0, \\ &\delta_{x_1}(\gamma) = \delta_1 \circ \eta_1. \end{aligned}$$

Extend the modules P and Q to $P \oplus A$ and $Q \oplus A$ and consider the following composition of the operator $\nabla \oplus \mathrm{id}_A$ and triangular operators

$$\begin{aligned} &\begin{pmatrix} \mathrm{id}_P & -\gamma \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \nabla & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_Q & 0 \\ \alpha \circ \delta_2 & 1 \end{pmatrix} \circ \begin{pmatrix} \Theta & 0 \\ 0 & 1 \end{pmatrix} = \\ &\qquad = \begin{pmatrix} (\nabla - \gamma \circ \alpha \circ \delta_2) \circ \Theta & -\gamma \\ \alpha \circ \delta_2 \circ \Theta & 1 \end{pmatrix}. \end{aligned} \tag{10}$$

The last operator which will play the role of ∇ has the order 1. In fact, α, δ_2 are operators of the order zero, the rest operators have the first order and

$$\begin{aligned} \delta_{x_1}^2((\nabla - \gamma \circ \alpha \circ \delta_2) \circ \Theta) &= 2(\delta_1 - \delta_{x_1}(\gamma) \circ \alpha \circ \delta_2) \circ \delta_{x_1}(\Theta) \\ &= 2(\delta_1 - \delta_{x_1}(\gamma) \circ \alpha \circ \delta_2) \circ (-\eta_1 \circ \beta \circ \delta_2) \\ &= -2(\delta_1 \circ \eta_1 - \delta_1 \circ \eta_1 \circ \alpha \circ \delta_2 \circ \eta_1) \circ \beta \circ \delta_2 = 0 \end{aligned}$$

(see the definitions Θ and α, and the formula for $\delta_{x_1}(\gamma)$),

$$\begin{aligned} &\delta_{x_2}^2((\nabla - \gamma \circ \alpha \circ \delta_2) \circ \Theta) = 2(\delta_2 - \delta_{x_2}(\gamma) \circ \alpha \circ \delta_2) \circ (-\xi_0 \circ \beta \circ \delta_2) = 0, \\ &(\delta_{x_2} \circ \delta_{x_1})((\nabla - \gamma \circ \alpha \circ \delta_2) \circ \Theta) = \\ &\qquad = (\delta_1 - \delta_{x_1}(\gamma) \circ \alpha \circ \delta_2) \circ \delta_{x_2}(\Theta) + (\delta_2 - \delta_{x_2}(\gamma) \circ \alpha \circ \delta_2) \circ \delta_{x_1}(\Theta) \\ &\qquad = (\delta_1 - \delta_{x_1}(\gamma) \circ \alpha \circ \delta_2) \circ (-\xi_0 \circ \beta \circ \delta_2) + \delta_2 \circ (-\eta_1 \circ \beta \circ \delta_2) \\ &\qquad = -(\delta_1 \circ \xi_0 + \delta_2 \circ \eta_1) \circ \beta \circ \delta_2 = 0 \end{aligned}$$

(see (8)).

Consider how the operator Δ changes under transformations (10). We have

$$\begin{pmatrix} \Theta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_Q & 0 \\ -\alpha\circ\delta_2 & 1 \end{pmatrix} \circ \begin{pmatrix} \Delta & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_P & \gamma \\ 0 & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} \Theta^{-1}\circ\Delta & 0 \\ -\alpha\circ\delta_2\circ\Delta & 1 \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_P & \gamma \\ 0 & 1 \end{pmatrix} = \tilde{\Delta}.$$

To find the order of the operator $\tilde{\Delta}$ with respect to x_2 we compute the operator $\delta_{x_2}^{k+1}(\tilde{\Delta})$. Since

$$\Theta^{-1} = \mathrm{id}_Q + \xi_0\circ\partial_2\circ\beta\circ\delta_2 + \eta_1\circ\partial_1\circ\beta\circ\delta_2$$

(see Example 3 and the definition of Θ), $\delta_{x_2}^k(\Delta) = \square, \delta_{x_2}(\gamma) = 0$ and $\delta_2\circ\square = 0$, then one has

$$\delta_{x_2}^{k+1}(\tilde{\Delta}) = \begin{pmatrix} \delta_{x_2}(\Theta^{-1})\circ\delta_{x_2}^k(\Delta) & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_P & \gamma \\ 0 & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} \xi_0\circ\beta\circ\delta_2\circ\square & 0 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_P & \gamma \\ 0 & 1 \end{pmatrix} = 0.$$

Hence, the order of $\tilde{\Delta}$ with respect to x_2 is the same as the one of Δ.

To see how the module T changes, we consider $\delta_{x_2}^k(\tilde{\Delta})$. Since $\delta_2\circ\square = 0, \delta_{x_2}(\gamma) = 0$ and $\delta_{x_2}(\Theta^{-1}) = \xi_0\circ\beta\circ\delta_2$ then

$$\delta_{x_2}^k(\tilde{\Delta}) = \begin{pmatrix} \delta_{x_2}(\Theta^{-1})\circ\delta_{x_2}^{k-1}(\Delta) + \Theta^{-1}\circ\square & 0 \\ \alpha\circ\delta_2\circ\square & 0 \end{pmatrix} \circ \begin{pmatrix} \mathrm{id}_P & \gamma \\ 0 & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} \xi_0\circ\beta\circ\delta_2\circ\delta_{x_2}^{k-1}(\Delta) + \square & (\xi_0\circ\beta\circ\delta_2\circ\delta_{x_2}^{k-1}(\Delta) + \square)\circ\gamma \\ 0 & 0 \end{pmatrix}.$$

From this and from the embeddings

$$\begin{aligned} &\square(P) \subset T, \\ &\beta\circ\delta_2\circ\delta_{x_2}^{k-1}(\Delta)(P) \subset A, \\ &\xi_0(A) \subset T \end{aligned}$$

it follows that

$$\delta_{x_2}^k(\tilde{\Delta})(P\oplus A) \subset T\oplus\{0\}. \tag{11}$$

We shall now show that the linear span of $\delta_{x_2}^k(\tilde{\Delta})(P\oplus A)$ does not coincide with $T\oplus\{0\}$. From the conditions imposed on ξ_0 (see Lemma 1) and from the equality $\beta(\delta_1(T)) = 0$ we obtain $\xi_0(1)\in T, \delta_2\circ\xi_0 = 0, \beta\circ\delta_1\circ\xi_0 = 0$ and hence

$$\mu(\beta)\circ\xi_0 = (\beta\circ\nabla - \partial_1\circ\beta\circ\delta_1 - \partial_2\circ\beta\circ\delta_2)\circ\xi_0 = \beta\circ\nabla\circ\xi_0.$$

For the same reason using (9) we obtain

$$\beta\circ\nabla\circ\xi_0 = \beta\circ(\nu(\xi_0) + \delta_1\circ\xi_0\circ\partial_1) = \beta\circ\nu(\xi_0) = 1.$$

Therefore, it follows from the equalities $\delta_2 \circ \Box = 0$, (4) and $\nabla \circ \Delta = \mathrm{id}_P$ that

$$\begin{aligned}\mu(\beta)\circ(\xi_0 \circ \beta \circ \delta_2 \circ \delta_{x_2}^{k-1}(\Delta) + \Box) &= \beta \circ \delta_2 \circ \delta_{x_2}^{k-1}(\Delta) + \mu(\beta) \circ \Box \\ &= \beta \circ \delta_2 \circ \delta_{x_2}^{k-1}(\Delta) + (\beta \circ \nabla - \partial_1 \circ \beta \circ \delta_1 - \partial_2 \circ \beta \circ \delta_2) \circ \Box \\ &= \beta \circ (\delta_2 \circ \delta_{x_2}^{k-1}(\Delta) + \nabla \circ \Box) \\ &= \beta \circ \delta_{x_2}^{k}(\nabla \circ \Delta) = 0.\end{aligned}$$

From here we obtain

$$(\mu(\beta) \oplus 0) \circ \delta_{x_2}^{k}(\tilde{\Delta}) = 0. \tag{12}$$

On the other hand,

$$(\mu(\beta) \oplus 0)(T \oplus \{0\}) = \mu(\beta)(T) \oplus \{0\} \neq \{0\} \tag{13}$$

since $\mu(\beta)(\xi_0(1)) = 1$ (see above) and $\xi_0(1) \in T$. Hence, we have reduced the dimension of the module T.

Part 3. The reasoning of Part 2 enables us, under the hypothesis of Lemma 1, to reduce $\mathrm{ord}_{x_2}\Delta$ to zero remaining in the situation (4). We shall now prove that the conditions $\mathrm{ord}_{x_2}\Delta = 0$ and (4) cannot be fulfilled simultaneously. If $\mathrm{ord}_{x_2}\Delta = 0$ and $\mathrm{ord}\Delta = k_1$, then

$$(\delta_{x_1}^{k_1} \circ \delta_{x_2})(\nabla \circ \Delta) = \delta_{x_2}(\nabla) \circ \delta_{x_1}^{k_1}(\Delta) = \delta_2 \circ \delta_{x_1}^{k_1}(\Delta) = 0$$

and

$$\delta_{x_1}^{k_1+1}(\nabla \circ \Delta) = \delta_1 \circ \delta_{x_1}^{k_1}(\Delta) = 0.$$

Therefore, $\mathrm{im}\delta_{x_1}^{k_1}(\Delta) \subset \ker \delta_1 \cap \ker \delta_2 \neq \{0\}$ and the condition $\mathrm{ord}_{x_2}\Delta = 0$ implies the fact that Δ is a composition of triangular operators (see the end of Part 1).

Part 4. Finally consider the case when the hypothesis of Lemma 1 is not realized and there exist a nonzero submodule $S \subset T$, such that $\nabla(S) \subset \delta_1(S)$. From (4) it follows that $\ker \delta_1\big|_S = 0$. Hence, $\delta_1\big|_S$ is an isomorphism from S into the module $\delta_1(S)$. Let us represent the module Q as $Q = S \oplus Q_1$ and let $P = \delta_1(S) \oplus P_1$. Since $\nabla(S) \subset \delta_1(S)$, the operator ∇ in this representation has the form

$$\nabla = \begin{pmatrix} \nabla_{11} & \nabla_{12} \\ 0 & \nabla_{22} \end{pmatrix}.$$

Let the operator Δ have the following block form

$$\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}.$$

Then

$$\nabla \circ \Delta = \begin{pmatrix} \nabla_{11} \circ \Delta_{11} + \nabla_{12} \circ \Delta_{21} & \nabla_{11} \circ \Delta_{12} + \nabla_{12} \circ \Delta_{22} \\ \nabla_{22} \circ \Delta_{21} & \nabla_{22} \circ \Delta_{22} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{\delta_1(S)} & 0 \\ 0 & \mathrm{id}_{P_1} \end{pmatrix}.$$

Hence, the operator of order one $\nabla_{22} : Q_1 \longrightarrow P_1$ is left-invertible and $\Delta_{22} : P_1 \longrightarrow Q_1$ is its inversion.

We can repeat the reasoning used in Part 2 for the operators ∇_{22} and Δ_{22}, i.e. to pass to the situation (4) for ∇_{22}, to reduce (if it is possible) the order with respect to x_2 of the operator Δ_{22}, etc. It is a finite procedure because every its step decreases either $\mathrm{ord}_{x_2}\Delta$ or the dimension of the module T (see (2), (3), (7), (11)-(13)). From this it follows that by ST-transformations the operator ∇ can be reduced to an operator $\tilde{\nabla}$ of the form (1), where $\delta_{x_2}(\Theta_i) = 0$ and $\delta_{x_1}(\Theta_i)$ is isomorphism for any $i = 1, \dots, r$. To obtain $\delta_{x_1}(\Phi_{i,j}) = 0$, $1 \le i < j \le r+1$, it is sufficient to change the operator $\tilde{\nabla}$ to the composition $\tilde{\nabla} \circ F^{-1}$, where

$$F = \begin{pmatrix} \delta_{x_1}(\Theta_1) & \delta_{x_1}(\Phi_{1,2}) & \cdots & \delta_{x_1}(\Phi_{1,r}) & \delta_{x_1}(\Phi_{1,r+1}) \\ 0 & \delta_{x_1}(\Theta_2) & \cdots & \delta_{x_1}(\Phi_{2,r}) & \delta_{x_1}(\Phi_{2,r+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \delta_{x_1}(\Theta_r) & \delta_{x_1}(\Phi_{r,r+1}) \\ 0 & 0 & \cdots & 0 & \mathrm{id}_{Q_{r+1}} \end{pmatrix}$$

is an isomorphism. Thus the conclusion of Theorem about left-invertible operators have proved.

We shall show that the case, when $\nabla(S) \subset \delta_1(S), S \subset T$ and ∇ is a two-sided invertible operator, is impossible. We have

$$\nabla\big|_S : S \longrightarrow \delta_1(S),$$
$$\delta_{x_2}(\nabla\big|_S) = \delta_2\big|_S = 0$$

since $S \subset T, \delta_2(T) = 0$. In addition, $\delta_{x_1}(\nabla\big|_S) = \delta_1\big|_S$ is an isomorphism. Let $q_i, i = 1, \dots, t$, be a basis of S and $\varepsilon_i, i = 1, \dots, t$, be operators from $Diff_0 Q$ such that $\varepsilon_i(1) = q_i, i = 1, \dots, t,$. Then

$$(\delta_1 \circ \varepsilon_i)(1) = \delta_1(q_i), i = 1, \dots, t,$$

is a basis in $\delta_1(S)$,

$$\delta_2 \circ \varepsilon_i = 0, i = 1, \dots, t,$$

and $(\nabla \circ \varepsilon_i)(1) = \nu(\varepsilon_i)(1)$ belongs to $\delta_1(S)$ for $i = 1, \dots, t$. Hence

$$\nu(\varepsilon_i)(1) = \sum_{j=1}^{t} a_{ij}\delta_1(q_j),$$

where $a_{ij} \in A, i = 1, \dots, t$. From here we obtain

$$\begin{aligned} \nabla(\sum_{i=1}^{t} b_i q_i) &= \nabla(\sum_{i=1}^{t} b_i \varepsilon_i(1)) = \nabla(\sum_{i=1}^{t} \varepsilon_i(b_i)) = \sum_{i=1}^{t} (\nabla \circ \varepsilon_i)(b_i) \\ &= \sum_{i=1}^{t} (\nu(\varepsilon_i) + \delta_1 \circ \varepsilon_i \circ \partial_1 + \delta_2 \circ \varepsilon_i \circ \partial_2)(b_i) \\ &= \sum_{i=1}^{t} (b_i \nu(\varepsilon_i)(1) + (\delta_1 \circ \varepsilon_i)(\partial_1(b_i))) \\ &= \sum_{i=1}^{t} [\sum_{j=1}^{t} a_{ij} b_i \delta_1(q_j) + \partial_1(b_i)\delta_1(q_i)]. \end{aligned}$$

Let us take for $b_i, i = 1, \dots, t$, a nonzero solution of the linear system of ordinary differential equations

$$\partial_1(b_i) + \sum_{j=1}^{t} a_{ji} b_j = 0,\ i = 1, \dots, t.$$

Then

$$\nabla(\sum_{i=1}^{t} b_i q_i) = 0,$$

$$\sum_{i=1}^{t} b_i q_i \neq 0$$

since the sections $q_i, i = 1, \dots, t$, are linearly independent in S. From here it follows that $\ker \nabla \neq \{0\}$, but it conflicts with the equality $\Delta \circ \nabla = \mathrm{id}_Q$ for two-sided invertible operators. Thus in this case the situation $\nabla(S) \subset \delta_1(S), S \subset T$ cannot occur. It means that by ST-transformations from Parts 1 and 2 we can achieve the case when $\mathrm{ord}_{x_2}\Delta = 0$ and $\mathrm{ord}\nabla = 1$. From Part 3 and 1 it follows that in this case the operators ∇ and Δ are compositions of triangular operators.▶

Concluding remark

Consider the following problem. For a given differential operator ∇ it is required to find its inversion or to show that the operator is not invertible. From the proof of Theorem it follows that this problem can be solved in the following way.

1) By ST-transformations from Part 1 of the proof the operator ∇ is reduced to an operator that satisfies the conditions (4).

2) To extract the submodule $Q_1 = \ker \delta_2 \subset Q$ and to check the condition $\nabla(Q_1) \subset \delta_1(Q_1)$.

3) If this condition is realized, then it means that the operator ∇ cannot be two-sided invertible. For testing its left-invertibility its component ∇_{22} should be extracted and investigated (see Part 4 of the proof).

4) If $\nabla(Q_1) \not\subset \delta_1(Q_1)$, then by ST-transformations from Part 3 of the proof the number of the components of ∇ with ∂_2 is decreased.

From the proof of Theorem it follows that this succession of steps gives one either an invertible operator of zero order, if the original operator is two-sided invertible, or an operator of the form (1), if it is only left-invertible. To test an operator of the form (1) for the left-invertibility other methods should be used.

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