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Boltzmanngasse 9
A-1090 Wien, Austria

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P. Kargaev
E. Korotyaev

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EFFECTIVE MASSES AND CONFORMAL MAPPINGS

P.Kargaev*, E.Korotyaev**

*Dep.of Math. Anal., Math.-Mech. Faculty, Univ. Bibliotechnaya pl.2, S.Petershof, St.Petersburg, 198904,

**Dep. of Math.2, Electrotech. Univ., ul. prof.Popov 5, St.Petersburg, 197376, Russia

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Abstract. Let $G_n, n \in \mathbf{N}$, denote the set of gaps of the Hill operator. We solve the following problems : 1) find the effective masses $M_n^{\pm}$, 2) compare the effective mass $M_n^{\pm}$ with the length of the gap G_n, and with the height of the corresponding slit on the quasimomentum plane (both with fixed number n and their sums) 3) consider the problems 1), 2) for more general cases (the Dirac operator with periodic coefficients, the Schrödinger operator with a limit periodic potential). To obtain 1)- 3) we use a conformal mapping corresponding to the quasimomentum of the Hill operator or the Dirac operator.

Introduction

Consider the Hill operator $H = -d^2/dt^2 + V(t)$ in $L^2(\mathbf{R})$ where V a is 1-periodic real potential from $L^1(0,1)$. It is well known that the spectrum of H is absolutely continuous and consists of the intervals $S_1, S_2, ...,$ and let

$$S_n = [A_{n-1}^+, A_n^-], ..., A_n^- \le A_n^+ < A_{n+1}^-, \quad , n = 1, 2, ..., A_0^+ = 0 < A_1^-, \quad A_0^- = -\infty.$$

The intervals are separated by the gaps $G_1, G_2, ...,$ where $G_n = (A_n^-, A_n^+)$. If a gap degenerates i.e. $G_n = \emptyset$ then the corresponding segments S_n, S_{n+1} merge. The spectrum of the Hill operator consists of closed non overlapping intervals which are called spectral bands. Instead of the spectral parameter E we introduce more convenient parameter $z, z^2 = E$, and numbers $a_n^{\pm} = \sqrt{A_n^{\pm}} \ge 0$ and gaps

$$g_n = (a_n^-, \ a_n^+), \quad g_{-n} = -g_n, \quad n \in \mathbf{N}, \quad g_0 = \emptyset.$$

Later on g_n will be called a gap and G_n a energy gap. Now we can define a quasimomentum function [11], [2]

$$k(z) = \arccos F(z), \quad z \in Z = \mathbf{C} \setminus \bar{g}, \quad g = \cup g_n,$$

where F is the Lyapunov function of the Hill operator (see Section 5). The function $k(z)$ is analytic and moreover $k(z)$ is a conformal mapping from Z onto a quasimomentum region $K = \mathbf{C} \setminus \cup\Gamma_n$, where Γ_n is an excised slit

$$\Gamma_n = \{\mathrm{Re}k = \pi n, |\mathrm{Im}k| \le h_n\}, \quad h_n = h_{-n} \ge 0, \quad n \in \mathbf{Z}, \quad h_0 = 0.$$

Any non degenerate (degenerate) slit Γ_n is connected in the some way with the non degenerate (degenerate) gap g_n and the energy gap G_n . With an edge of the energy gap G_n, having the length L_n, we associate the effective mass

$$M_0^- = 0, \quad M_0^+ = 1/E''(0), \quad M_n^\pm = 0, \quad \text{if} \quad L_n = 0, \text{ and } M_n^\pm = 1/E''(k(a_n^\pm)), \quad \text{if} \quad L_n \ne 0,$$

where $E(k) = z(k)^2$ and $z(k)$ is the inverse function for $k(z)$. It is well known that if $L_n \ne 0$ then

$$E(k) = A_n^\pm + (k - \pi n)^2(1/2M_n^\pm + \mathrm{o}(1)), \qquad \pm(k - \pi n) \downarrow 0.$$

Now we describe the main purpose of our paper.

Let we have only the set of gaps G_n, $n \in \mathbf{N}$, (or the set of segments S_n, $n \in \mathbf{N}$). Then we solve the following problems ;

a) find the effective masses,

b) compare the effective masses $M_n^\pm$ with the gap length L_n and with the height of the slit h_n (both with fixed number n and their sums), then compare such sums with a norm of the potential V in some space,

c) find asymptotics of $k(z)$ at large z,

d) consider the problems a)-c) for more general cases (the Dirac operator with periodic coefficients, the Schrödinger operator with a limit periodic potential).

The correlation between effective masses $M_n^\pm$, lengths L_n, heights h_n were studied in many articles. Firsova [3] found the relation between $M_n^\pm$, L_n, h_n and the Fourier coefficients of a potential V at large integer n. In [3] it was also shown that the sum of all effective masses is equal to the physical mass. In [2] Firsova has proved the asymptotics $k(z) = z + \mathrm{O}(z^{-1/3})$ as $|z| \to \infty$. Any Hill operator with finite band spectrum was described by explicit formulae in the work of Its, Matveev [5] (including inverse problem). In the book [10] Marchenco had obtained some inequalities between h_n, L_n and asymptotics $k(z)$ at large real $E, E = z^2$, (see also [11]). The main result of the paper [11] by Marchenco and Ostrovski is the solution of the inverse problem. It is shown that under some additional conditions on the slits $\Gamma_n, n \in \mathbf{Z}$, the region K corresponds to a periodic potential of the Hill operator. Later on the inverse problem and some properties of the function $k(z)$ have been considered in the paper of Garnett, Trubowitz [1]. In [8] Korotyaev has studied the propagation of the acoustic waves in a periodic media. It was shown that any spectral band (with number n) "creates" the wave with the velocity U_n(U_n is less than 1). The velocity U_n is equal to the maximum of the function $z'(k(z))$ when z^2 belongs to the energy band with the number n. Furthermore $2M_0^+U_1^2 = 1$ and

M_0^+ may be estimated in terms of the gap lengths and the edges of the bands. In [12] Pastur, Tkachenco have considered the direct and inverse problem for the operator with limit periodic potentials.

Let us write down the main results of the paper.

a) Simple formulae providing the possibility to find effective masses in terms of the edges of gaps $G_n, \quad n \in \mathbf{N}$, are found.

b) "The local estimates" (a number n is fixed) between the effective masses $M_n^{\pm}$, the height of slit h_n and the length of gap L_n are obtained.

c) We derive inequalities which relate the following quantities: the sum of squares (with weights) of the effective masses, the heights of the slits, the gaps lengths and a norm of a potential V in some Sobolev space.

d) Asymptotics of $k(z)$ for large $|z|$ are found.

e) There are some estimates about $U_n, n \in \mathbf{N}$.

f) We obtain the extension of a)-d) for more general cases (the Dirac operator with periodic coefficients, the Schrödinger operator with a limit periodic potential etc.).

It is necessary to note that the asymptotics of $k(z)$ for $E = z^2$ far from an energy gap differs from the case when E belongs some neighborhood of a energy gap.

To prove a)-f) we use a conformal mapping corresponding to quasimomentum of the Hill operator [11], [2] that makes possible to reformulate the problem for the differential operator as a problem of the conformal mapping theory. Thus we should study some "geometric properties" of conformal mappings from $\mathbf{C}_+$ onto"the comb" $K_+ = K \cap \mathbf{C}_+$. For solving these "new" problems we use some techniques from [11], [9] and we often use the Poisson integral for the domain $\mathbf{C}_+ \cup \mathbf{C}_- \cup (-1, 1)$, the Dirichlet integral for a function $k_p(z)$ (the definition of $k_p(z)$ see in Section 1) .In particular the Dirichlet integral for the function $k_0(z) \equiv k(z) - z$. The Dirichlet integral was used in Kargaev's work [6] to study the conformal mapping of the upper half plane to the comb.

1 . The main results

In this section we introduce the concepts and the facts needed to formulate the theorems, some results for the Hill operator, the Dirac operator with periodic coefficients and some results from the conformal mapping theory.

At first we give some definitions and facts from the theory of conformal mappings. We call the set $K_+ = \mathbf{C}_+ \setminus \cup \Gamma_n$ the "comb" where

$$\Gamma_n = \{\mathrm{Re}k = u_n, |\mathrm{Im}k| \leq h_n\}, \quad h_n \geq 0, \quad n \in \mathbf{Z}, \quad h_0 = 0,$$

while u_n is a strongly increasing sequence of real numbers such that $u_n \to \pm\infty$ as $n \to \pm\infty$. We call a conformal mapping $k(z)$ from the upper half plane $\mathbf{C}_+$ onto some comb K_+ a general quasimomentum (GQ) if 1) $k(0) = 0$, 2) $k(iy) = iy(1 + \mathrm{o}(1))$ as $y \to \infty$.

It is well known that a GQ $k(z)$ is a continuous function in $z \in \bar{\mathbf{C}}_+$. In this case we introduce the sets

$$g_n = (a_n^-, a_n^+), \quad s_n = [a_{n-1}^+,\ a_n^-] = k^{-1}([u_{n-1}, u_n]), \quad n \in \mathbf{Z}.$$

We call $\sigma = \cup s_n$ the spectrum of the corresponding general quasimomentum $k(z)$. We also denote by g_n a gap in the spectrum of GQ and we let $g = \cup g_n$. It is well known that the set σ can not be the spectrum of two different GQ [9]. Note that the function $k(z)$ may be continued onto the domain $Z = \mathbf{C} \setminus \bar{g}$ by the formula $k(\bar{z}) = \bar{k}(z),\ z \in Z$. If a gap g_n is empty then the components s_n, s_{n+1} merge. The spectrum σ consists of closed non overlapping intervals $s(n)$ with the lengths $r_n,\ n \in \mathbf{Z}$, and $\sigma = \cup s(n)$ where the point zero belongs to $s(0)$. We denote the length of the gap g_n by l_n. For GQ we introduce "reduced masses" (some analog of the effective masses)

$$\pm\mu_n^\pm = 1/z''(k(a_n^\pm)), \quad \text{if} \quad l_n \neq 0 \quad \text{and} \quad \pm\,\mu_n^\pm = 0, \quad \text{if} \quad l_n = 0.$$

It is clear that $\mu_n^\pm > 0$ if $l_n \neq 0$ and we shall often use the asymptotics

$$z(k) = a_n^\pm \pm (k - u_n)^2(1/2\mu_n^\pm + \mathrm{o}(1)), \quad \pm(k - u_n) \downarrow 0. \tag{1.1}$$

Later on p is an integer. We introduce the functions $u(z) = \mathrm{Re}k(z),\ v(z) = \mathrm{Im}k(z)$,

$$P_p(z) = \sum_0^p Q_{n-1} z^{-n}, k_p(z) = z^p \left\{k(z) - z + P_p(z)\right\}, \quad z \in\ Z, \quad p \geq 0,$$

where

$$Q_p = \frac{1}{\pi}\int x^p v(x)dx, \qquad Q_p^+ = \frac{1}{\pi}\int |x|^p v(x)dx, \quad p \geq -1.$$

Here and below an integral with no limits indicated denotes integration over $\mathbf{R}^d, d \geq 1$. For a non degenerate gap g_n we denote by $r_n^+(r_n^-)$ the distance between g_n and the nearest right (left) hand side non degenerate gap or the point zero. Analogously we denote by $s_n^+(s_n^-)$ the distance between g_n and the nearest right(left) non degenerate gap. Let us introduce the constants

$$\gamma_0 = \sup\ (l_n/\max_\pm s_n^\pm), \quad \text{if} \quad p = 0, \quad \text{and} \quad \gamma_1 = \sup\ (l_n/\max_\pm r_n^\pm), \quad \text{if} \quad p > 0,$$

and $r = \inf r_n^\pm$. We call a general quasimomentum

i) a normed quasimomentum if $Q_{-1}^+ < \infty$ *and* $Q_{-1} = 0$,

ii) a symmetric quasimomentum if $k(-z) = -k(z), \quad z \in Z$,

iii) a quasimomentum if $u_n = \pi n, \quad$ *for all* $n \in \mathbf{Z}$.

Note that for the case $Q_{-1}^+ < \infty$ we can normalize the general quasimomentum by some translation. We emphasize that a symmetric quasimomentum corresponds to the quasimomentum for the Hill operator, a quasimomentum corresponds to the quasimomentum for the Dirac operator with periodic coefficients. Furthermore a GQ is an integrated density of states (or the rotation number) for the Schrödinger operator with some limit periodic potential (see [12]).

We shall tell that GQ $k(z)$ has the moment of an order p if $Q_{2p} < \infty$. By Herglotz Theorem we have that GQ $k(z)$ has the moment of order $p \geq -1$.Later on we assume some conditions on the spectrum (or gaps).

Condition 1. *Let a GQ $k(z)$ have the moment of an order $p \geq 0$, if $p = 0$, then $\gamma_0 < \infty$ and if $p > 0$, then $\gamma_1 < \infty$.*

Condition A.*Let a GQ $k(z)$ have the moment of the order $p \geq 0$,*

i) if $p = 1$ then $k(z)$ is a normed GQ,

ii) if $p \geq 2$ then $k(z)$ is a symmetric quasimomentum.

Let us describe the connection between GQ and the Hill operator. Remember that the spectrum of H consists of the segments $S_n,\ n \in \mathbf{N}$, with the gaps G_n. In the case of the Hill operator the numbers $a_n^\pm$ satisfy $a_n^\pm = \sqrt{A_n^\pm} \geq 0,\quad a_{-n}^\pm = -a_n^\mp,\quad n = 0, 1, 2, 3, ...,$ and gaps g_n satisfy $g_n = (a_n^-, a_n^+),\quad g_{-n} = -g_n,\quad n \in \mathbf{Z},\quad g_0 = \emptyset$. For an energy gap G_n and a gap g_n we have the equality $L_n = A_n^+ - A_n^- = l_n(a_n^+ + a_n^-),\quad n = 1, 2, 3, ...$

The quasimomentum k is defined by $k(z) = \arccos F(z), z \in Z$, where F is the Lyapunov function for the Hill equation

$$-f'' + Vf = z^2 f, \quad z \in \mathbf{C}. \tag{1.2}$$

We note that the set g is symmetric with respect to the point zero and the function $k(-z) = -k(z), z \in Z$. In the case of the Hill operator the following equalities are valid

$$M_0^+ = k'(0)^2/2 = 1/2z'(0)^2, \quad \pm\mu_n^\pm = 2a_n^\pm M_n^\pm, \quad n \geq 1. \tag{1.3}$$

Moreover,for the Hill operator we have (see [10])

$$2Q_0 = \int_0^1 V(t)dt, \quad Q_1 = 0, \quad 8Q_2 = \int_0^1 V(t)^2 dt, ...$$

Let us formulate the main theorem.

Theorem 1.1.*Suppose $V \in L^1(0,1)$ and n=0, 1, 2.... Then*

$$M_{2n}^\pm = 2\sum_{m>0,q=\pm}(A_{2m-1}^q - A_{2n}^\pm)^{-1}, \qquad M_{2n+1}^\pm = 2\sum_{m\geq 0,q=\pm}(A_{2m}^q - A_{2n+1}^\pm)^{-1}, \tag{1.4}$$

$$\frac{1}{\pi}\int |k'(z) - 1|^2 dxdy = 2Q_0, \quad \textit{and} \quad \sum_{n\geq 1}(A_n^+ M_n^+ + A_n^- M_n^-) = Q_0, \quad \textit{if} \quad V \in L^2(0,1). \tag{1.5}$$

Furthermore, let $V \in L^2(0,1)$ and $p = 1$ then

$$\frac{1}{2}\int |(z(k(z) - z))'|^2 dxdy + \int v(x)u(x)xdx = 2\int x^2 v(x)dx = (\pi/4)\int_0^1 V^2(t)dt \tag{1.6}$$

$$and \quad \sum_{n\ge1}[(A_n^+)^2M_n^+ + (A_n^-)^2M_n^-] - Q_0^2/2 = (3/8)\int_0^1 V^2(t)dt, \quad if \quad V\in W_2^1(\mathbf{R}/\mathbf{Z}),$$

and etc. for V *belonging Sobolev space* $W_2^{p-1}(\mathbf{R}/\mathbf{Z})$ *and* $p=2,3,..$*All series converge absolutely.*

Now we present the main inequalities obtained in this paper. We define the Dirichlet integral $\pi d_p = \int |k_p'(z)|^2dxdy,\ z=x+iy$, and the constants $T=(\pi^2/48r^4)T^0\max L_n^2,\ T^0=1+Q_0r^{-2}$. For a sequence $f=\{f_n\}_1^\infty$ or a sequence $f=\{f_n\}_{-\infty}^\infty$, such that $f_{-n}=f_n, n=1,2,...,f_0=0$, we introduce a norm $\|f\|_{\pm,p}^2=\sum_{n>0}(A_n^\pm)^p|f_n|^2$. We have

Theorem 1.2. *a) Let* $V\in L^1(0,1)$. *Then* $r>0$ *and for any* $n\in\mathbf{N}$

$$l_n \le 2h_n \le l_n(1+Tn^{-2}), \tag{1.7}$$

$$l_n \le 2\mu_n^\pm \le l_n(1+Tn^{-2})^2. \tag{1.8}$$

b) Let $V\in L^1(0,1)$ *if* $p=0$ *and* $V\in W_2^{p-1}(\mathbf{R}/\mathbf{Z})$ *if* $p\ge1$. *Then for any* $p\ge0$ *there exist constants* $C_1,C_2,..,C_5$ *depending only on* p,γ_1 (γ_0 *if* $p=0$) *such that*

$$C_1Q_{2p} \le C_2\|L\|_{\pm,p-1}^2 \le C_3\|h\|_{\pm,p}^2 \le \|M^\pm\|_{\pm,p+1}^2 \le C_4d_p \le C_5Q_{2p}. \tag{1.9}$$

The exact representation of $C_1,C_2,..C_5$ will be given in Section 5. We note that in [10] there is the estimate $l_n\le 2h_n\le Cl_n$ for any $n=1,2,..$ and some $C>0$. Some analogs of Theorems 1.1, 1.2 for the Dirac operator with periodic coefficients will be considered in Theorems 1.3 - 1.5.

Let us consider the case of a general quasimomentum. We introduce the function $w_n(x)=|(x-a_n^-)(x-a_n^+)|^{1/2}, x\in\mathbf{R}$. We define numbers $a_n=\max|a_n^\pm|,\ b_n=\min|a_n^\pm|$ and the norm $\|f\|_p^2=\sum a_n^{2p}f_n^2$, with $\|f\|=\|f\|_0$, for a sequence of real numbers $f=\{f_n\}_{-\infty}^\infty$. The following statements hold true.

Theorem 1.3. *Let* $k(z)$ *be a general quasimomentum. Then for any* $n\in\mathbf{Z}$

$$v(x)=w_n(x)\left\{1+\frac{1}{\pi}\int_{\mathbf{R}\setminus g_n}\frac{v(t)dt}{w_n(t)|t-x|}\right\}, \quad x\in g_n, \tag{1.10}$$

$$2\mu_n^\pm = l_n\left\{1+\frac{1}{\pi}\int_{\mathbf{R}\setminus g_n}\frac{v(t)dt}{w_n(t)|t-a_n^\pm|}\right\}^2, \tag{1.11}$$

$$\frac{l_n}{2} \le\ h_n \le \pi\sqrt{\frac{l_n\mu_n^\pm}{2}}\ \le\ \pi\mu_n^\pm, \tag{1.12}$$

$$h_n^2 \le 2l_n\sqrt{\mu_n^+\mu_n^-}, \qquad l_n\le 2\mu_n^\pm. \tag{1.13}$$

At the same time for a general quasimomentum there are some "global estimates". We introduce the quantities

$$\mu_0^2 = \sum_{q=\pm, n\in\mathbf{Z}} \mu_n^q \min(\mu_n^q,\ s_n^q), \quad p=0, \quad \text{and} \quad \mu_p^2 = \sum_{q=\pm, n\in\mathbf{Z}} (a_n^q)^{2p} \mu_n^q \min(\mu_n^q,\ r_n^q), \quad p>0.$$

Let us present the theorem.

Theorem 1.4. *Let a GQ $k(z)$ have the moment of the order $p \geq 0$ and satisfy Condition A and Condition 1. Then there exist constants $C_1, C_2, .., C_5$ depending only on p and γ_1 (γ_0, if $p=0$), such that*

$$C_1\ \|l\|_p^2 \leq C_2\ \|h\|_p^2 \leq\ \mu_p^2 \leq C_3\ d_p \leq C_4\ Q_{2p} \leq C_5\ \|l\|_p^2.$$

Let us finally formulate now some equalities concerning a GQ and a quasimomentum(the Dirac operator).

Theorem 1.5. *Let $k(z)$ be a general quasimomentum.*

1) Suppose $\gamma_0 < \infty$, $\inf_{n,\pm}(b_n s_n^\pm) > 0$ and $\sum_{l_n \neq 0} b_n^{-2} < \infty$. Then

$$k'(z)^2 = 1 + \frac{1}{2}\sum\Big(\frac{\mu_n^+}{z-a_n^+} - \frac{\mu_n^-}{z-a_n^-}\Big), \quad z \in Z, \tag{1.14}$$

the series converges absolutely and uniformly on compact sets.

2) Suppose $\inf_{n,\pm} s_n^\pm > 0$ and $Q_p^+ < \infty$ for some $p \geq 0$. Then

$$4pQ_{p-1} + 2\sum_0^{p-3}(n+1)(p-2-n)Q_nQ_{p-3-n} = \sum_n(\mu_n^+(a_n^+)^p - \mu_n^-(a_n^-)^p). \tag{1.15}$$

and the series converges absolutely.

3) Suppose $Q_{2p} < \infty$ for some $p \geq 0$. Then

$$d_p/2 = (1+p)Q_{2p} - \frac{p}{\pi}\int x^{2p-1}u(x)v(x)dx - \sum_0^{p-1}(p-1-n)Q_nQ_{2p-2-n}. \tag{1.16}$$

4) Let $k(z)$ be a quasimomentum . Then for any $n \in \mathbf{Z}$ we have

$$\pm\mu_{2n}^\pm = 2V.P. \sum_{m\in\mathbf{Z}, q=\pm} \frac{1}{a_{2m+1}^q - a_{2n}^\pm}, \tag{1.17}$$

$$\pm\mu_{2n+1}^\pm = 2V.P. \sum_{m\in\mathbf{Z}, q=\pm} \frac{1}{a_{2m}^q - a_{2n+1}^\pm}.$$

We note that from (1.15) we have the equality $\sum(\mu_n^+ - \mu_n^-) = 0$.

2 . The local properties of the quasimomentum

In this chapter useful results will be presented. The main attention will be given to the analysis of the function $v(z)$. It is well-known that for any GQ $k = u + iv$ the function $u'_x(z) > 0, z = x+iy \in \mathbf{C}_+$ (see [9]). Hence there are two positive functions $v(z), z \in \mathbf{C} \setminus \sigma$ and $u'_x(z), z \in Z$. From the Herglotz theorem we have

$$v(z) = y(1 + \frac{1}{\pi} \int \frac{v(t)}{|t-z|^2} dt), \quad z \in \mathbf{C}_+, \tag{2.1}$$

therefore $u'_x(iy) = v'_y(iy)$ and

$$v'_y(iy) = 1 + \frac{1}{\pi} \int \frac{(t^2 - y^2)v(t)}{(t^2+y^2)^2} dt \leq 1 + \frac{1}{\pi} \int \frac{v(t)dt}{(t^2+y^2)} = 1 + o(1), \quad y \to \infty.$$

Proving some estimates in this chapter we use positive harmonic functions v, u'_x and asymptotics $v(iy) = y(1+o(1)), \quad u'_x(iy) = (1+o(1)), \quad y \to \infty$.

At first we shall consider harmonic functions in a domain $D(I) = \mathbf{C} \setminus \overline{(\mathbf{R} \setminus I)}$, where I is a closed interval. The word "local" is means that some properties are obtained as result that the function v (or u'_x) is positive and harmonic in a region $D(\bar{g_n})(D(s_n))$. Introduce the set $U = \{z : |z| < 1\}$. There is the Lemma

Lemma 2.1. *Let a function f be harmonic and positive in the domain $D = D(I), \ I = [-a, a], \ a > 0$. Then*

1. If $f(x)^2 = (a-x)(2\mu_+ + o(1))$, as $x \uparrow a$, then

$$f(x)^2 \leq \frac{(2a)(2\mu_+)(a-x)}{a+x}, \quad x \in I. \tag{2.2}$$

2.If $2(a-x)f(x)^2 = \mu_+ + o(1))$, as $x \uparrow a$, then

$$\mu_+ \leq \frac{2(2a)f(x)^2(a-x)}{a+x}, \quad x \in I. \tag{2.3}$$

3. Let $f(z) = f(\bar{z}), \ z \in D$. Suppose $f \in C(\bar{\mathbf{C}_+} \setminus \{t_n, n \in \mathbf{Z}\})$ where the sequence $\{t_n\}_{-\infty}^{\infty}$, such that $t_n \to \pm\infty$ as $n \to \pm\infty$ and $yf(t_n + iy) = o(1)$ as $y \to 0$. Then

$$f(x) = \sqrt{a^2 - x^2}(\beta + \frac{1}{\pi} \int_{\mathbf{R} \setminus I} \frac{f(t)dt}{|t-x|\sqrt{t^2-a^2}}), \quad x \in I, \tag{2.4}$$

$$\lim_{x \uparrow a} \frac{f(x)}{\sqrt{a-x}} = \sqrt{2a}(\beta + \frac{1}{\pi} \int_{\mathbf{R} \setminus I} \frac{f(t)dt}{|t-a|\sqrt{t^2-a^2}}), \tag{2.5}$$

where $\beta = \lim f(iy)/y$, as $y \to \infty$.

Proof. Take any $x \in I$. Let $W = W(z)$ be a conformal mapping from the region D onto the disk U. The function W is defined by conditions $W(x) = 0,\ W'(x) > 0$. Such mapping may be got by the composition of mappings

$$z_1 = \frac{b(z) - i}{b(z) + i}, \quad b(z) = \sqrt{\frac{z + a}{z - a}}, \quad z \in D, \quad W = \frac{z_1 - z_1(x)}{1 - z_1(x) z_1}, \quad z_1 \in U.$$

(here $\sqrt{1 + 0i} = 1$). Define the function f_1 from the equality $f_1(W(z)) = f(z),\ z \in D$. Using the Harnack inequality for the positive harmonic function f_1 we obtain

$$\frac{1 - r}{1 + r} f_1(0) \le f_1(r) \le \frac{1 + r}{1 - r} f_1(0), \quad 0 \le r < 1,$$

and hence

$$\frac{b(x) f(x)}{b(t)} \le f(t) \le \frac{b(t) f(x)}{b(x)}, \quad x \le t < a. \tag{2.6}$$

We rewrite the left hand side of (2.6) in the form

$$f(x) \le \frac{f(t)}{\sqrt{a - t}} \sqrt{\frac{(a - x)(a + t)}{a + x}}, \quad x \le t < a.$$

From this, as $t \uparrow a$, we get (2.2). Using the right hand side of (2.6) we obtain (2.3) by analogy.

The function $b(z)$ maps conformally the region D onto the upper half plane. For $-a < x < a,\ t < -a$ or $t > a$ we have the equalities

$$\mathrm{Im} \frac{1}{b(t) - b(x)} = \mathrm{Im} \frac{b(t) + b(x)}{b^2(t) - b^2(x)} = \frac{(t - a)\sqrt{a^2 - x^2}}{2a(t - x)}, \quad b'(t) = -\frac{a}{b(t)(t - a)^2}.$$

From here, using the property $f(z) = f(\bar{z}),\ z \in D$, we get the kernel of the Poisson integral for the domain D and hence (2.4).

By (2.4) we have (2.5). Q.E.D.

We have useful Corollary from Lemma 2.1.

Corollary 2.2. *Let function f be nonnegative, harmonic in the domain $D = D(I),\ I = [-a, a],\ a > 0$. Suppose $f(x)^2 = (a \pm x)(2\mu_\pm + o(1))$, as $\mp x \uparrow a$, then*

$$a f(x)^2 \le (\sqrt{\mu_+} + \sqrt{\mu_-})^2 (a^2 - x^2) \le 2(\mu_+ + \mu_-)(a^2 - x^2), \quad -a < x < a, \tag{2.7}$$

$$f(x)^2 \le 4a\sqrt{\mu_+ \mu_-}, \quad -a < x < a. \tag{2.8}$$

Proof. By (2.2)

$$\frac{f(x)^2}{a^2 - x^2} \le (2a) 2 \min\left\{ \frac{\mu_+}{(a - x)^2}, \frac{\mu_-}{(a + x)^2} \right\} \le \frac{(\sqrt{\mu_+} + \sqrt{\mu_-})^2}{a},$$

$-a < x < a$. Multiplying inequalities (2.2) for $\mu_\pm$ we obtain (2.8). Q.E.D.

Now we shall apply priveous results for GQ. Instead of a function f we shall use the functions $u'_x(z), z \in D(s_n)$, and $v(z), z \in D(\bar{g_n})$. In the case of a general quasimomentum we have asymptotics of $k(z)$ on any gap and band. For this case we have

Theorem 2.3. *Let k be a GQ. Then the statements (1.10)-(1.13) are valid. Furthermore*

$$w_n(x) \le v(x) \le \sqrt{2l_n\mu_n^\pm}\; w_n(x)/|x - a_n^\mp|, \quad x \in g_n, \tag{2.9}$$

$$l_n v(x)^2 \le 2(\sqrt{\mu_+} + \sqrt{\mu_-})^2 w_n(x)^2, \quad x \in g_n. \tag{2.10}$$

Let in addition $Q_0 < \infty$ and $\inf s_n^\pm \equiv s > 0$. *Then*

$$|\mu_n^+ - \mu_n^-| \le l_n^2\; Q_0(1 + Q_0/s^2)/s^3. \tag{2.11}$$

Proof of estimates (1.10), (1.11), (2.9), (2.10) follows immediately from the Theorem 2.1, the Corollary 2.2.

Multiplying (2.9) at $\mu_\pm$ we obtain the bound for h_n in (1.13), and by (1.11) we have last estimate in (1.13).

First inequality in (1.12) follows from (1.10). Let us prove the second inequality in (1.12). Integrating $v(x)$ on g_n, using (2.9) and the convexity of the function $v(x),\ x \in g_n$ we have

$$l_n h_n \le 2\int_{g_n} v(x)dx \le 2\sqrt{2l_n\mu_n^\pm}\int_{g_n} w_n(x)/|x - a_n^\pm|dx = \sqrt{2l_n\mu_n^\pm}\pi l_n.$$

Introduce

$$J_n^\pm = 1 + \frac{1}{\pi}\int_{\mathbf{R}\setminus g_n} \frac{v(t)dt}{w_n(t)|t - a_n^\pm|},$$

By (1.11) we have $2(\mu_n^- - \mu_n^+) = l_n(J_n^+ - J_n^-)(J_n^+ + J_n^-)$,

$$J_n^+ - J_n^- = \frac{l_n}{\pi}\int_{\mathbf{R}\setminus g_n} \frac{v(t)\mathrm{sign}(2t - a_n^+ - a_n^-)dt}{w_n(t)^3},$$

and hence (2.11).Q.E.D.

Now we present the result about the behavior of a general quasimomentum on the spectrum.

Theorem 2.4. *Let $S = [a_+, a_-]$ be a spectral component of GQ $k = u + iv$ and $\mu_\pm$ be the corresponding reduce masses. Then $u'_x(z)$ is a positive harmonic function in the domain $D(S)$ and*

$$u'_x(x) = 1 + \frac{1}{\pi}\int \frac{v(t)dt}{(t - x)^2} = \frac{f(x)}{\pi}\int_{\mathbf{R}\setminus S} \frac{u'_t(t)dt}{f(t)|t - x)|}, \quad x \in S, \tag{2.12}$$

$$\mu_{\pm} \le 2(u'_x(x))^2|S||x-a_{\pm}|/|x-a_{\mp}|, \qquad x \in S, \tag{2.13}$$

where $f(x) = |(x-a_-)(x-a_+)|^{1/2}$. *If* k *is a quasimomentum then*

$$\mu_{\pm}|S| \le 8n^2, \tag{2.14}$$

where n *is the number of the merged components which are composed the band* S.

Proof. The estimations (2.12), (2.13) follows from the Lemma 2.1. By (2.13) we have

$$\sqrt{\frac{\mu_{\pm}|x-a_{\mp}|}{2|S||x-a_{\pm}|}} \le u'_x(x), \quad x \in S.$$

Integrating it on S we obtain (2.14). Q.E.D.

Later on we shall need following results on the function v.

Lemma 2.5. *Let* k *be a GQ and* $z \in \mathbf{C}_+$. *Then*

$$k(z) = z + C + \frac{1}{\pi}\int v(t)\Big(\frac{1}{(t-z)} - \frac{t}{1+t^2}\Big)dt, \tag{2.15}$$

$$C = -\frac{1}{\pi}\int \frac{v(t)dt}{t(1+t^2)}.$$

If in addition $g = (a,b)$ *be a gap in the spectrum of a GQ and* $l = |g|$. *Then*

$$\int_g v(t)\Big(\frac{1}{(t-a)} + \frac{1}{(b-t)}\Big)dt = l\Big(\pi + \int_{\mathbf{R}\setminus g} \frac{v(t)dt}{(t-a)(t-b)}\Big). \tag{2.16}$$

$$2lv(x) \le 4\int_g v(t)dt \le l^2\Big(\pi + \int_{\mathbf{R}\setminus g} \frac{v(t)dt}{(t-a)(t-b)}\Big), \quad a < x < b. \tag{2.17}$$

Suppose that $Q_p^+ < \infty, \quad p \ge 0$ *then*

$$k_p(z) = \frac{1}{\pi}\int \frac{t^p v(t)}{t-z}dt, \qquad z \in Z. \tag{2.18}$$

Proof. We have (2.15) in the work [10]. Using $k(a) = k(b)$ and (2.15) we obtain (2.16). By $(t-a)^{-1} + (b-t)^{-1} \ge 4/l, \quad a < t < b$, and (2.16) and by the convexity of $v(t), a < t < b$, we have (2.17).

We rewrite (2.15) in the form

$$k(z) - z - Q_{-1} = \frac{1}{\pi}\int \frac{v(t)}{t-z}dt = \frac{1}{\pi z^p}\int \frac{(z^p - t^p)v(t)}{t-z}dt + \frac{1}{\pi z^p}\int \frac{t^p v(t)}{t-z}dt.$$

Hence by definition k_p we obtain (2.18). Q.E.D.

Later on we need some estimates.

Lemma 2.6. *Let a function f be analytic in the domain $D = \{|\mathrm{Re}z| < a\}, \quad a > 0$. Then for any $\pm\, t \in [0, a)$ we have*

$$\frac{16}{\pi}|f'(t)|^2 \leq \frac{1}{a^2\cos^2\frac{\pi t}{2a}}\int_D |f'(z)|^2 dxdy \leq \frac{1}{(a\pm t)^2}\int_D |f'(z)|^2 dxdy. \qquad (2.19)$$

Proof. Map the region D on the disk U by the function

$$b(z) = \frac{j(z) - j(t)}{j(z) + j(-t)}, \quad j(z) = \exp(\frac{\pi z}{2a}), \quad z \in D.$$

Then

$$b(t) = 0, \quad |b'(t)| = \frac{\pi}{4a\cos\frac{\pi t}{2a}} = \frac{\pi}{4a\sin\frac{\pi(a-t)}{2a}}.$$

Define the function f_1 by the relation $f_1(b(z)) = f(z),\ z \in D$. For the function $f_1(z_1), |z_1| < 1$, there exists the usual estimate

$$\pi|f_1'(t)|^2 \leq \int_U |f_1'(z_1)|^2 dx_1 dy_1, \quad z_1 = x_1 + iy_1.$$

Combining this with the inequality $\pi\sin t \geq 2t, \quad \pi \geq 2t \geq 0$, and with the equality

$$\int_U |f_1'(z_1)|^2 dx_1 dy_1 = \int_D |f'(z)|^2 dxdy.$$

we obtain (2.19). Q.E.D.

Now we present the main "local" results. We shall estimate a reduced mass through the Dirichlet integral from the GQ on some domain. Introduce the constant

$$A_p = \left\{\pi 2^p(1+p)(1+\sqrt{1+\frac{1}{2(1+p)^2}})\right\}^2,$$

and the integrals

$$I_p^2(D) = \frac{1}{\pi}\int_D |k_p'(z)|^2 dxdy.$$

and "the normalized integral"

$$j_p^q(D) = \frac{\pi 2^p I_p(D)}{4|a_q|^p(1+p)}, \quad D = \{a_+ < \mathrm{Re}z < a_-\}, \quad q = \pm.$$

We have

Theorem 2.7. *Let a GQ k satisfy the Condition A for some $p \geq 0$. Suppose an interval $S = (a_+, a_-)$ lies in some spectral band of k. Let $\mu_\pm$ be a corresponding reduced mass if $a_\pm$ coincides with the edge of the band and $s = |S|, \quad D = \{a_+ < \mathrm{Re}z < a_-\}$.*

1) Let $0 \notin S$, then

$$s\mu_q \le 8(1+p)^2 j_p^q(D)(s+j_p^q(D)), \quad q=\pm, \tag{2.20}$$

$$(a_q)^{2p}\mu_q \ \min(\mu_q, s) \le A_p I_p^2(D), \quad q=\pm, \tag{2.21}$$

$$(a_q)^{2p}(\mu_q)^2 \le A_p I_p^2(D), \ \ q=\pm, \ \ \text{if} \ \ s=\infty. \tag{2.22}$$

2) Let $p=0$. Then

$$\mu_q \min(\mu_q, s) \le A_0 I_0^2(D), \quad q=\pm. \tag{2.23}$$

3) Let $p \ge 1$ and $0 \in S$. Then

$$(a_q)^{2p}\mu_q \min(\mu_q, |a_q|) \le A_p I_p^2(D), \quad q=\pm. \tag{2.24}$$

Proof. We consider the case $S \subset \mathbf{R}_+$, the case $S \subset \mathbf{R}_-$ is considered by analogy. From the definition of k_p we have $k(z) = z - P_p(z) + z^{-p}k_p(z), \ z \in Z$. We obtain estimates for $x > 0, \ p \ge 2$ (the case $p = 0, 1$ is more simple)

$$0 < k'(x) = 1 - P_p'(x) + x^{-p}k_p'(x) - px^{-p-1}k_p(x) =$$
$$[1+p+x^{-p}k_p'(x)] - \frac{p}{x}[P_p(x) + \frac{x}{p}P_p'(x) + k(x)] \le 1+p+x^{-p}k_p'(x),$$

because $P_p(x) + \frac{x}{p}P_p'(x) > 0, \ \ k(x) > 0$ as $x > 0$. Hence we have

$$0 < k'(x) < 1+p+x^{-p}k_p'(x), \quad x \in S \subset \mathbf{R}_+. \tag{2.25}$$

Let $2b = a_- + a_+$, $2a = s$ and $x = b + t$. By (2.13), (2.25), (2.19) we obtain for $0 < t < a, \ c = 4(1+p)^2$,

$$(a+t)\mu_- \le 2s(a-t)u'(x)^2 \le 2s(a-t)[1+p+x^{-p}k_p'(x)]^2 \le$$
$$2s(a-t)[1+p+\frac{\pi I_p(D)}{4(a-t)b^p}]^2 \le \frac{sc(a-t)}{2}[1+\frac{j_p^-}{a-t}]^2.$$

The function

$$f(t) = \frac{(a-t)}{(a+t)}[1+\frac{j_p^-}{a-t}]^2, \ \ 0 < t < a,$$

has the minimum in the point $t_0 = a^2/(a+j_p^-)$ and $f(t_0) = j_p^-(j_p^- + 2a)/a^2$. Hence we have (2.20) for μ_-.

Consider two cases. 1). Let $\mu_- < s$. Then

$$\mu_-/j_p^- \le 2c(1+j_p^-/s) \le 2c(1+j_p^-/\mu_-).$$

For $R = j_p^-/\mu_-$ we obtain an inequality $R \leq 2c(1+1/R)$, which is truth under the condition $R < R_1 = c(1+\sqrt{1+2/c})$, i.e.

$$\mu_- \leq R_1 j_p^- \quad \text{if} \quad \mu_- < s. \tag{2.26}$$

2). Let $\mu_- \geq s$. Then

$$\mu_- s/(j_p^-)^2 \leq 2c(1+s/j_p^-) \leq 2c(1+\sqrt{s\mu_-}/j_p^-).$$

By analogy we obtain

$$\mu_- s \leq R_1^2 (j_p^-)^2, \qquad \mu_- \geq s. \tag{2.27}$$

Uniting (2.26), (2.27) we have got (2.21) for μ_-. In the case μ_+ we have

$$(a-t)\mu_+ \leq \frac{c(a+t)}{2}[1+\frac{j_p^+}{a+t}]^2, \quad -a < t < 0.$$

Repeating the proof for μ_- we obtain (2.21) for μ_+. From (2.21) for the interval $(a_+, a_+ + 2\mu_+)$ we obtain (2.22) for the case $q = +$. The case $q = -$ is proved by analogy.

2) The estimate (2.23) follows from (2.21) and from invariance (2.21) under translations.

3) Applying (2.21) for the intervals $(a_+, 0), (0, a_-)$ we have (2.24). Q.E.D.

Now we shall present the more exact result about the reduced masses for the quasimomentum. Define constants $h_+ = \sup h_n,\ \ l_+ = \sup l_n,\ \tau_0 = \pi/4(1+2\gamma_0)$. The function $f(t) = (2l_+/\pi t)\log\cot[(1-t)\tau_0],\ 0 < t < 1$, has the minimum at some point and denote such point by t^0. Later on we shall need the constants

$$\tau = [\cot(1-t^0)\tau_0]^{\frac{2l_+}{\pi t^0}}, \quad \nu = \frac{1}{8}(\tau^2 - \tau^{-2}).$$

The following statements hold true.

Theorem 2.8. *Let k be a quasimomentum. Then for any $q = \pm, n \in \mathbf{Z}$, we have*

$$h_+ \leq \log\tau, \tag{2.28}$$

$$\mu_n^q \leq \sinh h_+ \leq (\tau - \tau^{-1})/2, \tag{2.29}$$

$$|s_n| \geq 2\arcsin\frac{1}{\cosh h_+} \geq \frac{2}{\cosh h_+} \leq \frac{2}{\tau}, \tag{2.30}$$

$$\mu_n^q \leq \nu \inf_m |s_m|, \tag{2.31}$$

$$\gamma_0 \leq \sup_{q=\pm,n}(\frac{2\mu_n^q}{\max_\pm s_n^\pm}) \leq \frac{\sinh 2h_+}{2} \leq 2\nu. \tag{2.32}$$

Proof. The estimate (2.28) follows from (3.1).

Increase all slits $\Gamma_n, n \in \mathbf{Z}$, including degenerate until the height h_+. We obtain a new comb and a new quasimomentum k_1. From the Theorem 3.2 it follows that the reduced masses increase and the lengths of the bands decrease. It is very important that new reduced masses and the new lengths of the bands do not depend from number n. Denote the corresponding reduced masses by μ and the lengths of the bands by s. It is necessary to find μ, s. The Lyapunov function for k_1 has the form (see [9]) $F_1(z) = b\cos z = \cos k_1, \quad b = \cosh h_+$. From this formula it is easy to obtain the reduced mass in the point x_1 where $F_1(x_1) = 1 = b\cos x_1$:

$$\mu_n^{\pm} \leq \mu = -F_1'(x_1)/F_1(x_1) = b\sin x_1/b\cos x_1 = \sqrt{b^2-1} = \sinh h_+, \quad n \in \mathbf{Z}. \tag{2.33}$$

From this inequality and from (2.28) we obtain

$$2\mu_n^q \leq \exp(h_+) - \exp(h_+) \leq \tau - \tau^{-1}.$$

There are the equalities $\sin(\pi/2 - x_1) = \cos x_1 = 1/b$. From this it follows that

$$s = 2(\pi/2 - x_1) = 2\arcsin 1/b = 2\arcsin\frac{1}{\cosh h_+}.$$

Hence from the inequality $\arcsin t \geq t, 1 \geq t \geq 0$, we have

$$|s_n| \geq s = 2\arcsin\frac{1}{\cosh h_+} \geq \frac{2}{\cosh h_+} \geq \frac{2}{\tau}. \tag{2.34}$$

By (2.28), (2.34), (2.33) we obtain

$$8\frac{\mu_m^q}{|s_n|} \leq 2\sinh 2h_+ \leq \tau^2 - \tau^{-2} = 8\nu, \quad n, m \in \mathbf{Z}.$$

Hence, from (1.13), (2.33), (2.34) it follows that

$$\gamma_0 \leq \sup_n\left(\frac{l_n}{\max_\pm s_n^{\pm}}\right) \leq \sup_n\left(\frac{2\mu_n^{\pm}}{s}\right) \leq \frac{2\mu}{s} \leq \frac{\sinh 2h_+}{2} \leq 2\nu. \quad \mathbf{Q.E.D.}$$

Now we shall present the main result on the reduced masses in the case of a quasimomentum. We introduce the constant

$$B_p = \left\{\pi 2^p(1+p)\left(1+\sqrt{1+\frac{\nu}{2(1+p)^2}}\right)\right\}^2, p \geq 0.$$

Theorem 2.9. *Let a quasimomentum k satisfy the Condition A for some $p \geq 0$ and $D_n^- = \left\{a_{n-1}^+ < \mathrm{Re} z < a_n^-\right\}, D_n^+ = \left\{a_n^+ < \mathrm{Re} z < a_{n-1}^-\right\}, \ n \in \mathbf{Z}$. Then*

$$(|a_n^q|^p \mu_n^q)^2 \leq B_p I_p^2(D_n^q), \quad q = \pm. \tag{2.35}$$

Proof. We consider $q = -$. By (2.20), (2.31)

$$\mu_n^- s \le 8(1+p)^2 j(s+j), \quad \mu_n^- \le \nu s, \quad s = |s_n|, \quad j = \frac{\pi 2^p I_p(D_n^q)}{4|a_n^q|^p(1+p)}.$$

Hence $\mu_n^- \le s^0\nu$, where s^0 is the decision of the equation $8(1+p)^2 j(s+j) = \nu s^2$. It is easy to find

$$\nu s^0 = 4(1+p)^2 j B_p^{1/2}/2^p\pi = B_p^{1/2} I_p(D_n^-)/(a_n^-)^p.$$

The case $q = +$ is consider by analogy. Q.E.D.

3 . The identities and the "integral" estimates

In this chapter we shall present results about "global" properties of a general quasimomentum. Some of them we shall obtain using the previous proposals. We have

Theorem 3.1. *Let the set σ be such that $l_+ < \infty$, the point zero lies inside σ and $\gamma_0 < \infty$. Then σ is the spectrum of some GQ and*

$$h_+ \le \log \tau. \tag{3.1}$$

Proof. Suppose that I is arbitrary, fixed closed interval and $|I| > 2l_+$. Any gap, intersecting with I (but excluding two extreme gaps) lies in I together with the neighboring bands. Then

$$2\gamma_0|I\cap\sigma| + |I\cap\sigma| + 2l_+ \ \ge |I|. \tag{3.2}$$

First term on the left hand side estimates the sum of lengths of "inner" gaps. We take $a|I| = 2l_+$ where $a > 0$ and enough small. From (3.2) it follows that

$$(1+2\gamma_0)|I\cap\sigma| \ge (|I| - 2al_+) = (1-a)|I|.$$

We are needed the following facts (see [4], [9]).

Let $\mathcal{S}$ be a closed subset of a real axis such that for some values $L < \infty$ and $\delta > 0$ the Lebesgue measure of the intersection of $\mathcal{S}$ and any interval of length $2L$ is not less then δ. Then there exists the unique function $v(z)$ which is harmonic in the domain $\mathbf{C}\setminus\mathcal{S}$ and has the following properties:

i) a.e. on $\mathcal{S}$ the function $v(z)$ has zero limit values,

ii) for every $z \in \mathbf{C}$, $\quad 0 \le v(z) - |\mathrm{Im} z| \le \frac{L}{\pi}\log\cot\frac{\delta\pi}{4L}$.

We take $L = 2l_+/a, \delta = (1-a)2l_+/a(1+2\gamma_0)$ and $o < a < 1$. From last inequality and from Levin's work [9] we obtain that σ is the spectrum of GQ and we have (3.1).Q.E.D.

Now we shall prove the simple variational inequalities for effective masses (reduced masses).

Theorem 3.2. *Let $k_m(z)$ be GQ, $m = 1, 2$.*

1). Suppose that $u_{m,n} = u_n,\ m = 1,2,$ *and* $h_{1,n} \le h_{2,n}$ *for any* $n \in \mathbf{Z}$*. Then*

$$|s_{1,n}| \le |s_{2,n}|, \qquad \mu^{\pm}_{1,n} \le \mu^{\pm}_{2,n}. \tag{3.3}$$

2). Suppose that $s_{1,n} \subseteq s_{2,n}$ *for any* $n \in \mathbf{Z}$ *and* $a^+_{1,N} = a^+_{2,N}$ *for some* $N \in \mathbf{Z}$*. Then*

$$\mu^+_{1,N} \le \mu^+_{2,N}. \tag{3.4}$$

Proof. 1). Introduce the function $f(z) = \mathrm{Im}(z_1(k_2(z)))$. This function is harmonic, nonnegative in $\mathbf{C}_+$ and continuous in $\bar{\mathbf{C}}_+$. Suppose the inequality

$$f(z) \ge \mathrm{Im}(z_2(k_2(z))) = y, \quad z = x + iy,\ y > 0. \tag{3.5}$$

Then Im $z_1 \ge$ Im z_2 in the domain $k_2(\mathbf{C}_+)$ and

$$z_1'(u) = \frac{\partial}{\partial v}\mathrm{Im} z_1(u) \ge \frac{\partial}{\partial v}\mathrm{Im} z_2(u) = z_2'(u),\ u \in \mathbf{R}, u \ne u_n.$$

From this it follows the proposal of 1) because

$$|s_{m,n}| = \int_{u_{n-1}}^{u_n} z_m'(u)du,\ m = 1,2,\ n \in \mathbf{Z},$$

$$z_m(k) = a^{\pm}_{m,n} \pm (k - u_n)^2(1/2\mu^{\pm}_{m,n} + o(1)), \qquad \pm(k - u_n) \downarrow 0.$$

From the representation (2.15) we obtain that

$$k_m(z) = z(1 + o(1)), \quad z \in U(A) = \{z : y > A|x|\}, \quad |z| \to \infty.$$

But for any A there exists a constant $R = R(A) > 0$ such that $k_m(U(A)) \supset \{z : |z| > R\} \cap U(2A),\quad m = 1,2$. Hence $z_m(k) = k(1 + o(1)),\quad k \in U(2A),\quad |k| \to \infty$, and

$$z_1(k_2(iy))/iy = [z_1(k_2(iy))/(k_2(iy))][k_2(iy))/iy] \to 1, \quad \text{as} \quad y \to \infty.$$

From this it follows that $f(iy) = y(1 + o(1))$, as $y \to \infty$, and using the Herglotz theorem we obtain (3.5).

2). From the Phragmen-Lindelof theorem (for our case see [9]) we have the inequality $v_1(x) \le v_2(x),\quad x \in \mathbf{R}$. Then from the definition of the reduced mass we obtain

$$\mu^+_{1,N} = \lim_{x \uparrow a^+_{1,N}} \frac{v_1(x)^2}{2(a^+_{1,N} - x)} \le \lim_{x \uparrow a^+_{1,N}} \frac{v_2(x)^2}{2(a^+_{1,N} - x)} = \mu^+_{2,N}. \qquad \textbf{Q.E.D.}$$

Lemma 3.3. *Let* $k(z)$ *be a GQ.*

1). Suppose that $Q_0 < \infty$*. For* $t > 0, t \ne |a^{\pm}_n|,\ n \in \mathbf{Z}$*, introduce the functions*

$$S(t,z) = \frac{1}{2}\sum_{|a^{\pm}_n| < t} [\frac{\mu^+_n}{z - a^+_n} - \frac{\mu^-_n}{z - a^-_n}], \qquad f^2(t) = \frac{1}{\pi}\int_0^{2\pi} |k'(t\exp(i\varphi)) - 1|^2 d\varphi.$$

Then

$$\int_0^\infty tf^2(t)dt = d_0 < \infty, \tag{3.6}$$

$$|k'(z)^2 - 1 - S(t,z)| \le \frac{\pi t(f^2(t) + 2^{3/2}f(t))}{t-|z|}, |z| < t. \tag{3.7}$$

2). Let in addition $\gamma_0 < \infty$, $R \equiv \inf b_n r_n^\pm > 0$, *and* $\sum_{l_n \neq 0} b_n^{-2} < \infty$. *Then*

$$\sum_{n\neq 0} \frac{\mu_n^\pm}{|a_n^\pm|} < \infty. \tag{3.8}$$

Proof. From the Cauchy theorem about residues we obtain the equality

$$k'(z)^2 - 1 - S(t,z) = \frac{1}{2\pi i}\int_{|a|=t} \frac{k'(a)^2-1}{a-z}da,$$

and the inequality

$$|\int_{|a|=t} \frac{(k'(a)-1)^2 + 2(k'(a)-1)}{a-z}da| \le \frac{\pi t(f^2(t)+2^{3/2}f(t))}{t-|z|}.$$

and by (1.16) we have (3.6).

We have inequalities

$$(\sum_{n\neq 0, \mu_n^\pm < r_n^\pm} \frac{\mu_n^\pm}{|a_n^\pm|})^2 \le 2(\sum_{l_n \neq 0} b_n^{-2}) \ (\sum_{\mu_n^\pm < r_n^\pm} \mu_n^{\pm 2}), \quad \sum_{n\neq 0, \mu_n^\pm \ge r_n^\pm} \frac{\mu_n^\pm}{|a_n^\pm|} \le \sum_{\mu_n^\pm \ge r_n^\pm} \frac{\mu_n^\pm r_n^\pm}{R}.$$

From these inequalities and from the Theorem 1.4 we obtain the convergence (3.8). Q.E.D.

Now we prove the formulae for the reduced masses in the case of a quasimomentum and some equalities.

Proof of the Theorem 1.5. 1) From (3.6) it follows that we can take the sequence $\{t_n\}_1^\infty$ such that $t_n \to \infty$, $f(t_n) \to 0$ as $n \to \infty$. From this and from (3.7) we obtain (1.14).

2) The definition of k_p and its represention (2.18) result in following asymptotics

$$k'(iy) = 1 + \sum_0^p Q_{m-1} m (iy)^{-1-m} + O(y^{-2-p}), \quad y \to \infty. \tag{3.9}$$

Then for each term of the series in (1.14) we have

$$\frac{\mu_n^+}{z-a_n^+} - \frac{\mu_n^-}{z-a_n^-} = \sum_{m=0}^p [\mu_n^+(a_n^+)^m - \mu_n^-(a_n^-)^m] z^{-1-m} + F_n(z) z^{-1-p},$$

$$F_n(z) = \frac{\mu_n^+(a_n^+)^{p+1}}{z-a_n^+} - \frac{\mu_n^-(a_n^-)^{p+1}}{z-a_n^-}.$$

Suppose that

$$\sum_n |\mu_n^+(a_n^+)^m - \mu_n^-(a_n^-)^m| < \infty, \quad 0 \le m \le p, \tag{3.10}$$

$$\sup_{y\ge 1} \sum_n |F_n(iy)| < \infty. \tag{3.11}$$

Then by (1.14) we obtain

$$k'(iy)^2 = 1 + \frac{1}{2}\sum_{m=0}^{p}\sum_n [\mu_n^+(a_n^+)^m - \mu_n^-(a_n^-)^m](iy)^{-1-m} + O(y^{-2-p}), \tag{3.12}$$

$y \to \infty$. Hence we have (1.15) from the comparison of (3.9), (3.12).

Let us prove (3.10), (3.11). It is useful to note that from (1.11), (2.11) we have $\mu_n^\pm < Cl_n, \quad |\mu_n^- - \mu_n^+| < Cl_n^2, \quad n \in \mathbf{Z}$, for some $C > 0$. Hence

$$|\mu_n^+(a_n^+)^m - \mu_n^-(a_n^-)^m| < m|a_n^+ - a_n^-|a_n^{m-1}\mu_n^+ + |\mu_n^+ - \mu_n^-|a_n^m < Cl_n^2(m + a_n)a_n^{m-1},$$

and

$$|\frac{\mu_n^+(a_n^+)^{p+1}}{iy - a_n^+} - \frac{\mu_n^-(a_n^-)^{p+1}}{iy - a_n^-}| \le$$

$$|\mu_n^+(a_n^+)^{p+1} - \mu_n^-(a_n^-)^{p+1}|/a_n + |\mu_n^+(a_n^+)^{p-1} - \mu_n^-(a_n^-)^{p-1}| \le Cl_n^2(m + 1 + a_n)^2 a_n^{p-2},$$

and by (2.17), (1.12) we obtain $Q_p^+ \ge c\|l\|_{p/2}^2$ for some $c > 0$.

3) We can write $k_p = R + iJ$, where $J(x) = x^p v(x), \quad R(x) = x^p(u(x) - x + P_p(x)), \quad x \in \mathbf{R}$. For the domain $D = \{z : R_1 < |z| < R_2, y > 0\}$, $0 < R_1 < R_2 < \infty$, we have the Green formula

$$\pi I_p^2(D) = -\int_{R_1<|x|<R_2} R'(x)J(x)dx + (R_2 b'(R_2) - R_1 b'(R_1))/2 \tag{3.13}$$

where the function

$$b(t) = \int_0^\pi J^2(t\exp(i\varphi))d\varphi, \quad t > 0,$$

and we have got the equality

$$-R'(x)J(x) = x^p v(x)\left\{(p+1)x^p - px^{p-1}u(x) - \sum_0^{p-1} Q_m(p-1-m)x^{p-2-m}\right\}, \ x \in \mathbf{R}.$$

Introduce the set $\sigma^N = \sigma \cup (-\infty, N) \cup (N, \infty)$ and the variables corresponding σ^N denote by upper index N. It is well known (see [9]) that

$$v^N(x) \nearrow v(x), \quad |u^N(x)| \nearrow |u(x)|, \quad N \to \infty, \quad x \in \mathbf{R}.$$

From this and from Levy's theorem it follows that

$$Q_m^{N,+} \nearrow Q_m^+, \qquad \int v^N(x)|u^N(x)||x|^m dx \nearrow \int v(x)|u(x)||x|^m dx,$$

as $N \to \infty$, and by (2.18) we obtain that k_p^N converges to k_p uniformly on compact sets from $\mathbf{C} \setminus \sigma$. We also have from (2.18) that

$$k_p^N(z) = O(1/z), \quad (k_p^N(z))' = O(1/z^2), \quad \text{as} \quad |z| \to \infty.$$

Hence if $R_2 \to \infty$, $R_1 \to 0$ we obtain (1.16) for the case σ^N. Then by Fatou theorem

$$d_p/2 \leq (1+p)Q_{2p} - \frac{p}{\pi}\int x^{2p-1}u(x)v(x)dx - \sum_0^{p-1}(p-1-n)Q_nQ_{2p-2-n}.$$

But from this and from (3.13) we obtain that the limit $a = \lim \ tb'(t) \geq 0$, as $t \to \infty$, exists.Let us prove that $a = 0$. Suppose not.Then for some $C > 0$ we have $tb'(t) \geq C, \ \ t >> 1$. Hence $b(t) \to \infty$, as $t \to \infty$. Define the function

$$f^2(t) = \int_0^\pi |k_p'(t\exp(i\varphi))|^2 d\varphi, \quad t > 0,$$

where by the definition of d_p we have

$$\frac{2}{\pi}\int_0^\infty tf^2(t)dt = d_p < \infty.$$

There is a sequence $t_n \to \infty$, such that $t_n f(t_n) \to 0$, as $n \to \infty$. Suppose not. Then for some $c > 0$ we have $tf(t)^2 > c/t$ for large t and $d_p = \infty$. For this sequence t_n we obtain

$$|k_p(t_n \exp(i\varphi))| \leq |k_p(it_n)| + t_n \int_0^\pi |k_p'(t_n\exp(i\phi))|d\phi \leq |k_p(it_n)| + \pi t_n f(t_n) \to 0,$$

as $n \to \infty$, uniformly on $\phi \in [0, \pi]$, because by (2.18) $k_p(iy) \to 0$, as $y \to \infty$. So

$$b(t_n) \leq \int_0^\pi |k_p(t\exp(i\varphi))|^2 d\varphi, \to 0, \quad \text{as} \quad n \to \infty.$$

4). For the Lyapunov function $F(z) = \cos k(z), z \in \mathbf{C}$, there is the estimate $|F(x)| \leq \cosh v(x), x \in \mathbf{R}$. Then we obtain

$$\int \frac{\log^+|F(x)|}{1+x^2}dx \leq \int \frac{v(x)}{1+x^2}dx < \infty.$$

Hence the functions $F(z) \pm 1$ are entire functions of Cartwright class. Using the properties of this class (see [7]) and taking into account the fact that zeros of the function

$F(z)-1$ is the set $\left\{a_{2n}^{\pm}, n\in \mathbf{N}\right\}$ (if $a_{2n}^{-}=a_{2n}^{+}$ then the multiplicity equals to two) we obtain

$$F(z)-1=\exp(iaz)V.P.\prod_{n\in\mathbf{Z},q=\pm}(1-\frac{z}{a_{2n}^{q}}),\ z\in\mathbf{C}, \tag{3.14}$$

where $a\in\mathbf{C}$ and the multiplication in (3.14) converges uniformly on any compact set of the complex plane. Introduce the function $f_+(z)=F'(z)/(F(z)-1)$. From the Weierstrass theorem and from (3.14) we have

$$f_+(z)=ia+V.P.\sum_{m\in\mathbf{Z},q=\pm}\frac{1}{z-a_{2m}^{q}}, \tag{3.15}$$

where the series converges uniformly on any compact set lying in $\mathbf{C}\setminus\left\{a_{2n}^{\pm}\right\}$. Using (3.15) and the equality $\mathrm{Im}F(x)=0, x\in\mathbf{R}$, we have $a=0$. From $F(z)=\cos k(z), z\in\mathbf{C}$, we obtain $z'(k(z))F'(z)=-\sin k(z)$ and hence

$$-F(z)=z''(k(z))F'(z),\quad F(a_n^{\pm})=(-1)^n, z=a_n^{\pm}\quad . \tag{3.16}$$

From (3.15), (3.16) it follows that for $z=a_{2n+1}^{\pm}$

$$f_+(z)=F'(z)/2F(z)=-\pm\mu_{2n+1}^{\pm}/2=V.P.\sum_{m\in\mathbf{Z},q=\pm}\frac{1}{z-a_{2m}^{q}}.$$

Using $f_-(z)=F'(z)/(F(z)+1)$ we have (1.17) for $\mu_{2n}^{\pm}$. Q.E.D.

Let $\gamma_2=\max\{2,\gamma_0\}$. We shall prove "the global estimates" for GQ.

Theorem 3.4. *Let a GQ k satisfy the Condition A and the Condition 1 for some $p\geq 0$.*

1). Suppose $p=0$. Then

$$\|l\|^2/8\leq Q_0=d_0/2\leq\ \frac{1}{\pi}\|l\|\ \|h\|\leq\frac{2}{\pi}\|h\|^2\leq\pi\gamma_2\ \mu_0^2, \tag{3.17}$$

$$\mu_0^2\leq 2A_0d_0,\qquad d_0\leq 4\gamma_2A_0\|l\|^2.$$

2). Suppose $p\geq 1$. Then

$$2^{-4-2p}\|l\|_p^2\leq Q_{2p}\leq\frac{1}{\pi}\|l\|_p\|h\|_p\leq\frac{2}{\pi}\|h\|_p^2\leq 2\pi(1+\gamma_1)^{1+2p}\mu_p^2, \tag{3.18}$$

$$\mu_p^2\leq 2A_pd_p,\qquad d_p\leq 2(1+p)Q_{2p}, \tag{3.19}$$

$$Q_{2p}\leq 4(1+p)A_p\ (1+\gamma_1)^{1+2p}\ \|l\|_p^2. \tag{3.20}$$

Proof. 1). Prove successively all inequalities. Take any gap g_n. From (2.9) we obtain $v(x) \geq w_n(x), x \in g_n$. Integrating this inequality on g_n and adding in n we have first inequality. In (1.16) there is second equality.

In (1.12) we have the inequality $l_n \leq 2h_n$ and this gives

$$\pi Q_0 = \int v(t)dt \leq \sum \; h_n l_n \leq \|l\| \; \|h\| \leq \; 2\|h\|^2.$$

Let $Y_n = \min \mu_n^{\pm}$. By (1.13) we have $l_n \leq 2Y_n$. Hence

$$l_n Y_n = \min \left\{ l_n \mu_n^{\pm} \right\} = \min_{q=\pm} \left\{ \mu_n^q s_n^q l_n / s_n^q \right\} \leq \gamma_0 (\mu_n^- s_n^- + \mu_n^+ s_n^+),$$

and then

$$l_n Y_n \leq \min \left\{ 2Y_n^2, \gamma_0(\mu_n^- s_n^- + \mu_n^+ s_n^+) \right\} \leq \gamma_2 \sum_{q=\pm} \mu_n^q \min(\mu_n^q, s_n^q).$$

From this inequality and from $2h_n^2 \leq \pi^2 l_n Y_n$ (which follows from (1.12)) we obtain

$$\|h\|^2 \leq \frac{\pi^2}{2} \sum l_n Y_n \leq \frac{\gamma_2 \pi^2}{2} \sum_{q=\pm,n} \mu_n^q \min(\mu_n^q, s_n^q) = \frac{\gamma_2 \pi^2}{2} \mu_0^2.$$

Using (2.23) we have the estimate

$$\mu_0^2 = \sum_{q=\pm,n} \mu_n^q \min(\mu_n^q, s_n^q) \leq 2A_0 d_0.$$

By (3.17)

$$\pi^2 Q_0^2 \leq \|l\|^2 \|h\|^2 \leq \|l\|^2 \frac{\gamma_2 \pi^2}{2} \mu_0^2 \leq \|l\|^2 \gamma_2 \pi^2 2A_0 Q_0.$$

From this estimate it is easy to get the necessary inequality.

2). Consider the case $g_n \subset \mathbf{R}_+$ (the cases $g_n \subset \mathbf{R}_-$ or $0 \in g_n$ are proved by analogy). From (2.9) it follows that

$$\int_{g_n} t^{2p} v(t)dt \geq \int_{g_n} t^{2p} w_n(t)dt \geq \int_0^b \sqrt{b^2 - x^2} (x + \frac{a_n^+ + a_n^-}{2})^{2p} dx \geq$$
$$\frac{b^2 \pi}{4} (\frac{a_n^+ + a_n^-}{2})^{2p} \geq l_n^2 \pi 2^{-4-2p} a_n^{2p},$$

where $2b = l_n$. From this we have first inequality. By (1.12)

$$\int_{g_n} t^{2p} v(t)dt \leq a_n^{2p} l_n h_n \leq 2a_n^{2p} h_n^2, \quad n \in \mathbf{Z}.$$

Hence the three inequalities in (3.18) are proved. We have $l_n \leq \gamma_1 \max_{\pm} r_n^{\pm}$. There are two cases. First, let $l_n \leq \gamma_1 r_n^-$, then $a_n \leq (1 + \gamma_1)|a_n^-|$. By (1.12), (1.13)

$$h_n^2 \le \pi^2 l_n \mu_n^-/2 \le \pi^2 \mu_n^- \min\left\{\mu_n^-, \gamma_1 r_n^-/2\right\} \le \pi^2(1+\gamma_1)\mu_n^- \min\left\{\mu_n^-, r_n^-\right\}.$$

Hence

$$a_n^{2p} h_n^2 \le \pi^2(1+\gamma_1)^{1+2p}(a_n^-)^{2p}\mu_n^- \min\left\{\mu_n^-, r_n^-\right\}. \tag{3.21}$$

Second, let $l_n \le \gamma_1 r_n^+$, then by analogy

$$a_n^{2p} h_n^2 \le \pi^2(1+\gamma_1)^{1+2p}(a_n^+)^{2p}\mu_n^+ \min\left\{\mu_n^+, r_n^+\right\}. \tag{3.22}$$

By (3.21), (3.22)

$$a_n^{2p} h_n^2 \le \pi^2(1+\gamma_1)^{1+2p} \sum_{q=\pm} (a_n^q)^{2p} \mu_n^q \min(\mu_n^q, r_n^q).$$

From this it is easy to prove last estimate in (3.18).

To prove (3.19) we use (2.21) and then

$$\mu_p^2 = \sum_{q=\pm, n\in\mathbf{Z}} (a_n^q)^{2p} \mu_n^q \min(\mu_n^q, r_n^q) \le 2A_p d_p.$$

Last estimate in (3.19) follows from (1.16).

We shall prove (3.20). By (3.18), (3.19)

$$\pi^2 Q_{2p}^2 \le \|l\|_p^2 \|h\|_p^2 \le \|l\|_p^2 \pi^2 (1+\gamma_1)^{1+2p} \mu_p^2 \le \|l\|_p^2 \pi^2 (1+\gamma_1)^{1+2p} 4A_p(1+p)Q_{2p}.$$

From this estimate we obtain (3.20). Q.E.D.

4 . Asymptotics

Let $< A, B >$ be the distance between sets(numbers) A, B. Introduce the numbers $\xi > 0$,

$$\xi_n^\pm = \min(\xi, r_n^\pm/2), \quad \xi_n = \min_{\pm} \xi_n^\pm, B_n^\pm = a_n^\pm \pm \xi_n^\pm, \quad f_n^\pm = \frac{1}{\pi}\int_{g_n} \frac{v(t)dt}{|t-a_n^\pm|}, \quad n \in \mathbf{Z},$$

the domains $Z_n(\xi) = \{B_n^- < \mathrm{Re} z < B_n^+\}$, $\quad g_n(\xi) = \{|\mathrm{Im} z| < \xi\} \cap Z_n(\xi)$, and the functions

$$J(p,\xi,z) = \frac{2}{\pi}\int_{2|t-z|<z} \frac{|t|^p v(t)dt}{|t-z|+\xi}, \qquad f_p(z) = \frac{k_p(z)}{z^p}.$$

We present the theorem.

Theorem 4.1. *Let k be a GQ. Suppose that $Q_p^+ < \infty$ for some $p \ge 0$.*
1). Let $< z, g > \ge \xi > 0$. Then

$$|k_p(z)| \le 2Q_p^+/|z| + J(p,\xi,z), \tag{4.1}$$

and $J(p,\xi,z)\to 0,$ *as* $|z|\to\infty$.
2). Let $z\in g_n(\xi)$ *for some* $\xi\in(0,b_n)$*. Suppose* $P_p(x)'<1, z\in g_n$*. Then*

$$|f_p(z)|\le h_n+\frac{2Q_p^+}{(b_n-\xi)^{1+p}}+(b_n-\xi)^{-p}\left[a_n^p\max_{\pm} f_n^{\pm}+\max_{x\in\left\{a_n^{\pm},B_n^{\pm}\right\}} J(p,\xi_n,x)\right], \tag{4.2}$$

where

$$2h_n\le\pi(f_n^++f_n^-),\quad f_n^{\pm}\le\sqrt{2l_n\mu_n^{\pm}}, \tag{4.3}$$

$$f_n^++f_n^-\le\min\left\{\sqrt{4l_n(\mu_n^-+\mu_n^+)},\ \ l_n(1+\frac{Q_0}{\min_{\pm}(r_n^{\pm})^2})\right\}. \tag{4.4}$$

Remark. If $p\ge 0, |n|>>1$, then $P_p(x)'<1, x\in g_n$. Furthermore, if a GQ k satisfy the Condition A then $P_p(x)'<1$ for any $x\ne 0$.

Proof. 1) By (2.18) and by the inequality $2|z-t|\ge\xi+|z-t|$ if $<z,g>\ge\xi$ we have

$$|k_p(z)|\le\frac{1}{\pi}\int_{2|t-z|<|z|}\frac{|t|^pv(t)dt}{|t-z|}+\frac{1}{\pi}\int_{2|t-z|>|z|}\frac{|t|^pv(t)dt}{|t-z|}\le 2Q_p^+/|z|+J(p,\xi,z).$$

Since $Q_p^+<\infty$ we obtain that $J(p,\xi,z)\to 0$, as $|z|\to\infty$.

2) By the maximum principle enough to estimate f_p on the boundary of $Z_n(\xi)$. First we consider $f_p(z)$ when z belong to the upper side of the slit g_n (the case of the lower side is considered by analogy). By the definition of k_p, f_p we have

$$0\le \mathrm{Im} f_p(x+i0)=v(x)\le h_n,\quad x\in g_n. \tag{4.5}$$

Now we estimate the real part of $f_p(x+i0), x\in g_n$. We see $\mathrm{Re} f_p(x+i0)'=1-P_p(x)'>0$.

Then the function $-\mathrm{Re} f_p(x+i0)$ increases in $x\in g_n$ and $\sup_{x\in g_n}|\mathrm{Re} f_p(x+i0)|=\max_{\pm}|\mathrm{Re} f_p(a_n^{\pm})|$. Now we estimate the function $f_p(z),\ \mathrm{Re} z=B_n^{\pm}$. By (2.18)

$$|f_p(z)|\le\frac{1}{\pi|B_n^{\pm}|^p}\int\frac{|t|^pv(t)dt}{|t-B_n^{\pm}|},\quad \mathrm{Re} z=B_n^{\pm}.$$

Suppose $x\in\left\{a_n^{\pm},B_n^{\pm}\right\}$. Then

$$|x|^p|f_p(x)|\le\frac{1}{\pi}\int_{2|t-x|>|x|}\frac{|t|^pv(t)dt}{|t-x|}+a_n^pf_n^{\pm}+\frac{1}{\pi}\int_{\{2|t-x|<|x|\}\setminus g_n}\frac{|t|^pv(t)dt}{|t-z|}$$
$$\le 2Q_p^+/|x|+a_n^pf_n^{\pm}+J(p,\xi_n,z).$$

since $2|t-x|\ge|t-x|+\xi_n$ if $t\notin g_n$ and $|a_n^{\pm}-t|\le|B_n^{\pm}-t|$ if $t\in g_n$. By (2.16), (2.17) we have the first inequality in (4.3). By (2.9) we obtain

$$f_n^{\pm}\le\frac{\sqrt{2l_n\mu_n^{\pm}}}{\pi}\int_{g_n}\frac{1}{w_n(t)}dt\le\sqrt{2l_n\mu_n^{\pm}}.$$

Using the estimates for v from (2.10), (2.16) we obtain (4.4). Q.E.D.

We shall consider asymptotics for the Hill operator. We introduce the numbers

$$\pi v_n = \int_{g_n} v(t)dt, \quad T_n = \sum_{m\neq 0} v_{n+m}(mr)^{-2}, \quad \pi W_n = \int_{\mathbf{R}\setminus g_n} \frac{v(t)dt}{w_n(t)^2}, \quad n \in \mathbf{Z},$$

and the function

$$F_n(x) = \frac{1}{\pi}\int_{\mathbf{R}\setminus g_n} \frac{v(t)dt}{w_n(t)|t-x|}, \quad n \in \mathbf{Z}, \quad x \in g_n.$$

We present the theorem.

Theorem 4.2. *Let $k(z)$ be the quasimomentum of the Hill operator and $V \in L^1(0,1)$. Then for any $x \in g_n, n \in \mathbf{Z}$, the statements (1.7), (1.8) are valid. Furthermore*

$$\max\{W_n, F_n(x)\} < T_n \le Q_0 r^{-2}, \tag{4.6}$$

$$T_n \le T n^{-2}, \tag{4.7}$$

$$v(x) \le w_n(x)(1 + T n^{-2}). \tag{4.8}$$

Proof. We estimate W_n, the case of F_n is considered by analogy. We have the inequality $w_n(t)^2 \ge m^2 r^2, \quad t \in g_{n+m}$, and hence

$$W_n = \frac{1}{\pi}\sum_{m\neq n}\int_{g_m} \frac{v(t)dt}{w_n(t)^2} \le \frac{1}{\pi}\sum_{m\neq n}\int_{g_m} \frac{v(t)dt}{(m-n)^2 r^2} = T_n. \tag{4.9}$$

By $|m| \ge 1$ we have $T_n \le Q_0 r^{-2}$. In the case of the Hill operator

$$l_n = L_n/(a_n^+ + a_n^-) \le L_n/2nr, \ n > 0. \tag{4.10}$$

By (1.10), (4.9)

$$v(x) = w_n(x)(1 + F_n(x)) \le w_n(x)(1 + T_n), \quad x \in g_n, \quad n \in \mathbf{Z}. \tag{4.11}$$

We see from (4.10), (4.11), (4.6) that

$$v_n \le \frac{1}{\pi}\int_{g_n} w_n(t) T^0 dt \le (l_n/2)^2 T^0/2 \le \frac{T^0 L_n^2}{8(2nr)^2}.$$

Hence

$$T_n = \sum_{m\neq n, n\neq 0} v_{n-m}(mr)^{-2} \le \sum_{m\neq n, m\neq 0} \frac{T^0 L_m^2}{2(4m(n-m)r^2)^2} \le \frac{3T}{2\pi^2}\sum_{m\neq n, m\neq 0}\frac{1}{m^2(m-n)^2},$$

and by

$$\frac{n^2}{m^2(m-n)^2} = (\frac{1}{m-n} - \frac{1}{m})^2, \quad \sum_{m>0} 1/m^2 = \pi^2/6,$$

we have (4.7). By (4.6), (4.7), (4.11), (1.10), (1.11) we obtain (1.7),(1.8). Q.E.D.

Introduce the function

$$A(\beta,\xi,z) = 2\left\{\xi^{-1} + \xi^{-\frac{1}{\beta}}(\frac{\beta-1}{r})^{1-\frac{1}{\beta}}\right\}, \quad 1 \le \beta < \infty,$$

$$A(\beta,\xi,z) = 2\left\{\xi^{-1} + \frac{1}{r}\log(1+\frac{|z|}{2\xi})\right\}, \quad \beta = \infty, \quad \xi > 0.$$

We present

Theorem 4.3. *Let k be the quasimomentum for the Hill operator and $p \ge 0,\ \xi > 0,\ Q_p^+ < \infty,\ z \in \mathbf{C},\ \beta \ge 1$. Then*

$$\pi J(p,\xi,z) \le T^0 A(\beta,\xi,z)\left\{\sum_{2<g_n,z>\le|z|} a_n^{p\beta} l_n^{2\beta}\right\}^{\frac{1}{\beta}} \le T^0 A(\beta,\xi,z)\left\{\sum_{2<g_n,z>\le|z|} a_n^{(p-2)\beta} L_n^{2\beta}\right\}^{\frac{1}{\beta}}. \tag{4.12}$$

Proof. Introduce the function

$$B(\beta,\xi,z) = \sum_{2<g_n,z>\le|z|} (\xi + < g_n, z >)^{-\beta_1}, \quad 1/\beta_1 + 1/\beta = 1,$$

and a number $\pi Q(p,n) = \int_{g_n} |t|^p v(t)dt$. We have

$$J(p,\xi,z) \le 2\sum_{2<g_n,z>\le|z|} (\xi + < g_n, z >)^{-1} Q(p,n) \le 2(\sum_{2<g_n,z>\le|z|} Q(p,n)^\beta)^{1/\beta}\ B(\beta,\xi,z)^{1/\beta_1}.$$

We have to estimate B. We obtain

$$B(\beta,\xi,z) \le \sum_{2|n|\le|z|} (\xi + |n|r)^{-\beta_1} \le 2\left\{\xi^{-\beta_1} + \int_0^{|z|/2r} (\xi + |x|r)^{-\beta_1} dx\right\}$$

and

$$B(\beta,\xi,z) \le 2\left\{\xi^{-\beta_1} + \frac{\xi^{1-\beta_1}}{r(\beta_1-1)}\right\}, \quad 1 \le \beta < \infty,$$

$$B(\beta,\xi,z) \le 2\left\{\xi^{-1} + \frac{1}{r}\log(1+\frac{|z|}{2\xi})\right\}, \quad \beta = \infty,$$

and hence $B \le A^{\beta_1}$. By (1.7)

$$\pi Q(p,n) \le a_n^p \int_{g_n} v(t)dt \le a_n^p h_n l_n \le a_n^p T^0 l_n^2/2 \le a_n^{p-2} T^0 L_n^2/2. \quad Q.E.D.$$

5 . Applications

In this chapter we shall apply the previous results for the case both the Hill operator and the Dirac operator with periodic coefficients.

First we consider the Hill operator $H = -d^2/dt^2 + V(t)$ in $L^2(\mathbf{R})$ where V is 1-periodic real potential and $V \in L^1(0,1)$. Let $\varphi(t,z), \theta(t,z)$ be the solutions of (1.2), satisfying $\varphi_t'(0,z) = \theta(0,z) = 1, \quad \varphi(0,z) = \theta_t'(0,z) = 0,$ and the Lyapunov function $F(z) = (\varphi_t'(1,z)+\theta(1,z))/2$. The sequence $0 = A_0^+ < A_1^- \le A_1^+ < .....$ is the spectrum of equation (1.2) with periodic boundary conditions of period 2, i.e. $f(x+2) = f(x), x \in \mathbf{R}$. Here equality means that $A_n^- = A_n^+$ is a double eigenvalues. We remind that $a_n^\pm = \sqrt{A_n^\pm} \ge 0, \ a_{-n}^\pm = -a_n^\mp, \ n \in \mathbf{Z}_+$. Essentially that $F(a_{-n}^\pm) = (-1)^n, \quad n \in \mathbf{Z}$. The lowest eigenvalue A_0^+ is simple, $F(a_0^+) = 1$ and the corresponding eigenfunction has period 1. The eigenfunction corresponding to $A_n^\pm$ have period 1 when n is even and they are antiperiodic,$f(x+1) = -f(x), \ \ x \in \mathbf{R}$, when n is odd. We have the well-known estimate

$$A_n^\pm = (\pi n)^2 + \int_0^1 V(t)dt + O(1/n^2), \quad n \to \infty. \tag{5.1}$$

Later on we need the simple relations

$$\mu_n^\pm = \pm 2 a_n^\pm M_n^\pm, \quad M_n^\pm = M_{-n}^\mp, \ \ \mu_n^\pm = \mu_{-n}^\mp, \ \ n \in \mathbf{N}, \tag{5.2}$$

$$\frac{L_n}{2\sqrt{A_n^+}} \le l_n \le \frac{L_n}{\sqrt{A_n^+}}, \quad n \in \mathbf{N}. \tag{5.3}$$

There are some estimates for $l_n, \ h_n, \ \mu_n^\mp, \ v$, in Section 2 and the some series for the general quasimomentum in Section 3. For the Hill operator we can rewrite these results more simple.

Corollary 5.1.1)._Let k be GQ for the Hill operator and $V \in L^1(0,1)$. Then_

$$k'(z)^2 = 2E \sum_{n \ge 0, q=\pm} \frac{M_n^q}{E - A_n^q} = 1 + 2 \sum_{n>0, q=\pm} \frac{A_n^q M_n^q}{E - A_n^q}, \tag{5.4}$$

the series converges absolutely and uniformly on compact sets. The effective masses are expressed by (1.4).

2)._Let a potential $V \in W_2^p(\mathbf{R}/\mathbf{Z})$, $p \ge 0$. Then_

$$\sum_{n \ge 1} [(A_n^+)^{1+p} M_n^+ + (A_n^-)^{1+p} M_n^-] = (1+2p) Q_{2p} + \sum_0^{p-1} (1+2m)(p-m-\frac{1}{2}) Q_{2m} Q_{2(p-1-m)}, \tag{5.5}$$

and the series converges absolutely. If $p = 0 (p = 1)$ then we have (1.5) ((1.6)).

Proof. By (1.14), (5.2), (5.3)

$$2(k'(z)^2-1)=\sum_{n\in\mathbf{Z}}(\frac{\mu_n^+}{z-a_n^+}-\frac{\mu_n^-}{z-a_n^-})=\sum_{q=\pm,n>0}q[\frac{\mu_n^q}{z-a_n^q}+\frac{\mu_{-n}^q}{z-a_{-n}^q}]=$$

$$\sum_{q=\pm,n>0}q\mu_n^q[\frac{1}{z-a_n^q}-\frac{1}{z+a_n^q}]=\sum_{n>0,q=\pm}\frac{4A_n^qM_n^q}{E-A_n^q}.$$

and hence

$$(k'(z)^2-1)/2=\sum_{n>0,q=\pm}\frac{(A_n^q-E+E)M_n^q}{E-A_n^q}=$$

$$-\sum_{n>0,q=\pm}M_n^q+E\sum_{n>0,q=\pm}\frac{M_n^q}{E-A_n^q}=M_0^+-\frac{1}{2}+E\sum_{n>0,q=\pm}\frac{M_n^q}{E-A_n^q}$$

because $\sum_{n\geq 0,q=\pm}M_n^q=1/2$ (see (1.14) at $z=0$). Thus we obtain (5.4).

By (1.17), (5.2) we have (1.4) by analogy.

2). By (5.2)

$$A\equiv\sum q(a_n^q)^{1+2p}\mu_n^q=\sum_{n>0}q[(a_n^q)^{1+2p}\mu_n^q+(a_{-n}^q)^{1+2p}\mu_{-n}^q]=$$

$$\sum_{n>0}q(a_n^q)^{1+2p}\mu_n^q[1-(-1)^{1+2p}]=4\sum_{n>0}(A_n^q)^{1+p}M_n^q,\quad p\geq 0.$$

Using (1.15), (5.2) we obtain

$$A=4(1+2p)Q_{2p}+2\sum_0^{2(p-1)}(n+1)(2p-1-n)Q_nQ_{2p-2-n}=$$

$$4(1+2p)Q_{2p}+2\sum_0^{p-1}(2m+1)(2p-1-2m)Q_{2m}Q_{2(p-1-m)}.$$

By (3.17), (5.5) we have (1.5) and by analogy we get (1.6). Q.E.D.

Remind that for a sequence $f=\{f_n\}_1^\infty$ and a number p we introduced a norm $\|f\|_{\pm,p}^2=\sum_{n>0}(A_n^\pm)^p|f_n|^2$. If we define a number $\eta=\sup_{n>0}\{A_n^+/A_n^-\}>1$, then we have simple estimates $\|f\|_{-,p}^2\leq\ \|f\|_{+,p}^2\leq\ \eta^p\|f\|_{-,p}^2$. It is necessary to note that for an even sequence $f=\{f_n\}_{-\infty}^\infty$, i.e. such that $f_{-n}=f_n, n=1,2,3,...,f_0=0$, we have the equalities $2\|f\|_{\pm,0}^2=\|f\|^2,\ \ 2\|f\|_{+,p}^2=\|f\|_p^2$.

Now we present the theorem.

Theorem 5.2. *Let k be a quasimomentum of the Hill operator and $V\in L^1(0,1)$. Then*

$$\frac{1}{16}\|L\|_{+,-1}^2\leq\quad Q_0=d_0/2\leq\ \frac{2}{\pi}\|h\|^2,\tag{5.6}$$

$$\|h\|_{\pm,0}^2\leq\ \ 4\pi^2\|M^\pm\|_{\pm,1}^2\leq\quad\pi^2B_0d_0,\tag{5.7}$$

$$d_0\quad\leq 2B_0\ \|L\|_{\pm,-1}^2.\tag{5.8}$$

Suppose a potential $V \in W_2^{p-1}(\mathbf{R}/\mathbf{Z}), p \geq 1$. Then

$$2^{-5-2p}\|L\|^2_{+,p-1} \leq \quad Q_{2p} \leq \quad \frac{4}{\pi}\|h\|^2_{+,p}, \tag{5.9}$$

$$\|h\|^2_{\pm,p} \leq \quad 4\pi^2\|M^{\pm}\|^2_{\pm,1+p} \leq \quad \pi^2 B_p d_p/4, \tag{5.10}$$

$$d_p \leq \; 2(1+p)Q_{2p} \leq \; 8(1+p)B_p\|L\|^2_{+,p-1}. \tag{5.11}$$

Proof. By (5.3), (3.17)

$$d_0/2 = Q_0 \geq \quad \frac{2}{8}\sum_{n>0} l_n^2 \geq \quad \frac{1}{16}\sum_{n>0} L_n^2/A_n^+,$$

and again by (3.17) we have (5.6). Now we shall prove (5.7). By (5.2), (1.12)

$$h_n^2 \leq \pi^2(\mu_n^{\pm})^2 \leq 4\pi^2 A_n^{\pm}(M_n^{\pm})^2, \quad n \in \mathbf{N}. \tag{5.12}$$

Combining (5.12) with (2.35) we have (5.7). We see from (3.17), (5.3), (5.7) that

$$\pi^2 Q_0^2 \leq \|h\|^2\|l\|^2 \leq 2\|h\|^2_{+,0}\|l\|^2 \leq 4\pi^2 B_0 d_0\|l\|^2_{+,0} \leq 4\pi^2 B_0 d_0\|L\|^2_{\pm,-1},$$

and using $2Q_0 = d_0$ we have (5.8). The estimates (5.6)-(5.8) have been proved.

We rewrite (3.18) in the form $22^{-4-2p}\|l\|^2_{+,p} \leq Q_{2p} \leq 4\|h\|^2_{+,p}/\pi$, and by (5.3)

$$2^{-5-2p}\|L\|^2_{+,p-1} \leq Q_{2p} \leq \frac{4}{\pi}\|h\|^2_{+,p},$$

From (5.12) it follows that

$$\|h\|^2_{\pm,p} \leq 4\pi^2\sum_{n>0}(A_n^{\pm})^p|M_n^{\pm}|^2 A_n^{\pm} \leq 4\pi^2\|M^{\pm}\|^2_{\pm,p+1},$$

and by (2.35) $4\|M^{\pm}\|^2_{\pm,p+1} \leq B_p d_p$.

Now we shall prove (5.11). We have first inequality of (5.11) in (3.19). It is necessary to prove the second. By (3.18) and by the first estimate of (5.11) we obtain that

$$\pi^2 Q_{2p}^2 \leq 4\|h\|^2_{+,p}\|l\|^2_{+,p} \leq 4\|L\|^2_{+,p-1}\pi^2 B_p d_p \leq 8\|L\|^2_{+,p-1}\pi^2 B_p(1+p)Q_{2p},$$

and hence we have (5.11). Q.E.D.

Now we shall find asymptotics $k(z)$ as $|z| \to \infty$. We consider only the case $p = 0$. Suppose $\xi > 0$ and $< z, g > \geq \xi$. By (4.12) at $\beta = \infty$ we have

$$\pi J(0,\xi,z) \leq 2T^0\left\{\xi^{-1} + \frac{1}{r}\log(1+\frac{|z|}{2\xi})\right\}\sup_{2<z,g_n>\leq|z|} l_n^2, \tag{5.13}$$

and since

$$2|a_n^- + a_n^+| \geq |z|, \quad \text{as} \quad 2 < z, g_n > \leq |z|, \tag{5.14}$$

and by (5.3)

$$\sup_{2<z,g_n>\leq|z|} l_n^2 \leq \frac{4}{|z|^2} \sup_{2|a_n^-|\geq|z|} L_n^2, \tag{5.15}$$

then we see from (5.13)- (5.15) that

$$J(0,\xi,z) \leq J_1(\xi,z)J_2(\xi,z)/|z|^2, \tag{5.16}$$

$$\pi J_1(\xi,z) = 8T^0 \sup_{4A_n^-\geq|z|^2} L_n^2, \qquad J_2(\xi,z) = \xi^{-1} + \frac{1}{r}\log(1+\frac{|z|}{2\xi}).$$

Then we obtain

$$|k_(z) - z| \leq 2Q_0/|z| + J_1(\xi,z)J_2(\xi,z)/|z|^2. \tag{5.17}$$

Now we consider the case $< z, g >\leq \xi$. Let $m >> 1$ and such that $r_n^\pm \geq \pi/2$ as $|n| \geq m$. We take $4\xi < \pi$ and $z \in \{< z, g_n >\leq \xi\}$. By (1.10), (5.3)

$$2h_n \leq T^0 l_n \leq \frac{T^0 L_n}{|a_n^+ + a_n^-|} \leq \frac{T^0 L_n}{|z|}. \tag{5.18}$$

From (5.18), (1.10) it follows that

$$f_n^\pm \leq \quad T^0 l_n \leq \quad T^0 L_n/|z|, \tag{5.19}$$

and by (5.16) $J(0,\xi,x) \leq 4J_1(\xi,|z|/2)J_2(\xi,2|z|)/|z|^2$, $x \in \{a_n^\pm, a_n^\pm \pm \xi\}$. Finally, we obtain

$$|k(z) - z| \leq (4Q_0 + T^0 L_n)/|z| + 4J_1(\xi,z/2)J_2(\xi,2z)/|z|^2.$$

Now we shall consider some estimates about the velocity $U_n, n \in \mathbf{Z}$. Let the spectral band of the quasimomentum for the Hill operator $s(n) = [a(n), b(n)], r_n = |s(n)|, n \in \mathbf{Z}$. Suppose the point k_n such that $U_n = z'(k_n) = \max z'(k),\ z(k) \in s(n)$. We present

Corollary 5.3._Let_ $V \in L^1(0,1)$. _Then_

$$\sum r_n^2(1 - U_n^{-1})^2 \leq 4d_0.$$

Proof. Let $2x_n = a(n) + b(n)$ and the domain $D_n = \{\mathrm{Re} z \in s(n)\}, n \in \mathbf{Z}$. Then we have

$$\pi(r_n/2)^2|k'(x_n) - 1|^2 \leq \int_{D_n} |k'(z) - 1|^2 dxdy$$

and by $|k'(x_n) - 1| \geq |1 - U_n^{-1}|$ we obtain $(r_n/2)^2|1 - U_n^{-1}|^2 \leq I_0^2(D_n)$. Summing we have the estimate.Q.E.D.

Now we shall consider the Dirac operator H_D (with periodic coefficients) in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}) \oplus L^2(\mathbf{R})$

$$H_D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dt} + \begin{pmatrix} V_1(t) & 0 \\ 0 & V_2(t) \end{pmatrix}.$$

Later on we shall use the Dirac equation

$$f_2' + V_1 f_1 = z f_1, \quad -f_1' + V_2 f_2 = z f_2, \tag{5.20}$$

where V_1, V_2 are real 1-periodic functions in $t \in \mathbf{R}, V_1, V_2 \in L^1(1,0)$. For a vector -function $f(t) = \{f_1(t), f_2(t)\} \in \mathcal{H}$ we consider the following boundary conditions

$$f(0) = f(1), \tag{5.21}$$

$$f(0) = -f(1). \tag{5.22}$$

The boundary value problem (5.20), (5.21) is called by periodic and the boundary value problem (5.20), (5.22) is called by antiperiodic. We denote the eigenvalues of the periodic problem by $a_{2n}^{\pm}$ and the eigenvalues of the antiperiodic problem by $a_{2n+1}^{\pm}, n \in \mathbf{Z}$. It is well- known that

$$... < a_{2n-1}^- \le a_{2n-1}^+ < a_{2n}^- \le a_{2n}^+ < ....., \tag{5.23}$$

$$a_n^{\pm} = n(\pi + o(1)), \qquad |n| \to \infty.$$

Let $\varphi(t,z) = (\varphi_1(t,z), \varphi_2(t,z)), \theta(t,z) = (\theta_1(t,z), \theta_2(t,z))$ be the solutions of (5.20) satisfying $\varphi(0,z) = (0,1), \ \ \theta(t,z) = (1,0)$.

We introduce the Lyapunov function for the Dirac equation $2F_D(z) = \varphi_1(1,z) + \theta_2(1,z), z \in \mathbf{C}$. The properties of the Lyapunov function for the Dirac operator and for the Hill operator are similar. But there is one exception.The function $F_D(z)$ is not even in $z \in \mathbf{C}$. We have $F(a_{-n}^{\pm}) = (-1)^n, n \in \mathbf{Z}$. The spectrum of H_D is purely absolutely continuous and is given by the set $\cup s_n$, where a interval $s_n = [a_{n-1}^+, a_n^-]$. These intervals are separated by gaps $g_n = (a_n^-, a_n^+)$. If a gap g_n is degenerate, i.e. $g_n = \emptyset$ then the corresponding segments s_n, s_{n+1} merge.The spectrum of H_D falls into the components which are called spectral bands. Now we define the quasimomentum function $k(z) = \arccos F_D(z), z \in Z = \mathbf{C} \setminus \bar{g}, \ g = \cup g_n$. The function $k(z)$ is analytic and moreover k is a conformal map from Z onto the quasimomentum slit plane $K = \mathbf{C} \setminus \cup \Gamma_n$ where an excised slit is given by $\Gamma_n = \{\mathrm{Re}k = \pi n, |\mathrm{Im}k| \le h_n\}, \ \ h_n \ge 0, \ \ n \in \mathbf{Z}$. A lot of estimates for the Dirac operator repeat corresponding estimates for the Hill operator.

References

1. Garnett J., Trubowitz E.: Gaps and bands of one dimensional periodic Schrödinger operator. Comment. Math. Helv. 59, 258-312(1984)

2. Firsova N.: Direct and inverse problem of scattering for one dimensional perturbed Hill operator. Mat. Sb. 130(172), 3(7), 349- 385(1986), Russian.

3. Firsova N.: On the effective mass for one dimensional Hill operator. Vestnic LGU(fizika).10, 2,13-18(1979), Russian.

4. Fryntov A.: One extremal problem of potential theory. Dokl. Akad. Nauk SSSR. 300, no.4, (1988); English transl. in Soviet Math. Dokl. 37, 754-755.

5. Its A., Matveev V.: Schrödinger operators with the finite-gap spectrum and the N-soliton solutions of the Korteveg de Fries equation. Teoret. i Mat. Fiz. 23, 51-68(1975), Russian.

6. Kargaev P. : On the Martin boundary of a plane domain with the complement on the line. Mat. Zamet 47, 20-27(1990), Russian.

7. Koosis P.: The logarithmic integral,1.Cambridge: Cambridge University Press 1988.

8. Korotyaev E. : Propagation of the waves in the one-dimensional periodic media. Dokl. Akad. Nauk Russia.(to appear)

9. Levin B.: Majorants in the class of subharmonic functions.1-3. Theory of functions, functional analysis and their applications. 51, 3-17(1989); 52, 3-33 (1989). Russian.

10. Marchenco V.: Sturm-Liouville operator and applications. Basel: Birkhäuser 1986.

11. Marchenco V., Ostrovski I.:A characterization of the spectrum of the Hill operator. Mat Sb. 97(139), no.4(8), 540-606(1975), Russian.

12. Pastur L., Tkachenco V.: The spectral theory of some class of one dimensional Schrödinger operator with limit periodic potentials. Trudy of Moskow Math. Society. 51, 114-168(1988), Russian.