The Euler-Poincaré Equations and Double Bracket Dissipation

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Abstract

This paper studies the perturbation of a Lie-Poisson (or, equivalently an
Euler-Poincaré) system by a special dissipation term that has Brockett’s dou-
ble bracket form. We show that a formally unstable equilibrium of the un-
perturbed system becomes a spectrally and hence nonlinearly unstable equi-
librium after the perturbation is added. We also investigate the geometry of

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this dissipation mechanism and its relation to Rayleigh dissipation functions. This work complements our earlier work (Bloch, Krishnaprasad, Marsden and Ratiu [1991, 1993]) in which we studied the corresponding problem for systems with symmetry with the dissipation added to the internal variables; here it is added directly to the group or Lie algebra variables. The mechanisms discussed here include a number of interesting examples of physical interest such as the Landau-Lifschitz equations for ferromagnetism, certain models for dissipative rigid body dynamics and geophysical fluids, and certain relative equilibria in plasma physics and stellar dynamics.

1 Introduction

The purpose of this paper is to study the phenomenon of dissipation induced instabilities for Euler-Poincaré systems on Lie algebras or equivalently, for Lie-Poisson systems on the duals of Lie algebras. In our previous work (Bloch, Krishnaprasad, Marsden and Ratiu [1993]), we showed that if a mechanical system with symmetry has an indefinite second variation of the augmented Hamiltonian at a relative equilibrium, as determined by the energy-momentum method (Simo, Posbergh and Marsden [1990,1991], Simo, Lewis and Marsden [1991], Lewis [1992], and Wang and Krishnaprasad [1992]), then the system becomes spectrally unstable with the addition of a small amount of dissipation. This energy momentum method is an outgrowth of the energy-Casimir, or Arnold method that has its roots in original work going back to at least Routh [1877]; see Holm, Marsden, Ratiu and Weinstein [1985] and references therein. The dissipation that was considered in our earlier paper was of the standard Rayleigh dissipation type, but this dissipation was added to the internal variables of the system and the methods that were used to prove this were essentially those of linear analysis. In that paper, we did not consider dissipation terms in the group (or rotational) variables; that is the subject of the present work.

For systems on Lie algebras, or equivalently, for invariant systems on Lie groups, we show that one cannot have linear dissipative terms of Rayleigh dissipation type in the equations in the naive sense. However, when restricted to coadjoint orbits, we show that these dissipation terms are obtainable from a gradient structure that is similar in spirit to the way one gets dissipative terms from the gradient of a Rayleigh dissipation function. In this context, we prove that one gets dissipation induced instabilities, as one has in the case of internal dissipation. This means that the addition of dissipation to a state that is not formally stable forces at least one pair of eigenvalues into the right half plane, which one refers to as spectral instability, which of course implies nonlinear instability.

One of the interesting features of the present work is the method of construction of the nonlinear dissipative terms. We do this by linking the double bracket equation of Brockett (see Brockett [1988,1993]) to the present context. In fact, this form is well adapted to the study of dissipation on Lie groups since it was originally constructed as
a gradient system and it is well known in other contexts that this formalism plays an important role in the study of integrable systems (see, for example, Bloch, Flaschka and Ratiu [1990] and Bloch, Brockett and Ratiu [1992]).

We will also show that this type of dissipation can be described in terms of a symmetric Poisson bracket. Symmetric brackets for dissipative systems have been considered by Kaufman [1984, 1985], Grmela [1984,1993a,b], Morrison [1986], and Turski and Kaufman [1987]. It is not clear how the brackets of the present paper are related to those. Our brackets are more directly motivated by those in Vallis, Carnevale, and Young [1989], Shepherd [1992] and references therein.

We present a class of symmetric brackets that are systematically constructed in a general Lie algebraic context. We hope that our construction might shed light on possible general properties that these brackets might have. The general equations of motion that we consider have the following form:

\[ \dot{F} = \{F, H\}_{\text{skew}} + \{F, H\}_{\text{sym}} \]

where \( H \) is the total energy. In many cases however, especially those involving thermodynamics, one replaces \( H \) in the second bracket by \( S \), the entropy. We refer to the above references for this aspect; it remains for the future to link that work more closely with the present context and to see in what sense, if any, the combined bracket satisfies a graded form of Jacobi’s identity.

It is interesting that the type of dissipation described here is of considerable physical interest. For example, as we shall point out below, the Landau-Lifshitz (or Gilbert) dissipative mechanism in ferromagnetics is exactly of the type we describe and this dissipative mechanism is well accepted and studied (see O’Dell [1981] for example). In geophysical situations, one would like a dissipative mechanism that separates the different time scales of decay of the energy and the enstrophy. That is, one would like a dissipative mechanism for which the energy decays but the enstrophy remains preserved. This is exactly the sort of dissipative mechanism described here and that was described in Vallis, Carnevale, and Young [1989], Shepherd [1992] and references therein. Also, in plasma physics and stellar dynamics, one would like to have a dissipative mechanism that preserves the underlying conservation of particle number, yet has energy decay. Again, the general mechanism here satisfies these properties (see Kandrup [1991] and Kandrup and Morrison [1992]). We will discuss all of these examples in the body of the paper.

To get a concrete idea of the type of dissipative mechanism we have in mind, we now give a simple example of it for perhaps the most basic of Euler-Poincaré, or Lie-Poisson systems, namely the rigid body. Here, the Lie algebra in question is that of the rotation group; that is, Euclidean three space \( \mathbb{R}^3 \) interpreted as the space of body angular velocities \( \Omega \) equipped with the cross product as the Lie bracket. On this space, we put the standard kinetic energy Lagrangian \( L(\Omega) = \frac{1}{2}(I\Omega) \cdot \Omega \) (where \( I \) is the moment of inertia tensor) so that the general Euler-Poincaré equations (discussed below in §4) become the standard rigid body equations for a freely spinning rigid
body:

\[ I\dot{\Omega} = (I\Omega) \times \Omega, \quad (1) \]

or, in terms of the body angular momentum \( M = I\Omega \),

\[ \dot{M} = M \times \Omega. \]

In this case, the energy equals the Lagrangian; \( E(\Omega) = L(\Omega) \) and energy is conserved by the solutions of (1). Now we modify the equations by adding a term cubic in the angular velocity:

\[ \dot{M} = M \times \Omega + \alpha M \times (M \times \Omega), \quad (2) \]

where \( \alpha \) is a positive constant.

A related example is the 1935 Landau-Lifschitz equations for the magnetization vector \( M \) in a given magnetic field \( B \) (see, for example, O'Dell [1981], page 41):

\[ \dot{M} = \gamma M \times B + \frac{\lambda}{\|M\|}(M \times (M \times B)), \quad (3) \]

where \( \gamma \) is the magneto-mechanical ratio (so that \( \gamma \|B\| \) is the Larmour frequency) and \( \lambda \) is the damping coefficient due to domain walls.

In each case, it is well known that the equations without damping can be written in either Euler-Poincaré form or in Lie-Poisson (Hamiltonian) form. The equations are Hamiltonian with the rigid body Poisson bracket:

\[ \{F, K\}_{\text{rl}}(M) = -M \cdot [\nabla F(M) \times \nabla K(M)] \]

with Hamiltonians given respectively by \( H(M) = (M \cdot \Omega)/2 \) and \( H(M) = \gamma M \cdot B \).

One checks in each case that the addition of the dissipative term has a number of interesting properties. First of all, this dissipation is derivable from an \( SO(3) \)-invariant force field, but it is not induced by any Rayleigh dissipation function in the literal sense. However, it is induced by a Rayleigh dissipation function in the following restricted sense: It is a gradient when restricted to each momentum sphere (coadjoint orbit) and each sphere carries a special metric (later to be called the normal metric). Namely, the extra dissipative term in (2) equals the negative gradient of the Hamiltonian with respect to the following metric on the sphere. Take a vector \( \mathbf{v} \) in \( \mathbb{R}^3 \) and orthogonally decompose it in the standard metric on \( \mathbb{R}^3 \) into components tangent to the sphere \( \|M\|^2 = c^2 \) and vectors orthogonal to this sphere:

\[ \mathbf{v} = \frac{M \cdot \mathbf{v}}{c^2} M - \frac{1}{c^2} [M \times (M \times \mathbf{v})] \quad (4) \]

The metric on the sphere is chosen to be \( \|M\|^2 \alpha \) times the standard inner product of the components tangent to the sphere in the case of the rigid body model and just \( \lambda \) times the standard metric in the case of the Landau-Lifschitz equations.
Secondly, the dissipation added to the equations has the obvious form of a repeated Lie bracket, \textit{i.e.}, a double bracket, and it has the properties that the conservation law

\[
\frac{d}{dt} \|M\|^2 = 0
\]

is preserved by the dissipation (since the extra force is orthogonal to \(M\)) and the energy is strictly monotone except at relative equilibria. In fact, we have

\[
\frac{d}{dt} E = -\alpha \|M \times \Omega\|^2,
\]

for the rigid body and

\[
\frac{d}{dt} E = -\frac{\lambda}{\|M\|^2} \|M \times B\|^2,
\]

in the case of the Landau-Lifschitz equations, so that trajectories on the angular momentum sphere converge to the minimum (for \(\alpha\) and \(\lambda\) positive) of the energy restricted to the sphere, apart from the set of measure zero consisting of orbits that are relative equilibria or are the stable manifolds of the perturbed saddle point.

Another interesting feature of these dissipation terms is that they can be derived from a symmetric bracket in much the same way that the Hamiltonian equations can be derived from a skew symmetric Poisson bracket. For the case of the rigid body, this bracket is

\[
\{F, K\} = \alpha (M \times \nabla F) \cdot (M \times \nabla K).
\]

As we have already indicated, the same formalism can be applied to other systems as well. In fact, later in the paper we develop an abstract construction for dissipative terms with the same general properties as the above examples. When this method is applied to fluids one gets a dissipative mechanism related to that of Vallis, Carnevale, and Young [1989] and Shepherd [1992] as follows. One modifies the Euler equations for a perfect fluid, namely

\[
\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p
\]

where \(v\) is the velocity field, assumed divergence free and parallel to the boundary of the fluid container, and where \(p\) is the pressure. With dissipation, the equations become:

\[
\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + \alpha \mathbf{P} \left( (L_{u(v)}v)^t \right)
\]

where \(\alpha\) is a positive constant, \(\mathbf{P}\) is the Hodge projection onto the divergence free part, and where

\[
u(v) = \mathbf{P} \left( (L_v v^t)^t \right).
\]

The flat and sharp symbols denote the index lowering and raising operators induced by the metric; that is, the operators that convert vectors to one forms and vice versa.
Written in terms of the vorticity, these equations become
\[
\frac{d}{dt} \omega + {\mathcal L}_v \omega = \alpha {\mathcal L}_{u(v)} \omega.
\]
This dissipative term preserves the coadjoint orbits, that is, the isovortical surfaces (in either two or three dimensions, or in fact, on any Riemannian manifold), and with it, the time derivative of the energy is strictly negative (except at equilibria, where it is zero). As we shall see, there is a similar dissipative term in the case of the Vlasov-Poisson equation for plasma physics.

2 Dissipative Systems

For later use, it will be useful to recall some of the basic and essentially well known facts about dissipative mechanical systems. Let \( Q \) be a manifold, \( L : TQ \to \mathbb{R} \) be a smooth function, and let \( \tau : TQ \to Q \) be the tangent bundle projection. Let \( \mathbb{F}L : TQ \to T^*Q \) be the fiber derivative of \( L \); recall that it is defined by

\[
\langle \mathbb{F}L(v), w \rangle = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L(v + \epsilon w),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the pairing between the tangent and cotangent spaces. We also recall that the vertical lift of a vector \( w \in T_vQ \) along \( v \in T_vQ \) is defined by

\[
\text{vert}_v(w) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (v + \epsilon w) \in T_vTQ.
\]

The action and energy of \( L \) are defined by

\[
A(v) = \langle \mathbb{F}L(v), v \rangle
\]

and

\[
E(v) = A(v) - L(v).
\]

Let \( \Omega_L = (\mathbb{F}L)^*\Omega \) denote the pull back of the canonical symplectic form on \( T^*Q \) by the fiber derivative of \( L \); we also let \( \Theta \) denote the canonical one form on \( T^*Q \) with the sign conventions

\[
\Theta(\alpha) \cdot w = \langle \alpha, T\pi(w) \rangle
\]

where \( \alpha \in T^*Q, w \in T_\alpha(T^*Q) \) and \( \pi : T^*Q \to Q \) is the canonical cotangent bundle projection. In our conventions, \( \Omega = -d\Theta \) so that if \( \Theta_L \) denotes the pull back of \( \Theta \) by the fiber derivative, then \( \Omega_L = -d\Theta_L \).

A vector field \( Z \) on \( TQ \) is called a Lagrangian vector field for \( L \) if

\[
i_Z \Omega_L = dE
\]

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where $i_Z$ denotes the operation of interior multiplication (or contraction) by the vector field $Z$. In this generality, $Z$ need not exist, nor be unique. However, we will assume throughout that $Z$ is a second order equation; that is, $T\tau \circ Z$ is the identity on $TQ$. A second order equation is a Lagrangian vector field if and only if the Euler-Lagrange equations hold in local charts. We note that, by skew symmetry of $\Omega_L$, energy is always conserved; that is, $E$ is constant along an integral curve of $Z$. We also recall that the Lagrangian is called regular if $\Omega_L$ is a (weak) symplectic form; that is, if it is nondegenerate. This is equivalent to the second fiber derivative of the Lagrangian being, in local charts, also weakly nondegenerate. In the regular case, if the Lagrangian vector field exists, it is unique, and is given by the Hamiltonian vector field with energy $E$ relative to the symplectic form $\Omega_L$. If, in addition, the fiber derivative is a global diffeomorphism, then $Z$ is the pull back by the fiber derivative of the Hamiltonian vector field on the cotangent bundle with Hamiltonian $H = E \circ (\Omega L)^{-1}$. It is well known how one can pass back and forth between the Hamiltonian and Lagrangian pictures in this hyperregular case (see, for example, Abraham and Marsden [1978]).

We now turn to the definition of a dissipative system. Consider a general Lagrangian vector field $Z$ for a (not necessarily regular) Lagrangian on $TQ$. A vector field $Y$ on $TQ$ is called weakly dissipative provided that it is vertical (i.e., $T\tau \circ Y = 0$) and if, at each point of $TQ$,

$$\langle dE, Y \rangle \leq 0. \quad (17)$$

If the inequality is pointwise strict at each nonzero $v \in TQ$, then we say that the vector field $Y$ is dissipative. A dissipative Lagrangian system on $TQ$ is a vector field of the form $X = Z + Y$, where $Z$ is a (second order) Lagrangian vector field and $Y$ is a dissipative vector field. We use the word “weak” as above. It is clear by construction that the time derivative of the energy along integral curves of $X$ is nonpositive for weakly dissipative systems, and is strictly negative at nonzero points for dissipative systems. Define the one form $\Delta^Y$ on $TQ$ by

$$\Delta^Y = -i_Y \Omega_L.$$

**Proposition 2.1** If $Y$ is vertical, then $\Delta^Y$ is a horizontal one-form, i.e., $\Delta^Y(U) = 0$ for any vertical vector field $U$ on $TQ$. Conversely, given a horizontal one form $\Delta$ on $TQ$, and assuming that $L$ is regular, the vector field $Y$ on $TQ$ defined by $\Delta = -i_Y \Omega_L$, is vertical.

**Proof** This follows from a straightforward calculation in local coordinates. We use the fact that a vector field $Y(u, e) = (Y_1(u, e), Y_2(u, e))$ is vertical if and only if the first component $Y_1$ is zero, and the local formula for $\Omega_L$ (see, for example, Abraham and Marsden [1978], Section 3.5):

$$\Omega_L(u, e)(Y_1, Y_2, (U_1, U_2))$$

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This shows that \( (i_Y \Omega_L)(U) = 0 \) for all vertical \( U \) is equivalent to
\[
D_2 D_2 L(u, e)(U_2, Y_1) = 0.
\]

If \( Y \) is vertical, this is clearly true. Conversely if \( L \) is regular, and the last displayed equation is true, then \( Y \) must be vertical. \( \blacksquare \)

**Proposition 2.2** Any fiber preserving map \( F : TQ \rightarrow T^*Q \) over the identity induces a horizontal one-form \( \tilde{F} \) on \( TQ \) by
\[
\tilde{F}(v) \cdot V_v = \langle F(v), T_v \tau(V_v) \rangle
\]
where \( v \in TQ \), and \( V_v \in T_v(TQ) \). Conversely, formula (19) defines, for any horizontal one-form \( \tilde{F} \), a fiber preserving map \( F \) over the identity. Any such \( F \) is called a *force field* and thus in the regular case, any vertical vector field \( Y \) is induced by a force field.

**Proof** Given \( F \), formula (19) clearly defines a smooth one-form \( \tilde{F} \) on \( TQ \). If \( V_v \) is vertical, then the right hand side of formula (19) vanishes, and so \( \tilde{F} \) is a horizontal one-form. Conversely, given a horizontal one-form \( \tilde{F} \) on \( TQ \), and given \( v, w \in T_v(Q) \), let \( V_v \in T_v(TQ) \) be such that \( T_v \tau(V_v) = w \). Then define \( F \) by formula (19); i.e.,
\[
\langle F(v), w \rangle = \tilde{F}(v) \cdot V_v.
\]
Since \( \tilde{F} \) is horizontal, we see that \( F \) is well-defined, and its expression in charts shows that it is smooth. \( \blacksquare \)

**Corollary 2.3** A vertical vector field \( Y \) on \( TQ \) is dissipative if and only if the force field \( F^Y \) that it induces satisfies \( \langle F^Y(v), v \rangle < 0 \) for all nonzero \( v \in TQ \) (\( \leq 0 \) for the weakly dissipative case).

**Proof** Let \( Y \) be a vertical vector field. By Proposition 2.1, \( Y \) induces a horizontal one-form \( \Delta^Y = -i_Y \Omega_L \) on \( TQ \) and, by Proposition 2.2, \( \Delta^Y \) in turn induces a force field \( F^Y \) given by
\[
\langle F^Y(v), w \rangle = \Delta^Y(v) \cdot V_v = -\Omega_L(v)(Y(v), V_v),
\]
where \( T \tau(V_v) = w \) and \( V_v \in T_v(TQ) \). If \( Z \) denotes the Lagrangian system defined by \( L \), we get
\[
(dE \cdot Y)(v) = (i_Z \Omega_L)(Y)(v) = \Omega_L(Z, Y)(v) = -\Omega_L(v)(Y(v), Z(v)) = \langle F^Y(v), T_v \tau(Z(v)) \rangle = \langle F^Y(v), v \rangle
\]

since \( Z \) is a second order equation. We conclude that \( dE \cdot Y < 0 \) if and only if \( \langle F^Y(v), v \rangle < 0 \) for all \( v \in TQ \), which gives the result. \( \blacksquare \)
Definition 2.4 Given a dissipative vector field \( Y \) on \( TQ \), let \( F^Y : TQ \to T^*Q \) be the induced force field. If there is a function \( R : TQ \to \mathbb{R} \) such that \( F^Y \) is the fiber derivative of \(-R\), then \( R \) is called a Rayleigh dissipation function.

In this case, dissipativity of \( Y \) reads \( D^Y_R(q, v) \cdot v > 0 \). Thus, if \( R \) is linear in the fiber variable, the Rayleigh dissipation function takes on the classical form \( \langle R(q)v, v \rangle \) where \( \mathcal{R}(q) : TQ \to T^*Q \) is a bundle map over the identity that defines a symmetric positive definite form on each fiber of \( TQ \).

Treating \( \Delta^Y \) as the exterior force one-form acting on a mechanical system with a Lagrangian \( L \), we now will write the governing equations of motion. The basic principle is of course the Lagrange-d’Alembert principle. First, we recall the definition from Vershik and Faddeev [1981] and Wang and Krishnaprasad [1992].

Definition 2.5 The Lagrangian force associated with a given Lagrangian \( L \) and a given second order vector field \( X \) is the horizontal one form on \( TQ \) defined by

\[
\Phi_L(X) = \iota_X \Omega_L - dE.
\]  

(21)

Given a horizontal one form \( \omega \) (referred to as the exterior force one form), the local Lagrange d’Alembert principle states that

\[
\Phi_L(X) + \omega = 0.
\]  

(22)

It is easy to check that \( \Phi_L(X) \) is indeed horizontal if \( X \) is second order. Conversely, if \( L \) is regular and if \( \Phi_L(X) \) is horizontal, then \( X \) is second order. One can also formulate an equivalent principle in variational form.

Definition 2.6 Given a Lagrangian \( L \) and a force field \( F \) (as defined in Proposition 2.2), the integral Lagrange d’Alembert principle for a curve \( q(t) \) in \( Q \) is

\[
\delta \int_a^b L(q(t), \dot{q}(t)) dt + \int_a^b F(q(t), \dot{q}(t)) \cdot \delta q dt = 0,
\]  

(23)

where the variation is given by the usual expression

\[
\delta \int_a^b L(q(t), \dot{q}(t)) dt = \int_a^b \left( \frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \frac{d}{dt} \delta q^i \right) dt.
\]  

(24)

for a given variation \( \delta q \) (vanishing at the endpoints).

In this expression, we have employed coordinate notation so that the coordinates of \( q \) are denoted \( q^1, q^2, \ldots, q^n \) or \( q^i, i = 1, \ldots, n \), and there is an implied summation over repeated indices. However, it should be noted that this coordinate notation is
intended for the finite dimensional case, and one should note that the developments
here apply to infinite dimensional problems as well, such as fluids and plasmas.

The two forms of the Lagrange d’Alembert principle are equivalent. This fol-

low from the fact that both give the Euler-Lagrange equations with forcing in local
coordinates (provided that $Z$ is second order). We shall see this in the following
development.

**Proposition 2.7** Let the exterior force one-form $\omega$ be associated to a vertical vector
field $Y$, i.e., let $\omega = \Delta^Y = -i_Y \Omega_L$. Then $X = Z + Y$ satisfies the local Lagrange-
d’Alembert principle. Conversely, if, in addition, $L$ is regular, the only second order
vector field $X$ satisfying the local Lagrange-d’Alembert principle is $X = Z + Y$.

**Proof** For the first part, the equality $\Phi_L(X) + \omega = 0$ is a simple verification. For
the converse, we already know that $X$ is a solution, and uniqueness is guaranteed by
regularity.

To develop the differential equations associated to $X = Z + Y$, we take $\omega = -i_Y \Omega_L$
and note that, in a coordinate chart, $Y(q, v) = (0, Y_2(q, v))$ since $Y$ is vertical, i.e.,
$Y_1 = 0$. From the local formula for $\Omega_L$, we get

$$
\omega(q, v) \cdot (u, w) = D_2 D_2 L(q, v) \cdot Y_2(q, v) \cdot u. 
$$

Letting $X(q, v) = (v, X_2(q, v))$, one finds that

$$
\Phi_L(X)(q, v) \cdot (u, w) = (-D_1(D_2 L(q, v)\cdot v - D_2 D_2 (q, v) \cdot X_2(q, v) + D_1 L(q, v)) \cdot u. 
$$

Thus, the local Lagrange-d’Alembert principle becomes

$$
-D_1(D_2 L(q, v)\cdot v - D_2 D_2 L(q, v) \cdot X_2(q, v) + D_1 L(q, v) + D_2 D_2 L(q, v) \cdot Y_2(q, v) = 0
$$

Setting $v = dq/dt$ and $X_2(q, v) = dv/dt$, the preceding relation and the chain rule
gives

$$
\frac{d}{dt}(D_2 L(q, v)) - D_1 L(q, v) = D_2 D_2 L(q, v) \cdot Y_2(q, v)
$$

which, in finite dimensions, reads,

$$
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^j} \right) - \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} Y^j(q^k, \dot{q}^k). 
$$

The force one-form $\Delta^Y$ is therefore given by

$$
\Delta^Y(q^k, \dot{q}^k) = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} Y^j(q^k, \dot{q}^k) dq^i
$$

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and the corresponding force field is
\[ F^Y = \left( q^i, \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} Y^j(q^k, \dot{q}^k) \right). \] (31)

Thus, the condition for an integral curve takes the form of the standard Euler-Lagrange equations with forces:
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = F^Y_i(q^k, \dot{q}^k). \] (32)

Since the integral Lagrange-d'Alembert principle gives the same equations, it follows that the two principles are equivalent. From now on, we will refer to either one as simply the Lagrange-d'Alembert principle.

Finally, if the force field is given by a Rayleigh dissipation function \( R \), then the Euler-Lagrange equations with forcing become:
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = -\frac{\partial R}{\partial \dot{q}^i}. \] (33)

Combining Corollary 2.3 with the fact that the differential of \( E \) along \( Z \) is zero, we find that under the flow of the Euler-Lagrange equations with forcing of Rayleigh dissipation type,
\[ \frac{d}{dt} E(q, v) = F(v) \cdot v = -\nabla R(q, v) \cdot v < 0. \] (34)

### 3 Equivariant Dissipation

In this section we study Lagrangian systems that are invariant under a group action and we will add to them, in the sense of the preceding section, dissipative vector fields that are equivariant. This invariance property will yield dissipative mechanisms that preserve the basic conserved quantities, yet dissipate energy, as we shall see.

Let \( G \) be a Lie group that acts on the configuration manifold \( Q \) and assume that the lifted action leaves the Lagrangian \( L \) invariant. In this case, the fiber derivative \( \mathbb{F}L : TQ \to T^*Q \) is equivariant with respect to this action on \( TQ \) and the dual action on \( T^*Q \). Evidently, the action \( A \), the energy \( E \), and the Lagrangian two form \( \Omega_L \) are all invariant under the action of \( G \) on \( TQ \). Let \( Z \) be the Lagrangian vector field for the Lagrangian \( L \), which we assume to be regular. Because of regularity, the vector field \( Z \) is also invariant under \( G \). If the action is free and proper, so that \((TQ)/G \) is a manifold, then the vector field and its flow \( F_t \) drop to a vector field \( Z^G \) and flow \( F^G_t \) on \((TQ)/G \). The determination of this dropped vector field and flow is the subject of Lagrangian reduction (see Marsden and Scheurle [1993a,b]).
Let $J : TQ \to \mathfrak{g}^*$ be the momentum map associated with the $G$ action, given by

$$J(v_q) \cdot \xi = \langle \mathbb{P}L(v_q), \xi_Q(q) \rangle$$

for $v_q \in T_qQ$ and for $\xi \in \mathfrak{g}$, where $\xi_Q$ denotes the infinitesimal generator for the action on $Q$. The infinitesimal generator for the action on the tangent bundle will be likewise denoted by $\xi_{TQ}$ and for later use, we note the relation $T \tau \circ \xi_{TQ} = \xi_Q \circ \tau$. If $v(t)$ denotes an integral curve of the vector field with an equivariant dissipation term $Y$ added, as in the preceding section, and we let $J^\xi(v) = \langle J(v), \xi \rangle$ be the $\xi$-component of the momentum mapping, then we have

$$\frac{d}{dt}J^\xi(v(t)) = \mathsf{d}J^\xi(v(t)) \cdot Z(c(t)) + \mathsf{d}J^\xi(v(t)) \cdot Y(v(t)).$$

The first term vanishes by conservation of the momentum map for the Lagrangian vector field $Z$. From (20) and the definition of the momentum map, we get

$$\mathsf{d}J^\xi(v) \cdot Y(v) = (i_{\xi_{TQ}} \Omega_L)(Y)(v)
= - (i_{\mathsf{d}L}(\xi_{TQ}))(v)
= \left\langle F^Y(v), T_v \tau(\xi_{TQ}(v)) \right\rangle
= \left\langle F^Y(v), \xi_Q(\tau(v)) \right\rangle$$

and therefore

$$\frac{d}{dt}J^\xi(v(t)) = \left\langle F^Y, \xi_Q \circ \tau \right\rangle(v(t)).$$

In particular, if $F$ is determined by a Rayleigh dissipation function, we get

$$\frac{d}{dt}J^\xi(v(t)) = - \left\langle \mathbb{P}R, \xi_Q \circ \tau \right\rangle(v(t)).$$

We summarize this discussion as follows.

**Proposition 3.1** The momentum map $J : TQ \to \mathfrak{g}^*$ is conserved under the flow of a $G$-invariant dissipative vector field $Z + Y$ if and only if $\left\langle F^Y, \xi_Q \circ \tau \right\rangle = 0$ for all Lie algebra elements $\xi \in \mathfrak{g}$. If the force field $F^Y$ is given by a Rayleigh dissipation function $R : TQ \to \mathbb{R}$, i.e., $F^Y = -\mathbb{P}R$, then this condition becomes $\left\langle \mathbb{P}R, \xi_Q \circ \tau \right\rangle = 0$ for all $\xi \in \mathfrak{g}$. Moreover, $G$-invariance of $Y$ is equivalent to $G$-equivariance of $\mathbb{P}R$ and if $R$ is $G$-invariant, then $\mathbb{P}R$ is $G$-equivariant.

We note that equivariance of $\mathbb{P}R$ need not imply invariance of $R$. (Consider, for example, $G = S^1$ and $Q = S^1$ with $R(\theta, \dot{\theta}) = (\dot{\theta})^2/2 + f(\theta)$ where $f$ is any non-invariant function of $\theta$ such as $f(\theta) = \sin \theta$.) Also note that if the action of $G$ on $Q$ is transitive, then conservation of $J$ along the flow of $Z + Y$ implies that the force field $F^Y$ vanishes and hence, if $L$ is regular, that $Y$ also vanishes. Thus, in the regular
case and for a transitive group action, there is no dissipative vector field preserving the momentum map.

In this paper we shall consider dissipative vector fields for which the flow drops to the reduced spaces. Thus, a first requirement is that $Y$ be a vertical $G$-invariant vector field on $TQ$. A second requirement is that all integral curves $v(t)$ of $Z + Y$ preserve the sets $J^{-1}(O)$, where $O$ is an arbitrary coadjoint orbit in $\mathfrak{g}^*$. Under these hypotheses the vector field $Z + Y$ induces a vector field $Z^G + Y^G$ on $(TQ)/G$ that preserves the symplectic leaves of this Poisson manifold, namely all reduced spaces $J^{-1}(O)/G$.

The condition that $v(t) \in J^{-1}(O)$ is equivalent to $J(v(t)) \in O$, i.e., to the existence of an element $\eta(t) \in \mathfrak{g}$ such that $dJ(v(t))/dt = \text{ad}^*_{\eta(t)}J(v(t))$, or

$$\frac{dJ^\xi(v(t))}{dt} = J^\xi_{\eta(t)}(v(t)),$$

(40)

for all $\xi \in \mathfrak{g}$. In view of (38), we get the following:

**Corollary 3.2** The integral curves of the vector field $Z + Y$, for $Y$ a vertical $G$-invariant vector field on $TQ$ and $Z$ the Lagrangian vector field of a $G$-invariant Lagrangian function $L : TQ \to \mathbb{R}$, preserve the inverse images of the coadjoint orbits in $\mathfrak{g}^*$ by the momentum map $J$ if and only if for each $v \in TQ$ there is some $\eta(v) \in \mathfrak{g}$ such that

$$\langle F^Y, \xi_Q \circ \tau \rangle(v) = J^\xi_{\eta(v)}(v)$$

(41)

for all $\xi \in \mathfrak{g}$. As before, $F^Y$ denotes the force field induced by $Y$.

We will see in section 5 how to construct such force fields in the case $Q = G$. As we mentioned in the introduction, these force fields do not literally come from a Rayleigh dissipation function in the naive sense, but rather come from a Rayleigh dissipation function (the energy itself!) in a more sophisticated sense.

### 4 The Euler-Poincaré equations

In this section we present complete proofs for the reduction theory of the Euler-Poincaré equations (see Poincaré [1901], Arnold [1988], Chetaev [1961]). These results were stated in Marsden and Scheurle [1993b], but proofs were given only for the case of matrix groups. We will use these methods in conjunction with the results above to discuss the forced Euler-Poincaré equations.

A key step in the reduction of the Euler-Lagrange equations from the tangent bundle $TG$ of a Lie group $G$ to its Lie algebra $\mathfrak{g}$ is to understand how to drop the variational principle to the quotient space. To do this we need to characterize variations of curves in $TG$ purely in terms of variations of curves in $\mathfrak{g}$. To accomplish this, we use a method of Alekseevski and Michor [1993] that constructs a large class of connections on a bundle of the form $G \times M$ with explicit formulae for the curvature.
We begin with their general construction. Assume that the Lie group \( G \) acts on
the left on a manifold \( M \) and let \( \alpha \in \Omega^1(M; \mathfrak{g}) \) be a given smooth \( \mathfrak{g} \)-valued one-form on \( M \). For \( u_g \in T_gG \) and \( v_m \in T_mM \), define
\[
\Gamma^i(g, m)(u_g, v_m) = T_gR_{g^{-1}}(u_g) - \text{Ad}_g(\alpha(m) \cdot v_m).
\]
(42)
Then \( \Gamma^i \in \Omega^1(G \times M; \mathfrak{g}) \). The left action of \( G \) on \( G \times M \) makes \( \text{pr}_2 : G \times M \to M \)
into a principal left \( G \)-bundle and if \( \xi \in \mathfrak{g} \), the infinitesimal generator it defines equals
\( \xi_{G \times M}(g, m) = (T_gR_g(\xi), 0) \). Therefore, by (42) we see that \( \Gamma^i(\xi_{G \times M}) = \xi \) and
\[
\Gamma^i(hg, m)(T_gL_h(u_g), v_m) = T_{hg} \cdot R_{(hg)^{-1}}(T_gL_h(u_g)) - \text{Ad}_{hg}(\alpha(m) \cdot v_m)
\]
\[
= \text{Ad}_h(T_gR_{g^{-1}}(u_g)) - \text{Ad}_g(\alpha(m) \cdot v_m)
\]
\[
= \text{Ad}_h(\Gamma^i(g, m)(u_g, v_m))
\]
so that \( \Gamma^i \) defines a left principal connection one-form on the trivial bundle \( \text{pr}_2 : G \times M \to M \). The horizontal subbundle \( H \subset T(G \times M) \) is therefore given by
\[
H^i_{(g, m)} = \{(T_gL_g(\alpha(m) \cdot v_m), v_m) \mid v_m \in T_mM\}.
\]
(43)
To compute the curvature of this connection, we recall that if \( \lambda, \rho \in \Omega^1(G; \mathfrak{g}) \) are
defined by
\[
\lambda(u_g) = T_gL_{g^{-1}}(u_g), \quad \rho(u_g) = T_gR_{g^{-1}}(u_g),
\]
then the Maurer-Cartan structure equations state that
\[
d\lambda + \frac{1}{2}[\lambda, \lambda]^\wedge = 0, \quad d\rho - \frac{1}{2}[\rho, \rho]^\wedge = 0,
\]
(45)
where \( d \) is the exterior derivative and \( [\cdot, \cdot]^\wedge \) is the exterior product induced on \( \mathfrak{g} \) by
its Lie algebra bracket. Our coefficient conventions for \( [\cdot, \cdot]^\wedge \) are the following: if \( \alpha, \beta \in \Omega^1(M; \mathfrak{g}) \) then
\[
[\alpha, \beta]^\wedge(u, v) = [\alpha(u), \beta(v)] - [\alpha(v), \beta(u)] = [\beta, \alpha]^\wedge(u, v).
\]
(46)
Finally, recall that for left principal bundles, the structure equations state that the curvature \( \Omega^i \) is given by
\[
\Omega^i = d\Gamma^i - \frac{1}{2}[\Gamma^i, \Gamma^i]^\wedge.
\]
(47)
To compute the curvature, it is convenient to rewrite \( \Gamma^i \) given by (42) intrinsically as
\[
\Gamma^i = \text{pr}_1^*\rho - (\text{pr}_1^*\text{Ad}.)(\text{pr}_2^*\alpha).
\]
(48)
where the dot indicates a blank variable. Then we get
\[
\Omega^i = d\Gamma^i - \frac{1}{2}[\Gamma^i, \Gamma^i]^\wedge
\]
\[
= d\text{pr}_1^*\rho - \frac{1}{2}[\text{pr}_1^*\rho, \text{pr}_1^*\rho]^\wedge - d(\text{pr}_1^*\text{Ad}.(\text{pr}_2^*\alpha))
\]
\[
+ [\text{pr}_1^*\rho, \text{pr}_1^*\text{Ad}.(\text{pr}_2^*\alpha)]^\wedge - \frac{1}{2}[\text{pr}_1^*\text{Ad}.(\text{pr}_2^*\alpha), \text{pr}_1^*\text{Ad}.(\text{pr}_2^*\alpha)]^\wedge.
\]
(49)
The first two terms equal \( pr_1^*(d\rho - \frac{1}{2}[\rho, \rho]^\flat) = 0 \) by (45). The third term equals

\[
-d(pr_1^*\text{Ad.}(pr_2^*\alpha)) = -(pr_1^*d\text{Ad.}) \wedge pr_2^*\alpha - (pr_1^*\text{Ad.})(pr_2^*d\alpha) \tag{50}
\]

However, if \( \xi \in \mathfrak{g} \), we have

\[
(d\text{Ad.})(g) \cdot T_e R_g \xi = \frac{d}{dt} \bigg|_{t=0} \text{Ad}_{\exp(t\xi)_g} = \text{ad}_\xi \circ \text{Ad}_g \tag{51}
\]

and therefore if \((u^1_g, v^1_m), (u^2_g, v^2_m) \in T_g G \times T_m M\) we get by (50)

\[
-d(pr_1^*\text{Ad.}(pr_2^*\alpha))(g, m)((u^1_g, v^1_m), (u^2_g, v^2_m))
\]

\[
= -(\text{ad}_{pr_1^*\rho}(u^1_g) \circ \text{Ad}_g)((\alpha(m) \cdot v^2_m) + (\text{ad}_{pr_2^*\rho}(u^2_g) \circ \text{Ad}_g)((\alpha(m) \cdot v^1_m) \nonumber
\]

\[
- \text{Ad}_g(d\alpha(m)(v^1_m, v^2_m))
\]

\[
= -[\rho(u^1_g), \text{Ad}_g(\alpha(m) \cdot v^2_m)] + [\rho(u^2_g), \text{Ad}_g(\alpha(m) \cdot v^1_m)]
\nonumber
\]

\[
- \text{Ad}_g(d\alpha(m)(v^1_m, v^2_m))
\]

\[
= -[pr_1^*\rho, (pr_1^*\text{Ad.})(pr_2^*\alpha)](g, m)((u^1_g, v^1_m), (u^2_g, v^2_m))
\]

\[
- (pr_1^*\text{Ad.})(pr_2^*d\alpha))(g, m)((u^1_g, v^1_m), (u^2_g, v^2_m)) \tag{52}
\]

Therefore, the first summand in (52) of the third term in (49) cancels the fourth term in (49) and we get

\[
\Omega^r = -(pr_1^*\text{Ad.} \left( pr_2^* \left( d\alpha + \frac{1}{2}[\alpha, \alpha]^\flat \right) \right) \tag{53}
\]

**Proposition 4.1** The curvature of the connection one-form \( \Gamma^l \in \Omega^1(G \times M; \mathfrak{g}) \) given by (42) has the expression (53).

If we assume that \( M \times G \to M \) is a right action then \( \Gamma^r \in \Omega^1(M \times G, \mathfrak{g}) \) given by

\[
\Gamma^r(m, g)(v_m, u_g) = T_g L_{g^{-1}}(u_g) - \text{Ad}_{g^{-1}}((\alpha(m) \cdot v_m) \tag{54}
\]

is a right connection one-form whose curvature is given by

\[
\Omega^r = -(pr_1^*\text{Ad.} \circ \text{Inv} \left( pr_2^* \left( d\alpha + \frac{1}{2}[\alpha, \alpha]^\flat \right) \right) \tag{55}
\]

Here, Inv denotes the inversion map. The relative sign change occurs, since for right bundles and right connections, the structure equations are \( d\Gamma^r + (1/2)[\Gamma^r, \Gamma^r]^\flat = \Omega^r \).

**Corollary 4.2** The connection \( \Gamma^l \) (respectively \( \Gamma^r \)) is flat if and only if \( d\alpha + (1/2)[\alpha, \alpha]^\flat = 0 \) (respectively \( d\alpha - (1/2)[\alpha, \alpha]^\flat = 0 \)).
Now recall that a principal connection is flat if and only if its horizontal subbundle is integrable. If \( f : M \to G \), we will denote, following Kolár, Michor, and Slovák [1993], by \( \delta^l f, \delta^r f \in \Omega^1(M; g) \) the left and right logarithmic derivatives of \( f \):

\[
\delta^l f(m) = T_{f(m)}L_{f(m)^{-1}} \circ T_m f, \quad \delta^r f(m) = T_{f(m)}R_{f(m)^{-1}} \circ T_m f.
\]  

(56)

**Corollary 4.3** For any smooth map \( f : M \to G \), its logarithmic derivatives satisfy

\[
d\delta^l f + \frac{1}{2} [\delta^l f, \delta^l f]^* = 0
\]

(57)

\[
d\delta^r f - \frac{1}{2} [\delta^r f, \delta^r f]^* = 0.
\]

(58)

Conversely, given a one-form \( \alpha \in \Omega^1(M; g) \) satisfying \( d\alpha + (1/2) [\alpha, \alpha]^* = 0 \) (respectively \( d\alpha - (1/2) [\alpha, \alpha]^* = 0 \)) for every \( m \in M \) there is an open set \( U \subset M, m \in U \) and a smooth function \( f : U \to G \) such that \( \delta^l f = \alpha[U \) (respectively \( \delta^r f = \alpha[U \). If \( M \) is simply connected we can take \( U = M \).

**Proof** Given \( f : M \to G \) consider the left principal connection \( \Gamma^l \) defined by \( \alpha = \delta^l f \) on the trivial bundle \( \text{pr}_2 : G \times M \to M \). By (43), its horizontal subbundle \( H^l \) equals

\[
H^l_{(g,m)} = \{(T_m L_{g \cdot f(m)^{-1}} \circ f)(v_m), v_m) \mid v_m \in T_m M \}.
\]

(59)

This is, however, obviously integrable, the leaf through \( (g, m) \) being

\[
H^l_{(g,m)} = \{(g f(m)^{-1} f(x), x) \mid x \in M \}.
\]

(60)

Therefore the curvature \( \Omega^l \) vanishes and (57) holds by Corollary 4.2. Note that \( H^l_{(f(m),m)} = \text{graph} \ f \).

Conversely, assume \( \alpha \in \Omega^1(M; g) \) satisfies \( d\alpha + (1/2) [\alpha, \alpha]^* = 0 \). By Corollary 4.2 the connection \( \Gamma^l \) it defines is flat and therefore its horizontal subbundle \( H \), given by (43), is integrable. Let \( \mathcal{H} \) be one of the leaves of the induced foliation. Then \( \text{pr}_2 : \mathcal{H} \to M \) is a smooth covering space, so in particular, if \( m \in M \) there are open sets \( U \subset M, m \in U \), and \( V \subset \mathcal{H} \) such that \( \text{pr}_2 : V \to U \) is a diffeomorphism. Let \( x \in U \mapsto (f(x), x) \in V \) be its inverse, which thus defines a smooth map \( f : U \to G \). We claim that \( \delta^l f = \alpha[U \). Indeed, \((T_x f(v_x), v_x) \in T_{(f(x), x)}, \mathcal{H} = H^l_{(f(x), x)} \) so by (43),

\[
(T_x f(v_x), v_x) = (T_x f(v_x), \alpha(x) \cdot v_x), v_x), \text{whence } \alpha(x) = T_{f(x)} f(v_x) = \delta^l f.
\]

If \( M \) is simply connected the covering \( \text{pr}_2 : \mathcal{H} \to M \) is necessarily a homeomorphism and hence a diffeomorphism. The open set \( U \) can therefore be chosen to equal \( M \).

**Corollary 4.4** Let \( g : U \subset \mathbb{R}^2 \to G \) be a smooth map and denote by \( \xi(t, \varepsilon) = TL_{g(t, \varepsilon)}^{-1} (\partial g(t, \varepsilon)/\partial t) \) and \( \eta(t, \varepsilon) = TL_{g(t, \varepsilon)}^{-1} (\partial g(t, \varepsilon)/\partial \varepsilon) \). Then

\[
\frac{\partial \xi}{\partial \varepsilon} - \frac{\partial \eta}{\partial t} = [\xi, \eta]
\]

(61)
Conversely, if \( U \) is simply connected and \( \xi, \eta : U \to \mathfrak{g} \) are smooth functions satisfying (61) then there exists a smooth function \( g : U \to G \) such that \( \xi(t, \varepsilon) = TL_{g(t,\varepsilon)^{-1}}(\partial g(t,\varepsilon)/\partial t) \) and \( \eta(t, \varepsilon) = TL_{g(t,\varepsilon)^{-1}}(\partial g(t,\varepsilon)/\partial \varepsilon) \).

**Proof** Take in Corollary 4.3, \( M = U, f = g, \) and evaluate (57) on the basis vector fields \((\partial/\partial t, \partial/\partial \varepsilon)\). Since \([\partial/\partial t, \partial/\partial \varepsilon] = 0\) we get

\[
\frac{\partial}{\partial t} \left[ \delta^l_g \left( \frac{\partial}{\partial \varepsilon} \right) \right] - \frac{\partial}{\partial \varepsilon} \left[ \delta^l_g \left( \frac{\partial}{\partial t} \right) \right] + \frac{1}{2} \left[ \delta^l_g \left( \frac{\partial}{\partial \varepsilon} \right), \delta^l_g \left( \frac{\partial}{\partial t} \right) \right] - \frac{1}{2} \left[ \delta^l_g \left( \frac{\partial}{\partial \varepsilon} \right), \delta^l_g \left( \frac{\partial}{\partial \varepsilon} \right) \right] = 0. \tag{62}
\]

However, by (56)

\[
\delta^l_g \left( \frac{\partial}{\partial t} \right) = TL_{g(t,\varepsilon)^{-1}} \frac{\partial g(t,\varepsilon)}{\partial t} = \xi(t, \varepsilon)
\]

and similarly \( \delta^l_g(\partial/\partial \varepsilon) = \eta(t, \varepsilon) \), so that (62) becomes

\[
\frac{\partial \eta}{\partial t} - \frac{\partial \xi}{\partial \varepsilon} + \frac{1}{2} [\xi(t, \varepsilon), \eta(t, \varepsilon)] - \frac{1}{2} [\eta(t, \varepsilon), \xi(t, \varepsilon)] = 0
\]

which is equivalent to (61).

Conversely, given \( U \subset \mathbb{R}^2 \) simply connected and \( \xi, \eta : U \to \mathfrak{g} \) satisfying (61), define \( \alpha \in \Omega^1(U; \mathfrak{g}) \) by \( \alpha = \xi(t, \varepsilon) dt + \eta(t, \varepsilon) d\varepsilon \). Then by (61)

\[
d\alpha + \frac{1}{2} [\alpha, \alpha] = \left( -\frac{\partial \xi}{\partial \varepsilon} + \frac{\partial \eta}{\partial t} \right) dt \wedge d\varepsilon + \left( \frac{1}{2} [\xi, \eta] - \frac{1}{2} [\eta, \xi] \right) dt \wedge d\varepsilon = 0.
\]

By Corollary 4.3 there is a function \( g : U \to G \) such that \( \delta^l_g = \alpha \) which, in view of the computations above, is equivalent to \( \xi(t, \varepsilon) = TL_{g(t,\varepsilon)^{-1}}(\partial g(t,\varepsilon)/\partial t) \) and \( \eta(t, \varepsilon) = TL_{g(t,\varepsilon)^{-1}}(\partial g(t,\varepsilon)/\partial \varepsilon) \). \( \blacksquare \)

**Remarks**

1. For matrix groups, formula (61) is an easy verification.

2. Formula (61) can also be deduced from the expression of the complete left trivialization of elements of TTG using the ideas in Marsden, Ratiu, and Raugel [1991]. If \( V \in TTG \) is represented as an element of \( G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \), its expression equals \( (g, \eta, T_g L_{\gamma^{-1}} \dot{g}(0), \dot{\gamma}(0) + [T_g L_{\gamma^{-1}} \dot{g}(0), \eta(0)]) \) where \( V \) is represented as \( V = (d/ds)_{s=0} (d/dt)_{t=0} g(t) \exp s \eta(t) \) for curves \( g(t) \) in \( G \), \( g(0) = g \), and \( \eta(t) \) in \( \mathfrak{g} \). Formula (61) is then the fourth component of \( V \) in this trivialization.

Next, we turn to the formulation of the Euler-Poincaré equations and the reduced variational principle.

**Theorem 4.5** Let \( G \) be a Lie group and \( L : TG \to \mathbb{R} \) a left invariant Lagrangian. Let \( l : \mathfrak{g} \to \mathbb{R} \) be its restriction to the tangent space at the identity. For a curve \( g(t) \in G \), let \( \xi(t) = g(t)^{-1} \cdot \dot{g}(t) \); i.e., \( \xi(t) = T_{g(t)} L_{g(t)^{-1}} \dot{g}(t) \). Then the following are equivalent
i g(t) satisfies the Euler-Lagrange equations for L on G

ii the variational principle

$$\delta \int_a^b L(g(t), \dot{g}(t)) dt = 0$$  \hspace{1cm} (63)

holds, for variations with fixed endpoints

iii the Euler-Poincaré equations hold:

$$\frac{d}{dt} \frac{\delta L}{\delta \xi} = \text{ad}_\xi \frac{\delta L}{\delta \xi}$$  \hspace{1cm} (64)

iv the variational principle

$$\delta \int_a^b l(\xi(t)) dt = 0$$  \hspace{1cm} (65)

holds on g, using variations of the form

$$\delta \xi = \dot{\eta} + [\xi, \eta]$$  \hspace{1cm} (66)

where \( \eta \) vanishes at the endpoints.

In coordinates, the Euler-Poincaré equations read as follows

$$\frac{d}{dt} \frac{\partial l}{\partial \xi^a} = \epsilon^b_{\text{ad}} \frac{\partial l}{\partial \xi^b} \xi^a,$$  \hspace{1cm} (67)

where \( \epsilon^b_{\text{ad}} \) are the structure constants of \( g \) relative to a given basis and \( \xi^a \) are the components of \( \xi \) relative to this basis.

**Proof**  The equivalence of i and ii holds for any configuration manifold \( Q \) and so, in particular, for \( Q = G \).

Next, we prove that ii and iv are equivalent. First, note that \( l : g \rightarrow \mathbb{R} \) determines uniquely a function \( L : TG \rightarrow \mathbb{R} \) by left translation of the argument and conversely. Thus, the equivalence of ii and iv comes down to proving that all variations \( \delta g(t) \in TG \) of \( g(t) \) with fixed endpoints induce and are induced by variations \( \delta \xi(t) \) of \( \xi(t) \) of the form \( \delta \xi = \dot{\eta} + [\xi, \eta] \), where \( \eta(t) \) vanishes at the endpoints. This, however, is precisely the content of Corollary 4.4.

To complete the proof, we show the equivalence of iii and iv. Indeed, using the definitions and integrating by parts,

$$\delta \int l(\xi) dt = \int \frac{\delta l}{\delta \xi} \delta \xi dt$$

$$= \int \frac{\delta l}{\delta \xi} (\dot{\eta} + \text{ad}_\xi \eta) dt$$

$$= \int \left[ \frac{d}{dt} \left( \frac{\delta l}{\delta \xi} \right) + \text{ad}_\xi \frac{\delta l}{\delta \xi} \right] \eta dt$$

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and so the result follows.

Since the Euler-Lagrange and Hamilton equations on $TQ$ and $T^*Q$ are equivalent if the fiber derivative of $L$ is a diffeomorphism from $TQ$ to $T^*Q$, it follows that the Lie-Poisson and Euler-Poincaré equations are also equivalent under similar hypotheses. To see this directly, we make the following Legendre transformation from $\mathfrak{g}$ to $\mathfrak{g}^*$:

$$\mu = \frac{\delta l}{\delta \xi}, \quad h(\mu) = \langle \mu, \xi \rangle - l(\xi)$$

and assume that $\xi \mapsto \mu$ is a diffeomorphism. Note that

$$\frac{\delta h}{\delta \mu} = \xi + \left\langle \mu, \frac{\delta \xi}{\delta \mu} \right\rangle - \left\langle \frac{\delta l}{\delta \xi}, \frac{\delta \xi}{\delta \mu} \right\rangle = \xi$$

and so it is now clear that the Euler-Poincaré equations are equivalent to the Lie-Poisson equations on $\mathfrak{g}^*$, namely

$$\frac{d\mu}{dt} = \text{ad}^*_{\delta h/\delta \mu}$$

which is equivalent to $\hat{F} = \{F, h\}$ relative to the Lie-Poisson bracket (see Marsden [1992] for more information and references).

As an example, let us consider the free rigid body equations. Here $G = SO(3), \mathfrak{g} = (\mathbb{R}^3, \times)$ and $l(\Omega) = (1/2)\|\Omega\| \cdot \Omega$, where $\| = \text{diag}(I_1, I_2, I_3)$. For an arbitrary vector $\delta \Omega \in \mathbb{R}^3$ we have

$$\left\langle \frac{\delta l}{\delta \Omega}, \delta \Omega \right\rangle = Dl(\Omega) \cdot \delta \Omega = \Omega \cdot \delta \Omega,$$

so that identifying $\mathbb{R}^3$ with itself relative to the dot product we get $\delta l/\delta \Omega = \Omega$. Moreover,

$$\left\langle \text{ad}^*_\Omega \frac{\delta l}{\delta \Omega}, \delta \Omega \right\rangle = \left\langle \frac{\delta l}{\delta \Omega}, \Omega \times \delta \Omega \right\rangle = \frac{\delta l}{\delta \Omega} \cdot (\Omega \times \delta \Omega) = (\Omega \times \Omega) \cdot \delta \Omega,$$

so that

$$\text{ad}^*_\Omega \frac{\delta l}{\delta \Omega} = \Omega \times \Omega$$

and therefore the Euler-Poincaré equations are

$$\hat{\Omega} = \Omega \times \Omega$$

which are the classical Euler equations in the body representation.
5 Dissipation for Euler-Poincaré and Lie-Poisson Equations

Now we are ready to synthesize our discussions of forces and of the Euler-Poincaré equations and to transfer this forcing to the Lie-Poisson equations by means of the Legendre transform. We begin with a formulation of the Lagrange-d’Alembert principle.

**Theorem 5.1** Let $G$ be a Lie group, $L : TG \to \mathbb{R}$ a left invariant Lagrangian, and $F : TG \to T^*G$ a force field equivariant relative to the canonical left actions of $G$ on $TG$ and $T^*G$ respectively. Let $l : \mathfrak{g} \to \mathbb{R}$ and $f : \mathfrak{g} \to \mathfrak{g}^*$ be the restriction of $L$ and $F$ to $T_eG = \mathfrak{g}$. For a curve $g(t) \in G$, let $\xi(t) = T_{g(t)}L_{g(t)}\dot{g}(t)$. Then the following are equivalent:

i) $g(t)$ satisfies the Euler-Lagrange equations with forcing for $L$ on $G$

ii) the integral Lagrange-d’Alembert principle

$$\delta \int_a^b L(g(t), \dot{g}(t)) dt = \int_a^b F(g(t), \dot{g}(t)) \cdot \delta g(t) dt$$

holds for all variations $\delta g(t)$ with fixed endpoints

iii) the Euler-Poincaré equations with forcing are valid:

$$\frac{d}{dt} Dl(\xi) - \text{ad}^*_\xi Dl(\xi) = f(\xi)$$

(iv) the variational principle

$$\delta \int_a^b l(\xi(t)) dt = \int_a^b f(\xi(t)) \cdot \delta \xi(t) dt$$

holds on $\mathfrak{g}$, using variations of the form

$$\delta \xi = \eta + [\xi, \eta]$$

where $\eta$ vanishes at the endpoints.

**Proof** We have already seen that i and ii are equivalent for any configuration manifold $Q$ in section 2. The equivalence of ii and iv and of iii and iv repeats the proof of theorem 4.5. 

The Euler-Poincaré equations with forcing have the following expression in local coordinates

$$\frac{d}{dt} \frac{\partial l}{\partial \xi^a} - c^{\xi^a}_{ba} \frac{\partial l}{\partial \xi^b} = f_a$$

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where $c_{ba}^d$ are the structure constants of the Lie algebra $\mathfrak{g}$.

The condition that the integral curves of the dissipative vector field preserve the inverse images of coadjoint orbits by the momentum map and hence the integral curves of (69) preserve the coadjoint orbits of $\mathfrak{g}^*$ is given by (73). Since $\xi_G(g) = T_{\varepsilon} R_g(\xi)$ and $J(v_g) = T_{\varepsilon}^* R_g \mathbb{F} L(v_g)$, we get

$$\langle F, \xi_G \circ \tau \rangle(v_g) = \langle F(v_g), T_{\varepsilon} R_g(\xi) \rangle = T_{\varepsilon}^* R_g F(v_g) \cdot \xi$$

and

$$J^{\eta(v_g), \xi} (v_g) = T_{\varepsilon}^* R_g \mathbb{F} L(v_g) \cdot [\eta(v_g), \eta] = (\text{ad}_{\eta(v_g)}^* \circ T_{\varepsilon}^* R_g \circ \mathbb{F} L)(v_g) \cdot \xi.$$

Since $F$ and $\mathbb{F} L$ are equivariant,

$$T_{\varepsilon}^* R_g F(v_g) = \text{Ad}_{g^{-1}}^* F(T_g L_{g^{-1}} v_g),$$


and

$$(\text{ad}_{\eta(v_g)}^* \circ T_{\varepsilon}^* R_g \circ \mathbb{F} L)(v_g) = (\text{ad}_{\eta(v_g)}^* \circ \text{Ad}_{g^{-1}}^* \circ \mathbb{F} L)(T_g L_{g^{-1}} v_g).$$

However, $\text{Ad}_{g^{-1}} \circ \text{ad}_{\eta(v_g)} = \text{ad}_{\text{Ad}_{g^{-1}} \eta(v_g)} \circ \text{Ad}_{g^{-1}}$ and thus we get

$$J^{\eta(v_g), \xi} (v_g) = (\text{Ad}_{g^{-1}}^* \circ \text{ad}_{\text{Ad}_{g^{-1}} \eta(v_g)}^* \circ \mathbb{F} L)(T_g L_{g^{-1}} v_g)$$

and the identity (73) thus becomes

$$F(T_g L_{g^{-1}} v_g) = (\text{ad}_{\text{Ad}_{g^{-1}} \eta(v_g)}^* \circ \mathbb{F} L)(T_g L_{g^{-1}} v_g).$$

Letting $\zeta = T_g L_{g^{-1}} v_g$, this becomes

$$f(\zeta) = \text{ad}_{\text{Ad}_{g^{-1}} \eta(v_g)}^* \mathcal{D} l(\zeta).$$

The left hand side is independent of $g$ and thus the right hand side must be also $g$-independent. Thus taking $g = e$, the criterion (73) becomes: for every $\zeta \in \mathfrak{g}$, there is some $\eta(\zeta) \in \mathfrak{g}$ such that

$$f(\zeta) = \text{ad}_{\eta(\zeta)}^* \mathcal{D} l(\zeta).$$

In other words, the force field $f$ (and hence $F$) is completely determined by an arbitrary map $\eta : \mathfrak{g} \to \mathfrak{g}$ via formula (73) and we conclude the following.

**Corollary 5.2** The solutions of the Euler-Poincaré equations with forcing (69) preserve the coadjoint orbits of $\mathfrak{g}^*$ provided the force field $f$ is given by (73) for some smooth map $\eta : \mathfrak{g} \to \mathfrak{g}$. 

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Next, we want to restore Rayleigh dissipation functions as much as possible. As we have mentioned in the introduction, the force field terms we want for the rigid body cannot literally come from such a function. Relaxing this slightly, we will ask that they be gradient relative to a metric on the orbit.

We begin with transforming the Euler-Poincaré equations with the forcing by means of the Legendre transform, namely

$$\mu = Dl(\xi), \quad h(\mu) = \langle \mu, \xi \rangle - l(\xi). \quad (74)$$

Then the functional derivative of the Hamiltonian $h : g^* \to \mathbb{R}$ equals $\delta h/\delta \mu = \xi$ and (69) with the force field term (73) becomes

$$\frac{d\mu}{dt} - \text{ad}^*_{\delta h/\delta \mu} \mu = -\text{ad}^*_{\eta(\mu)} \mu \quad (75)$$

where $\eta : g^* \to g$. (We have changed $\eta$ to $-\eta$ for later convenience.) The requirement on the map $\eta$ is that the right hand side of (75) be a gradient relative to a certain metric on the orbit.

This Riemannian metric is usually defined on adjoint orbits of semi-simple compact Lie algebras in the following manner. The negative of the Killing form defines by left translation a left-invariant metric on the group $G$. Given the adjoint orbit $O$ containing the element $\mu \in g$, it is diffeomorphic to $G/G_\mu$, where $G_\mu$ is the isotropy subgroup of the adjoint action at $\mu$. The Riemannian metric drops to the quotient $G/G_\mu$ and therefore the above mentioned diffeomorphism pushes it forward to a Riemannian metric on $O$, called the normal metric. In general, this metric is not Kähler but, due to bi-invariance of the Killing form, it is $G$-invariant. An explicit formula for this metric is as follows. If $[\mu, \eta], [\mu, \zeta] \in T_\mu O$, their inner product is

$$\langle [\mu, \eta], [\mu, \zeta] \rangle_N = -\kappa(\eta^\mu, \zeta^\mu)$$

where $\kappa$ is the Killing form of $g$ and $\eta^\mu, \zeta^\mu$ are the $g^\mu$-components of $\eta$ and $\zeta$ respectively in the direct sum orthogonal decomposition

$$g = g_\mu \oplus g^\mu$$

for $g_\mu = \text{ker}(\text{ad}_\mu), g^\mu = \text{range}(\text{ad}_\mu)$.

To generalize this metric to coadjoint orbits of the dual $g^*$ of a general Lie algebra $g$, we introduce a symmetric positive definite bilinear form $\Gamma : g^* \times g^* \to \mathbb{R}$. We also refer to Brockett [1993] for a related generalization in the compact case.

Denote by $\Gamma : g^* \to g$ the induced map given by $\Gamma(\alpha, \beta) = \langle \beta, \Gamma \alpha \rangle$ for all $\alpha, \beta \in g^*$, where $\langle \cdot, \cdot \rangle : g^* \times g \to \mathbb{R}$ denotes the pairing between $g^*$ and $g$. Symmetry of $\Gamma$ is equivalent to symmetry of $\Gamma$, i.e. $\Gamma^* = \Gamma$. We introduce the following new inner product on $g$:

$$\langle \xi, \eta \rangle_{\Gamma^{-1}} = \langle \Gamma^{-1} \eta, \xi \rangle$$
for all $\xi, \eta \in \mathfrak{g}$, and call it the $\Gamma^{-1}$-inner product. Let $\mathfrak{g}_\mu$ denote the coadjoint isotropy subalgebra of $\mu$, i.e. the kernel of the map $\xi \mapsto \text{ad}_\xi^* \mu$, and denote by $\mathfrak{g}^\mu$ its orthogonal complement relative to the $\Gamma^{-1}$-inner product. For an element $\xi \in \mathfrak{g}$ we denote by $\xi_\mu$ and $\xi^\mu$ the components of $\xi$ in the orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{g}^\mu$.

Let $C$ be a positive Casimir function on $\mathfrak{g}^*$ and let $O_{\mu_0}$ be the coadjoint orbit through $\mu_0 \in \mathfrak{g}^*$. If $\mu \in O_{\mu_0}$, then $\text{ad}_\xi^* \mu \in T_\mu O_{\mu_0}$ and we define the $(C, \Gamma^{-1})$-normal metric on $O_{\mu_0}$ by

$$\langle \text{ad}_\xi^* \mu, \text{ad}_{\eta}^* \mu \rangle_N = C(\mu) \langle \eta^\mu, \xi^\mu \rangle_{\Gamma^{-1}} = C(\mu) \langle \Gamma^{-1} \eta^\mu, \xi^\mu \rangle.$$  \hspace{1cm} (76)

We will regard $C$ and $\Gamma$ as fixed in the following discussion and just refer to this metric as the normal metric. Let $k : \mathfrak{g}^* \rightarrow \mathbb{R}$ be a smooth function. We will compute the gradient vector field of $k|_{O_{\mu_0}}$ relative to this normal metric. For this purpose denote by $\delta k / \delta \mu \in \mathfrak{g}$ the functional derivative of $k$ at $\mu$ and by grad $k(\mu)$ the gradient of $k|_{O_{\mu_0}}$. Since grad $k(\mu) \in T_\mu O_{\mu_0}$, we can write grad $k(\mu) = \text{ad}_{\eta}^* \mu$ for some $\eta \in \mathfrak{g}$. Since $\xi_\mu$ and $\eta^\mu$ are orthogonal in the $\Gamma^{-1}$-inner product, we get

$$-\langle \text{ad}_{\xi_{\delta k/\delta \mu}}^* \xi, \xi \rangle = \langle \mu, \left[ \xi, \frac{\delta k}{\delta \mu} \right] \rangle = \langle \text{ad}_{\xi_{\delta k/\delta \mu}}^* \xi, \frac{\delta k}{\delta \mu} \rangle = Dk(\mu) \cdot \text{ad}_{\xi_{\delta k/\delta \mu}}^* \xi$$

$$= \langle \text{grad} k(\mu), \text{ad}_{\xi_{\delta k/\delta \mu}}^* \xi \rangle = \langle \text{ad}_{\eta}^* \mu, \text{ad}_{\xi_{\delta k/\delta \mu}}^* \xi \rangle_N$$

$$= C(\mu) \langle \Gamma^{-1} \xi^\mu, \eta^\mu \rangle = C(\mu) \langle \Gamma^{-1} (\xi^\mu + \xi_\mu), \eta^\mu \rangle$$

$$= C(\mu) \langle \Gamma^{-1} \xi, \eta^\mu \rangle = C(\mu) \langle \Gamma^{-1} \eta^\mu, \xi \rangle$$

for any $\xi \in \mathfrak{g}$. Therefore $C(\mu) \Gamma^{-1} \eta^\mu = -\text{ad}_{\xi_{\delta k/\delta \mu}}^* \mu$, or

$$\eta^\mu = -\frac{1}{C(\mu)} \Gamma \left( \text{ad}_{\xi_{\delta k/\delta \mu}}^* \mu \right).$$

Thus grad $k(\mu) = \text{ad}_{\eta}^* \mu = \text{ad}_{\eta^\mu} \mu = -(1/C(\mu)) \text{ad}_{(\text{ad}_{\xi_{\delta k/\delta \mu}})^* \mu} \mu$ and the equation of motion for the gradient vector field of $k|_{O_{\mu_0}}$ relative to the normal metric on the coadjoint orbit $O_{\mu_0}$ is

$$\frac{d\mu}{dt} = -\frac{1}{C(\mu)} \text{ad}_{\mu}^* \Gamma (\text{ad}_{\xi_{\delta k/\delta \mu}}^* \mu).$$

Therefore, in (75), we will choose $\eta(\mu) = \Gamma (\text{ad}_{\xi_{\delta k/\delta \mu}}^* \mu)$ and the Lie-Poisson equations with forcing (75) become

$$\frac{d\mu}{dt} - \text{ad}_{\xi_{\delta k/\delta \mu}}^* \mu = \frac{1}{C(\mu)} \text{ad}_{\xi_{\delta k/\delta \mu}}^* \mu,$$  \hspace{1cm} (77)

If $\mathfrak{g}$ is a compact Lie algebra, let $\langle \cdot, \cdot \rangle$ be a bi-invariant inner product on $\mathfrak{g}$; if $\mathfrak{g}$ is in addition semisimple we could let $\langle \cdot, \cdot \rangle = -\kappa (\cdot, \cdot)$, where $-\kappa (\cdot, \cdot)$ is the Killing form. This inner product identifies $\mathfrak{g}$ with its dual, coadjoint orbits with adjoint orbits, so
that $\text{ad}_{\xi}^* \mu = [\mu, \xi]$, and $\delta k/\delta \mu = \nabla k(\mu)$, where $\nabla k(\mu)$ is the gradient of $k$ on $\mathfrak{g}$ at $\mu$ relative to the bi-invariant inner product $\langle \cdot, \cdot \rangle$. The formula for the gradient vector field on the adjoint orbit $O_{\mu_0}$ becomes

$$\frac{d\mu}{dt} = -\frac{1}{C(\mu)} [\mu, \Gamma([\nabla k(\mu), \mu])].$$

where $\Gamma : \mathfrak{g} \to \mathfrak{g}$ defines the symmetric positive definite bilinear form $(\xi, \eta) \mapsto \langle \Gamma \xi, \eta \rangle$. Thus, in this case the Lie-Poisson equations with forcing become

$$\frac{d\mu}{dt} = [\nabla h(\mu), \mu] - \frac{1}{C(\mu)} [\mu, \Gamma([\mu, \nabla k(\mu)])].$$

(78)

Taking $C(\mu) = 1$ and $\Gamma$ to be the identity, the dissipative term in (78) is in Brockett double bracket form.

Let us return to the general case. The condition that the forcing terms be dissipative is $dh/dt < 0$ (see section 2). This will impose conditions on the choice of the function $k : \mathfrak{g}^* \to \mathbb{R}$. We have

$$\frac{d}{dt} h(\mu(t)) = \left\langle \dot{\mu}(t), \frac{\delta h}{\delta \mu} \right\rangle$$

$$= \frac{1}{C(\mu)} \left\langle \text{ad}_{\mu}^* \left( \text{ad}_{\delta h/\delta \mu}^* \mu \right), \frac{\delta h}{\delta \mu} \right\rangle$$

$$= \frac{1}{C(\mu)} \langle \mu, \left[ \Gamma(\text{ad}_{\delta h/\delta \mu}^* \mu), \frac{\delta h}{\delta \mu} \right] \rangle$$

$$= -\frac{1}{C(\mu)} \left\langle \text{ad}_{\delta h/\delta \mu}^* \mu, \Gamma(\text{ad}_{\delta h/\delta \mu}^* \mu) \right\rangle$$

$$= -\frac{1}{C(\mu)} \tilde{\Gamma}(\text{ad}_{\delta h/\delta \mu}^* \mu, \text{ad}_{\delta h/\delta \mu}^* \mu).$$

(79)

Thus, since $\tilde{\Gamma}$ is positive definite and $C$ is positive, the choice $k = h$ will render $dh/dt < 0$.

6 The Lie-Poisson Instability Theorem

We will now prove an instability theorem in the Lie-Poisson context. However, with little added effort, we can prove a somewhat more general theorem for dissipative systems on Poisson manifolds suggested by the constructions we have given for Lie-Poisson systems and by the work of Vallis, Carnevale, and Young [1989].

We assume that we are given a Poisson manifold $(P, \{,\})$ with Poisson tensor denoted by $\Lambda$, so that at each point $z \in P$, we have $\Lambda_z : T^*_z P \to T_z P$ given by $\Lambda(\text{d} H) = X_H$, i.e. $\langle \text{d} F, \Lambda(\text{d} H) \rangle = \{ F, H \}$. By skew-symmetry of the Poisson bracket
we have $\Lambda^* = -\Lambda$. We also assume that there is a Riemannian metric $\alpha$ defined on each symplectic leaf of $P$. We will use the same notation $\alpha_z$ for the induced map $T_z S \to T^*_z S$, where $S$ is the symplectic leaf through $z$. For a Hamiltonian $H : P \to \mathbb{R}$ we will consider perturbations of the Hamiltonian vector field $X_H$ of the form

$$\frac{dz}{dt} = \Lambda^* dH(z) + \Lambda_z \alpha_z \Lambda_z dH(z).$$

The second term on the right hand side defines a vector field equivalently given by

$$\hat{F} = \{ F, H \}_{\text{sym}}$$

for any $F : P \to \mathbb{R}$, where

$$\{ F, H \}_{\text{sym}} = \langle dF, \Lambda^* \Lambda dH \rangle = -\alpha(X_F, X_H).$$

Thus the full equations can be written as

$$\hat{F} = \{ F, H \} + \{ F, H \}_{\text{sym}}$$

for any $F : P \to \mathbb{R}$.

As an example, take $P = \mathfrak{g}^*$ and

$$\alpha_\mu(\text{ad}_\xi^* \mu, \text{ad}_\eta^* \mu) = \frac{1}{C(\mu)} \tilde{\Gamma} \langle \text{ad}_\xi^* \mu, \text{ad}_\eta^* \mu \rangle.$$

This formula defines on each coadjoint orbit the induced metric given by $\tilde{\Gamma}$, up to the factor $1/C(\mu)$. For $f, h : \mathfrak{g}^* \to \mathbb{R}$ we get

$$\langle d f(\mu), \Lambda_\mu \alpha_\mu \Lambda_\mu d h(\mu) \rangle = -\alpha_\mu(X_f(\mu), X_h(\mu))$$

$$= -\alpha_\mu(\text{ad}^*_{\delta f/\delta \mu} \mu, \text{ad}^*_{\delta h/\delta \mu} \mu)$$

$$= -\frac{1}{C(\mu)} \tilde{\Gamma} (\text{ad}^*_{\delta f/\delta \mu} \mu, \text{ad}^*_{\delta h/\delta \mu} \mu)$$

$$= -\frac{1}{C(\mu)} \langle \text{ad}^*_{\delta f/\delta \mu} \mu, \tilde{\Gamma} (\text{ad}^*_{\delta h/\delta \mu} \mu) \rangle$$

$$= \frac{1}{C(\mu)} \langle \text{ad}^*_{\Gamma(\text{ad}^*_{\delta h/\delta \mu} \mu)} \mu, \delta f/\delta \mu \rangle$$

$$= \langle d f(\mu), \frac{1}{C(\mu)} \text{ad}^*_{\Gamma(\text{ad}^*_{\delta h/\delta \mu} \mu)} \mu \rangle,$$

so that

$$\Lambda_\mu \alpha_\mu \Lambda_\mu d h(\mu) = \frac{1}{C(\mu)} \text{ad}^*_{\Gamma(\text{ad}^*_{\delta h/\delta \mu} \mu)} \mu$$

which coincides with the right hand side of equation (77). Thus the dissipative term considered in the previous section is exactly of this form. The symmetric bracket is hence in this case equal to

$$\{ f, h \}_{\text{sym}}(\mu) = -\frac{1}{C(\mu)} \tilde{\Gamma} (\text{ad}^*_{\delta f/\delta \mu} \mu, \text{ad}^*_{\delta h/\delta \mu} \mu).$$

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It is also interesting to note that this symmetric bracket is the negative of the Beltrami bracket given by the normal metric. The Beltrami bracket of two functions on a Riemannian manifold is the inner product of the gradients of the two functions relative to this metric (see Crouch [1981] and references therein). In our case, if \( f : \mathfrak{g}^* \to \mathbb{R} \) we saw in the previous section that the normal metric on the coadjoint orbit has the expression

\[
\text{grad} f(\mu) = -\frac{1}{C(\mu)} \text{ad}^*_\mu [\text{ad}^*_h [\text{ad}^*_f [\mu]]]
\]

Since \( \mathfrak{g}_\mu \) and \( \mathfrak{g}_\mu \) are orthogonal in the \( \Gamma^{-1} \)-inner product, we get

\[
\langle \text{grad} f(\mu), \text{grad} h(\mu) \rangle_N = \frac{1}{C(\mu)} \left\langle [\Gamma(\text{ad}^*_h [\mu])]^\mu, [\Gamma(\text{ad}^*_h [\mu])]^\mu \right\rangle_{\Gamma^{-1}}
\]

\[
= \frac{1}{C(\mu)} \left\langle \text{ad}^*_h [\mu], [\Gamma(\text{ad}^*_h [\mu])]^\mu \right\rangle.
\]

Denoting \( \Gamma(\text{ad}^*_h [\mu]) = \zeta \), this expression equals

\[
\frac{1}{C(\mu)} \left\langle \text{ad}^*_h [\mu], \zeta \right\rangle = \frac{1}{C(\mu)} \left\langle \text{ad}^*_\mu [\zeta], \frac{\delta f}{\delta \mu} \right\rangle
\]

\[
= \frac{1}{C(\mu)} \left\langle \text{ad}^*_\mu [\zeta], \frac{\delta f}{\delta \mu} \right\rangle
\]

\[
= \frac{1}{C(\mu)} \left\langle \text{ad}^*_h [\mu], \zeta \right\rangle
\]

\[
= \frac{1}{C(\mu)} \left\langle \text{ad}^*_h [\mu], \text{ad}^*_h [\mu] \right\rangle
\]

\[
= \frac{1}{C(\mu)} \left\langle \{ f, h \}_{\text{sym}}(\mu) \right\rangle.
\]

Let us now return to the general case. An important point is that the added dissipation terms of the above form do not destroy the equilibrium. In other words:

**Proposition 6.1** If \( z_\varepsilon \) is an equilibrium for a Hamiltonian system with Hamiltonian \( H \) on a Poisson manifold, then it is also a relative equilibrium for the system with added dissipative term of the form \( -\varepsilon A \text{ad} H \) as above, or of double bracket form on the dual of a Lie algebra.

**Proof** An equilibrium \( z_\varepsilon \) is characterized by the fact that \( X_H(z_\varepsilon) = 0 \) and the added term is \( \varepsilon A X_H(z) \). In the case of duals of Lie algebras, this can be said this way: the added dissipation does not destroy a given relative equilibrium because it is the gradient of the Hamiltonian on the orbit relative to the normal metric, and the differential of the Hamiltonian restricted to the orbit is zero at a relative equilibrium.
Theorem 6.2 Assume that \( z_e \) is an equilibrium of a Hamiltonian system on a Poisson manifold (or, specifically, on the dual of a Lie algebra with the Lie Poisson bracket). Assume that the second variation of the Hamiltonian restricted to the symplectic leaf \( S(z_e) \) (or coadjoint orbit in the case of the dual of a Lie algebra) through \( z_e \) is nonsingular but indefinite. Then with a dissipative term of the form \( \lambda \alpha X_H(z) \) described above added to the equations, the equilibrium becomes nonlinearly unstable; if the dissipation is small, it is spectrally and hence nonlinearly unstable on the leaf.

Proof As is well known and easily verified (see, for example, Marsden, Ratiu and Raugel [1991]), the second variation of the Hamiltonian in the space tangent to the leaf (coadjoint orbit) generates the linearized equations (restricted to the leaf or coadjoint orbit). With dissipation added, we look at the equation

\[
\dot{H}(z) = -\alpha(X_H(z), X_H(z)).
\]

(For the specific case of Lie Poisson systems, this is equation (79) with \( h = k \)). Notice that the relative equilibrium is isolated in the leaf (coadjoint orbit), which follows from our nondegeneracy assumption. Thus, we see that in the leaf (coadjoint orbit), \( \dot{H} \) is strictly negative in a deleted neighborhood of the equilibrium. The Liapunov instability now follows from Liapunov’s instability theorem (see Lemma 3.2 of Bloch, Krishnaprasad, Marsden and Ratiu [1993]) and thus we get the first part of the theorem. We get the second part of the theorem by Proposition 4.1 of the same paper (which is based on Hahn [1967]).

Note that in this theorem we do not need to modify the Hamiltonian to the Chetaev function as was done in Bloch, Krishnaprasad, Marsden and Ratiu [1993]; that is, \( \dot{H} \) is already positive definite, being (in the dual of the Lie algebra case) the square norm of the gradient of the Hamiltonian relative to the normal metric. However, when we do couple the Lie algebra case to that of internal variables below, it will indeed be necessary to modify the Hamiltonian to a Chetaev like functional.

7 Lie-Poisson Examples

7.1 The Rigid Body and the Landau-Lifshitz equations

The calculations needed to show that the general theory applied to the dual of the Lie algebra of the rotation group gives the dissipative terms given in the introduction are straightforward following the outline given. We can omit the details.

7.2 Ideal Fluids

We now give the calculations for the results stated in the introduction.
For incompressible fluids moving in a region $\Omega$ of $\mathbb{R}^d$, or, more generally a smooth oriented Riemannian manifold, the phase space is $\mathfrak{x}_{\text{div}}(\Omega)^*$ which we identify with $\mathfrak{x}_{\text{div}}(\Omega)$, the Lie algebra of vector fields that are divergence free and parallel to the boundary by the $L^2$-inner product. The (+) Lie-Poisson bracket is

$$\{F,H\}(v) = -\int_{\Omega} g\left(v, \frac{\delta F}{\delta v}, \frac{\delta H}{\delta v}\right) dx$$

where $g$ is the Riemannian metric on $\Omega$ and $dx$ is the associated volume element. There is a minus in front of the integral sign because the Jacobi-Lie bracket of vector fields is the \textit{right} Lie algebra bracket for the group of volume preserving diffeomorphisms on $\Omega$. In general, Hamilton’s equations for the (+) Lie-Poisson structure are

$$\frac{d\mu}{dt} = -\text{ad}^*_{\delta H/\delta \mu} \mu.$$

We compute the $\text{ad}^*$-action in our case. Let $u, v, w \in \mathfrak{x}_{\text{div}}(\Omega)$. Then

$$\langle -\text{ad}^*_u v, w \rangle = -\langle v, [u, w] \rangle = -\int_{\Omega} g(v, [u, w]) dx = -\int_{\Omega} v^i \cdot (\mathcal{L}_u w) dx,$$

where $^i$ denotes the index lowering action defined by the metric $g$ on $\Omega$. However,

$$\mathcal{L}_u (v^i \cdot w) dx = (\mathcal{L}_u v^i) \cdot w dx + v^i \cdot (\mathcal{L}_u w) dx + (v^i \cdot w) \mathcal{L}_u (dx).$$

The last term vanishes since $u \in \mathfrak{x}_{\text{div}}(\Omega)$. Thus the above relation becomes:

$$\langle -\text{ad}^*_u v, w \rangle = \int_{\Omega} (\mathcal{L}_u v^i) \cdot w dx - \int_{\Omega} \mathcal{L}_u (v^i \cdot w) dx.$$

The second integral vanishes:

$$\int_{\Omega} \mathcal{L}_u (v^i \cdot w) dx = \int_{\Omega} \mathcal{L}_u (v^i \cdot w) dx = \int_{\partial \Omega} (v^i \cdot w)(u \cdot n) da = 0$$

where $n$ is the outward unit normal to $\partial \Omega$ and $da$ is the induced volume on the boundary. Denoting by $\mathbf{P} : \mathfrak{x}(\Omega) \to \mathfrak{x}_{\text{div}}(\Omega)$ the Hodge projection and by $^i$ the index raising action defined by the Riemannian metric $g$, we get

$$\langle -\text{ad}^*_u v, w \rangle = \int_{\Omega} (\mathcal{L}_u v^i) \cdot w dx = \int_{\Omega} g((\mathcal{L}_u v^i)^i, w) dx \quad = \int_{\Omega} g(\mathbf{P}((\mathcal{L}_u v^i)^i), w) dx = \langle \mathbf{P}((\mathcal{L}_u v^i)^i), w \rangle,$$

whence

$$-\text{ad}^*_u v = \mathbf{P}((\mathcal{L}_u v)^i).$$
Consequently, denoting by $\Lambda : T^*\mathfrak{x}_{\text{div}}(\Omega) \to T\mathfrak{x}_{\text{div}}(\Omega)$ the Poisson structure defined by

$$\langle dH(v), \Lambda_v(dF(v)) \rangle = \{F, H\}(v)$$

$$= \left\langle \text{ad}_{\frac{\partial}{\partial \nu}} v, \frac{\delta H}{\delta v} \right\rangle$$

$$= \left\langle -P((L_{\frac{\partial}{\partial \nu}} v^i)^i), \frac{\delta H}{\delta v} \right\rangle,$$

we get

$$\Lambda_v(u) = -P((L_u v^i)^i).$$

For example, if we choose $\Gamma = \text{identity}$ and $C(v) = 1/\alpha$ for $\alpha$ a strictly positive constant, the dissipative forcing term has the expression

$$\alpha \text{ad}_{\frac{\partial}{\partial \nu}} v = -\alpha P((L_{\text{ad}_{\frac{\partial}{\partial \nu}} v^i)^i)$$

$$= \alpha P((L_u v^i)^i)$$

where $u(v) = P((L_u v^i)^i)$.

It is instructive to verify directly that $dH/dt < 0$ on the solutions of the dissipative system

$$\frac{\partial v}{\partial t} + \nabla v = -\nabla p + \alpha P((L_u v^i)^i).$$

Recall the formula

$$L_v(v^i) = (\nabla v^i) + \frac{1}{2} d\|v\|^2.$$ 

Therefore the equation above becomes

$$\frac{\partial v^i}{\partial t} + L_v(v^i) = -d \left( p + \frac{1}{2} \|v\|^2 \right) + \alpha [P((L_u v^i)^i)^i]$$

and so we get:

$$\frac{dH}{dt} = d \frac{1}{dt} \int_\Omega \|v\|^2 dx = \frac{d}{dt} \int_\Omega v^i \cdot v dx$$

$$= \int_\Omega \frac{dv^i}{dt} \cdot v dx + \alpha \int_\Omega [P((L_u v^i)^i)^i] \cdot v dx$$

$$= \alpha \int_\Omega L_{u(v)}(v^i) \cdot v dx = -\alpha \int_\Omega v^i \cdot L_{u(v)} v dx$$

$$= \alpha \int_\Omega v^i \cdot L_v u(v) dx = -\alpha \int_\Omega L_v(v^i) \cdot u(v) dx$$

$$= -\alpha \int_\Omega \|P((L_u v^i)^i)^i\|^2 dx < 0.$$ 

The vorticity form of the equations, as stated in the introduction are readily verified by taking the differential of the dissipative equations for $v^i$ and recalling that $\omega = dv^i$. 

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7.3 The Vlasov-Poisson Equations

The equations of motion for a one species collisionless plasma moving in a background static ion field in $\mathbb{R}^n$ are given by the Vlasov-Poisson equations

$$\frac{df}{dt} + v \cdot \frac{\partial f}{\partial x} - \frac{q}{m} \frac{\partial \phi_f}{\partial x} \cdot \frac{\partial f}{\partial v} = 0,$$

$$\nabla^2 \phi_f(x) = -\rho_f(x) = q \left( \int f(x, v) dv - 1 \right),$$

where $\partial/\partial x, \partial/\partial v$ denote the gradients with respect to $x$ and $v$ respectively, $\nabla^2$ is the Laplacian in the $x$-variable, $f(x, v)$ is the phase space density satisfying

$$\iint f(x, v) dx dv = 1,$$

$q$ is the charge, $m$ is the mass, and $\rho_f(x)$ is the total charge density of the plasma. We assume that $f$ is either periodic in $x$ or has appropriate asymptotic behavior as $x$ tends to infinity and that $f$ decays for $v$ approaching infinity.

For two functions $g(x, v), h(x, v)$ define

$$\{g, h\} = \frac{1}{m} \left( \frac{\partial g}{\partial x} \cdot \frac{\partial h}{\partial v} - \frac{\partial h}{\partial x} \cdot \frac{\partial g}{\partial v} \right),$$

the canonical Poisson bracket in $(x, v)$-space. Under the above hypotheses on the functions considered, it can be shown by integration by parts that the $L^2$-inner product is invariant on the Lie algebra $\mathfrak{g}$ of functions of $(x, v)$ endowed with the above Poisson bracket.

The Vlasov-Poisson equations can be equivalently written in the form

$$\frac{\partial f}{\partial t} = \{\mathcal{H}_f, f\},$$

where

$$\mathcal{H}_f = \frac{1}{2} m^2 \|v\|^2 + q \phi_f(x)$$

is the one particle Hamiltonian. The total energy of the system has the expression

$$H(f) = \frac{1}{2} \int m \|v\|^2 f(x, v) dx dv + \frac{1}{2} \int q \phi_f(x) \rho_f(x) dx,$$

and one has $\delta H/\delta f = \mathcal{H}_f$.

The Vlasov-Poisson equations are Hamiltonian on the dual of the Lie algebra $\mathfrak{g}$ of functions of $(x, v)$ under the canonical Poisson bracket. We identify $\mathfrak{g}$ with its dual by identifying functions with densities using the Liouville volume element, denoted by $dx dv$ (see Morrison [1980,1982] and Marsden and Weinstein [1982]). The $(\pm)$ Lie-Poisson bracket has the expression

$$\{F, K\}_{LP} = \int f \left\{ \frac{\delta F}{\delta f}, \frac{\delta K}{\delta f} \right\} dx dv.$$
The Hamiltonian vector field of a functional $F$ evaluated at a plasma density function $f \in \mathfrak{g}^*$ is given by

$$X_F(f) = \left\{ \frac{\delta F}{\delta f}, f \right\} dx dv.$$

Since $\delta H/\delta f = \mathcal{H}_f$, the Vlasov-Poisson equations are equivalent to $\dot{F} = \{F, H\}_{LF}$ for $H$, the total energy of the plasma.

The equations with dissipation have the usual form

$$\dot{F} = \{F, H\}_{LF} + \{F, H\}_{sym}.$$

where the symmetric bracket is given by

$$\{F, K\}_{sym} = \alpha \int \langle X_F, X_K \rangle dx dv.$$

Due to the invariance of the $L^2$-inner product, $\Gamma$ is the identity in this case. Thus the symmetric bracket is given by

$$\{F, K\}_{sym} = \int_P \left\{ \frac{\delta F}{\delta f}, f \right\} \cdot \left\{ \frac{\delta K}{\delta f}, f \right\} dx dv$$

$$= \int_P \frac{\delta F}{\delta f} \cdot \left\{ f, \left\{ \frac{\delta K}{\delta f}, f \right\} \right\} dx dv.$$

and hence the Vlasov-Poisson equation with dissipation is

$$\dot{f} + \{f, \mathcal{H}_f\} = \alpha \left\{ f, \{f, \mathcal{H}_f\} \right\},$$

where $\mathcal{H}_f$ is the one particle Hamiltonian and $\alpha$ is a strictly positive constant. Since the equations of stellar dynamics are identical in form to this system (with attractive gravitational rather than repulsive electrical forces), the same formalism applies to them as well. See Kandrup [1991] and Kandrup and Morrison [1992].

### 7.4 The Heavy Top

It is known from Lewis, Ratiu, Simo and Marsden [1992] that there are equilibria for the heavy top with a fixed point that exhibit gyroscopic stabilization, and these equilibria are thus interesting from the point of view of dissipation induced instabilities.

We recall that the equations are of Lie-Poisson form on the dual of the Lie algebra of the Euclidean group of $\mathbb{R}^3$. They are given by

$$\dot{\Pi} = \Pi \times \Omega + g \gamma \times M$$

$$\dot{\gamma} = \gamma \times \Omega$$
where $\Pi = I \Omega$, $I$ is the moment of inertia tensor, $M$ is the constant center of mass vector, $\gamma$ is the direction of gravity as seen from the body and $g$ is the acceleration due to gravity. The Hamiltonian is

$$H = \frac{1}{2} \Pi \cdot \Omega + g \gamma \cdot M$$

and the Lie-Poisson bracket is

$$\{ F, K \}(\Pi, \gamma) = -(\Pi, \gamma) \cdot (\nabla_\Pi F \times \nabla_\Pi K, \nabla_\Pi F \times \nabla_\gamma K + \nabla_\gamma F \times \nabla_\Pi K).$$

Computing the double bracket from the general theory above, with $\Gamma$ being the identity, one finds that the dissipative equations are:

$$\dot{\Pi} = \Pi \times \Omega + g \gamma \times M + \alpha [\Pi \times (\Pi \times \Omega + g \gamma \times M) + \gamma \times (\gamma \times \Omega)]$$

$$\dot{\gamma} = \gamma \times \Omega + \alpha [\gamma \times (\Pi \times \Omega + g \gamma \times M)].$$

This form of the dissipation automatically preserves the coadjoint orbits; that is, it preserves the length of $\gamma$ and the orthogonality of $\gamma$ and $\Pi$. Thus, this dissipation will have the property that when it is added to the equations, it will preserve relative equilibria and any equilibrium that is energetically a saddle point but which has eigenvalues on the imaginary axis will become spectrally (and hence linearly and nonlinearly) unstable when the dissipation is added; equilibria with this property are exhibited in Lewis, Ratiu, Simo and Marsden [1992].

8 A General Instability Theorem

In our previous paper Bloch, Krishnaprasad, Marsden and Ratiu [1993], we considered more general mechanical systems on configuration spaces $Q$ that are invariant under the action of a group $G$ on $Q$. As before, the Lie algebra of $G$ will be denoted $\mathfrak{g}$. In this context, the variables in the problem divide into group (sometimes called rigid) variables and into internal variables. We considered the effect of adding dissipation to the internal variables and showed that if the second variation of the augmented energy is indefinite, and if the rigid-internal coupling matrix $C$ satisfies a nondegeneracy condition (namely that $C$ be surjective as a map from the internal space to the rigid space, i.e., that its transpose $C^T$ is injective), then the addition of this internal dissipation induced a spectral instability in the equations linearized at a relative equilibrium. Here we show that there is a similar theorem for the case of the addition of double bracket Lie-Poisson dissipation of the sort considered in this paper. We also allow a combination of internal and Lie-Poisson dissipation. Interestingly, the details of the argument in the present case are different than in the purely internal dissipative case, and so we will give the proof.
We will need to recall the form of the linearized equations at a relative equilibrium with internal dissipation. By making use of the block diagonalization theory of Simo, Lewis and Marsden [1991], they are shown in Bloch, Krishnaprasad, Marsden and Ratiu [1993] to be the following:

\[
\begin{align*}
\dot{r} &= -L_\mu^{-1}A_\mu r - L_\mu^{-1}CM^{-1}p \\
\dot{q} &= M^{-1}p \\
\dot{p} &= -C^TL_\mu^{-1}A_\mu r - \Lambda q - \tilde{S}M^{-1}p - RM^{-1}p.
\end{align*}
\] (81)

Here, the variable \( r \) is a dynamic variable in the linear space \( V_{\text{RIG}} \), which is isomorphic to the tangent space to the coadjoint orbit \( \text{Orb}_\mu \subset \mathfrak{g}^* \) that passes through the value \( \mu \) of the momentum of the relative equilibrium in question. The operator \( L_\mu \) is the Kirillov-Kostant-Souriau symplectic operator on the coadjoint orbit evaluated at \( \mu \), so that it is skew symmetric. Thus, its inverse is the Poisson tensor. The symmetric operator \( A_\mu \) is the linearized energy operator for the rigid variables. The operator \( C \) is the coupling matrix, coupling the internal variables and the rigid variables, and \( M \) is the positive definite symmetric mass matrix. The variables \( q \) and \( p \) are the (linearized) internal configuration and momentum variables. The matrix \( \Lambda \) is the linearized internal amended potential energy (so it includes the centrifugal energy), \( \tilde{S} \) is a skew symmetric gyroscopic term and \( R \) is the symmetric Rayleigh dissipation matrix for the internal variables. See Bloch, Krishnaprasad, Marsden and Ratiu [1993] for the explicit expression for these equations. In that paper, we assumed full dissipation in the sense that the matrix \( R \) was assumed to be positive definite and that the coupling matrix \( C \) was surjective; in this section, we assume only that the matrix \( R \) is positive semidefinite. In fact, provided a condition spelled out below is satisfied, the matrix \( R \) can be allowed to be zero. In the case that \( R \) is zero, the condition reduces to the condition that the matrix \( C \) is injective (rather than surjective as before). Thus, \textit{we allow partial internal dissipation} in this theorem. We modify the above linearized equations and consider the system

\[
\begin{align*}
\dot{r} &= -L_\mu^{-1}A_\mu r - L_\mu^{-1}CM^{-1}p - G^{-1}A_\mu r \\
\dot{q} &= M^{-1}p \\
\dot{p} &= -C^TL_\mu^{-1}A_\mu r - \Lambda q - \tilde{S}M^{-1}p - RM^{-1}p.
\end{align*}
\] (82)

Here, the matrix \( G \) will be assumed to be symmetric and positive definite. Note that this extra term is dissipation of the form that we considered earlier where \( G \) represents the normal metric on the coadjoint orbit. With the dissipative terms \( R \) and \( G \) omitted, these equations are Hamiltonian with the Hamiltonian function given by the second variation of the augmented Hamiltonian \( \delta^2H_\xi \) (where \( \xi \) is the Lie
algebra element defining the underlying relative equilibrium); this second variation is the quadratic form associated to the block diagonal matrix

\[
\begin{bmatrix}
A_\mu & 0 & 0 \\
0 & \Lambda & 0 \\
0 & 0 & M^{-1}
\end{bmatrix}.
\]  

(83)

One can check directly that the following dissipation equation holds:

\[
\frac{d\delta^2 H_\xi}{dt} = -(M^{-1} p)^T R (M^{-1} p) - (A_\mu r)^T G^{-1} (A_\mu r).
\]

(84)

Of course, because the right hand side is only semidefinite in the variables \((r, q, p)\), one cannot directly use the energy equation alone to conclude instability. This is a central difficulty that was addressed in the work of Chetaev and generalized in Bloch, Krishnaprasad, Marsden and Ratiu [1993]. We consider the following nondegeneracy hypothesis:

\text{(D) If} \ v \ \text{is a vector in the internal space such that} \ Cv = 0 \ \text{and} \ Rv = 0, \\
\text{then} \ v = 0. \\

Note that this hypothesis is equivalent to saying that the matrix \(C^T C + R\) is positive definite.

**Theorem 8.1** Assume that \(G\) is symmetric and positive definite, and that either \(A_\mu\) or \(\Lambda\) has at least one negative eigenvalue. Also assume that \(G\) is positive definite, \(R\) is positive semidefinite and condition (D) holds. Then the system (82) is Liapunov unstable. If, in addition, the dissipation added is sufficiently small, then the equilibrium is spectrally unstable as well (i.e., it has some eigenvalues in the right half plane). Thus, if the dissipation of a given nonlinear system is such that the linearized equations at a relative equilibrium have the form (81), and the dissipation is sufficiently small, then the relative equilibrium is nonlinearly unstable.

As discussed in Bloch, Krishnaprasad, Marsden and Ratiu [1993], spectral instability of the linearized system implies that the underlying relative equilibrium becomes nonlinearly unstable.

**Proof** We will be writing various matrices using block form; when doing so, we will write them consistently in the order \((r, q, p)\). We consider the Chetaev-type function defined as follows:

\[
W(r, q, p) = \frac{1}{2} p \cdot M^{-1} p + \frac{1}{2} q \cdot \Lambda q + \frac{1}{2} r \cdot A_\mu r + \beta B q \cdot M^{-1} p + \alpha Dr \cdot M^{-1} p.
\]

(85)

We do not make any assumptions about the nonsingularity of the matrix \(\Lambda\), but the same remarks as in Bloch, Krishnaprasad, Marsden and Ratiu [1993] (see the proof of Theorem 3.1 of that paper) allow one to reduce to the case in which \(\Lambda\) is nonsingular,
so we will make this assumption. We choose a positive definite matrix $K$ on the internal configuration variables (the freedom to choose $K$ is important only to deal with the possibility that $\Lambda$ is degenerate; if $\Lambda$ is nondegenerate, one can take $K$ to be the identity) and let $D = C^T K L_{\mu}$ and $B = MK^{-1}A$. Note that the choice of $D$ here is not the same as in the case of purely internal dissipation; in that case, we chose $\alpha = \beta$ and had a third term in the definition of $W$—this will not be the case here. We choose $\beta = \alpha^{3/2}$ and choose $\alpha$ to be sufficiently small. As in Bloch, Krishnaprasad, Marsden and Ratiu [1993], a straightforward but somewhat lengthy computation shows that the time derivative of $-W$ is given in block partitioned form by

$$-\dot{W} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix},$$

(86)

where the matrices in this array are given by:

$$A_{11} = \frac{\alpha}{2} (D^T M^{-1} C^T L_{\mu}^{-1} A_{\mu} - A_{\mu} L_{\mu}^{-1} C M^{-1} D) + A_{\mu} G^{-1} A_{\mu}$$

(87)

$$A_{22} = \frac{\beta}{2} (B^T M^{-1} \Lambda + \Lambda M^{-1} B)$$

(88)

$$A_{33} = M^{-1} R M^{-1} - \frac{\beta}{2} \left( M^{-1} B^T M^{-1} + M^{-1} B M^{-1} \right)$$

$$+ \frac{\alpha}{2} \left( -M^{-1} C^T L_{\mu}^{-1} D^T M^{-1} + (M^T)^{-1} D L_{\mu}^{-1} C M^{-1} \right)$$

(89)

$$A_{12} = \frac{\alpha}{2} D^T M^{-1} \Lambda - \frac{\beta}{2} A_{\mu} L_{\mu}^{-1} C M^{-1} B$$

(90)

$$A_{13} = \frac{\alpha}{2} (D^T M^{-1} (R + \tilde{S}) M^{-1} - A_{\mu} (L_{\mu}^{-1} + G^{-1}) D^T M^{-1})$$

(91)

$$A_{23} = \frac{\beta}{2} B^T M^{-1} (R + \tilde{S}) M^{-1}$$

(92)

We now show that $-\dot{W}$ is positive definite for $\alpha$ sufficiently small. To do this, it is sufficient to show that the matrices $A_{11}$, $\bar{A}_{22} = A_{22} - A_{12}^T A_{11}^{-1} A_{12}$ and

$$\bar{A}_{33} = A_{33} - A_{12}^T A_{11}^{-1} A_{13} - (A_{23}^T - A_{13} A_{11}^{-1} A_{12})(A_{22} - A_{12}^T A_{11}^{-1} A_{12})^{-1} (A_{23} - A_{12}^T A_{11}^{-1} A_{13})$$

(93)

are positive definite. This is proved in Bloch, Krishnaprasad, Marsden and Ratiu [1993]; see Lemma 2.11 and equation (3.15). However, by direct inspection of the forms of these matrices, one finds that

$$A_{11} = A_{\mu} G^{-1} A_{\mu} + O(\alpha)$$

(94)

$$\bar{A}_{22} = (\alpha)^{3/2} \Lambda K^{-1} \Lambda + O(\alpha^2)$$

(95)

$$\bar{A}_{33} = \alpha M^{-1} (C^T K C + R) M^{-1} + O((\alpha)^{3/2})$$

(96)
Thus, under the given condition (D), these matrices are all positive definite if $\alpha$ is small enough. Clearly $W$ itself is indefinite if $\alpha$ is small enough, and so by Liapunov's instability theorem (see Lemma 3.2) of Bloch, Krishnaprasad, Marsden and Ratiu [1993]) we get the first part of the theorem. We get the second part of the theorem by Proposition 4.1 of the same paper. 

One can ask in this context, what form of dissipation should be added to the original nonlinear system so that its linearization at a relative equilibrium will have the stated form. We believe that the answer to this is that the force function should be divided into a vertical and a horizontal part and that the vertical part should be of Lie-Poisson double bracket form and that the horizontal part should be of Rayleigh dissipation type. Here, the horizontal and vertical decomposition should be done relative to a connection as in the reduced Euler-Lagrange equations in Marsden and Scheurle [1993b]. We plan to investigate the global aspects of such splittings in another publication, but we will see how this works in the specific example of the rigid body with internal rotors below.

9 The Rigid body with rotors

Here we illustrate Theorem 8.1 using a rigid body with two or three symmetric internal rotors. In the case of two rotors, we will require no internal dissipation, i.e., we can choose $R = 0$. As we will see, if there are three internal rotors, then the rotor about the axis of rotation must have its own internal dissipation for hypothesis (D) to hold.

We first consider the case of three rotors subject to internal friction with the overall rotation subject to double bracket dissipation. We will shortly specialize to the case of two rotors with no internal dissipation. A steady spin about the minor axis of the locked inertia tensor ellipsoid (i.e., the long axis of the body), is a relative equilibrium. Without friction, this system can experience gyroscopic stabilization and the second variation of the augmented Hamiltonian can be indefinite. We will show that this is an unstable relative equilibrium with double bracket dissipation added.

The full equations of motion with both internal and double bracket dissipation are (see Krishnaprasad [1985] and Bloch, Krishnaprasad, Marsden, and Sanchez de
Alvarez [1992]):

\[
(\mathbb{I}_\text{lock} - \mathbb{I}_\text{rotor})\dot{\Omega} = (\mathbb{I}_\text{lock} \Omega + \mathbb{I}_\text{rotor} \Omega_r \times \Omega \\
+ \alpha (\mathbb{I}_\text{lock} \Omega + \mathbb{I}_\text{rotor} \Omega_r) \times ((\mathbb{I}_\text{lock} \Omega + \mathbb{I}_\text{rotor} \Omega_r) \times \Omega - R \Omega_r) \\
\dot{\Omega}_r = -((\mathbb{I}_\text{lock} - \mathbb{I}_\text{rotor})^{-1}(\mathbb{I}_\text{lock} \Omega + \mathbb{I}_\text{rotor} \Omega_r) \times \Omega - R \Omega_r) \\
\dot{\theta}_r = \Omega_r.
\]

(97)

Here, \(\alpha\) is a positive constant, \(Q = SO(3) \times S^1 \times S^1\) (three factors if there are three rotors) and \(G = SO(3)\). Also \(A \in SO(3)\) denotes the attitude/orientation of the carrier rigid body relative to an inertial frame, \(\Omega \in \mathbb{R}^3\) is the body angular velocity of the carrier, \(\Omega_r \in \mathbb{R}^3\) is the vector of angular velocities of the rotors in the body frame (with third component set equal to zero) and \(\theta_r\) is the ordered set of rotor angles in body frame (again, with third component set equal to zero). Further, \(\mathbb{I}_\text{lock}\) denotes the moment of inertia of the body and locked rotors in the body frame and \(\mathbb{I}_\text{rotor}\) is the \(3 \times 3\) diagonal matrix of rotor inertias. Finally, \(R = \text{diag}(R_1, R_2, R_3)\) is the matrix of rotor dissipation coefficients, \(R_i \geq 0\).

In Hamiltonian form, these equations read:

\[
\dot{\Pi} = \Pi \times \Omega + \alpha \Pi \times (\Pi \times \Omega) \\
\dot{\ell} = -\mathbb{I}_\text{rotor} R \Omega_r
\]

(98)

where \(\Pi = \mathbb{I}_\text{lock} \Omega + \mathbb{I}_\text{rotor} \Omega_r\) and \(\ell = \mathbb{I}_\text{rotor} (\Omega + \Omega_r)\). Here, \(\Omega = \mathbb{J}^{-1}(\Pi - \ell)\), where \(\mathbb{J} = \mathbb{I}_\text{lock} - \mathbb{I}_\text{rotor}\). The Hamiltonian is

\[
H = \frac{1}{2} \langle \mathbb{J}^{-1}(\Pi - \ell), \Pi - \ell \rangle + \frac{1}{2} \langle \mathbb{J}^{-1}_\text{rotor} \ell, \ell \rangle.
\]

Notice by direct calculation that

\[
\frac{d}{dt} \|\Pi\|^2 = 0
\]

and that

\[
\frac{d}{dt} H = -\alpha \|\Pi \times \Omega\|^2 - \langle R \Omega_r, \Omega_r \rangle.
\]

We let

\[
\mathbb{I}_\text{lock} = \text{diag}(B_1, B_2, B_3), \\
\mathbb{I}_\text{rotor} = \text{diag}(J^1_1, J^2_2, J^3_3), \\
\mathbb{I}_\text{lock} - \mathbb{I}_\text{rotor} = \text{diag}(A_1, A_2, A_3).
\]

(99)

37
Assume that $B_1 > B_2 > B_3$.

Now we specialize to the case of two rotors and no internal dissipation. We set $R_i = 0$ and $J_3^2 = 0$. Consider the relative equilibrium for (97) defined by, $\Omega^e = (0, 0, \omega)^T$; $\Omega_i^e = (0, 0, 0)^T$ and $\dot{\theta}_e = \theta_e^c$ an arbitrary constant. This corresponds to a steady minor axis spin of the rigid body with the two rotors non-spinning. Linearization about this equilibrium yields,

$$
(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}}) \delta \dot{\Omega} = (\mathbb{I}_{\text{lock}} \delta \Omega + \mathbb{I}_{\text{rotor}} \delta \Omega_r) \times \Omega^e + (\mathbb{I}_{\text{lock}} \Omega^e) \times \delta \Omega
$$

$$+ \alpha (\mathbb{I}_{\text{lock}} \Omega^e) \times ((\mathbb{I}_{\text{lock}} \Omega^e) \times \delta \Omega) + \alpha (\mathbb{I}_{\text{lock}} \Omega^e) \times ((\mathbb{I}_{\text{lock}} \Omega^e + \mathbb{I}_{\text{rotor}} \delta \Omega_r) \times \Omega^e)
$$

$$
\delta \dot{\Omega}_r = - (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1} [(\mathbb{I}_{\text{lock}} \delta \Omega + \mathbb{I}_{\text{rotor}} \delta \Omega_r) \times \Omega^e]
$$

$$+ (\mathbb{I}_{\text{lock}} \Omega^e) \times \delta \Omega] - R \delta \Omega_r$

$$\delta \dot{\theta}_e = \delta \Omega_r. \tag{100}
$$

It is easy to verify that $\delta \dot{\Omega}_3 = 0$, which reflects the choice of relative equilibrium. Similarly $\delta \dot{\Omega}_r = 0$. We will now apply Theorem 8.1 in the case of $\Lambda = 0$.

Dropping the kinematic equations for $\delta \dot{\theta}_e$ we have the “reduced” linearized equations

$$
\begin{bmatrix}
\delta \dot{\Omega}_{r1} \\
\delta \dot{\Omega}_{r2} \\
\delta \dot{\Omega}_1 \\
\delta \dot{\Omega}_2
\end{bmatrix} =
\begin{bmatrix}
0 & -\frac{J_2^2 \omega}{A_1} & 0 & \frac{B_3 - B_2}{A_1} \\
\frac{J_1 \omega}{A_2} & 0 & \frac{B_1 - B_3}{A_2} & 0 \\
0 & \frac{J_2^2 \omega}{A_1} & \frac{B_3 \omega^2 (B_3 - B_1)}{A_2} & \frac{B_2 - B_3}{A_2} \\
-\frac{J_1 \omega}{A_2} & 0 & \frac{B_3 - B_1}{A_2} & \frac{B_3 \omega^2 (B_2 - B_3)}{A_2}
\end{bmatrix}
\begin{bmatrix}
\delta \Omega_{r1} \\
\delta \Omega_{r2} \\
\delta \Omega_1 \\
\delta \Omega_2
\end{bmatrix}. \tag{101}
$$

Assume that $\omega \neq 0$ (nondegeneracy of the relative equilibrium). Then the above equations are easily verified to be in the normal form (8.2) with $R = 0$, upon making
the identifications, \( p = (\delta \Omega_1, \delta \Omega_2) \), \( q = (\delta \theta_{1}, \delta \theta_{2}) \), \( r = (\delta \Omega_1, \delta \Omega_2) \), and

\[
L_{\mu} = \begin{pmatrix} 0 & -1/\omega \\ 1/\omega & 0 \end{pmatrix}; \quad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \tilde{S} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}
\]

\[
A_{\mu} = \begin{pmatrix} B_3 - B_1 \\ \frac{B_3 - B_2}{A_2} \end{pmatrix}; \quad M^{-1} = \begin{pmatrix} \frac{J_1}{A_1} & 0 \\ 0 & \frac{J_2}{A_1} \end{pmatrix};
\]

\[
G^{-1} = \begin{pmatrix} B_3 & 0 \\ 0 & B_3 \end{pmatrix}.
\]

Since \( B_1 > B_2 > B_3 \), \( A_{\mu} \) is negative definite. Also, \( M \) and \( G \) are positive definite, and \( C \) is injective and thus all the hypotheses of Theorem 8.1 are satisfied. Thus the linearized system (100) or (101) displays dissipation-induced instability. That is, for \( \alpha \) sufficiently small, the system will have at least one pair of eigenvalues in the right half plane.

For three rotors, the matrix \( C \) will have three columns, with zeros in its last column and the first two columns as above; however, dissipation in the third rotor will reinstate the validity of hypothesis (D).

### 10 Conclusions and Comments

In this paper we have given a general method of constructing dissipative mechanisms that have the property that they preserve symplectic leaves of reduced spaces and dissipate energy. The most important case is that of the dual of a Lie algebra, in which case the dissipative term is shown to be given by a double bracket form considered by Brockett. We have shown that such dissipative terms induce linear instabilities. For systems that come up in the energy-momentum method, we have given a general dissipation induced instability theorem that couples the double bracket form of instability with internal dissipation, thereby complementing our previous results in Bloch, Krishnaprasad, Marsden and Ratiu [1993]. We have shown that this theory applies to a number of interesting examples from ferromagnetics, ideal fluid flow and plasma dynamics.

Other systems beside Lagrangian and Hamiltonian systems also exhibit phenomena similar to dissipation induced instabilities. In particular, one gets this phenomena in reversible systems (see O'Reilley [1993]) and when one breaks the symmetry of a system (see Guckenheimer and Mahalov [1992] and Knobloch, Marsden and Mahalov [1994]).

In the future, we would like to give an infinite dimensional analysis for problems such as fluids and the Richardson number example of Abarbanel, Holm, Marsden
and Ratiu [1986]. The Richardson number criterion for stability of shear flows in stratified fluids is especially interesting because one knows there that the ideal dissipationless flow is energetically a saddle point yet is spectrally stable for Richardson number between $1/4$ and $1$. Another candidate would be a case like an ABC Euler flow on the sphere, as in Chern and Marsden [1990]. In the case of Euler flow, the techniques of Ebin and Marsden [1970] together with invariant manifold theory for infinite dimensional dynamical system should allow one to rigorously prove nonlinear instability from spectral instability.

Other examples that might be treated are damping mechanisms in planetary physics using the theory of rotating gravitational fluid masses of Riemann [1860], Poincaré [1885, 1892, 1910], Chandrasekhar [1977], Lewis and Simo [1990], and Touma and Wisdom [1992]. We also expect that there will be a more detailed theory in the context of the semidirect product theory of Marsden, Ratiu and Weinstein [1984]. For example, one can treat the heavy top as either a Lie Poisson system or as a system with group $S^1$ and the rest of the variables internal variables. Comparison of the two methods would undoubtedly be of interest.

We also expect that one can develop eigenvalue movement formula for the present context, as we did in Bloch, Krishnaprasad, Marsden and Ratiu [1993]. References relevant for this and other aspects of the general dissipation induced instability phenomenon include Thomson and Tait [1879], Poincaré [1885], Krein [1950], Ziegler [1956], Taussky [1961], Namachchivaya and Ariaratnam [1985], MacKay [1991], Haller [1992], and Pego and Weinstein [1992].

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References


Kandrup, H.E. [1991] The secular instability of axisymmetric collisionless star clus-

system and the problem of stability for spherical relativistic star clusters. preprint..

Letters A 100, 419–422.

Kaufman, A.N. [1985] Lorentz-covariant dissipative Lagrangian systems. Physics
Letters A 109, 87–89.

Springer-Verlag.

Knobloch, E., Mahalov and J.E. Marsden [1993], Normal forms for three-dimensional

Krein, M.G. [1950] A generalization of some investigations of linear differential equa-

Krishnaprasad, P.S. [1985] Lie-Poisson structures, dual-spin spacecraft and asymp-


MacKay, R. [1991] Movement of eigenvalues of Hamiltonian equilibria under non-

note series, 174, Cambridge University Press.

Marsden, J.E., T.S. Ratiu and G. Raugel [1991] Symplectic connections and the


