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Abstract

We study the generalizations of the well-known Lieb-Thirring inequality for the magnetic Schrödinger operator with nonconstant magnetic field. Our main result is the naturally expected magnetic Lieb-Thirring estimate on the moments of the negative eigenvalues for a certain class of magnetic fields (including even some unbounded ones). We develop a localization technique in path space of the stochastic Feynman-Kac representation of the heat kernel which effectively estimates the oscillatory effect due to the magnetic phase factor.

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Contents

1 Introduction

In this paper we discuss generalizations of the magnetic Lieb-Thirring inequality obtained in [LSY-II] for the constant magnetic field. The main goal is to obtain reasonable estimates for the moments of the negative eigenvalues of the three-dimensional Pauli operator with external potential (describing a nonrelativistic spin-1/2 electron in an electromagnetic field). The basic difference from the previous related works is that we focus on nonhomogeneous magnetic field.

For the possible applications of this inequality, especially its role played in the proof of the semiclassical formulas, we refer to the papers [LSY-I] and [LSY-II]. Here we just note two requirements that a useful Lieb-Thirring type estimate is expected to fulfil:

- it must be comparable (up to universal constants) with the corresponding semiclassical formula;

- apart from the necessary integrability conditions (which make the semiclassical formula finite) no extra condition should be imposed on the external potential (since in the applications the electric potential is usually chosen to be an effective potential whose detailed properties might not be known).

In addition to these basic requirements we mention that in the related works ([Sob], [LSY-I], [LSY-II], etc.) special attention is devoted to the case of a strong magnetic field. We also found it physically interesting, and at the same time mathematically difficult, and hence challenging, to treat strong nonhomogeneous magnetic fields.

There is a vast literature of various spectral studies in the case of the homogeneous magnetic field, but results, especially quantitative ones, are fairly rare for nonhomogeneous field (see [AHS], [CdV], [AC], [Mat-1990], [Mat-1991], [T]). The technical reason for this (apart from the obvious physical relevance of the constant magnetic field) is twofold. First, the Schrödinger operator with constant magnetic field (without electric potential) is exactly solvable, and after decomposing the operator according to the Landau levels one obtains a simplified (lower dimensional) setup, so the additional effect of the external potential becomes easier. Some version of this strategy has almost always been used in any work concerning homogeneous magnetic field.

The second technical difficulty is that perturbations of the magnetic field can be much less controlled than that of the external potential. Naively, one would expect that a local change of the magnetic field does not have a large effect on local quantities observed far away, but the magnetic vector potential, appearing in the operator is a nonlocal quantity (i.e. it undergoes a nonlocal change with a long tail even under local perturbation of the field itself). This is the source of the Aharonov-Bohm effect.

Our basic method is stochastic via the Feynman-Kac formula, which is valid under fairly general conditions on the magnetic field. The analysis of the stochastic oscillatory integral in the Feynman-Kac formula involves a new localization technique in path space which enables us to estimate the heat kernel of the Pauli operator (without electric potential). The key idea of this technique has been presented in the simplest possible setup in [E-1993(b)] yielding new pointwise estimates on the magnetic heat kernel. In the present paper we refine this technique to obtain a stronger estimate (unfortunately under more restrictive conditions) which can be combined with the Birman-Schwinger principle (in order to include the external potential) to obtain the desired Lieb-Thirring inequality.

2 Definitions, conjectures, results

The three-dimensional Pauli Hamiltonian is

$$H_{Pauli} := [(\mathbf{p} - \mathbf{A}) \cdot \boldsymbol{\sigma}]^2 + V \cdot \mathbf{I} \tag{1}$$

acting on $L^2(\mathbf{R}^3, \mathbf{C}^2)$, the Hilbert space of a spin-1/2 particle. Here $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ stands for the vector of Pauli matrices, $\mathbf{A}$ is the vector potential of the underlying magnetic field $\mathbf{B}(\mathbf{x}) = \mathrm{curl}\mathbf{A}(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^3$, $\mathbf{I}$ is the 2×2 identity matrix and $\mathbf{p} = -i\nabla$.

Throughout our work we consider nonhomogeneous magnetic field with *constant direction*, i.e. assume that $\mathbf{B}(\mathbf{x}) = (0, 0, B(\mathbf{x})) \in \mathbf{R}^3$, where $B(\mathbf{x}) \geq 0$. By $\mathrm{div}\mathbf{B} = 0$ the function $B(\mathbf{x})$ depends only on the first two coordinates of $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, which we will denote by $x := (x_1, x_2) \in \mathbf{R}^2$. We do not specify the gauge $\mathbf{A}(\mathbf{x})$ here, but we will always restrict ourselves to an appropriate two-dimensional gauge, i.e. $\mathbf{A}(\mathbf{x}) = (A_1(x), A_2(x), 0) =: (A(x), 0)$ (depending only on x). We will use the convention that $\mathbf{A} = (A_1, A_2, A_3)$ denotes a vectorfield in $\mathbf{R}^3$ and $A = (A_1, A_2)$ denotes the associated two-dimensional vectorfield, similarly to the convention on the points $\mathbf{x} \in \mathbf{R}^3$ and $x \in \mathbf{R}^2$. We assume that A and $\mathrm{div}A$ are continuous and that $B(x)$ is continuously differentiable.

Under these conditions H_{Pauli} decouples into two operators of the form $(\mathbf{p}-\mathbf{A})^2 \pm B + V$ acting on the spin-up and spin-down subspaces, respectively. For upper bounds on the moments of the negative eigenvalues it is clearly enough to study

$$H_0 := (\mathbf{p}-\mathbf{A})^2 - B + V, \tag{2}$$

since the contribution from the other operator yields a factor of at most 2 in the estimate for H_{Pauli} by the variational principle. The negative eigenvalues of H_0 are denoted by $E_1 \le E_2 \le \ldots \le 0$.

Remark. We are not aware of any general theorem that would ensure apriori (after imposing some L^p-bound on V) the selfadjointness of H_{Pauli} or H_0. The usual theorems about the perturbation of a selfadjoint operator do not seem to work if B is unbounded (which will be our main concern). Nevertheless, the way we will prove our Lieb-Thirring inequalities implies almost immediately that the operator is semibounded (so it has a selfadjoint extension) and has no negative essential spectrum. The details are found in Appendix A.

The naive conjecture for the moments of the negative eigenvalues is the following:

Naive conjecture. For any $\gamma > 1/2$ there exist two absolute constants $C_1(\gamma)$ and $C_2(\gamma)$ such that

$$\sum_i |E_i|^\gamma \le C_1(\gamma)\int_{\mathbf{R}^3} B(x)|V(\mathbf{x})|_-^{\gamma+1/2} d\mathbf{x} + C_2(\gamma)\int_{\mathbf{R}^3} |V(\mathbf{x})|_-^{\gamma+3/2} d\mathbf{x} \tag{3}$$

($|V|_-$ denotes the negative part of V).

Remark 1. This conjecture is based on the following heuristic argument. The two-dimensional unperturbed operator

$$H := (p_1 - A_1)^2 + (p_2 - A_2)^2 - B = (p-A)^2 - B \tag{4}$$

is nonnegative and has a nontrivial zero energy spectral projection P_0 with a kernel $P_0(x,y)$, whose diagonal element $P_0(x,x)$ is more or less equal to $B(x)/2\pi$. For more precise statements see [E-1993(a)]. Recall that $H_0 = H + p_2^3 + V$, so over each point $x \in \mathbf{R}^2$ the operator $p_3^2 + V$ acting on a one-dimensional fiber gives rise to $\int_{\mathbf{R}} |V(x,x_3)|_-^{\gamma+1/2} dx_3$ as a contribution to the eigenvalue moment. Multiplying it by the density of states $\sim B(x)/2\pi$ and integrating over $x \in \mathbf{R}^2$ one obtains the first term in (3). The second term comes from the contribution of the strictly positive part of the spectrum of H and it has the form as of the usual Lieb-Thirring inequality. The reason for it is that $(1-P_0)H$ can be estimated from below by the two-dimensional free Laplacian (in some suitable sense).

Remark 2. The conjecture above is not true without any further condition on B. A counterexample is provided in Appendix B. The spirit of this counterexample suggests a simple but necessary modification in (3), namely $B(x)$ on the right hand side must be replaced by a screened version of $B(x)$ with screening length $\sim B(x)^{-1/2}$, i.e. by

$$\tilde{B}(x) := \left(B * F_{B(x)^{-1/2}}\right)(x), \tag{5}$$

where $F \geq 0$ is a C^∞-function supported on the unit disc with $\int F = 1$, and $F_\varepsilon(x) := \varepsilon^2 F(\varepsilon x)$.

Our methods are too weak to deal with magnetic fields if there is a substantial difference between $B(x)$ and $\tilde{B}(x)$; more precisely, whenever we are able to prove (3), the conditions will automatically imply that $B(x)$ and $\tilde{B}(x)$ are comparable, uniformly in x. Therefore we will concentrate on proving (3). The discussion of a different (much rougher) modification of B is found in [E-1994].

First we present a simple Lieb-Thirring type estimate.

Theorem 2.1 *For the negative eigenvalues of H_0 we have*

$$\sum_i |E_i|^\gamma \leq C_1 \cdot \|B\|_\infty \int_{\mathbf{R}^3} |V(\mathbf{x})|_-^{\gamma+1/2} d\mathbf{x} + C_2 \int_{\mathbf{R}^3} |V(\mathbf{x})|_-^{\gamma+3/2} d\mathbf{x}, \tag{6}$$

where

$$C_1 := \frac{2^{\gamma-1}e}{\pi(4\gamma^2-1)}\frac{\mu+1}{1-\lambda} \quad \textit{and} \quad C_2 := \frac{2^{\gamma-1}}{\pi(4\gamma^2-1)}\frac{1}{\lambda^2}\left(1+\frac{1}{\mu}\right)^{3/2} \tag{7}$$

with $0 < \lambda < 1$ *and* $\mu > 0$ *being free parameters and* $\gamma > 1/2$.

Remark. This theorem does not impose any condition on B apart from the uniform boundedness. But the estimate is weaker than (3) unless we have positive lower bound $0 < B_0 \le B(x)$ (in which case C_1 and C_2 in (3) will depend on B_0). For magnetic fields that are close to zero on some domain (3) is definitely stronger than (6). At the same time the counterexample in Appendix B shows that the vanishing magnetic field might cause troubles in the original conjecture. Therefore we will impose a uniform positive lower bound on B, and we then address the question of eliminating the condition on the upper bound. Although the necessary conditions given in Theorem 2.2 below are still very restrictive and the proof of this theorem requires a conceptually new approach, we do obtain the original form, (3), of the naive conjecture by imposing these conditions.

Theorem 2.2 *Assume that the magnetic field has a positive lower bound* $0 < B_0 \le B(x)$ *and for some constant* c *it satisfies*

$$|B(x) - B(y)| \le c \cdot d(x)|x-y| \tag{8}$$

where

$$d(x) := B_0^{3/2} \cdot \left(\frac{B_0}{B(x)}\right)^{31/6} \tag{9}$$

for any $x, y \in \mathbf{R}^2$. *Then there exist two constants* C_1 *and* C_2 *depending only on* c *such that the estimate (3) is valid with*

$$C_1(\gamma) := \frac{C_1 \cdot 2^{\gamma-1}}{(1-\lambda)(4\gamma^2-1)} \quad \textit{and} \quad C_2(\gamma) := \frac{C_2 \cdot 2^{\gamma+3/2}}{\lambda^2(4\gamma^2-1)}, \tag{10}$$

where $0 < \lambda < 1$ *is a free parameter.*

Remark 1. The conditions (8), (9) essentially impose a condition on the size of the gradient of B. Only a small gradient is allowed on the regions where $B(x)$ is large. Nevertheless, the theorem applies to any magnetic field with positive lower bound B_0 and asymptotic behaviour not bigger than $|x|^{6/37}$ at infinity (with the corresponding asymptotics $\leq |x|^{-1+6/37}$ for the gradient). The exponent 31/6, which appears in (9) and determines the maximal growth rate of B at infinity, is necessary for the following proof, but, as we remarked above, the conjecture (3) is expected to hold under much more general circumstances. Therefore this exponent only expresses the limitations of our method and does not have any physical meaning.

Remark 2. The conditions (8) and (9) are almost homogeneous in the magnetic field, therefore we have a semiclassical statement as well. If we include the Planck constant in the original Pauli Hamiltonian, $[(h\mathbf{p} - \mathbf{A}) \cdot \boldsymbol{\sigma}]^2 + V \cdot \mathbf{I}$, then H_0 becomes

$$h^2 \left[\left(\mathbf{p} - \frac{\mathbf{A}}{h} \right)^2 - \frac{B}{h} + \frac{V}{h^2} \right], \tag{11}$$

so the magnetic field must be rescaled by h^{-1}. Notice that this change makes the conditions even weaker (moreover they become irrelevant in the $h \to 0$ limit). The estimate for the eigenvalue moment is

$$\sum_i |E_i|^\gamma \leq C_2(\gamma) \cdot h^{-2\gamma-1} \int |V|_-^{\gamma+3/2} + C_1(\gamma) \cdot h^{-2\gamma} \int B|V|_-^{\gamma+1/2}. \tag{12}$$

Since $C_2(\gamma)$ is not the semiclassical constant, the second term becomes relevant only for large magnetic field.

Before going into the details of the proofs, we would like to mention very briefly two other results related to the naive conjecture (3) which can be found in the author's Ph.D. Thesis [E-1994].

One can try to check the naive conjecture directly for exactly solvable models. It turns out that for the "Coulombic" magnetic field, $B(x) := b/|x|$ (with $b > 0$), the operator (with the natural gauge choice) $H := (p - A)^2 - B$ is exactly solvable (and actually it has dense point spectrum in the interval $[0, B^2]$, similar to a phenomenon investigated qualitatively in [MS]). Using explicit formulas for the eigenfunctions, one can estimate the spectral density of H with a precision that is sufficient to prove (3). The proof involves various estimates on the asymptotic behaviour of the Laguerre polynomials. The significance of this result is that the Coulombic magnetic field is neither bounded from above nor has a positive lower bound (so none of the previous theorems apply), and it shows that the conjecture can be valid even for magnetic fields with a singularity.

The second, related result deals with cylindrically symmetric situation.

Proposition 2.3 *Assume that $B(x) = B(|x|) \geq 0$ and $V(\mathbf{x}) = V(|x|, x_3)$ (where $|x| := \sqrt{x_1^2 + x_2^2}$) and let $a(r) := (1/r)\int_0^r B(s)ds$ be the absolute value of the natural radial gauge. Then for any $\gamma > 0$ there exists a universal constant $c(\gamma)$ such that*

$$\sum_i |E_i|^\gamma \leq c(\gamma) \sum_{n\in\mathbf{Z}} \int_{\mathbf{R}} \int_0^\infty \left| -B(r) + V(r, x_3) + \left(\frac{n}{r} - a(r)\right)^2 \right|_-^{\gamma+1} r\, dr\, dx_3. \tag{13}$$

The main idea of the proof is that one can investigate the problem separately in each angular momentum sector (so that the magnetic field becomes an effective potential) and apply a modified version of the idea of [L-1980]. This estimate is not comparable directly to the original form of the conjecture, but imposes no condition on the magnetic field apart from the symmetry, so it might be useful in some situations when Theorems 2.1 and 2.2 do not apply.

3 Separation of the external potential

For the proof of both theorems we follow the method of [LSY-II]. The key idea is to split the Birman-Schwinger kernel

$$K_E := \left|V + \frac{E}{2}\right|_-^{1/2} \left(H + p_3^2 + \frac{E}{2}\right)^{-1} \left|V + \frac{E}{2}\right|_-^{1/2} \tag{14}$$

$(E > 0)$ into a lower and an upper part at level L, $K_E = K^{<}_{E,L} + K^{>}_{E,L}$, defined as

$$K^{<}_{E,L} := \left|V + \frac{E}{2}\right|_-^{1/2} \Pi_L \left(H + p_3^2 + \frac{E}{2}\right)^{-1} \Pi_L \left|V + \frac{E}{2}\right|_-^{1/2} \tag{15}$$

$$K^{>}_{E,L} := \left|V + \frac{E}{2}\right|_-^{1/2} (1 - \Pi_L) \left(H + p_3^2 + \frac{E}{2}\right)^{-1} (1 - \Pi_L) \left|V + \frac{E}{2}\right|_-^{1/2}, \tag{16}$$

where let P_L be the spectral projection onto $[0, L]$ in the spectrum of H (which is nonnegative), and let $\Pi_L := P_L \otimes \mathrm{Id}$ be its natural extension to $L^2(\mathbf{R}^3)$. (According to the heuristic argument outlined in the previous section we should choose $L = 0$, but sometimes the splitting is technically more convenient at a positive L.) Using Lemma 2.3 in [LSY-II] (where we do not really have to assume that the operators in question are of trace class) we have

$$N_E \le (1 - \lambda)^{-1} \mathrm{Tr}(K^{<}_E) + \lambda^{-2} \mathrm{Tr}[(K^{>}_E)^2], \tag{17}$$

for N_E, the number of eigenvalues of $H_0 := H + p_3^2 + V$ less than $-E$ $(0 < \lambda < 1$ is a free parameter). Naturally

$$\sum_i |E_i|^\gamma = \int_0^\infty N_E E^{\gamma-1} dE \le \tag{18}$$

$$\le (1 - \lambda)^{-1} \int_0^\infty \mathrm{Tr}(K^{<}_{E,L}) E^{\gamma-1} dE + \lambda^{-2} \int_0^\infty \mathrm{Tr}[(K^{>}_{E,L})^2] E^{\gamma-1} dE.$$

Using that

$$P_L \le e^{tL} \cdot e^{-tH} \tag{19}$$

(for any $t \ge 0$) and that $\mathrm{Tr}(CA) \le \mathrm{Tr}(CB)$ in case of $0 \le A \le B$ and $C \ge 0$, a simple calculation, similar to (2.15) in [LSY-II], shows that

$$\mathrm{Tr}(K^{<}_{E,L}) \le \mathrm{Tr}\left[\left|V + \frac{E}{2}\right|_- \Pi_L \left(p_3^2 + \frac{E}{2}\right)^{-1}\right] \le \tag{20}$$

$$\le \mathrm{Tr}\left[\left|V + \frac{E}{2}\right|_- e^{tL} e^{-tH} \left(p_3^2 + \frac{E}{2}\right)^{-1}\right] = \frac{1}{\sqrt{2E}} \int_{\mathbf{R}^3} \left|V(\mathbf{x}) + \frac{E}{2}\right|_- e^{tL} e^{-tH}(x,x) d\mathbf{x},$$

where $e^{-tH}(x,x)$ denotes the diagonal element of the heat kernel of H (its existence will be discussed later). Then, still following the technique of [LSY-II], we obtain

$$\int_0^\infty \mathrm{Tr}(K_{E,L}^{<})E^{\gamma-1}dE \leq \frac{2^{\gamma+1}}{4\gamma^2-1}\int_{\mathbf{R}^3}|V(\mathbf{x})|_-^{\gamma+1/2}e^{tL}e^{-tH}(x,x)d\mathbf{x} \tag{21}$$

(recall that for the moment t and L are free parameters).

If we choose $L=0$, then we can see that P_0 has a kernel with well defined diagonal function $P_0(x,x)$ (defined via the zero energy eigenfunctions of H exactly as it was done in [E-1993(a)]), and in this case the estimate (21) can be replaced by

$$\int_0^\infty \mathrm{Tr}(K_{E,L}^{<})E^{\gamma-1}dE \leq \frac{2^{\gamma+1}}{4\gamma^2-1}\int_{\mathbf{R}^3}|V(\mathbf{x})|_-^{\gamma+1/2}P_0(x,x)d\mathbf{x} \tag{22}$$

(use directly the kernel of P_0 in (20)).

For the contribution of the upper part for each $E>0$, $L\geq 0$ we will present an operator $M_{E,L}$ satisfying

$$(1-\Pi_L)\left(H+p_3^2+\frac{E}{2}\right)^{-1}(1-\Pi_L)\leq(1-\Pi_L)M_{E,L}(1-\Pi_L), \tag{23}$$

and such that $M_{E,L}^2$ has a continuous kernel. Then using the following trace estimates for nonnegative operators:

- $0\leq A\leq B \Longrightarrow \mathrm{Tr}A^2\leq \mathrm{Tr}B^2$,
- $\mathrm{Tr}(PA)\leq\mathrm{Tr}(A)$ for any projection P and $A\geq 0$,
- $\mathrm{Tr}(CDCD)\leq\mathrm{Tr}(C^2D^2)$,

we obtain

$$\mathrm{Tr}[(K_{E,L}^{>})^2]\leq \mathrm{Tr}\left(\left|V+\frac{E}{2}\right|_-^{1/2}M_{E,L}\left|V+\frac{E}{2}\right|_-^{1/2}\right)^2\leq \tag{24}$$

$$\leq \mathrm{Tr}\left(\left|V+\frac{E}{2}\right|_-^{2}M_{E,L}^2\right)=\int_{\mathbf{R}^3}\left|V(\mathbf{x})+\frac{E}{2}\right|_-^2 M_{E,L}^2(\mathbf{x},\mathbf{x})d\mathbf{x}.$$

As we will show later, $M_{E,L}^2(\mathbf{x},\mathbf{x}) \leq C(L)/\sqrt{E}$ uniformly in $\mathbf{x}$ for some L-dependent number $C(L)$, therefore

$$\int_0^\infty \mathrm{Tr}[(K_{E,L}^{>})^2] E^{\gamma-1} dE \leq C(L) \int_{\mathbf{R}^3} \int_0^\infty \left| V(\mathbf{x}) + \frac{E}{2} \right|^2 E^{\gamma-3/2} dE d\mathbf{x} \leq \tag{25}$$

$$\leq C(L) \frac{2^{\gamma+3/2}}{4\gamma^2 - 1} \int_{\mathbf{R}^3} |V(\mathbf{x})|_-^{\gamma+3/2} d\mathbf{x}.$$

Combining (18) (21) and (25) we can reduce the proof of Theorems 2.1 and 2.2 to the following two Propositions.

Proposition 3.1 *In the case of the bounded magnetic field we have*

$$e^{tL} e^{-tH}(x,x) \leq \frac{e}{4\pi}(L + \|B\|_\infty) \tag{26}$$

for $t := (L + \|B\|_\infty)^{-1}$*. Furthermore, the operator*

$$M_{E,L} := \left(\frac{L}{L + \|B\|_\infty} (\mathbf{p} - \mathbf{A})^2 + \frac{E}{2} \right)^{-1}, \tag{27}$$

satisfies (23) and

$$M_{E,L}^2(\mathbf{x},\mathbf{x}) \leq \frac{C(L)}{\sqrt{E}} \quad \textit{with} \quad C(L) := \frac{1}{2^{5/2}\pi} \left(1 + \frac{\|B\|_\infty}{L} \right)^{3/2}. \tag{28}$$

Remark. After proving this Proposition, we choose $L := \mu \|B\|_\infty$ to finish the proof of Theorem 2.1. $\square$

Proposition 3.2 *Under the conditions of Theorem 2.2 there are two constants* C_1 *and* C_2 *depending only on* c*, and there is an operator* $M_{E,0}$ *satisfying (23) (with* $L = 0$*) such that*

$$P_0(x,x) \leq C_1 \cdot B(x), \tag{29}$$

and

$$M_{E,0}^2(\mathbf{x},\mathbf{x}) \leq \frac{C_2}{\sqrt{E}}, \tag{30}$$

which imply Theorem 2.2 via (17), (22) and (25).

The proof of these Propositions relies on the magnetic Feynman-Kac formula for the Pauli operator, which we present in the most general form recalling the statement of Appendix B in [E-1993(a)]. For the rest of the paper $\mathbf{E}_{0,x}^{2t,y}$ denotes the expectation for the two-dimensional Brownian bridge $W(s)$ $(0 \le s \le 2t)$ under the conditions $W(0) = x$ and $W(2t) = y$.

Proposition 3.3 *Let A be a vectorfield on $\mathbf{R}^2$ such that $A \in L^2_{loc}$, $\mathrm{div} A \in L^2_{loc}$; A and $\mathrm{div} A$ do not grow faster than some polynomial at infinity, furthermore for $B := \mathrm{rot} A$ assume that $0 \le B(x) \le c_0(|x|^{2-\varepsilon} + 1)$ with some positive ε and c_0. Then the heat operator $\exp(-tH)$ of the two-dimensional operator $H := (p - A)^2 - B$ has a kernel $D^{(t)}(x, y)$ defined by*

$$D^{(t)}(x,y) := \frac{1}{4\pi t} \cdot e^{-\frac{(x-y)^2}{4t}} \mathbf{E}_{0,x}^{2t,y} \exp \Psi(W), \tag{31}$$

where

$$\Psi(W) := -i \int_0^{2t} A(W(s)) \circ dW(s) + \frac{1}{2} \int_0^{2t} B(W(s)) ds = \tag{32}$$

$$= -i \int_0^{2t} A(W(s)) dW(s) + \frac{i}{2} \int_0^{2t} \mathrm{div} A(W(s)) ds + \frac{1}{2} \int_0^{2t} B(W(s)) ds$$

(as usual, $\int F(W)dW$ denotes the Ito integral, while $\int F(W) \circ dW$ is the Stratonovich integral), and there exists a continuous function $\Gamma(t,x) : [0, \infty) \times \mathbf{R}^2 \to \mathbf{R}_+$ such that

$$|D^t(x,y)| \le \frac{\Gamma(t,x)}{t} \cdot e^{-\frac{(x-y)^2}{8t}}. \tag{33}$$

If, in addition, B is continuously differentiable, and A and $\mathrm{div} A$ are continuous then $D^{(t)}(x,y)$ is continuous and e^{-tH} maps $L^2(\mathbf{R}^2)$ into $C(\mathbf{R}^2)$.

Remark 1. The growth condition on B is clearly satisfied for the fields investigated in this paper. The conditions on A are very weak and will always be satisfied when we explicitly specify the gauge.

Remark 2. The only difference between this Proposition and the statement in Appendix B of [E-1993(a)] is that in (33) we take into account the $1/t$ singularity, so $\Gamma(t,x)$ is continuous for $t \ge 0$, while the corresponding function in Equation (71) in [E-1993(a)] was continuous only for $t > 0$.

4 Bounded magnetic field

In this section we prove Proposition 3.1. By Proposition 3.3 we have

$$e^{-tH}(x,x) = \frac{1}{4\pi t}\mathbf{E}_{0,x}^{2t,x} \exp\left(-i\int_0^{2t} A(W(s))\circ dW(s) + \frac{1}{2}\int_0^{2t} B(W(s))ds\right). \tag{34}$$

Estimating the oscillatory part trivially (diamagnetic inequality) and using $\|B\| := \|B\|_\infty \geq B(x)$ we have

$$e^{tL}e^{-tH}(x,x) \leq e^{tL}e^{t\|B\|}\frac{1}{4\pi t}. \tag{35}$$

Choosing the optimal $t := \frac{1}{L+\|B\|}$ we obtain that $e^{tL}e^{-tH}(x,x) \leq \frac{e}{4\pi}(L + \|B\|)$.

For the upper part we use the fact (see [LSY-II]) that

$$(1-P_L)H(1-P_L) = (1-P_L)[(p-A)^2 - B](1-P_L) \geq L \tag{36}$$

and $B \leq \|B\|$ imply that

$$(1-\Pi_L)[(\mathbf{p}-\mathbf{A})^2 - B](1-\Pi_L) \geq \frac{L}{L+\|B\|}(I-\Pi_L)(\mathbf{p}-\mathbf{A})^2(I-\Pi_L), \tag{37}$$

that is

$$(1-\Pi_L)\left(H + p_3^2 + \frac{E}{2}\right)^{-1}(1-\Pi_L) \leq \tag{38}$$

$$\leq (I-\Pi_L)\left(\frac{L}{L+\|B\|}(\mathbf{p}-\mathbf{A})^2 + \frac{E}{2}\right)^{-1}(I-\Pi_L),$$

showing that $M_{E,L}$ chosen in Proposition 3.1 satisfies (23).

To estimate the kernel of $M_{E,L}^2$ we use the diamagnetic inequality as in [LSY-II], and one can easily show that

$$M_{E,L}^2(\mathbf{x},\mathbf{x}) \leq \frac{1}{\sqrt{E}}\frac{1}{2^{5/2}\pi}\left(1 + \frac{\|B\|}{L}\right)^{3/2} \tag{39}$$

which finishes the proof of Proposition 3.1. $\square$

We briefly remark that the trick to estimate $(\mathbf{p}-\mathbf{A})^2 - B$ by $(const)\cdot(\mathbf{p}-\mathbf{A})^2$ from below (see (37) without the projections Π_L) works for general (not necessarily bounded) magnetic field if the so-called electron g factor is smaller than 2 (see [FLL]). In this case one considers $H_g := (\mathbf{p}-\mathbf{A})^2 - \frac{1}{2}g\cdot B$ instead of H, and clearly

$$H_g = \frac{g}{2}((\mathbf{p}-\mathbf{A})^2 - B) + \left(1-\frac{g}{2}\right)(\mathbf{p}-\mathbf{A})^2 \geq \left(1-\frac{g}{2}\right)(\mathbf{p}-\mathbf{A})^2. \tag{40}$$

By the diamagnetic inequality the usual (non-magnetic) Lieb-Thirring inequality can be used and we obtain

$$\sum_i |E_i|^\gamma \leq \frac{2^{\gamma-1}}{\pi(4\gamma^2-1)}\left(\frac{2}{2-g}\right)^{3/2}\int_{\mathbf{R}^3}|V(\mathbf{x})|_-^{\gamma+3/2}d\mathbf{x}. \tag{41}$$

5 Unbounded magnetic field; reduction to the Main Lemma

In the rest of the paper we present the proof of Proposition 3.2. The crucial estimate (37) in Section 4 relied on the global boundedness of B. If we do not want to assume this, or we wish to obtain the "real" estimate (3) instead of (6), then we have to analyse the local behaviour of $H = (p-A)^2 - B$.

The first inequality in Proposition 3.2 is a straightforward consequence of Lemma 5.1 below. At this point we still do not make use of the oscillation effect in the magnetic Feynman-Kac formula due to the $-i\int A\circ dW$ term. The proof of the second inequality (30) is much more difficult because we have to exploit the full power of this oscillation. We will compare the heat kernel of H with that of the operator with constant magnetic field. This is the content of the Main Lemma 5.2, formulated at the end of this section.

First we prove the following technical estimate which will be used throughout our stochastic analysis.

Lemma 5.1 *Let $F : \mathbf{R}^2 \to \mathbf{R}$ be a measurable function with $|F(w)| \le d|w|$, furthermore assume that $0 < t \le 1/B_0$, $d \le cB_0^{3/2}$ for some positive B_0 and c. Then there exist two constants $C^{(0)} = C^{(0)}(c)$ and $C^{(1)} = C^{(1)}(c)$ depending only on c such that the following estimates hold for $z \in \mathbf{R}^2$:*

$$\mathbf{E}_{0,0}^{2t,z} \exp\left(\frac{1}{2}\int_0^{2t} F(W(s))ds\right) \le C^{(0)} \exp\left(\frac{z^2}{40t}\right) \tag{42}$$

and

$$\mathbf{E}_{0,0}^{2t,z} \left(\frac{1}{2t}\int_0^{2t} F(W(s))ds\right) \exp\left(\frac{1}{2}\int_0^{2t} F(W(s))ds\right) \le C^{(1)} d(1+t^2) \exp\left(\frac{z^2}{20t}\right). \tag{43}$$

Proof. Consider the absolute value process $r(s) := |W(s)|$ (Bessel process) and use the upper bound for $|F|$ to transform (42) and (43) into inequalities about $r(s)$. There is an explicit formula for the exponential moment of the integral of $r^2(s)$, so we estimate $r(s)$ from above by $K + Mr^2(s)$ where $K := 100t^2 d/\pi^2$ and $M := 1/(4K)$. Therefore

$$\mathbf{E}_{0,0}^{2t,z} \exp\left(\frac{1}{2}\int_0^{2t} F(W(s))ds\right) \le e^{Kdt} \cdot \mathbf{E} \exp\left(\frac{b^2}{2}\int_0^{2t} r^2(s)ds\right) =$$

$$= e^{Kdt} \cdot \frac{2bt}{\sin 2bt} \exp\left(\frac{z^2}{4t}(1 - 2bt\cot(2bt))\right), \tag{44}$$

where $b := \sqrt{Md} \le \frac{\pi}{20t}$, i.e $2bt \le \pi/10$, and $\mathbf{E}$ denotes the expectation for the process $r(s)$. Here we used the analytic extension of the Laplace transform of $\int_0^{2t} r^2(s)ds$ given, for example, in [Y-1992, p.17] or in [E-1993(b)]. The analytic extension is possible for $2bt < \pi$. Using that $(1 - 2bt\cot(2bt)) \le 1/10$ and $2bt \le 2\sin(2bt)$ for $2bt \le \pi/10$, we easily obtain (42).

For (43) one uses Hölder's inequality

$$\mathbf{E}_{0,0}^{2t,z} \left(\frac{1}{2t}\int_0^{2t} F(W(s))ds\right) \exp\left(\frac{1}{2}\int_0^{2t} F(W(s))ds\right) \le$$

$$\le \frac{d}{2}\mathbf{E}_{0,0}^{2t,z} \left(1 + \left(\frac{1}{2t}\int_0^{2t} |W(s)|ds\right)^2\right) e^{\frac{d}{2}\int_0^{2t} |W(s)|ds} \le \tag{45}$$

$$\le \frac{C^{(0)}d}{2} \exp\left(\frac{z^2}{40t}\right) + \frac{d}{2}\mathbf{E}\left(\frac{1}{2t}\int_0^{2t} r^2(s)ds\right) e^{Kdt} \cdot e^{\frac{Md}{2}\int_0^{2t} r^2(s)ds}$$

using (42) and $r(s) \le K + Mr^2(s)$ as above. Differentiating the explicit formula (see (44)) for $\mathbf{E}\exp(b^2/2\int r^2(s)ds)$ with respect to b one can estimate the obtained expression for $2bt \le \pi/10$ as follows

$$\mathbf{E}\left(b\int_0^{2t} r^2(s)ds\right)\exp\left(\frac{b^2}{2}\int_0^{2t} r^2(s)ds\right) \le (const)\cdot t^2\exp\left(\frac{z^2}{20t}\right) \tag{46}$$

(by *(const)* we shall denote universal constants, not necessary the same ones). Combining this with (45) and with the conditions on t and d, one obtains (43). □

Now we can easily prove (29):

$$P_0(x,x) \le e^{-tH}(x,x) \le \frac{e^{tB(x)}}{4\pi t}\mathbf{E}_{0,x}^{2t,x}\exp\left(\frac{1}{2}\int_0^{2t}(B(W(s)) - B(x))ds\right). \tag{47}$$

Choose $t := 1/B(x)$ and apply the estimate (42) from Lemma 5.1 with $F(w) := B(x+w) - B(x)$ using (9). □

To treat the contribution from the upper part of the spectrum of H first we have to present an operator $M_{E,0}$ satisfying (23) and the continuity requirement for the kernel of $M_{E,0}^2$, and prove (30).

The first trick is to realize that $(I - P_0)H(I - P_0) \ge 2B_0$ because of the spectral gap (the spectrum of H has a gap of size at least $2B_0$ above 0, for details see [CFKS]). On the other hand for $u \ge 2B_0$

$$e^{-tu} \le (const)\left(e^{-tu} - e^{-(t+\beta)u}\right) \tag{48}$$

with $\beta := 1/(2B_0)$, therefore

$$(I - P_0)e^{-tH}(I - P_0) \le (const)\left(e^{-tH} - e^{-(t+\beta)H}\right) \tag{49}$$

as operators on $L^2(\mathbf{R}^2)$. (Note that it is enough to check this inequality on $\mathrm{Ran}(I - P_0)$ where $H \ge 2B_0 = \beta^{-1}$, since on $\mathrm{Ran}(P_0)$ both sides are 0.) Extending this inequality to $L^2(\mathbf{R}^3)$ and multiplying with $\exp\left[-t(p_3^2 + \frac{E}{2})\right]$ we obtain

$$(I - \Pi_0)e^{-t\left(H + p_3^2 + \frac{E}{2}\right)}(I - \Pi_0) \le (const)\cdot e^{-t\left(p_3^2 + \frac{E}{2}\right)}\left(e^{-tH} - e^{-(t+\beta)H}\right) \tag{50}$$

(here we use that if $0 \leq A \leq B$, and $C \geq 0$ commutes with A and B then $AC \leq BC$). The next simple trick is to realize that

$$\frac{1}{u+\frac{E}{2}} \leq (const) \cdot \int_0^\beta e^{-t\left(u+\frac{E}{2}\right)} dt \tag{51}$$

if $u \geq \beta^{-1}$.

Therefore by (50) and (51)

$$(I-\Pi_0)\left(H+p_3^2+\frac{E}{2}\right)^{-1}(I-\Pi_0) \leq (const) \cdot \int_0^\beta (I-\Pi_0) e^{-t\left(H+p_3^2+\frac{E}{2}\right)}(I-\Pi_0) \leq M_{E,0}, \tag{52}$$

where

$$M_{E,0} := (const) \cdot \int_0^\beta e^{-t\left(p_3^2+\frac{E}{2}\right)}\left(e^{-tH}-e^{-(t+\beta)H}\right) dt, \tag{53}$$

so (23) is satisfied.

By Proposition 3.3 (and especially by the estimate (33)) it is clear that $M_{E,0}(\mathbf{x},\mathbf{y})$ exists for $\mathbf{x} \neq \mathbf{y}$, and

$$M_{E,0}(\mathbf{x},\mathbf{y}) \leq (const) \cdot \int_0^\beta \frac{1}{t^{3/2}} \cdot e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{8t}} \Gamma(t,x) dt \tag{54}$$

Therefore $M_{E,0}^2$ has a kernel even for $\mathbf{x}=\mathbf{y}$, since using the estimate (54)

$$M_{E,0}^2(\mathbf{x},\mathbf{x}) = \int_{\mathbf{R}^3} M_{E,0}(\mathbf{x},\mathbf{z}) M_{E,0}(\mathbf{z},\mathbf{x}) d\mathbf{z} \leq \tag{55}$$

$$\leq (const) \cdot \int_0^\beta \int_0^\beta \frac{\Gamma(t,x)\Gamma(s,x)}{(s+t)^{3/2}} ds\, dt < \infty$$

(the existence of $M_{E,0}(\mathbf{x},\mathbf{y})$ for $\mathbf{x} \neq \mathbf{y}$ is even more obvious).

Calculating $M_{E,0}^2(\mathbf{x},\mathbf{x})$ from (53) one obtains

$$M_{E,0}^2(\mathbf{x},\mathbf{x}) = (const) \cdot \int_0^\beta dt \int_0^\beta ds \int_{\mathbf{R}^3} d\mathbf{y} \left[e^{-t\left(p_3^2+\frac{E}{2}\right)}\left(e^{-tH}-e^{-(t+\beta)H}\right)\right](\mathbf{x},\mathbf{y}) \times$$

$$\times \left[e^{-s\left(p_3^2+\frac{E}{2}\right)}\left(e^{-sH}-e^{-(s+\beta)H}\right)\right](\mathbf{y},\mathbf{x}). \tag{56}$$

The main point in this computation was that we wanted to estimate the projected resolvent kernel by the heat kernel (so that we could use the Feynman-Kac formula). On the other hand the heat kernel is always larger than the ground state projection kernel P_0, which grows linearly with B (this is why we had to treat its contribution separately), but we need a B-independent estimate. Therefore we have to deal with the difference of two heat kernels, so that P_0 be cancelled.

The next problem is that we will be able to estimate effectively the heat kernel from above (via Feynman-Kac), but obtaining lower bound is much harder. The best thing we can do is to introduce an approximating Hamiltonian H_c with constant magnetic field, use that $e^{-tH_c} - e^{-(t+\beta)H_c}$ can be exactly calculated, and try to prove that $e^{-tH} - e^{-tH_c}$ is small. This last statement can be proved only for small t, this is why the truncation in the limit of integration in (51) was needed (this step shows implicitely that the positive lower bound on $B(x)$ is necessary for the proof). So we anticipate the following Main Lemma:

Main Lemma 5.2 *Assume the conditions of Theorem 2.2 and choose a gauge $\mathbf{A} = (A, 0)$ so that A satisfies the conditions of Proposition 3.3. Fix $x, y \in \mathbf{R}^2$ and let $B := B(x)$. For any $z = (z_1, z_2) \in \mathbf{R}^2$ define the following divergence free gauge (written as a 1-form on $\mathbf{R}^2$)*

$$A^z(u) := \frac{B(z)}{2}[(u_1 - z_1)du_2 - (u_2 - z_2)du_1] \tag{57}$$

generating the constant $B(z)$ magnetic field: $rot A^z(u) = B(z)$. Let $H_c := (p - A^x)^2 - B$ be the operator with constant $B = B(x)$ magnetic field. Then there exist a real number $\varphi = \varphi(x, y)$ and a constant $C = C(c)$ depending only on c (the constant appearing in (8) and (9) in Theorem 2.2), such that for any $t \leq 1/B_0$ we have

$$\left| e^{-tH}(x, y) - e^{i\varphi} e^{-tH_c}(x, y) \right| \leq \frac{C}{4\pi t} e^{-\frac{(x-y)^2}{8t}}, \tag{58}$$

or, equivalently,

$$\left| \mathbf{E}_{0,x}^{2t,y} \left(e^{-i\int_0^{2t} A(W(s))\circ dW(s) + \frac{1}{2}\int_0^{2t} B(W(s))ds} - e^{i\varphi} e^{-i\int_0^{2t} A^x(W(s))\circ dW(s) + Bt} \right) \right| \leq C \cdot e^{\frac{(x-y)^2}{8t}}. \tag{59}$$

Estimating $(e^{-(t+\beta)H_c} - e^{-tH_c})(x,y)$ is relatively easy using the explicit formula (see e.g. [S-1979]):

$$e^{-tH_c}(x,y) = \frac{Be^{Bt}}{4\pi \sinh Bt} \exp\left(-B\coth(Bt)\frac{(x-y)^2}{4}\right) \tag{60}$$

and its derivative with respect to t. Notice that

$$\left|\left(\frac{d}{d\tau}e^{-\tau H_c}\right)(x,y)\right| \le \frac{const}{\tau^2}e^{-\frac{(x-y)^2}{8\tau}} \tag{61}$$

independently of B, although $e^{-\tau H_c}$ itself grows linearly with B. Therefore using the Main Lemma 5.2 above for $t \le \beta = 1/(2B_0)$, we have

$$\left|\left(e^{-tH} - e^{-(t+\beta)H}\right)(x,y)\right| \le \left[(const)\int_t^{t+\beta}\frac{d\tau}{\tau^2}e^{-\frac{(x-y)^2}{8\tau}} + \frac{C}{4\pi t}e^{-\frac{(x-y)^2}{8t}} + \frac{C}{4\pi(t+\beta)}e^{-\frac{(x-y)^2}{8(t+\beta)}}\right]. \tag{62}$$

Now we plug this estimate into (56). The $d\mathbf{y}$ integration is done explicitly, but then we arrive at the following complicated ds and dt integral

$$\begin{aligned}
(56) \le C'\int_0^\beta dt\int_0^\beta ds\frac{1}{\sqrt{t+s}}\cdot e^{-(s+t)\frac{E}{2}}\Bigg[\int_t^{t+\beta}d\tau\int_s^{s+\beta}d\zeta\frac{1}{\tau\zeta(\tau+\zeta)}+\\
+\int_t^{t+\beta}\frac{d\tau}{\tau(\tau+s)}+\int_t^{t+\beta}\frac{d\tau}{\tau(\tau+s+\beta)}+\int_s^{s+\beta}\frac{d\zeta}{\zeta(\zeta+t)}+\int_s^{s+\beta}\frac{d\zeta}{\zeta(\zeta+t+\beta)}+\\
+\frac{1}{t+s}+\frac{2}{t+s+\beta}+\frac{1}{t+s+2\beta}\Bigg] \le \\
\le C'\int_0^\beta dt\int_0^\beta ds\ \frac{1}{\sqrt{s+t}}\cdot e^{-(t+s)\frac{E}{2}}\Bigg[\int_t^{t+\beta}d\tau\int_s^{s+\beta}d\zeta\ \frac{1}{\tau\zeta(\tau+\zeta)}+\\
+2\int_t^{t+\beta}\frac{d\tau}{\tau(\tau+s)}+2\int_s^{s+\beta}\frac{d\zeta}{\zeta(\zeta+t)}+\frac{4}{t+s}\Bigg],
\end{aligned} \tag{63}$$

where C' depends on the constant C obtained in the Main Lemma 5.2. The right hand side of (63) is monotone increasing in β and we need a $\beta = 1/(2B_0)$-independent estimate for (30), so we can take immediately $\beta = \infty$:

$$(56) \le C'\int_0^\infty dt\int_0^\infty ds\ \frac{1}{\sqrt{t+s}}\cdot e^{-(s+t)\frac{E}{2}}\left[\int_t^\infty d\tau\int_s^\infty d\zeta\ \frac{1}{\tau\zeta(\tau+\zeta)}+\right.$$

$$+2\int_t^\infty \frac{d\tau}{\tau(\tau+s)} + 2\int_s^\infty \frac{d\zeta}{\zeta(\zeta+t)} + \frac{4}{t+s}\Bigg] \le$$

$$\le C' \int_0^\infty dt \int_0^\infty ds \, \frac{1}{\sqrt{t+s}} e^{-(s+t)\frac{E}{2}} \left[\int_t^\infty \frac{d\tau}{\tau^2} \log\left(1+\frac{\tau}{s}\right) + \frac{2}{s}\log\left(1+\frac{s}{t}\right) + \right. \tag{64}$$

$$\left. + \frac{2}{t}\log\left(1+\frac{t}{s}\right) + \frac{4}{t+s}\right] \le C'\cdot(const)\int_0^\infty dt \int_0^\infty ds \frac{1}{\sqrt{t+s}} \cdot e^{-(s+t)\frac{E}{2}} \left[\frac{1}{\sqrt{ts}} + \frac{1}{t+s}\right] \le$$

$$\le \frac{C'\cdot(const)}{\sqrt{E}} \int_0^\infty dT \int_0^\infty dS \, \frac{e^{-(T+S)}}{\sqrt{TS(T+S)}} = \frac{C_2}{\sqrt{E}}$$

(using that $\log(1+u) \le \sqrt{u}$), which proves (30), and so Theorem 2.2.

6 Proof of the Main Lemma

This section contains the essence of the whole proof; we compare the heat kernel of the operator with nonconstant field with that of an operator with frozen constant field. We use a localization technique in path space; a similar method has been used in [E-1993(b)], but the present setup is more complicated and we need better estimates. Nevertheless, the intuitive idea of the method outlined in Section 2 of [E-1993(b)] might help to understand the present proof.

Introduce the following notations

$$F^z(w) := B(z+w) - B(z) \tag{65}$$

$$G^z(w) := \int_0^1 tF^z(tw)dt(w_1dw_2 - w_2dw_1), \tag{66}$$

then G^z is a 1-form generating F^z, i.e. $\mathrm{d}G^z(w) = F^z(w)$ (we use the canonical identification between 1-forms $A = A_1dx_1 + A_2dx_2$ and vectorfields $A = (A_1, A_2)$ without any further comment). Let

$$A_*^z(u) := A(u) - G^z(u-z), \tag{67}$$

then $\mathrm{d}A^z_*(u) = B(u) - F^z(u-z) = B(z)$, so A^z_* and A^z both generate the constant $B(z)$ field. Therefore there is a function $\varphi^z : \mathbf{R}^2 \to \mathbf{R}$ such that $A^z_* = A^z + \mathrm{d}\varphi^z$. The phase difference $\varphi = \varphi(x,y)$ in the Main Lemma 5.2 will be given as $\varphi := \varphi^x(x) - \varphi^x(y)$.

We will not give the exact value of $C = C(c)$, but it is explicitly computable from the proof below. Also we will use the same letter C for various positive constants depending only on c.

First we eliminate some extreme cases.

Case 1. (short time): If $Bt \le 1$ (recall that $B := B(x)$) then by the roughest estimate

$$\text{LHS of (59)} \le e^{Bt} + \mathbf{E}^{2t,y}_{0,x} e^{\frac{1}{2}\int_0^{2t} B(W(s))ds} \le e^{Bt}\left(1 + \mathbf{E}^{2t,y}_{0,x} e^{\frac{1}{2}\int_0^{2t} F^x(W(s)-x)ds}\right) \le$$

$$\le e^{Bt}\left(1 + C^{(0)} \cdot e^{\frac{(x-y)^2}{40t}}\right) \le C \cdot e^{\frac{(x-y)^2}{8t}} \tag{68}$$

using Lemma 5.1.

Case 2. (large distance): If $(x-y)^2 \ge 16Bt^2$ then $Bt \le (x-y)^2/(16t)$, so one can use the same rough estimate (68) as above to obtain

$$\text{LHS of (59)} \le e^{\frac{(x-y)^2}{16t}}\left(1 + C^{(0)} e^{\frac{(x-y)^2}{40t}}\right) \le C \cdot e^{\frac{(x-y)^2}{8t}}. \tag{69}$$

So from now on we can assume that $Bt \ge 1$, $(x-y)^2 \le 16Bt^2$ and we have to bound the expression (for brevity we use a straightforward shorthand notation when it makes no confusion)

$$I := \left|\mathbf{E}\left(e^{-i\int A \circ dW + \frac{1}{2}\int B} - e^{i\varphi} e^{-i\int A^x \circ dW + Bt}\right)\right| \tag{70}$$

from above.

Let $\varepsilon := 2t/([Bt]+1)$ (here [ ] denotes the integer part) and we define a sequence of stopping times τ_i inductively as follows. Let $\tau_0 := 0$, $x_j := W(\tau_j)$ and for $j \ge 0$ let

$$\tau_{j+1} := \varepsilon\left[\frac{s_{j+1}}{\varepsilon} + 1\right],$$

where

$$s_{j+1} := \inf\left\{ r \,:\, \tau_j < r \le 2t \text{ and } \left| \int_{\tau_j}^{r} A^{x_j}(W(s))dW(s) \right| = \frac{\pi}{2} \right\}, \tag{71}$$

which is also a stopping time (if there is no such r then we stop defining the sequence τ_j). The crucial idea is that s_{j+1} is the first time when the flux of the frozen constant magnetic field $B(x_j)$ between the Brownian curve starting at time τ_j from the point x_j and the corresponding chord reaches $\pi/2$ in absolute value. After that, we look for the next stopping time with the same property, but for technical reasons we have to discretize the set of the starting times, this is why we introduce the τ's.

Let $\tau_{n(W)}$ be the last stopping time defined above ($n(W) \le [tB] + 1 = 2t/\varepsilon$ is an integer valued random variable). Define $\mathcal{H}_n := \{W \,:\, n(W) = n\}$, then clearly $\mathbf{P}(\cup_n \mathcal{H}_n) = 1$, and define the j^{th} reflection $T_j : \mathcal{H}_n \to \mathcal{H}_n$ $(0 \le j \le n-1)$ in the following way. It will affect only the $\{W(s) \,:\, \tau_j \le s \le s_{j+1}\}$ part of the Brownian bridge, so let $[T_j(W)](s) := W(s)$ for $s < \tau_j$ or $s > s_{j+1}$. For $\tau_j \le s \le s_{j+1}$ let $[T_j(W)](s)$ be the geometric reflection of $W(s)$ onto the segment $[W(\tau_j), W(s_{j+1})]$. By the strong Markov property T_j preserves the probability measure and the sequence of stopping times τ_j, and T_j is an involution. These last two statements follow from the crucial relation:

$$\int_{\tau_j}^{r} A^{x_j}(W(s))dW(s) = -\int_{\tau_j}^{r} A^{x_j}(\overline{W}(s))d\overline{W}(s) \tag{72}$$

for any $\tau_j \le r \le s_{j+1}$, where $\overline{W} := T_j(W)$ for simplicity.

Define the following stochastic integrals for $0 \le j \le n = n(W)$ for paths W belonging to $\mathcal{H}_n$:

$$N_j(W) = N_j := -i \int_{\tau_j}^{\tau_{j+1}} G^{W(\tau_j)}(W(s) - W(\tau_j)) \circ dW(s) + \frac{1}{2}\int_{\tau_j}^{\tau_{j+1}} F^{W(\tau_j)}(W(s) - W(\tau_j))ds, \tag{73}$$

$$M_j(W) = M_j := -\int_{\tau_j}^{\tau_{j+1}} A_*^{W(\tau_j)}(W(s)) \circ dW(s), \tag{74}$$

$$L_j(W) = L_j := \frac{1}{2}B(W(\tau_j))(\tau_{j+1} - \tau_j), \tag{75}$$

where $\tau_{n+1} := 2t$ (it might be that $\tau_n = 2t$, then $N_n = M_n = L_n = 0$.) Let $\overline{N}_j := N_j(T_jW)$ and $\overline{M}_j := M_j(T_jW)$ be the same quantities for the T_j-reflected path. Then for $0 \leq j \leq n-1$ we have $|M_j - \overline{M}_j| = \pi$ by the careful definition of the stopping times, since

$$|M_j - \overline{M}_j| = \left| \int_{\tau_j}^{\tau_{j+1}} A_*^{W(\tau_j)}(W(s)) \circ dW(s) - \int_{\tau_j}^{\tau_{j+1}} A_*^{W(\tau_j)}(\overline{W}(s)) \circ d\overline{W}(s) \right| = \tag{76}$$

$$= \left| \int_{\tau_j}^{s_{j+1}} A^{W(\tau_j)}(W(s)) \circ dW(s) - \int_{\tau_j}^{s_{j+1}} A^{W(\tau_j)}(\overline{W}(s)) \circ d\overline{W}(s) \right| = \pi,$$

using first the fundamental theorem of calculus for the Stratonovich integral, then the fact that A^z is divergence free, so its Ito and Stratonovich integrals are the same, and finally the relation (72) and the definition of s_{j+1}.

We shall decompose the quantity I to be estimated (see (70)) according to the disjoint events $\mathcal{H}_n$:

$$I = \left| \sum_{n=0}^{2t/\varepsilon} I_n \right| \tag{77}$$

with

$$I_n := \mathbf{E}_{0,x}^{2t,y} \chi(\mathcal{H}_n) \left(e^{-i\int A \circ dW + \frac{1}{2}\int B} - e^{i\varphi} e^{-i\int A^x \circ dW + Bt} \right). \tag{78}$$

The $n = 0$ case must be treated separately; on this event the contribution from both terms in (70) is proportional to B, but they will cancel each other. In the operator language this corresponds to the ground states; we know that the heat kernel contains the ground state projection, which is proportional to B, but we need a B-independent estimate. On the other hand we wanted to estimate the heat kernel only on the subspace orthogonal to the ground states, which allowed us to subtract an other heat kernel (namely that of with constant magnetic field) having more or less the same ground state projection as H. This was the essence of the calculation in Section 5.

Case $n = 0$. Using the notations above we have

$$|I_0| \le e^{Bt}\mathbf{E}_{0,x}^{2t,y}\left(\chi(\mathcal{H}_0)\left|e^{N_0}-1\right|\right). \tag{79}$$

We have to treat the largely deviating bridges separately. Let $R := 8t\sqrt{B}$ and let

$$E := \{W(s) : \sup_{0\le s\le 2t} |W(s)-x| \le R\} \tag{80}$$

be a measurable subset of the path space. By standard large deviation estimate (see e.g. [S-1984]) for the complement of E we have

$$\mathbf{P}_{0,x}^{2t,y}(E^c) \le 4e^{-\frac{R^2}{16t}} \le 4e^{-4Bt}, \tag{81}$$

and on the subset E^c the left hand side of (79) can be easily estimated by

$$e^{Bt}\mathbf{E}_{0,x}^{2t,y}\left(\chi(E^c)\chi(\mathcal{H}_0)\left|e^{N_0}-1\right|\right) \le e^{Bt}\left(\mathbf{P}_{0,x}^{2t,y}(E^c) + \mathbf{E}_{0,x}^{2t,y}\chi(E^c)e^{\frac{1}{2}\int_0^{2t}F^x(W(s)-x)ds}\right) \le \tag{82}$$

$$\le e^{Bt}\left(\mathbf{P}_{0,x}^{2t,y}(E^c) + \left(\mathbf{P}_{0,x}^{2t,y}(E^c)\cdot \mathbf{E}_{0,0}^{2t,y-x}e^{\int_0^{2t}F^x(W(s))ds}\right)^{1/2}\right) \le C\cdot e^{-\frac{(x-y)^2}{80t}}$$

using Lemma 5.1 again (now for $2F^x$ instead of F^x, so the constant C obtained here is essentially $C^{(0)}(2c)$ with the notations of Lemma 5.1).

On the subset E, by the general estimate $|e^{iX+Y}-1| \le |X| + |Y|e^{|Y|}$ for real numbers X and Y, we have

$$e^{Bt}\mathbf{E}_{0,x}^{2t,y}\left(\chi(E\cap\mathcal{H}_0)\left|e^{N_0}-1\right|\right) \le$$

$$\le e^{Bt}\mathbf{E}_{0,x}^{2t,y}\left(\chi(E\cap\mathcal{H}_0)\left(\left|\int_0^{2t}G^x(W(s)-x)dW(s)\right| + \right.\right. \tag{83}$$

$$\left.\left.+\frac{1}{2}\left|\int_0^{2t}\mathrm{div}G^x(W(s)-x)ds\right| + \frac{1}{2}\int_0^{2t}|F^x(W(s)-x)|ds\cdot e^{\frac{1}{2}\int_0^{2t}F^x(W(s)-x)ds}\right)\right).$$

Notice that we have replaced the Stratonovich integral by the Ito integral plus the divergence term. The reason for it is that the Ito integral is a martingale so the calculations become easier.

We estimate each term in (83) separately. For the last term use that

$$|F^x(W(s)-x)| \le cd(x)|W(s)-x| \le cd(x)R \tag{84}$$

on the event E to obtain that

$$\text{Last term on the RHS of (83)} \le e^{Bt}\mathbf{P}_{0,x}^{2t,y}(\mathcal{H}_0)cd(x)Rt \cdot e^{cd(x)Rt}. \tag{85}$$

For estimating the probability of $\mathcal{H}_0$ we recall Lemma 4.1 from [E-1993(b)] (in a simplified form)

Lemma 6.1 *For any $\overline{x}, \overline{y} \in \mathbf{R}^2$ let*

$$\overline{\xi}(s) := \frac{\overline{B}}{2}\int_0^s (W_1(u)-\overline{x}_1)dW_2(u) - (W_2(u)-\overline{x}_2)dW_1(u) \tag{86}$$

be the random flux process of the two dimensional Brownian bridge under the constraints $W(0) = \overline{x}$, $W(2\overline{t}) = \overline{y}$ for the constant magnetic field $\overline{B}$. Assume that

$$\overline{Bt} \ge \overline{c} \tag{87}$$

for some positive $\overline{c}$, then there exists a constant $\overline{C} = \overline{C}(\overline{c})$ depending only on $\overline{c}$ such that

$$\mathbf{P}_{0,\overline{x}}^{2\overline{t},\overline{y}}\left(\sup_{0\le s\le 2\overline{t}} |\overline{\xi}(s)| < \frac{\pi}{2}\right) \le \overline{C}(1+\overline{Bt})e^{-\overline{Bt}}. \tag{88}$$

By the definition of A^x (see (57)) and the sequence of stopping times τ_j we have

$$\mathbf{P}_{0,x}^{2t,y}(\mathcal{H}_0) = \mathbf{P}_{0,x}^{2t,y}\left(\sup_{0\le s\le 2t}\left|\int_0^s A^x(W(u))dW(u)\right| < \frac{\pi}{2}\right) \le (const)(1+Bt)e^{-Bt} \tag{89}$$

after applying Lemma 6.1 (recall that $Bt \ge 1$). Plugging (89) and the value $R := 8t\sqrt{B}$ into (85) we have

$$\text{Last term in (83)} \le (const)Bt \cdot \sqrt{B}t^2cd(x) \cdot e^{8\sqrt{B}t^2cd(x)} \le C \tag{90}$$

by (9) and $t \leq 1/B_0$.

The estimate of the divergence term in (83) is similar. By the definition of G^x (see (66)) clearly

$$|\mathrm{div} G^x(w)| \leq |w| \cdot \sup_{[0,w]} |\nabla F^x| = |w| \cdot \sup_{u \in [x,x+w]} |\nabla B(u)| \leq c|w| \cdot \sup_{u \in [x,x+w]} d(u) \tag{91}$$

since the function $cd(u)$ clearly dominates $|\nabla B(u)|$ by (8) ($[x, x+w] \subset \mathbf{R}^2$ denotes the segment joining x and $x+w$). For $|u-x| \leq R$ we have

$$|B(x) - B(u)| \leq cB_0^{3/2} \sqrt{\frac{B_0}{B(x)}} R \leq 8cB_0, \tag{92}$$

especially $B(u) \geq B(x) - 8cB_0$ which implies in particular that $B(u) \geq B(x)/(1+8c)$ (recall that $B(u) \geq B_0$), therefore

$$|u - x| \leq R \Longrightarrow B(u) \geq CB(x) \quad \text{and} \quad d(u) \leq Cd(x) \tag{93}$$

(recall that C denotes different positive constants depending only on c). Therefore on the event E

$$|\mathrm{div} G^x(W(s) - x)| \leq |W(s) - x| \cdot \sup_{u\,:\,|u-x| \leq R} d(u) \leq CRd(x), \tag{94}$$

which allows us to estimate the divergence term in (83) as

$$\text{Second term on the RHS of (83)} \leq Ce^{Bt} tRd(x) \mathbf{P}_{0,x}^{2t,y}(\mathcal{H}_0) \leq C. \tag{95}$$

Finally we have to estimate the first term on the right hand side of (83). This is much harder since one cannot plug an upper estimate on the integrand into a stochastic integral. First we have to estimate the stochastic integral by an ordinary integral using the Kolmogorov inequality for martingales. This is the content of Lemma 4.3 in [E-1993(b)] which we recall here for convenience:

Lemma 6.2 *Let W be the Brownian bridge in $\mathbf{R}^2$ with $W(0) = 0$ and $W(2\theta) = z$. Then for any function $H : \mathbf{R}^2 \to \mathbf{R}^2$ with at most polynomial growth and $\mu \geq 1$ integer*

$$\mathbf{E}\left(\int_0^{2\theta} H(W(s))dW(s)\right)^{2\mu} \leq (const)^\mu \mu^{4\mu/3}(z^{2\mu} + \theta^\mu) \cdot \mathbf{E}^{1/4}\frac{1}{2\theta}\int_0^{2\theta} |H(W(s))|^{8\mu} ds. \tag{96}$$

Before applying this lemma we have to separate the $\chi(\mathcal{H}_0 \cap E)$ factor since on a restricted set the stochastic integral is not a martingale.

Let $\mu := [Bt]$ (integer part) and use Hölder's inequality

$$e^{Bt}\mathbf{E}_{0,x}^{2t,y}\left(\chi(\mathcal{H}_0 \cap E)\left|\int_0^{2t} G^x(W(s) - x)dW(s)\right|\right) \leq$$

$$\leq e^{Bt}\left(\mathbf{P}_{0,x}^{2t,y}(\mathcal{H}_0)\right)^{1-\frac{1}{2\mu}}\left(\mathbf{E}_{0,x}^{2t,y}\left(\int_0^{2t} G^x(W(s) - x)dW(s)\right)^{2\mu}\right)^{\frac{1}{2\mu}}. \tag{97}$$

For the probability of $\mathcal{H}_0$ we use the same estimate as before, notice that this factor still essentially cancels e^{Bt}, since

$$\left(\mathbf{P}_{0,x}^{2t,y}(\mathcal{H}_0)\right)^{1-\frac{1}{2\mu}} \leq \left((const)(1 + Bt)e^{-Bt}\right)^{1-\frac{1}{2\mu}} \leq (const)(1 + Bt)e^{-Bt} \tag{98}$$

since $Bt \leq 2\mu$ by $Bt \geq 1$. For the other term in (97) we use Lemma 6.2 to obtain

$$\text{RHS of (97)} \leq (const)(1 + Bt)\mu^{2/3}((x - y)^{2\mu} + t^\mu)^{\frac{1}{2\mu}}\left(\mathbf{E}_{0,x}^{2t,y}\frac{1}{2t}\int_0^{2t} |G^x(W(s) - x)|^{8\mu} ds\right)^{\frac{1}{8\mu}} \leq$$

$$\leq (const)cd(x)\sqrt{t}(Bt)^{5/3}\left(1 + \frac{(x - y)^2}{t}\right)^{1/2}\left(\mathbf{E}_{0,x}^{2t,y}\frac{1}{2t}\int_0^{2t} |(W(s) - x)|^{16\mu} ds\right)^{\frac{1}{8\mu}} \tag{99}$$

where in addition to some arithmetic estimates we have used that

$$|G^x(w)| \leq |w| \cdot \sup_{u\in[0,w]} |F^x(u)| \leq |w| \cdot \sup_{u\in[0,w]} |B(u + x) - B(x)| \leq cd(x)|w|^2 \tag{100}$$

based upon (66).

Estimating the expectation in (99) is standard, one uses the following representation for the Brownian bridge:

$$W(s) = x + (y-x)\cdot\frac{s}{2t} + \sqrt{2t}\,b\left(\frac{s}{2t}\right), \tag{101}$$

where $b(\tau)$ is the standard two-dimensional Brownian loop under constraints $b(0)=b(1)=0$. We denote by $\mathbf{E}_b$ the expectation with respect to the measure of $b(\tau)$. Therefore

$$\begin{aligned}\mathbf{E}_{0,x}^{2t,y}\frac{1}{2t}\int_0^{2t}|W(s)-x|^{16\mu}ds \le \\ \le (const)^{\mu}\left(\frac{1}{2t}\int_0^{2t}\left(|x-y|\cdot\frac{s}{2t}\right)^{16\mu}ds + \mathbf{E}_b\frac{1}{2t}\int_0^{2t}(2t)^{8\mu}\left|b\left(\frac{s}{2t}\right)\right|^{16\mu}ds\right) \le \\ \le (const)^{\mu}\left(|x-y|^{16\mu}+(2t)^{8\mu}(8\mu)!\right)\end{aligned} \tag{102}$$

using the explicit formula for the moments of $b(\tau)$:

$$\mathbf{E}_b|b(\tau)|^{2m} = (2m-1)!!\cdot(2\tau(1-\tau))^m \tag{103}$$

for any positive integer m.

Plugging (102) and $\mu \le Bt$ into (99), using Stirling's formula to estimate the factorial we get

$$\begin{aligned}\text{RHS of (97)} \le (const)cd(x)\sqrt{t}(Bt)^{5/3}\left(1+\frac{(x-y)^2}{t}\right)^{1/2}((x-y)^2+Bt^2) \le \\ \le (const)cd(x)(Bt)^{8/3}t^{3/2}\cdot e^{\frac{(x-y)^2}{8t}} \le Ce^{\frac{(x-y)^2}{8t}},\end{aligned} \tag{104}$$

which finishes the estimate of I_0.

Case $n \ge 1$.

We first note that for $n \ge 1$ the contribution to (78) from the operator H^c (with frozen constant field) is zero, since

$$\mathbf{E}\left(\chi(\mathcal{H}_n)e^{-i\int_0^{2t}A^x(W(s))\circ dW(s)}\right) = \frac{1}{2}\mathbf{E}\left(\chi(\mathcal{H}_n)\left(e^{-i\int_0^{2t}A^x(W(s))\circ dW(s)} + e^{-i\int_0^{2t}A^x(\overline{W}(s))\circ d\overline{W}(s)}\right)\right) = 0 \tag{105}$$

(where $\overline{W} := T_0 W$), because the difference of the phase factors is exactly π (see (76) with $j = 0$). We use the shorthand notation $\mathbf{E} = \mathbf{E}_{0,x}^{2t,y}$ and similarly $\mathbf{P} = \mathbf{P}_{0,x}^{2t,y}$. Therefore only

$$\mathbf{E}\left[(1-\chi(\mathcal{H}_0))e^{-i\int_0^{2t} A(W(s))\circ dW(s)+\frac{1}{2}\int_0^{2t} B(W(s))ds}\right] = \sum_{n=1}^{2t/\varepsilon} \mathbf{E}\left(\chi(\mathcal{H}_n)\prod_{j=0}^{n} e^{N_j+iM_j+L_j}\right) \tag{106}$$

remains to be estimated.

The trick is that on the set $\mathcal{H}_n$ we consider all the 2^n paths of the form $T^{\underline{\sigma}}W$ together, where $\underline{\sigma} \in \{0,1\}^n$ and $T^{\underline{\sigma}} = T_0^{\sigma_0} T_1^{\sigma_1} \dots T_{n-1}^{\sigma_{n-1}}$. Therefore

$$\sum_{n\geq 1} I_n = \sum_{n=1}^{2t/\varepsilon} \frac{1}{2^n}\mathbf{E}\left(\chi(\mathcal{H}_n)\sum_{\underline{\sigma}\in\{0,1\}^n}\prod_{j=0}^{n-1}\left(e^{iM_j(T^{\underline{\sigma}}W)+N_j(T^{\underline{\sigma}}W)}\right) e^{iM_n(T^{\underline{\sigma}}W)+N_n(T^{\underline{\sigma}}W)}\prod_{j=0}^{n} e^{L_j(T^{\underline{\sigma}}W)}\right). \tag{107}$$

Clearly M_n, N_n and L_n do not depend on $\underline{\sigma}$, and

$$\sum_{\underline{\sigma}\in\{0,1\}^n}\prod_{j=0}^{n-1}\left(e^{iM_j(T^{\underline{\sigma}}(W))+N_j(T^{\underline{\sigma}}W)}\right) = \pm\prod_{j=0}^{n-1} e^{iM_j(W)}\prod_{j=0}^{n-1}\left(e^{N_j} - e^{\overline{N}_j}\right) \tag{108}$$

using (76). Putting (106), (107) and (108) together, (59) will follow from

$$\sum_{n=1}^{2t/\varepsilon}\frac{1}{2^n}\mathbf{E}\left(\chi(\mathcal{H}_n\cap\tilde{E})\prod_{j=0}^{n-1}\left(|e^{N_j} - e^{\overline{N}_j}|e^{L_j}\right)|e^{N_n}|e^{L_n}\right) \leq C\cdot e^{\frac{(x-y)^2}{8t}} \tag{109}$$

and from

$$\sum_{n=1}^{2t/\varepsilon}\frac{1}{2^n}\mathbf{E}\left(\chi(\mathcal{H}_n\cap\tilde{E}^c)\prod_{j=0}^{n-1}\left(|e^{N_j} - e^{\overline{N}_j}|e^{L_j}\right)|e^{N_n}|e^{L_n}\right) \leq C\cdot e^{\frac{(x-y)^2}{8t}}, \tag{110}$$

where

$$\tilde{E} := \left\{W(s)\,:\,\forall\underline{\sigma}\in\{0,1\}^{n(W)}\quad\sup_{0\leq s\leq 2t}|T^{\underline{\sigma}}W(s) - x| \leq R\right\}. \tag{111}$$

The left hand side of (110) is estimated very crudely as follows (using that $\tilde{E}$ is invariant under the reflections)

$$\text{LHS of (110)} \leq \sum_{n=1}^{2t/\varepsilon}\frac{1}{2^n}\mathbf{E}\left(\chi(\mathcal{H}_n\cap\tilde{E}^c)\prod_{j=0}^{n-1}\left(|e^{N_j}| + |e^{\overline{N}_j}|\right)e^{L_j}|e^{N_n}|e^{L_n}\right) \leq \tag{112}$$

$$\leq \mathbf{E}\left(\chi(E^c)e^{\frac{1}{2}\int_0^{2t}B(W(s))ds}\right) \leq e^{Bt}\mathbf{P}^{1/2}(\tilde{E}^c)\cdot \mathbf{E}^{1/2}e^{\int_0^{2t}F^x(W(s)-x)ds}.$$

Now use Lemma 5.1 and that

$$\mathbf{P}(\tilde{E}^c) \leq 2^{[Bt]+1}\cdot \mathbf{P}(E^c) \leq (const)e^{-3Bt} \tag{113}$$

(since $n(W) \leq [Bt]+1$ for each path) to obtain (110).

For the proof of (109) we are going to split the path $W : [0,2t] \to \mathbf{R}^2$ into pieces according to the stopping times $\tau_j = \varepsilon k_j$ (integer k_j is defined as τ_j/ε). First we split the event $\mathcal{H}_n \cap \tilde{E}$ by defining

$$E_j := \left\{ W(s)\ :\ j < n(W) \text{ and for } \sigma \in \{0,1\} \sup_{\varepsilon k_j \leq s \leq \varepsilon k_{j+1}} |T_j^\sigma W(s) - x| \leq R \right\} \tag{114}$$

with the remark that if $\sup|\xi(s)| < \pi/2$ for $\varepsilon k_j \leq s \leq \varepsilon k_{j+1}$ (i.e. $\varepsilon k_{j+1} = 2t$ and there is no reflection T_j) then only $\sigma = 0$ should be considered in the definition of E_j. These events clearly depend only on the corresponding part of the path $W(s)$, and

$$\bigcap_{0\leq j\leq n} (E_j \cap \mathcal{H}_n) = \tilde{E} \cap \mathcal{H}_n. \tag{115}$$

Furthermore, let $x_j := W(\tau_j) = W(\varepsilon k_j)$, and let

$$\xi_j(s) := \int_{\varepsilon k_j}^s A^{x_j}(W(u))dW(u)$$

be the flux process starting at $\tau_j = \varepsilon k_j$ from the point x_j. We decompose the path (and the corresponding measure) at times τ_j using the strong Markov property of the Brownian bridge to get the following formula ($\delta := \varepsilon/2$)

$$\text{RHS of (109)} = \sum_{n=1}^{2t/\varepsilon} \frac{1}{2^n}\cdot(4\pi t)\cdot e^{\frac{(x-y)^2}{4t}} \int_{|x-x_j|\leq R} dx_1 dx_2 \dots dx_n$$

$$\sum_{0<k_1<\ldots<k_n\leq 2t/\varepsilon} \left(\prod_{j=0}^{n-1} \frac{\exp\left(-\frac{(x_{j+1}-x_j)^2}{2\varepsilon(k_{j+1}-k_j)}\right)}{(2\pi\varepsilon(k_{j+1}-k_j))} \right) \cdot \frac{\exp\left(-\frac{(y-x_n)^2}{2(2t-\varepsilon k_n)}\right)}{(2\pi(2t-\varepsilon k_n))} \times$$

$$\times \prod_{j=0}^{n-1} \left(e^{\delta(k_{j+1}-k_j)B(x_j)} \mathbf{E}_{\varepsilon k_j, x_j}^{\varepsilon k_{j+1}, x_{j+1}} \left\{ \chi(E_j) |e^{N_j} - e^{\overline{N}_j}| \times \right. \right. \tag{116}$$

$$\left. \left. \times \boldsymbol{\chi} \left(\sup_{\varepsilon k_j \le s \le \varepsilon(k_{j+1}-1)} |\xi_j(s)| < \frac{\pi}{2} \le \sup_{\varepsilon k_j \le s \le \varepsilon k_{j+1}} |\xi_j(s)| \right) \right\} \right) \times$$

$$\times e^{(t-\delta k_n)B(x_n)} \mathbf{E}_{\varepsilon k_n, x_n}^{2t, y} \left\{ \chi(E_n) \cdot |e^{N_n}| \cdot \boldsymbol{\chi} \left(\sup_{\varepsilon k_n \le s \le 2t} |\xi_n(s)| < \frac{\pi}{2} \right) \right\},$$

where in the case of $\tau_n = \varepsilon k_n = 2t$ the exponential factor containing $2t - \varepsilon k_n$ in the denominator is considered 1.

Estimating the last line of (116) is easy. If $2t = \varepsilon k_n$ then it is simply 1, so we can assume that $2t - \varepsilon k_n \ge \varepsilon$ (recall that $2t/\varepsilon$ is an integer). Use that on the event E_n we have (for $\varepsilon k_n \le s \le 2t$)

$$|F^{x_n}(W(s) - x_n)| = |B(W(s)) - B(x_n)| \le c d(x_n) |W(s) - x_n| \le C d(x) R \tag{117}$$

by (93) and $|W(s) - x_n| \le |W(s) - x| + |x - x_n| \le 2R$. Now use Lemma 6.1 with $\overline{B} := B(x_n)$ and

$$2\overline{t} := 2t - k_n \varepsilon \ge \varepsilon \ge 1/B \ge C/B(x_n) = C/\overline{B} \tag{118}$$

(by (93)) to estimate the probability in the last line of (116) and combine it with (117) and the definition of N_n to obtain

$$\text{Last line of (116)} \le e^{(t-\delta k_n)B(x_n)} e^{Cd(x)Rt} C(1 + B(x_n)(t - k_n\delta)) e^{-(t-\delta k_n)B(x_n)} \le CBt \tag{119}$$

(at the last estimate we used again (93)).

To estimate the third and fourth line of (116) we use Hölder's inequality for each fixed j with exponents P and $P/(P-1)$ where $P := 2[tB(x_j)] + 2$ (depending on j), and we omit the part of the conditions on $\xi_j(s)$. Therefore

$$\mathbf{E}_{\varepsilon k_j, x_j}^{\varepsilon k_{j+1}, x_{j+1}} \left\{ \chi(E_j) |e^{N_j} - e^{\overline{N}_j}| \cdot \boldsymbol{\chi} \left(\sup_{\varepsilon k_j \le s \le \varepsilon(k_{j+1}-1)} |\xi_j(s)| < \frac{\pi}{2} \le \sup_{\varepsilon k_j \le s \le \varepsilon k_{j+1}} |\xi_j(s)| \right) \right\} \le \tag{120}$$

$$\leq \left\{ \mathbf{E}_{\varepsilon k_j, x_j}^{\varepsilon k_{j+1}, x_{j+1}} \chi(E_j)\chi(T_j \text{ exists})|e^{N_j} - e^{\overline{N}_j}|^P \right\}^{1/P} \left\{ \mathbf{P}_{\varepsilon k_j, x_j}^{\varepsilon k_{j+1}, x_{j+1}} \left(\sup_{\varepsilon k_j \leq s \leq \varepsilon(k_{j+1}-1)} |\xi_j(s)| < \frac{\pi}{2} \right) \right\}^{1-1/P} .$$

The second factor is estimated by Lemma 6.1 if $k_{j+1} - 1 > k_j$ (in which case $B(x_j)(k_{j+1} - 1 - k_j)\varepsilon \geq C$, so the condition (87) is satisfied), otherwise it is simply 1. So

$$\text{Second factor on the RHS of (120)} \leq \tag{121}$$

$$\left\{ C(1 + tB(x_j))e^{-B(x_j)(k_{j+1}-1-k_j)\delta} \right\}^{1-1/P} \leq CtB \cdot e^{-B(x_j)(k_{j+1}-k_j)\delta}$$

by (93) and by the definition of P.

For the first factor in (120) use that

$$\left| e^{N_j} - e^{\overline{N}_j} \right| \leq \left| e^{N_j} - 1 \right| + \left| e^{\overline{N}_j} - 1 \right| \tag{122}$$

and recalling the definition of N_j (see (73) with $W(\tau_j) = x_j$) we have

$$\left| e^{N_j} - 1 \right| \leq \frac{1}{2} \left| \int_{\tau_j}^{\tau_{j+1}} F^{x_j}(W(s) - x_j) ds \right| \cdot e^{\frac{1}{2} \int_{\tau_j}^{\tau_{j+1}} F^{x_j}(W(s)-x_j)ds} +$$

$$+ \left| \int_{\tau_j}^{\tau_{j+1}} G^{x_j}(W(s) - x_j) dW(s) \right| + \frac{1}{2} \left| \int_{\tau_j}^{\tau_{j+1}} \mathrm{div} G^{x_j}(W(s) - x_j) ds \right|, \tag{123}$$

and similarly for $\overline{N}_j$, using the simple estimate $|e^{iX+Y} - 1| \leq |Y|e^Y + |X|$ as before. On the event E_j we have $|W(s) - x_j| \leq 2R$ and $|\overline{W}(s) - x_j| \leq 2R$ for the reflected path $\overline{W} := T_j W$. Therefore (using (93))

$$|F^{x_j}(W(s) - x_j)| \leq 2cd(x_j)R \leq Cd(x)R, \tag{124}$$

$$|\mathrm{div} G^{x_j}(W(s) - x_j)| \leq Cd(x)R \tag{125}$$

$$|G^{x_j}(W(s) - x_j)| \leq cd(x_j)|W(s) - x_j|^2 \leq Cd(x)|W(s) - x_j|^2 \tag{126}$$

similarly to (84), (94) and (100), and the same estimates are valid for $\overline{W}(s)$ as well. Remark that (126) is valid for any path, while the first two inequalities are valid only for paths in E_j.

So to estimate the first factor on the right hand side of (120) first separate the six different terms obtained in (122) and (123) using the Minkowski inequality then treat each of them separately. The $|\frac{1}{2}\int F^{x_j}\,ds|^P \exp\left(\frac{P}{2}\int F^{x_j}\,ds\right)$ and the $|\frac{1}{2}\int \mathrm{div}G^{x_j}\,ds|^P$ terms are estimated directly by $(Cd(x)tR)^P \cdot e^{Cd(x)tRP}$ and $(Cd(x)Rt)^P$, respectively ($\tau_{j+1}-\tau_j$ is roughly overestimated by $2t$).

For the stochastic integral we use Lemma 6.2 (here with $\mu = P/2$, recall that P is even) and (126) to obtain a bound

$$\mathbf{E}^{\tau_{j+1},x_{j+1}}_{\tau_j,x_j}\left|\int_{\tau_j}^{\tau_{j+1}} G^{x_j}(W(s)-x_j)dW(s)\right|^P \leq (Cd(x))^P \cdot P^{2P/3} \times \tag{127}$$

$$\times \left(|x_j - x_{j+1}|^P + \eta_j^{P/2}\right)\left[\mathbf{E}^{2\eta_j,x_{j+1}}_{0,x_j}\frac{1}{2\eta_j}\int_0^{2\eta_j}|W^*(s)-x_j|^{8P}ds\right]^{1/4}$$

with $2\eta_j := \tau_{j+1}-\tau_j$ and $W^*(s) := W(s+\tau_j)$. Recall that we had to omit $\chi(E_j)\chi(T_j \text{ exists})$ since the martingale technique of Lemma 6.2 is not valid for restricted processes. Using the crudest estimates $\eta_j \leq t$, $|x_j - x_{j+1}| \leq 2R$ and the scaling

$$W^*(s) - x_j := \frac{(x_{j+1}-x_j)s}{2\eta_j} + \sqrt{2\eta_j}\cdot b\left(\frac{s}{2\eta_j}\right), \tag{128}$$

where $b(u)$ is the standard Brownian loop, we have

$$\text{LHS. of (127)} \leq \tag{129}$$

$$\leq \left(Cd(x)P^{2/3}\right)^P\left(R^P + t^{P/2}\right)\left(|x_j - x_{j+1}|^{2P} + t^P\mathbf{E}^{1/4}\int_0^1 |b(u)|^{8P}du\right) \leq$$

$$\leq \left(Cd(x)P^{2/3}\right)^P\left(R^P + t^{P/2}\right)\left(R^{2P} + ((const)tP)^P\right)$$

using the moments of the standard Brownian loop.

Collecting the estimates for the terms in (123) we have

First factor on the RHS of (120) $\leq$

$$\leq \left[(Cd(x)tR)^P \left(e^{Cd(x)tRP} + 1 \right) + \left(Cd(x)P^{2/3} \right)^P \left(R^P + t^{P/2} \right) \left(R^{2P} + ((const)tP)^P \right) \right]^{1/P} \leq$$

$$\leq Cd(x)t^{3/2}(Bt)^{13/6} \leq \frac{C}{(Bt)^3}, \tag{130}$$

where in the calculation we used that $Bt \geq 1$, the explicit value of $R = 8t\sqrt{B}$ and $P = 2[tB(x_j)] + 2 \leq CBt$ (by (93)). The last line shows that the critical exponent in the definition of $d(x)$ in (9) determining the maximal growth rate of $B(x)$ at infinity must be at least 31/6 in order to obtain the $C(Bt)^{-3}$ estimate which is necessary for the rest of the proof.

Finally after estimating the last three lines of (116) by quantities independent of x_j's (see (119), (120), (121) and (130)), we can drop the condition $|x - x_j| \leq R$ on the range of integration and use the semigroup property of the heat kernel to perform the x_j integrations. Therefore

$$\text{RHS of (109)} \leq \sum_{n=1}^{2t/\varepsilon} \frac{1}{2^n} \sum_{0<k_1<\ldots<k_n\leq 2t/\varepsilon} \left(\frac{C}{(Bt)^2} \right)^n \cdot CBt. \tag{131}$$

Finally use that $k_n \leq 2t/\varepsilon = [Bt] + 1$, therefore the sum over all possible $0 < k_1 < \ldots < k_n$ contains altogether $\begin{pmatrix} [Bt]+1 \\ n \end{pmatrix}$ choices. So eventually we have

$$\text{RHS of (109)} \leq \sum_{n=1}^{[Bt]+1} \frac{1}{2^n} \begin{pmatrix} [Bt]+1 \\ n \end{pmatrix} \left(\frac{C}{(Bt)^2} \right)^n \cdot CBt \leq$$

$$\leq \left[\left(1 + \frac{C}{2(Bt)^2} \right)^{[Bt]+1} - 1 \right] \cdot CBt \leq C \tag{132}$$

using $Bt \geq 1$, $[Bt] + 1 \leq 2Bt$ and the fact that the function

$$X \to \left[\left(1 + \frac{C}{X^2} \right)^X - 1 \right] \cdot X \tag{133}$$

is bounded uniformly for $X \geq 1$ by a constant depending only on C (use for $X := 2Bt$). The estimate (132) finishes the proof of Main Lemma 5.2 and Theorem 2.2. □

Appendices

A Selfadjointness and negative essential spetrum

Here we prove a statement about the self-adjointness, the negative essential spectrum of H_0 (see (2)) and their connection with the Lieb-Thirring inequality. (The conclusions for H_{Pauli} are obvious.)

Proposition A.1 *Suppose that for some $L \geq 0$ and $\gamma > 1/2$*

$$\int_0^\infty Tr(K^{<}_{E,L})E^{\gamma-1}dE < \infty \quad \textit{and} \quad \int_0^\infty Tr[(K^{>}_{E,L})^2]E^{\gamma-1}dE < \infty. \tag{134}$$

Then H_0 is self-adjoint and it has no negative essential spectrum.

Remark. Since all Lieb-Thirring type inequalities in this paper are proved via the Birman-Schwinger kernel, the conditions of this Proposition are automatically satisfied if for a given B and V any of the Lieb-Thirring inequalities proved in this paper gives a finite bound.

Proof: The selfadjointness is trivial, since the conditions via (17) and the Birman-Schwinger principle imply that H_0 is bounded from below.

For the essential spectrum we note that $Tr(K^{<}_{E,L})$ and $Tr[(K^{>}_{E,L})^2]$ are monotone decreasing functions of E (for fixed L). Therefore it follows from the conditions that they are finite for any $E > 0$, thus

$$\left|V + \frac{E}{2}\right|_- \Pi_L \left(H + p_3^2 + \frac{E}{2}\right)^{-1} \Pi_L \quad \text{and} \quad \left|V + \frac{E}{2}\right|_- (I - \Pi_L) \left(H + p_3^2 + \frac{E}{2}\right)^{-1} (I - \Pi_L) \tag{135}$$

are compact, so it is their sum $|V + \frac{E}{2}|_-(H + p_3^2 + \frac{E}{2})^{-1}$.

Using Corollary 2 in [RS, Ch. X.] we have that for $U_E := H + p_3^2 + \frac{E}{2} - |V + \frac{E}{2}|_-$

$$\sigma_{ess}(U_E) = \sigma_{ess}\left(H + p_3^2 + \frac{E}{2}\right) \subset \left[\frac{E}{2}, \infty\right]. \tag{136}$$

Clearly

$$H_0 = U_E + \left| V + \frac{E}{2} \right|_+ - E \geq U_E - E \geq H + p_3^2 - |V|_- - \frac{E}{2}. \tag{137}$$

On the other hand, when proving the Lieb-Thirring inequality via the Birman-Schwinger principle we basically give a bound to the negative eigenvalues of $H + p_3^2 - |V|_-$, and since by the conditions their γ^{th} moment is finite, we have that $H + p_3^2 - |V|_-$ is bounded from below.

Now we use the simple fact that if two operators $X \leq Y$ are bounded from below then $\inf \sigma_{ess}(X) \leq \inf \sigma_{ess}(Y)$, we have

$$\inf \sigma_{ess}(H_0) \geq \inf \sigma_{ess}(U_E - E) \geq -\frac{E}{2} \tag{138}$$

by (137). Since this is true for any $E > 0$, the proof is finished. □

B Counterexample

For any $\gamma \geq 0$ and given constants C_1 and C_2 we construct a special magnetic field $B(x)$ and potential $V(\mathbf{x})$ such that the γ^{th} moment of the negative eigenvalues of H_0 is not bounded by

$$C_1 \int_{\mathbf{R}^3} B(x)|V(\mathbf{x})|_-^{\gamma+1/2} d\mathbf{x} + C_2 \int_{\mathbf{R}^3} |V(\mathbf{x})|_-^{\gamma+3/2} d\mathbf{x}. \tag{139}$$

The key idea is that we will choose B and V such that their supports are disjoint, so the first term disappears in the possible bound (139). Then we will show that the sum of the negative eigenvalues behaves at least like $(const)N$ if we rescale the magnetic field by N^2, but this rescaling does not effect the bound (139).

For the proof, choose a one-dimensional potential v with $|v|_- \in L^{5/2}(\mathbf{R})$ such that $p_3^2 + v(x_3)$ has a negative eigenvalue $-\lambda$ and let $\psi(x_3)$ be the corresponding normalized eigenfunction. (E.g. $v(x_3) = x_3^2 - 2$, $-\lambda = -1$, $\psi(x_3) = \pi^{-1/4} e^{-x_3^2/2}$.) Let $V(\mathbf{x}) := v(x_3)\chi(|x| \leq 1)$, i.e. the potential is supported in a cylinder built over the unit disc in $\mathbf{R}^2$; and let $B(x) = N^2 \chi(|x| \geq 1)$ where

N is a free positive parameter, $B := N^2$. In this case the conjecture (3) says that

$$\sum_i |E_i|^\gamma \le C_2\pi \int_{\mathbf{R}} |v(x_3)|_-^{\gamma+3/2} dx_3 \tag{140}$$

independently of $B = N^2$. On the other hand we will show that $\sum_i |E_i|^\gamma \ge (const)\cdot N$. Notice that $B(x)$ is not continuous since the calculation happens to be simpler in this way, but the same idea easily provides a counterexample with a C^∞ magnetic field which is sufficiently close to $B(x)$.

We will use the complex notation $z := x_1 + ix_2$ for $x = (x_1, x_2)$ in the plane of the first two coordinates. As it is explained in the proof of the Aharonov-Casher theorem (see [AC], [CFKS, Section 6.4]) the ground state eigenfunctions of $H = (p - A(x))^2 - B(x)$ can be found in the form of $e^{h(z)}g(z)$ where h satisfies $-\Delta h(z) = B(z)$ and $g(z)$ is analytic.

In our case let $h(z)$ be the following function:

$$h(z) := \begin{cases} -B/4 & \text{for} \quad |z| \le 1 \\ -B(\log|z|^2 - |z|^2)/4 & \text{for} \quad |z| > 1, \end{cases} \tag{141}$$

then clearly $-\Delta h(z) = B(z)$ for the $B(z) = B(x)$ defined above. The functions

$$f_n(z) := z^n \cdot \begin{cases} e^{-B/4} & \text{for} \quad |z| \le 1 \\ |z|^{B/2} e^{-\frac{B}{4}|z|^2} & \text{for} \quad |z| > 1 \end{cases} \tag{142}$$

$(n = 0, 1, 2\ldots)$ are ground states of H, and they are orthogonal in $L^2(\mathbf{C}) = L^2(\mathbf{R}^2)$. Define $F_n(\mathbf{x}) := f_n(x)\psi(x_3)$, then $\|F_n\|_{L^2(\mathbf{R}^3)} = \|f_n\|_{L^2(\mathbf{R}^2)}$, and they are all linearly independent. By variational principle

$$\sum_i |E_i|^\gamma \ge \sum_{n=0}^{K_N - 1} \left| \frac{(F_n,\ H_0 F_n)}{\|F_n\|^2} \right|^\gamma, \tag{143}$$

where K_N is any integer (to be determined later) not greater than the number of negative eigenvalues (with multiplicity). Computing

$$(F_n,\ H_0 F_n) = \|\nabla\psi\|^2_{L^2(\mathbf{R})} \int_{|x|\ge 1} |f_n(x)|^2 dx - \lambda \cdot \int_{|x|<1} |f_n(x)|^2 dx \tag{144}$$

(using $H_0 = H + p_3^2 + V$ and $Hf_n = 0$), we have that

$$\sum_i |E_i|^\gamma \geq \sum_{n=0}^{K_N-1} \left| \frac{\lambda - T_n \|\nabla \psi\|^2_{L^2(\mathbf{R})}}{T_n + 1} \right|^\gamma, \tag{145}$$

where

$$T_n := \frac{\|f_n\|^2_{out}}{\|f_n\|^2_{in}} := \frac{\int_{|x|\geq 1} |f_n(x)|^2 dx}{\int_{|x|<1} |f_n(x)|^2 dx} \tag{146}$$

is the ratio of the norms of f_n outside and inside the unit disc.

The inside norm is easily computed: $\|f_n\|^2_{in} = \frac{\pi}{n+1} e^{-B/2}$. The outside norm can be estimated from above as follows (in polar coordinates):

$$\|f_n\|^2_{out} = 2\pi \int_1^\infty r^{2n+B+1} e^{-\frac{B}{2}r^2} dr \leq 2\pi \int_0^\infty r^{2n+B+1} e^{-\frac{B}{2}r^2} dr = \tag{147}$$

$$= \frac{2\pi}{B} \left(\frac{2}{B}\right)^{n+B/2} \int_0^\infty t^{n+B/2} e^{-t} dt.$$

The gamma integral is estimated by the Stirling formula, yielding

$$\|f_n\|^2_{out} \leq \frac{const}{\sqrt{B}} e^{-B/2} \tag{148}$$

for $n \leq (e-2)B$, so $T_n \leq c_0 \frac{n+1}{\sqrt{B}} = c_0 \frac{n+1}{N}$ with some universal $c_0 > 0$.

Choose

$$K_N := \left[\frac{N\lambda}{2c_0 \|\nabla \psi\|^2_{L^2(\mathbf{R})}} \right] \tag{149}$$

($[x]$ denotes the integer part), then by (144) there are at least K_N negative eigenvalues, since $(F_n, H_0 F_n) < 0$ for $0 \leq n \leq K_N - 1$ (and $n \leq (e-2)B$ is also satisfied). So by (145) and (149)

$$\sum_i |E_i|^\gamma \geq \sum_{n=0}^{K_N-1} \left(\frac{\lambda N - c_0(n+1)\|\nabla \psi\|^2_{L^2(\mathbf{R})}}{c_0(n+1) + N} \right)^\gamma \geq (const) \cdot N, \tag{150}$$

where this last positive constant depends on everything except N. $\square$

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