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M. Sh. Birman
D. R. Yafaev

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SCATTERING MATRIX FOR A PERTURBATION OF A PERIODIC SCHRÖDINGER OPERATOR BY A DECAYING POTENTIAL

M. Sh. Birman (St.Petersburg University),
D. R. Yafaev* (Rennes University)

International Erwin Schrödinger Institute for Mathematical Physics

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Abstract. We consider a perturbation $H = H_0 + V$ of a periodic Schrödinger (or more general) operator H_0 by a short-range potential V. A strong form of the limiting absorption principle for the operator H is established. The stationary scattering theory for the pair H_0, H is developed. The results obtained allow us to give a representation for the scattering matrix in terms of the spectral representation of H_0 and of the resolvent of H. The asymptotics of the spectrum of the scattering matrix is calculated for asymptotically homogeneous V.

Introduction

We discuss the stationary approach to the scattering problem for a pair H_0, H, where H_0 is an elliptic periodic operator of second order in $L_2(\mathbb{R}^d)$, $H = H_0 + V$, $V = V(x) = O(|x|^{-\rho})$, $|x| \to \infty$, $\rho > 1$. In particular, H_0 may be a periodic Schrödinger operator. We justify the limiting absorption principle in its strong form. Then we obtain an "explicit" stationary representation for the scattering matrix. This representation (see formula (5.5) below) is well known in scattering theory, but every time its concrete interpretation requires some efforts. The representation obtained forms the basis for calculation of asympotics of phases (i.e. of spectrum) of S-matrix under the assumption that asymptotically V is a homogeneous function of order $-\rho$, $\rho > 1$. This calculation was done by the authors earlier (see [BY1], and also [BY2] and [BY3]) for the operator H_0 with a constant symbol. In this case the smooth version of the stationary scattering theory is well elaborated which facilitates a study of the asymptotics of phases.

A construction of advanced scattering theory for the case of the periodic operator H_0 meets with some difficulties if $d > 1$. Restrictive assumptions on eigenvalues and eigenfunctions of quasiperiodic problems on a basic cell are required. These

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*On leave from the Institute of Mathematics, University of St. Petersburg, St. Petersburg, Russia assumptions are local in energy. Therefore we construct *local* wave operators which are sufficient for a definition of the scattering matrix in the corresponding energy interval. Similar additional assumptions were imposed in the papers [Be], [Si] where H_0 is a periodic Schrödinger operator. Note, however that a proof of the absolute continuity of H_0 (which demands a study of some boundary values of the resolvent) was given in [Th] without any superfluous assumptions. In [Be] the stationary approach (for $d = 3$, $\rho > 2$) was used but for a construction of S-matrix it was not developed far enough. The paper [Si], where $\rho > 1$, relies on the time-dependent construction of wave operators which is not particularly adapted for a study of S-matrix.

Relying on the limiting absorption principle, we develop consecutively the stationary scattering theory in its smooth version (§3–5). In particular, we obtain an Agmon–type [A] result that a point, where the resolvent of H does not have a limit, is its eigenvalue. Moreover, eigenvalues of H do not have interior accumulation points. We simplify considerably the original Agmon approach worked out by him for the case $H_0 = -\Delta$.

A representation for S-matrix obtained in §5 allows us to apply the scheme of [BY1] for calculation of phase asymtotics. The problem reduces to a study of the asymptotics of spectrum of some integral operator on a surface of constant energy. This operator is treated as a pseudodifferential operator of negative order, but its amplitude contains an additional (compared to [BY1]) periodic factor. A similar factor impedes also a justification of the limiting absorption principle. In both cases these difficulties are overcome with the help of the theory of *multiplicators for integral kernels*. Necessary information and references on their behalf are given in §1. In the long run we show that the periodic factor does not intervene into a formula for asymptotic coefficients. They depend only on eigenvalues of quasiperiodic problems (dispersive functions) but not on their eigenfunctions. The asymtotics of phases is calculated in §6. It remains to say that spectral decomposition of the operator H_0 is discussed in §2. In particular, we formulate there additional assumptions mentioned above.

By necessity, our presentation is not homogeneous. We give only indications and references when following some samples. On the contrary, complete proofs are exposed when new features arise.

Double numeration is used for references to paragraphs (for example, p. 3.2 means p. 2 of §3). The following agreements are adopted. For a compact self-adjoint operator T we denote by $v_k^+(T)$ (by $-v_k^-(T)$) its consecutive positive (negative) eigenvalues. An integral without specification of integration domain is taken over a whole space; C and c are different constants in estimates; Sobolev spaces are denoted by H^s, $s > 0$; $\mathbb{Q}^d$ is the unit cube in $\mathbb{R}^d$. Further, $\Omega_* := [-\pi, \pi]^d$ and Ω is obtained from Ω_* if its opposite sides are identified. If not specified otherwise, the torus $\Pi^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ is standardly realized as Ω

1. An auxiliary information. Multiplicators

1. Classes of compact operators. For separable Hilbert spaces $\mathfrak{H}_1, \mathfrak{H}_2$ we denote by $\mathcal{R} = \mathcal{R}(\mathfrak{H}_1, \mathfrak{H}_2)$ the Banach space of bounded linear operators mapping $\mathfrak{H}_1$ into $\mathfrak{H}_2$; $\mathfrak{S}_\infty \subset \mathcal{R}$ is the subspace of all compact operators. Every operator $T \in \mathfrak{S}_\infty$

admits a canonical decomposition

$$T = \sum_k s_k(\cdot, \alpha_k)\beta_k, \tag{1.1}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are orthonormal sequences in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. The numbers $s_k = s_k(T) > 0$ (singular numbers) do not increase and tend to zero as $k \to \infty$. They equal positive eigenvalues of the operator $(T^*T)^{1/2}$.

A class $\mathfrak{S}_r$, $0 < r < \infty$, consists of all $T \in \mathfrak{S}_\infty$ such that the functional

$$||T||_r := (\sum_k s_k^r(T))^{1/r} \tag{1.2}$$

is finite. The space $\mathfrak{S}_r$ is complete and separable with respect to the norm (quasi-norm, for $r < 1$) (1.2). The triangle inequality

$$||T_1 + T_2||_r^\kappa \le ||T_1||_r^\kappa + ||T_2||_r^\kappa, \quad \kappa = \min\{r, 1\}, \tag{1.3}$$

holds. In particular, $\mathfrak{H}_2$ is the Hilbert-Schmidt class and $\mathfrak{H}_1$ is the trace class.

We also use classes Σ_r, $0 < r < \infty$, of compact operators T with finite quasinorm

$$|T|_r := \sup_k (k^{1/r} s_k(T)). \tag{1.4}$$

The classes Σ_r are complete but *not separable* spaces with respect to the quasinorm (1.4). Let us formulate an analog of the inequality (1.3). If $T = \sum_n T_n$, then

$$|T|_r^\kappa \le c \sum_n |T_n|_r^\kappa. \tag{1.5}$$

Here $\kappa = 1$, $c = c(r)$ for $r > 1$; $\kappa = r$, $c = c(r)$ for $r < 1$; $\forall \kappa < 1$, $c = c(\kappa)$ for $r = 1$. Denote by Σ_r^0 the separable subspace of Σ_r obtained as the closure in Σ_r of all operators of finite rank. This subspace can be defined directly:

$$\Sigma_r^0 = \{T \in \mathfrak{S}_\infty : s_k(T) = o(k^{-1/r})\}.$$

2. Multiplicators in classes of kernels of integral operators. Considerable technical simplification in the main part of the paper are due to a concept of a multiplicator. Some necessary information about it is collected here. A much more detailed presentation (in a more general situation) of properties of multiplicators is given in [BS1], [BS2].

Let X, Y be two seperable spaces with (σ-finite) measures dx, dy, respectively, and let $\mathfrak{H}_1 = L_2(Y)$, $\mathfrak{H}_2 = L_2(X)$. Operators T of the class $\mathfrak{S}_2$ are integral operators,

$$(Tu)(x) = \int_Y t(x, y)u(y)dy,$$

with square integrable kernels and

$$||T||_2^2 = \int_{X\times Y} |t(x, y)|^2 dxdy.$$

Let now $b \in L_\infty(X \times Y)$ and let $T(b)$ be the operator with kernel $b(x,y)t(x,y)$. The linear mapping (transformation)

$$\mathbf{M}(b) : T \mapsto T(b) \tag{1.6}$$

is continuous in $\mathfrak{S}_2$ and its norm equals the norm of b in L_∞:

$$||\mathbf{M}(b)||_{\mathfrak{S}_2} = v.\sup |b(x,y)|.$$

Suppose now that a function b (a multiplicator) is such that the transformation (1.6) is a continuous mapping of $\mathfrak{S}_1$ into $\mathfrak{S}_1$. By duality (the definition of $\mathbf{M}(b)$ on $\mathcal{R}$ relies already on the duality between $\mathcal{R}$ and $\mathfrak{S}_1$), this is equivalent to continuity of $\mathbf{M}(b)$ as a mapping of $\mathcal{R}$ into $\mathcal{R}$ and of $\mathfrak{S}_\infty$ into $\mathfrak{S}_\infty$. In this case

$$||\mathbf{M}(b)||_{\mathfrak{S}_1} = ||\mathbf{M}(b)||_{\mathfrak{S}_\infty} = ||\mathbf{M}(b)||_{\mathcal{R}}. \tag{1.7}$$

The set of multiplicators b for which (1.6) is a linear mapping of $\mathfrak{S}_1$ into itself is denoted by $\mathfrak{M}$.

Remark 1.1. Suppose that $b \in \mathfrak{M}$ and let $\mathfrak{S}$ be any class $\mathfrak{S}_r$, $\sum_r$ or $\sum_r^0$ for $r > 1$. Then the transformation (1.6) is continuous in $\mathfrak{S}$ and

$$||b||_{L_\infty} \le ||\mathbf{M}(b)||_{\mathfrak{S}} \le ||\mathbf{M}(b)||_{\mathcal{R}}.$$

3. Let us now give a sufficient condition (see [BS1]) for $b \in \mathfrak{M}$.

Theorem 1.2. *Let $\mathcal{T}$ be a separable space with (σ-finite) measure $d\tau$. Assume that*

$$b(x,y) = \int_{\mathcal{T}} f(x,\tau) g(y,\tau) d\tau \tag{1.8}$$

for almost every (a.e.) $x \in X$, $y \in Y$ and

$$< f >^2 := v.\sup_x \int_{\mathcal{T}} |f(x,\tau)|^2 d\tau < \infty,$$

$$< g >^2 := v.\sup_y \int_{\mathcal{T}} |g(y,\tau)|^2 d\tau < \infty.$$

Then $b \in \mathfrak{M}$ and

$$||\mathbf{M}(b)||_{\mathcal{R}} \le < f >< g > . \tag{1.9}$$

For the reader's convenience, we give here a *proof* of this theorem. By virtue of (1.7), it suffices to estimate the norm of $\mathbf{M}(b)$ in $\mathfrak{S}_1$. Suppose first that T is a normalized rank-one operator, i.e. $t(x,y) = \alpha(x)\beta(y)$ and

$$||\alpha||_{L_2(X)} = ||\beta||_{L_2(Y)} = 1.$$

According to (1.8), the operator $T(b)$ is a product of two operators of the class $\mathfrak{S}_2$ with kernels $\alpha(x) f(x,\tau)$ and $g(y,\tau)\beta(y)$. Therefore

$$||T(b)||_{\mathfrak{S}_1}^2 \le \int_{X \times \mathcal{T}} |\alpha f|^2 dx d\tau \int_{Y \times \mathcal{T}} |\beta g|^2 dy d\tau \le < f >^2 < g >^2 . \tag{1.10}$$

For an arbitrary $T \in \mathfrak{S}_1$ use the representation (1.1). Taking into account (1.10), we obtain the bound

$$||T(b)||_{\mathfrak{S}_1} \le < f >< g > \sum_k s_k(T),$$

which is equivalent to (1.9). □

Remark 1.3. Denote by $g_\tau (f_\tau)$ multiplication by $g(y,\tau)$ $(f(x,\tau))$ in $\mathfrak{H}_1$ $(\mathfrak{H}_2)$. Under the assumptions of Theorem 1.2, for every $T \in \mathcal{R}$ the operator $T(b)$ admits the representation

$$T(b) = \int_{\mathcal{T}} f_\tau T g_\tau d\tau, \tag{1.11}$$

where the integral is strongly convergent. The bound (1.9) can be deduced from (1.11) directly, avoiding the duality between $\mathfrak{S}_1$ and $\mathcal{R}$.

4. Let us formulate now a concrete condition of continuity of $\mathbf{M}(b)$ in classes $\mathfrak{S}_r$ or Σ_r. Suppose that for a domain $X \subset \mathbb{R}^d$, $d \geq 1$, and some $s > 0$

$$b(\cdot, y) \in H^s(X), \quad a.e. \quad y \in Y, \tag{1.12}$$

$$||b||_{(s)} := v.\sup_y ||b(\cdot, y)||_{H^s(X)} < \infty. \tag{1.13}$$

Theorem 1.4. *Assume that X is a bounded domain with a Lipschitz boundary and let the conditions (1.12), (1.13) be fulfilled for $2s > d$ if $r \geq 1$ and for $s > (r^{-1} - 1/2)d$ if $r < 1$. Denote by $\mathfrak{S}^{(r)}$ any class $\mathfrak{S}_r$, Σ_r or Σ_r^0. Then the transformation (1.6) is continuous in $\mathfrak{S}^{(r)}$ and*

$$||\mathbf{M}(b)||_{\mathfrak{S}^{(r)}} \leq C(X, r)||b||_{(s)}. \tag{1.14}$$

A proof for the classes $\mathfrak{S}_r$ is given in [BS2], Theorem 9.2. By means of the real interpolation it can easily be extended to the classes Σ_r and Σ_r^0.

Remark 1.5. The conclusion of Theorem 1.4 holds true if the roles of variables x and y are interchanged in its conditions. For the proof it suffices to consider adjoint operators.

Remark 1.6. Theorem 1.4 remains to be valid if X admits a decomposition into a finite union of Lipschitz domains $X_1, \dots, X_N$ and the conditions (1.12), (1.13) are fulfilled for each of them. In this case the right hand side of (1.14) is a sum of the corresponding functionals (1.13).

5. Here we formulate a simple assertion not related to multiplicators. Let $G \subset \mathbb{R}^d, d > 1$, be a compact smooth manifold (perhaps, with a boundary) of codimension 1 and let dG be Euclidean measure on G (induced by that on $\mathbb{R}^d$). Denote by ω_r the Fourier transform of the function $(1 + |x|^2)^{-r}, r > 0$, in $\mathbb{R}^d$. We consider integral operators

$$(A_m v)(\eta) = (2\pi)^{-d/2} \int_G \omega_r(\eta - \tilde{\eta} - 2\pi m) v(\tilde{\eta}) dG(\tilde{\eta}), \ m \in \mathbb{Z}^d, \tag{1.15}$$

in the space $L_2(G)$.

Lemma 1.7. *Let $2r > 1$. Then the operators A_m are compact and*

$$||A_m|| \le C(N)(1+|m|)^{-N}, \quad \forall N > 0. \tag{1.16}$$

Proof. Clearly, $\omega_r \in C^\infty(\mathbb{R}^d \setminus 0)$, $D^\alpha \omega_r(\eta) = O(|\eta|^{-N})$, $\forall \alpha, \forall N$, as $|\eta| \to \infty$ and $\omega_r(\eta) = O(|\eta|^{2r-d})$ as $|\eta| \to 0$. The last condition implies that A_0 has only a week singularity if $2r > 1$ so that $A_0 \in \mathfrak{S}_\infty$. Moreover, operators A_m are compact for $m \neq 0$. For sufficiently large m the bound (1.16) holds true even in the class $\mathfrak{S}_2$. □

Remark 1.8. Lemma 1.7 remains true if ω_r in (1.15) is replaced by the Fourier transform of any function which belongs to $C^\infty(\mathbb{R}^d)$ and is homogeneous of order $-2r < -1$ outside of some ball. Furthermore, the kernel is small with its derivatives up to any fixed order if $|m|$ is large enough. Therefore, for large $|m|$, $||A_m||$ can be replaced in (1.16) by $||A_m||_t$, i.e. by the quasinorm in the class Σ_t for arbitrary $t > 0$.

2. Spectral properties of a periodic operator H_0

1. A periodic operator of the second order. Let us consider a self-adjoint operator of the second order

$$H_0 u = -\mathrm{div}(a(x)\mathrm{grad} u) + i(q(x)\nabla + \nabla q(x))u + p(x)u \tag{2.1}$$

in the space $L_2(\mathbb{R}^d), d > 1$. Here a matrix a, a vector q and a function p are periodic:

$$a(x+n) = a(x),\ q(x+n) = q(x),\ p(x+n) = p(x),\ n \in \mathbb{Z}^d. \tag{2.2}$$

Moreover, we assume that $a > 0$, $p = \bar{p}$.

$$a + a^{-1} \in L_\infty(\mathbb{R}^d),\ q \in L_\infty(\mathbb{R}^d),\ p \in L_\infty(\mathbb{R}^d). \tag{2.3}$$

Under such assumptions a precise definition of the operator H_0 can be given in terms of the corresponding quadratic form defined on the class $H^1(\mathbb{R}^d)$:

$$h_0[u] = \int (a\nabla u \overline{\nabla u} + 2\mathrm{Re} i(q\nabla u)\bar{u} + p|u|^2)dx,\ u \in H^1(\mathbb{R}^d). \tag{2.4}$$

Operators (2.1) are invariant with respect to linear transformations in $\mathbb{R}^d$, i.e. after such a transformation we obtain again an operator of the type (2.1). Therefore, the assumption that the lattice of periods is cubic, does not diminish a generality. We call H_0 the *Schrödinger operator* if the matrix a is constant and $q = 0$.

Let us introduce a subspace $\hat{H}^1(\mathbb{Q}^d) \subset H^1(\mathbb{Q}^d)$ of *periodic* functions. We consider a family of quadratic forms $h(\xi), \xi \in \mathbb{R}^d$,

$$h(\xi)[u] = \int_{\mathbb{Q}^d} (a\nabla u \overline{\nabla u} + 2\mathrm{Re} i(q\nabla u)\bar{u} + p|u|^2)dx, \tag{2.5}$$

$$e^{-i\xi x}u(x) \in \hat{H}^1(\mathbb{Q}^d), \tag{2.6}$$

in the space $L_2(\mathbb{Q}^d)$. Denote by $H(\xi)$ the self-adjoint operator in $L_2(\mathbb{Q}^d)$ generated by the form (2.5), (2.6). We emphasize that the boundary condition (2.6) is not changed if ξ is replaced by $\xi + 2\pi n$, $n \in \mathbb{Z}^d$. Therefore one may assume that the quasimomentum $\xi \in \mathbb{T}^d$.

The operator $H(\xi)$ has a semibounded from below discrete spectrum

$$E_1(\xi) \leq E_2(\xi) \leq \cdots \leq E_l(\xi) \leq E_{l+1}(\xi) \leq \ldots. \tag{2.7}$$

Functions $E_l(\xi)$ are continuous in $\xi \in \mathbb{T}^d$. Every $E_l(\xi)$ is an analytic function of ξ in a neighbourhood of any point $\xi_0 \in \mathbb{T}^d$ such that $E_l(\xi_0)$ is a *simple* eigenvalue of $H(\xi_0)$.

The spectrum of the operator H_0 corresponding to the form (2.4) consists of closed intervals (bands) Λ_l which are images of functions E_l, $l = 1, 2 \ldots.$ Bands may overlap and gaps may be absent in the spectrum.

The spectrum of the Schrödinger operator is absolutely continuous [Th]. It is known [Sk1], [Sk2] that for $d = 2, 3$ every sufficiently large E belongs to the image of more than one function (2.7) (presumably, this is also true for $d > 3$).

Suppose that $q = 0$ in (2.1). Then $E_1(0) = \min E_1(\xi) < \min E_2(\xi)$, $E_1(\xi) > E_1(0)$ if $\xi \neq 0$ and $\xi = 0$ is a *non-degenerate* minimum of E_1. It follows that for sufficiently small positive values of $\lambda - E_1(0)$ the surface $\{\xi : E_1(\xi) = \lambda\}$ is an analytic surface without critical points and the set $\{\xi : E_2(\xi) = \lambda\}$ is empty.

Let $\psi_l(\xi, x)$ be eigenfunctions corresponding to eigenvalues (2.7). We assume that functions ψ_l are normalized:

$$\int_{\mathbb{Q}^d} |\psi_l(\xi, x)|^2 dx = 1. \tag{2.8}$$

If $E_l(\xi_0)$ is a simple eigenvalue of $H(\xi_0)$, then the function $\psi_l(\xi, x)$, can be chosen *analytic* in ξ in a neighbourhood of $\xi_0 \in \mathbb{T}^d$. By virtue of (2.6), ψ_l admits a representation

$$\psi_l(\xi, x) = e^{i\xi x}\varphi_l(\xi, x),\ \xi \in \Omega_*, \tag{2.9}$$

where the function φ_l is periodic in x. In contrast to ψ_l the function φ_l is not periodic in ξ. The condition $\xi \in \Omega_*$ (in place of $\xi \in \mathbb{T}^d$) determines a choice of φ_l. Sometimes, however, we use the relation (2.9) for ξ from some neighbourhood $\Omega_* \subset \mathbb{R}^d$ (see p. 3.2 and below).

Remark 2.1. According to known results [LU] on elliptic operators of second order with bounded (see (2.3)) coefficients, the functions ψ_l, φ_l satisfy in x the Hölder condition with some positive exponent.

2. The expansion theorem for the operator H_0. (see e.g. [RS], [Sk1]) Kernels ψ_l give rise to integral transformations similar to the Fourier transform. Set

$$(\Psi_l u)(\xi) = (2\pi)^{-d/2} \int \overline{\psi_l(\xi, x)} u(x) dx, \quad \xi \in \mathbb{T}^d. \tag{2.10}$$

The operator (2.10) is defined first on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and then it is extended by continuity to the whole space $L_2(\mathbb{R}^d)$. *The operator Ψ_l is partially isometric on $L_2(\mathbb{R}^d)$ and its image equals $L_2(\mathbb{T}^d)$.* The projectors

$$P_l := \Psi_l^* \Psi_l \tag{2.11}$$

satisfy the conditions of orthogonality and completeness

$$P_k P_l = \delta_{kl} P_l, \quad \sum_l P_l = I. \tag{2.12}$$

The intertwining property holds:

$$\Psi_l H_0 = [E_l]\Psi_l, \quad l = 1, 2 \ldots, \tag{2.13}$$

where $[E_l]$ is the multiplication by the function E_l in the space $L_2(\mathbb{T}^d)$. Thus the functions ψ_l form a complete set of eigenfunctions of the continuous spectrum of the operator H_0.

Let $\Delta \subset \mathbb{R}$ now be a bounded interval, let $\mathcal{E}_0(\cdot)$ be the spectral measure of H_0 and set $\tilde{f}_l = \Psi_l f$. The equalities (2.11)–(2.13) ensure that

$$(\mathcal{E}_0(\Delta) f, g) = \sum_l \int_{\xi : E_l(\xi) \in \Delta} \tilde{f}_l(\xi) \overline{\tilde{g}_l(\xi)} d\xi. \tag{2.14}$$

The sum in (2.14) contains, of course, only a finite number of terms.

3. The main assumption. A direct integral. Below we always fix a number λ which is an interior point of some band Λ_l and accept

Assumption 2.2. *a) A number $\lambda \in \Lambda_l$ does not belong to Λ_k if $k \neq l$. b) The surface*

$$G(\lambda) := \{\xi \in \mathbb{T}^d : E_l(\xi) = \lambda\}$$

does not have critical points, i.e. $(\nabla E_l(\xi)) \neq 0$ for $\xi \in G(\lambda)$.

This assumption is rather restrictive. It does not hold for λ large enough. On the other hand, it is fulfilled for λ close to $E_1(0)$. A possibility to relax the conditions of Assumption 2.2 is discussed in p. 6.5.

Note that the surface $G(\lambda)$ is not supposed to be connected. Below we omit the index l, that is we write $E, \psi, \varphi, \tilde{f} = \Psi f$ in place of $E_l, \psi_l, \varphi_l, \tilde{f}_l = \Psi_l f$ and so forth. Under Assumption 2.2 $E(\xi)$ is a *real analytic* function in a neighbourhood of the (analytic) surface $G(\lambda)$. Clearly, Assumption 2.2 is automatically fulfilled for points $\mu \in [\lambda - \varepsilon, \lambda + \varepsilon] =: \delta(\lambda, \varepsilon)$ if ε is small enough. We fix an interval $\delta(\lambda, \varepsilon)$ but it can be diminished if necessary. There is a canonic diffeomorphism between surfaces $G(\mu), \mu \in \delta$, and $G(\lambda) = G$

$$\zeta(\mu) : G \to G(\mu)$$

generated by the field of normals. Observe that we omit λ in the notation $G(\mu)$ if $\mu = \lambda$; the same is true with respect to different objects defined on $G(\mu)$ whenever

$\mu = \lambda$. We define the measure dG_μ on the surface $G(\mu)$ induced by the (Euclidean) measure on $\mathbb{R}^d$ and set $d\sigma_\mu(\xi) = |\nabla E(\xi)|^{-1} dG_\mu(\xi)$,

$$e(\mu, \xi) = \frac{d\sigma_\mu(\zeta(\mu)\xi)}{d\sigma(\xi)}, \quad \xi \in G. \tag{2.15}$$

Furthermore, we define a mapping

$$(J(\mu)v)(\xi) = \nu(\zeta(\mu)\xi)\sqrt{e(\mu, \xi)}, \quad \xi \in G. \tag{2.16}$$

Let $< \cdot, \cdot >$ be the scalar product in $\mathfrak{N} := L_2(G; d\sigma)$. It follows from (2.14)–(2.16) that

$$\frac{d(\mathcal{E}_0(\mu)f, g)}{d\mu} =< J(\mu)\tilde{f}, J(\mu)\tilde{g} >, \quad \mu \in \delta. \tag{2.17}$$

Set $\mathfrak{H} = L_2(\mathbb{R}^d)$. By (2.17), *the spectrum of H_0 in $\mathcal{E}_0(\delta)\mathfrak{H}$ is absolutely continuous.* Moreover, (2.17) allows us to decompose $\mathcal{E}_0(\delta)\mathfrak{H}$ in *a direct integral such that H_0 reduces to multiplication by independent variable $\mu \in \delta$.* Indeed, put $\mathcal{H} = L_2(\delta; \mathfrak{N})$ and consider an operator $\mathcal{U}$ mapping a function $f \in \mathcal{E}_0(\delta)\mathfrak{H}$ to the vector-function $J(\mu)\tilde{f}$ with values in $\mathfrak{N}$. Then $\mathcal{U}$ is a unitary mapping of $\mathcal{E}_0(\delta)\mathfrak{H}$ onto $\mathcal{H}$ and the operator $\mathcal{U}H_0\mathcal{U}^*$ is multiplication by independent variable in $\mathcal{H}$.

4. Let us discuss some technicalities which we use in §3. Set

$$\eta = \eta(\mu, \xi) = \zeta(\mu)\xi, \quad \xi \in G.$$

Then

$$|\eta(\mu, \xi) - \xi| \leq C|\mu - \lambda|, \quad \xi \in G, \tag{2.18}$$

$$|e(\mu, \xi) - 1| \leq C|\mu - \lambda|, \quad \xi \in G. \tag{2.19}$$

Let us add a lower index to objects arising naturally when $\mathbb{T}^d$ is replaced by Ω_*. For example, $G_*(\mu)$ is a surface which one obtains from $G(\mu) \subset \mathbb{T}^d$ (realized as Ω) if the identification of opposite sides of Ω is removed. Sometimes, however, we need more complicated agreements due to the fact that surfaces G_* and $G_*(\mu)$ can lose their closeness in $\mathbb{R}^d$. Actually, for points $\xi \in G_*$ we define the field of normals to G_* in $\mathbb{R}^d$. A shift in the direction of this field leads to points $\eta(\mu, \xi) \in G(\mu) \subset \mathbb{T}^d$ not belonging, in general, to Ω_*. We denote corresponding points of $\mathbb{R}^d$ by $\eta_*(\mu, \xi)$. Then (2.18) should be replaced by the inequality

$$|\eta_*(\mu, \xi) - \xi| \leq C|\mu - \lambda|, \quad \xi \in G_* \tag{2.20}$$

5. Let us now discuss a dependence of an eigenfunction $\psi(\xi, x)$ on ξ in a neighbourhood of G. Let $\{\mathcal{G}_j\}$ be a finite *covering* of the surface G by open connected sets and let $\{G_j\}$ be a *decomposition* of G into domains with piece-wise smooth boundary such that $\operatorname{clos} G_j \subset \mathcal{G}_j$. Set

$$\hat{\mathcal{G}}_j = \bigcup_{\mu\in\delta} \zeta(\mu)\mathcal{G}_j, \quad \hat{G}_j = \bigcup_{\mu\in\delta} \zeta(\mu)G_j.$$

If the covering $\{\mathcal{G}_j\}$ is sufficiently "fine", then $\psi(\xi,\cdot)$ can be chosen as a *real analytic vector-function* (with values in $L_2(\mathbb{Q}^d)$) for $\xi \in \hat{\mathcal{G}}_j$. We choose $\psi(\xi,\cdot)$ as a piece-wise analytic function, defining it on $\hat{G}_j$ as a restriction of the function analytic in $\hat{\mathcal{G}}_j$. The values of ψ on ∂G_j (and on $\widehat{\partial G_j}$) are irrelevant. The bound (2.18) ensures that for such choice of ψ

$$\int_{\mathbb{Q}^d} |\psi(\eta, x) - \psi(\xi, x)|^2 dx \leq C|\mu - \lambda|^2, \quad \xi \in G. \tag{2.21}$$

We emphasize that there are, possibly, topological obstructions to a construction of a function ψ analytic in a full neighbourhood of the surface G.

3. Estimates on the derivative of the spectral measure of the operator H_0

We will show here that the spectral measure $\mathcal{E}_0$ is differentiable in a suitable sense for $\mu \in \delta$ and its derivative satisfies a Hölder condition. The boundary points of δ need not be excluded since Assumption 2.2 holds for a somewhat extended interval. Estimates obtained (Theorems 3.1, 3.2) are crucial for construction of a scattering theory for the pair $H_0, H = H_0 + V$. Everywhere below X_r is *the operator of multiplication by the function* $(1 + |x|^2)^{-r/2}$, $r \in \mathbb{R}$, $x \in \mathbb{R}^d$.

1. We start with

Theorem 3.1. *Let $2r > 1$. For every $\mu \in \delta$ there exists the weak derivative*

$$\mathcal{F}_r(\mu) := \frac{d}{d\mu}(X_r \mathcal{E}_0(\mu) X_r) \in \mathfrak{S}_\infty, \tag{3.1}$$

$$||\mathcal{F}_r(\mu)|| \leq C, \quad \mu \in \delta. \tag{3.2}$$

Proof. Denote by $\mathcal{I}(\mu)$ the operator of restriction on the surface $G(\mu)$ and let $< \cdot, \cdot >_\mu$ be the scalar product in $L_2(G(\mu), d\sigma_\mu)$. It is convenient to rewrite (2.17) as

$$\frac{d(\mathcal{E}_0(\mu)f.g)}{d\mu} = < \mathcal{I}(\mu)\tilde{f}, \mathcal{I}(\mu)\tilde{g} >_\mu, \quad \mu \in \delta. \tag{3.3}$$

The relations (3.1), (3.2) are equivalent to compactness of the operator

$$B = \mathcal{I}(\mu)\Psi X_r : L_2(\mathbb{R}^d) \to L_2(G(\mu), d\sigma_\mu)$$

and to a bound for its norm. This operator can be replaced by the operator BB^* with kernel

$$(2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(\tilde{\xi}-\xi)x}\overline{\varphi(\xi,x)}\varphi(\tilde{\xi},x)(1+|x|^2)^{-r}dx. \tag{3.4}$$

We suppose that $\xi,\tilde{\xi}\in G_*(\mu)$ and consider the operator with kernel (3.4) in the space $L_2(G_*(\mu))$.

Let us decompose a periodic in x function into the Fourier series

$$\overline{\varphi(\xi,x)}\varphi(\tilde{\xi},x)=\sum_m b_m(\xi,\tilde{\xi})\exp(2\pi imx). \tag{3.5}$$

Then the kernel (3.4) equals the sum

$$(2\pi)^{-d/2}\sum_{m\in\mathbb{Z}^d} b_m(\xi,\tilde{\xi})\omega_r(\xi-\tilde{\xi}-2\pi m), \tag{3.6}$$

where ω_r is the Fourier transform of $(1+|x|^2)^{-r/2}$ and

$$b_m(\xi,\tilde{\xi})=\int_{\mathbb{Q}^d}\overline{\varphi(\xi,x)}\varphi(\tilde{\xi},x)\exp(-2\pi imx)dx. \tag{3.7}$$

The last equality is exactly a representation (1.8). According to Theorem 1.2 and the normalization (2.8), the multiplicator norm in $L_2(G_*(\mu))$ of each kernel b_m is bounded by 1. Lemma 1.7 can be applied to operators with kernels $\omega_r(\xi-\tilde{\xi}-2\pi m)$. The estimate (1.16) ensures that the series of operators corresponding to (3.6) converges in the sense of the norm. This convergence is, obviously, uniform in $\mu\in\delta$. □

2. Now we will prove the Hölder continuity of the operator-function (3.1).

Theorem 3.2. *Under the assumptions of Theorem 3.1*

$$||\mathcal{F}_r(\mu)-\mathcal{F}_\mu(\mu_0)||\le C(\theta)|\mu-\mu_0|^\theta,\quad \mu,\mu_0\in\delta,\ \forall\theta<\min\{1,r-1/2\}. \tag{3.8}$$

The proof of this theorem is comparatively cumbersome, although, basically it relies on the same ideas as that of Theorem 3.1. To simplify the notation, we set $\mu_0=\lambda$ in (3.8). Since all points of δ are, in reality, equivalent, this assumption does not reduce a generality. The operator

$$\mathcal{L}(\mu):=\mathcal{F}_r(\mu)-\mathcal{F}_r(\lambda)$$

is self-adjoint. Hence it suffices to estimate its quadratic form.

We define now $\varphi(\xi,x)$ using the notation of p. 2.4:

$$\varphi(\eta_*,x)=\psi(\eta,x)\exp(-i\eta_*x),\quad \eta=\eta(\mu,\xi),\ \eta_*=\eta_*(\mu,\xi),\ \xi\in G_*. \tag{3.9}$$

By virtue of (3.9), a function φ is a piece-wise analytic in a neighbourhood of G_* (when G is replaced by G_* new jumps due to the phase factor can appear). The inequality

$$\int_{\mathbb{Q}^d} |\varphi(\eta_*, x) - \varphi(\xi, x)|^2 dx \le C|\mu - \lambda|^2, \quad \xi \in G_*, \tag{3.10}$$

follows from (2.20),(2.21).

Set $v = \Psi X_r f$. According to (2.17),

$$\begin{aligned}(\mathcal{L}(\mu)f, f) &= < J(\mu)v, J(\mu)v > - < Jv, Jv > \\ &= \int_G (|v(\eta)|^2 e(\mu,\xi) - |v(\xi)|^2) d\sigma(\xi).\end{aligned} \tag{3.11}$$

By (2.19), (3.3) and (3.2),

$$\begin{aligned}|\int_G |v(\eta)|^2 (e(\mu,\xi) - 1) d\sigma(\xi)| &\le C|\mu - \lambda| \int_G |v(\eta)|^2 d\sigma(\xi) \\ &\le C_1|\mu - \lambda| \int_G |v(\eta)|^2 d\sigma_\mu(\xi) = C_1|\mu - \lambda|(\mathcal{F}_r(\mu)f, f) \\ &\le C_2|\mu - \lambda| \, ||f||^2.\end{aligned} \tag{3.12}$$

Thus we can replace e by 1 in (3.11). So we need to estimate

$$\begin{aligned}|\int_G (|v(\eta)|^2 - |v(\xi)|^2) d\sigma(\xi)| &\le (\int_G |v(\eta) - v(\xi)|^2 d\sigma(\xi))^{1/2} \\ &\times (2\int_G (|v(\eta)|^2 + |v(\xi)|^2) d\sigma(\xi))^{1/2}.\end{aligned} \tag{3.13}$$

Since the last factor is bounded by $C\,||f||$, it suffices to estimate the norm in $L_2(G, \sigma)$ of the difference

$$\begin{aligned}v(\eta) - v(\xi) &= (2\pi)^{-d/2} \int (e^{-ix\eta_*}\overline{\varphi(\eta_*, x)} \\ &- e^{-ix\xi}\overline{\varphi(\xi, x)})(1 + |x|^2)^{-r/2} f(x) dx = (2\pi)^{-d/2}(z + w),\end{aligned} \tag{3.14}$$

where

$$z = \int e^{-ix\eta_*}(\overline{\varphi(\eta_*, x)} - \overline{\varphi(\xi, x)})(1 + |x|^2)^{-r/2} f(x) dx, \quad \xi \in G_*,$$

$$w = \int (e^{-ix\eta_*} - e^{-ix\xi})\overline{\varphi(\xi, x)}(1 + |x|^2)^{-r/2} f(x) dx, \quad \xi \in G_*.$$

The operator $B^0 : f \mapsto z$ has the same structure as the operator B of p. 1 but the role of kernel $\varphi(\xi, x)$ is played by the kernel $\varphi(\eta_*, x) - \varphi(\xi, x)$. Repeating arguments of p. 1 and taking into account (3.10), we arrive at the estimate

$$\int_{G_*} |z(\xi)|^2 d\sigma(\xi) \le C|\mu - \lambda|^2 \, ||f||^2. \tag{3.15}$$

3. It remains to estimate the norm of the function w. Let us introduce an operator $K_r(\mu) : L_2(\mathbb{R}^d) \to L_2(G_*)$, $2r > 1$,

$$(K_r(\mu)f)(\xi) = \int e^{-i\eta_* x}(1+|x|^2)^{-r/2} f(x)dx, \quad \xi \in G_*.$$

Set $\vartheta = r - 1/2$ if $2r < 3$; $\vartheta < 1$ if $2r = 3$; $\vartheta = 1$ if $2r > 3$. The estimate

$$||K_r(\mu) - K_r(\lambda|| \le C|\mu - \lambda|^{\vartheta} \tag{3.16}$$

is equivalent to a standard Sobolev theorem about traces in $L_2(G_*(\mu))$ of functions belonging to $H^r(\mathbb{R}^d)$.

The expression for w can be rewritten as

$$w = (Q_r(\mu) - Q_r(\lambda))f, \quad Q_r : L_2(\mathbb{R}^d) \to L_2(G_*),$$
$$(Q_r(\mu)f)(\xi) = \int e^{-i\eta_* x}\overline{\varphi(\xi, x)}(1+|x|^2)^{-r/2} f(x)dx, \quad \xi \in G_*.$$

So we need to estimate the norm of the operator $\mathcal{P}(\mu) := Q_r(\mu) - Q_r(\lambda)$ or, which is more convenient, the norm of the operator $\mathcal{P}(\mu)\mathcal{P}^*(\mu)$. Its kernel equals

(3.17)
$$\int (e^{-i\eta_* x} - e^{-i\xi x})\overline{\varphi(\xi, x)}\varphi(\tilde{\xi}, x)(1+|x|^2)^{-r}(e^{i\tilde{\eta}_* x} - e^{i\tilde{\xi} x})dx$$
$$= \sum_{m \in \mathbb{Z}^d} b_m(\xi, \tilde{\xi})\gamma_m(\xi, \tilde{\xi}), \quad \tilde{\eta}_* = \eta_*(\tilde{\xi}, \mu),$$

where b_m are defined (see(3.5)) by formulas (3.7), and

$$\gamma_m(\xi, \tilde{\xi}) = \int (e^{-i\eta_* x} - e^{-i\xi x})e^{2\pi i m x}(1+|x|^2)^{-r}(e^{i\tilde{\eta}_* x} - e^{i\tilde{\xi} x})dx.$$

An operator Γ_m in $L_2(G_*)$ with kernel γ_m equals

$$\Gamma_m = (K_r(\mu) - K_r(\lambda))[e^{2\pi i m x}](K_r(\mu) - K_r(\lambda))^*,$$

where $[\beta]$ is the operator of multiplication by β. We have seen already that the multiplicator norm of each kernel β_m is bounded by 1. Thus an estimate for the operator $\mathcal{P}(\mu)\mathcal{P}^*(\mu)$ reduces to that for norms of the operators Γ_m. Fix $\nu > 0$. If $|m| \le \nu$, then, by (3.16),

$$||\Gamma_m|| \le C|\mu - \lambda|^{2\vartheta}, \quad |m| \le \nu. \tag{3.19}$$

An estimate is different for $|m| > \nu$. Actually, the kernel of the operator (3.18) is a linear combination of kernels $\omega_r(\eta_* - \tilde{\eta}_* - 2\pi m)$, $\omega_r(\eta_* - \tilde{\xi} - 2\pi m)$, $\omega_r(\xi - \tilde{\eta}_* - 2\pi m)$ and $\omega_r(\xi - \tilde{\xi} - 2\pi m)$. Operators with such kernels are of the type (1.15) and satisfy the bound (1.16). Therefore

$$||\Gamma_m|| \le C(N)(1+|m|)^{-N}, \quad |m| > \nu. \tag{3.20}$$

It follows from (3.17), (3.19), (3.20) that

$$||\mathcal{P}(\mu)||^2 = ||\mathcal{P}(\mu)\mathcal{P}^*(\mu)|| \leq C(N)\nu^d(|\mu-\lambda|^{2\vartheta} + \nu^{-N}). \tag{3.21}$$

Choosing ν as $\nu^{-N} = |\mu-\lambda|^{2\theta}$, we obtain from (3.21) that

$$||\mathcal{P}(\mu)|| \leq C(N)|\mu-\lambda|^{\theta}, \quad \theta = \vartheta(N-d)/N. \tag{3.22}$$

The last inequality implies that

$$\int_{G_*} |w(\xi)|^2 d\sigma(\xi) \leq C(N)|\mu-\lambda|^{2\theta}\, ||f||^2. \tag{3.23}$$

Now to arrive at (3.8) it remains to put together (3.11)-(3.15), (3.23) and to take into account that N is arbitrary large. This concludes the proof of the Theorem 3.2.

Remark 3.3. The results mentioned in Remark 2.1 ensure that the kernel $\varphi(\xi, x)$ is a multiplicator of the class $\mathfrak{M}$. Together with (3.16), this implies (3.22) for $\theta = \vartheta$. We preferred, however, a proof which does not rely on a "difficult" result of the elliptic theory.

4. It is usefull to supplement the estimate (3.8) by the following

Theorem 3.4. *Let assumptions of Theorem 3.1 hold and let θ be the same as in* (3.8)*. If*

$$(\mathcal{F}_r(\mu_0)f, f) = 0, \tag{3.24}$$

then uniformly in $\mu, \mu_0 \in \delta$

$$(\mathcal{F}_r(\mu)f, f) \leq C(\theta)|\mu-\mu_0|^{2\theta}\, ||f||^2. \tag{3.25}$$

Proof. Let again $\mu_0 = \lambda$. Recall that

$$(\mathcal{F}_r(\mu)f, f) = \int_G |v(\eta)|^2 e(\mu, \xi) d\sigma(\xi) \leq C \int_G |v(\eta)|^2 d\sigma(\xi).$$

The condition (3.24) for $\mu_0 = \lambda$ means that $v(\xi) = 0$. Therefore

$$\int_G |v(\eta)|^2 d\sigma(\xi) = \int_G |v(\eta) - v(\xi)|^2 d\sigma(\xi)$$

and it remains to take (3.14), (3.15), (3.23) into account. $\square$

4. The limiting absorption principle

1. In this section we consider, together with the "free" operator H_0, a "perturbed" operator $H = H_0 + V(x)$, where

$$|V(x)| \le C(1+|x|)^{-\rho}, \quad \rho > 1. \tag{4.1}$$

In view of applications to the scattering theory for the pair H_0, H we discuss here properties of the resolvents

$$R_0(z) = (H_0 - zI)^{-1}, \quad R(z) = (H - zI)^{-1}.$$

Set $\mathfrak{H} = L_2(R^d)$, $\mathfrak{H}^{(s)} := X_s\mathfrak{H}$, $||f||_s = ||X_{-s}f||$. An interval $\delta = \delta(\lambda, \varepsilon)$ is the same as in §2,3.

The limiting absorption principle for the operators H_0, H and the interval δ is formulated in the following way.

Theorem 4.1. *If $2r > 1$, then the operator–function $X_r R_0(z) X_r$ with values in $\mathfrak{S}_\infty(\mathfrak{H})$ is continuous in z for* $\mathrm{Re} z \in \delta$, $\pm\mathrm{Im} z \ge 0$. *Under the assumption* (4.1) *the same is true with respect to $X_r R(z) X_r$ if* $\mathrm{Re} z \in \delta \backslash \mathcal{N}$, $\pm\mathrm{Im} z \ge 0$, *where $\mathcal{N}$ is some closed set of Lebesgue measure zero.*

Let us give only some hints on a proof of this assertion. Details can be found e.g. in [K], [Ya1]. By the spectral theorem and properties of Cauchy–type integrals, the result on $R_0(z)$ is a consequence of Theorems 3.1 and 3.2. When considering the "full" resolvent $R(z)$ we use a factorization

$$V = X_s \mathcal{V} X_r, \quad 1 < 2r < 2\rho - 1, \quad s = \rho - r > 1/2, \tag{4.2}$$

where $\mathcal{V}$ is multiplication by a bounded function. Resolvents are connected by the identity

$$X_r R(z) X_r = (I + X_r R_0(z) X_s \mathcal{V})^{-1} X_r R_0(z) X_r, \quad \mathrm{Im} z \ne 0. \tag{4.3}$$

The operator $X_r R_0(z)$ is compact (for any $z > 0$). Therefore, by a study of continuity as $z \to \mu \pm i0$ of the first factor in the right–hand side of (4.3), one can apply the analytic Fredholm alternative. Thus (4.3) implies that an "exceptional' set $\mathcal{N}$ consists of those and only those points μ_0 where at least one of two equations

$$f + X_r R_0(\mu_0 \pm i0) X_s \mathcal{V} f = 0, \quad \mu_0 \in \delta, \tag{4.4}$$

has a non–trivial solution. Note that equations (4.4) corresponding to different admissible factorizations have non–trivial solutions simultaneously.

2. Theorem 4.1 ensures that *the spectrum of the operator H is absolutely continuous on the set $\delta\backslash\mathcal{N}$.* The remaining part of this section is mainly devoted to a proof of the following

Theorem 4.2. *Under the assumptions of Theorem 4.1 the set $\mathcal{N}$ consists of eigenvalues of the operator H. In particular, the operator H does not have the singular continuous spectrum on the interval δ.*

We start a proof with the following remarks. In terms of $g = X_s \mathcal{V} f$ (4.4) can be equivalently rewritten as

$$g + VR_0(\mu_0 \pm i0)g = 0, \quad g \in \mathfrak{H}^{(s)}. \tag{4.5}$$

For any $h \in \mathfrak{H}$ we set

$$\pi(h;\mu) = d(\mathcal{E}_0(\mu)h,h)/d\mu, \quad \mu \in \delta. \tag{4.6}$$

The function (4.6) is integrable. If $g \in \mathfrak{H}^{(s)}$, $2s > 1$, then, by (3.8), $\pi(g;\mu)$ is Hölder continuous. If g satisfies (4.5), then

$$\pi(g;\mu_0) = 0. \tag{4.7}$$

(One arrives at (4.7) taking the imaginary part of the scalar product of (4.5) with $R_0(\mu_0 \pm i0)g$.) In particular, (4.7) implies that equations (4.5) and hence (4.4) have the same solutions for the signs "+" and "−".

If g is a solution of (4.5), then the element $\omega = R_0(\mu_0 \pm i0)g$ satisfies *formally* the equation $H\omega = \mu_0\omega$. Thus for the proof that μ_0 is an eigenvalue of the operator H it suffices to check that

$$R_0(\mu_0 \pm i0)g \in \mathfrak{H}. \tag{4.8}$$

We start with the case $s > 1$ which, by (4.2), settles the problem for $\rho > 3/2$. Let us prove the following

Proposition 4.3. *Let g satisfy the equation* (4.5) *for $s > 1$. Then* (4.8) *is fulfilled.*

Proof. According to (4.7) and (3.25),

$$\pi(g;\mu) \le C|\mu - \mu_0|^{2\theta}\, ||g||_s^2, \ \mu \in \delta, \ \theta < \min\{1, s - 1/2\}, \tag{4.9}$$

where $C = C(\theta)$ does not depend on $\mu_0 \in \delta$. The spectral theorem and (4.9) ensure that

$$||\mathcal{E}_0(\delta)R_0(\mu_0 \pm i0)g||^2 \le C||g||_s^2 \int_\delta |\mu - \mu_0|^{2\theta - 2} d\mu. \tag{4.10}$$

One can suppose that $2\theta > 1$ in (4.9) if $s > 1$. Then the integral in (4.10) converges which yields (4.8). □

3. A verification of the inclusion (4.8) under the assumption (4.1), for arbitrary $\rho > 1$, is not so simple. The corresponding result for the case $H_0 = -\Delta$ is due to S. Agmon [A]. Probably, our arguments simplify considerably his approach.

The heart of the matter is a consecutive improvement of estimates on solutions of the equation (4.5). Finally, we will prove

Proposition 4.4. *Let* (4.1) *hold. If* g *satisfies* (4.5) *for* $2s > 1$, *then* $g \in \mathfrak{H}^{(\bar{s})}$ *for some* $\bar{s} > 1$.

Combined with Proposition 4.3, this implies (4.8) and hence μ_0 is an eigenvalue of the operator H. Thus a proof of Theorem 4.2 reduces to that of Proposition 4.4.

4. We start with some preliminary technical considerations. In complement to (4.6) we define

$$\pi(h_1, h_2; \mu) = d(\mathcal{E}_0(\mu)h_1, h_2)/d\mu, \quad h_1, h_2 \in \mathfrak{H}, \quad \mu \in \delta. \tag{4.11}$$

Clearly,

$$|\pi(h_1, h_2; \mu)|^2 \leq \pi(h_1; \mu)\pi(h_2; \mu) \tag{4.12}$$

$$\int_\delta |\pi(h_1, h_2; \mu)| d\mu \leq ||h_1|| \, ||h_2||. \tag{4.13}$$

Let now $h_1, h_2 \in \mathfrak{H}^{(s)}$. Then (3.1), (3.2) ensure that

$$\sup_{\mu \in \delta} |\pi(h_1, h_2; \mu)| \leq C ||h_1||_s \, ||h_2||_s, \quad 2s > 1. \tag{4.14}$$

The formula (4.11) defines a (continuous) bilinear mapping Π of a pair $\{h_1, h_2\}$ into $\pi(h_1, h_2; \mu)$ such that

$$\Pi : \mathfrak{H} \times \mathfrak{H} \to L_1(\delta), \quad \Pi : \mathfrak{H}^{(s)} \times \mathfrak{H}^{(s)} \to L_\infty(\delta), \ 2s > 1. \tag{4.15}$$

The mapping Π satisfies the conditions of *a complex bilinear interpolation theorem* of Calderon (see [C], [Tr]). Therefore (4.15) ensures that

$$\Pi : \mathfrak{H}^{(\beta)} \times \mathfrak{H}^{(\beta)} \to L_\tau(\delta), \ \beta = (1 - \sigma)s, \ \tau\sigma = 1,$$

for any $\sigma \in (0, 1]$. Furthermore,

$$||\pi(h_1, h_2; \cdot)||_{L_\tau} \leq C(\sigma) ||h_1||_\beta \, ||h_2||_\beta. \tag{4.16}$$

In particular, for $h_1 = h_2 = h$ this yields

Lemma 4.5. *Let* $h \in \mathfrak{H}^{(\beta)}$, $0 < \beta \leq 1/2$. *Then*

$$\int_\delta |\pi(h; \mu)|^\tau d\mu \leq C(\beta, \tau) \, ||h||_\beta^{2\tau}, \quad \forall \tau < (1 - 2\beta)^{-1}. \tag{4.17}$$

We need also

Lemma 4.6. *Let $g \in \mathfrak{H}^{(s)}$, $s \in (1/2, 1]$ and let (4.7) be satisfied. Then*

$$\omega := R_0(\mu_0 \pm i0) g \in \mathfrak{H}^{(-\beta)} \tag{4.18}$$

for any $\beta \in (1/2, 1]$ such that $\beta + s > 1$.

Proof. It suffices to check that

$$|(\mathcal{E}_0(\delta) R_0(\mu_0 \pm i0) g, h)| \leq C ||g||_s \, ||h||_\beta \tag{4.19}$$

for any function $h \in \mathcal{S}(\mathbb{R}^d)$. By the spectral theorem and (4.12), the left-hand side of (4.19) is bounded by

$$\int_\delta |\mu - \mu_0|^{-1} \pi(g; \mu)^{1/2} \pi(h; \mu)^{1/2} d\mu.$$

According to (4.9), the last integral does not exceed

$$C(\theta) \, ||g||_s \int_\delta |\mu - \mu_0|^{\theta - 1} \pi(h; \mu)^{1/2} d\mu, \quad \forall \theta < s - 1/2. \tag{4.20}$$

The integral in (4.20) can be estimated by

$$(\int_\delta |\mu - \mu_0|^{-p(1-\theta)} d\mu)^{1/p} \, (\int_\delta \pi(h; \mu)^{q/2} d\mu)^{1/q}, \; p^{-1} + q^{-1} = 1. \tag{4.21}$$

The condition $p(1 - \theta) < 1$ of convergence of the first integral in (4.21) implies that

$$q^{-1} < \theta < s - 1/2. \tag{4.22}$$

By virtue of (4.17), the second factor in (4.21) is bounded by $C||h||_\beta$ whenever $q/2 < (1 - 2\beta)^{-1}$, i.e. $q^{-1} > 1/2 - \beta$. The last inequality and (4.22) are compatible if $\beta + s > 1$. □

Now we can finish *the proof of Proposition 4.4.* Let (4.5) hold for $s \in (1/2, 1]$. Then (4.17) and, by Lemma 4.6, (4.18) are fulfilled. According to (4.5), $g = -V\omega \in \mathfrak{H}^{(s_1)}$, where $s_1 = \rho - \beta$, $\beta \in (0, 1/2]$, $\beta > 1 - s$, and hence $s_1 < s + \rho - 1$. By a choice of β we can make s_1 arbitrary close to $s + \rho - 1$. If $s + \rho - 1 > 1$, then $s_1 > 1$, and we can set $\bar{s} = s_1$. Otherwise, we repeat this argument, replacing the original s by $s_1 > s$. Let k be the smallest integer such that $s + k(\rho - 1) > 1$. Then after k steps, we obtain that $g \in \mathfrak{H}^{(\bar{s})}$ where $\bar{s} = s_k > 1$. □

5. Theorem 4.2 can be complemented by some useful additional information. Note, first, that it is reversible: every eigenvalue $\mu_0 \in \delta$ of the operator H belongs to the "exceptional" set $\mathcal{N}$. Indeed, if $H\omega = \mu_0 \omega$, then $g = -V\omega \in \mathfrak{H}^{(\rho)}$ satisfies the equation (4.5) and the condition (4.7).

Let us show now that the operator H may have only a finite number of eigenvalues of finite multiplicity. Suppose to the contrary. Then there exists an orthonormalized in $\mathfrak{H}$ sequence $\{\omega_k\}$ such that $H\omega_k = \mu_k \omega_k$, $\mu_k \in \delta$ and $\mu_k \to \mu_0 \in \delta$ ($\mu_k = \mu_0, \forall k$, is not excluded). Set $g_k = -V\omega_k$. Then $g_k \in \mathfrak{H}^{(\rho)}$, $||g_k||_\rho \leq C$ and

$$g_k + V R_0(\mu_k \pm i0) g_k = 0, \tag{4.23}$$

$$\pi(g_k;\mu_k) = 0. \tag{4.24}$$

According to (4.24) and (3.25),

$$|\pi(g_k;\mu)| \leq C(\theta)|\mu - \mu_k|^{2\theta}, \quad \mu \in \delta, \quad \theta < \min\{1, \rho - 1/2\}. \tag{4.25}$$

Theorem 4.1 implies that the operator

$$VR_0(\mu_k \pm i0) : \mathfrak{H}^{(r)} \to \mathfrak{H}^r, \quad 2r = \rho,$$

is compact and depends continuously on $\mu \in \delta$. Therefore equation (4.23) shows that the sequence $\{g_k\}$ is compact in $\mathfrak{H}^{(r)}$ and consequently, in $\mathfrak{H}$.

Let us show now that the set of elements $\omega_k = R_0(\mu_k \pm i0)g_k$ is compact in $\mathfrak{H}$ which leads to contradiction. Set $\Delta_\eta = [\mu_0 - \eta, \mu_0 + \eta] \cap \delta$, $\Delta'_\eta = \mathbb{R}\backslash\Delta_\eta$, where $\eta > 0$. Compactness of $\{\mathcal{E}_0(\Delta'_\eta)\omega_k\}$ is a consequence of compactness of $\{g_k\}$. It remains to check that $\mathcal{E}(\Delta_\eta)\omega_k$ get small as $\eta \to 0$ uniformly in k. By virtue of (4.25),

$$||\mathcal{E}_0(\Delta_\eta)\omega_k||^2 = \int_{\Delta_\eta} |\mu - \mu_k|^2 \pi(g_k;\mu)d\mu \leq C(\theta)\int_{\Delta_\eta} |\mu - \mu_k|^{-2+2\theta} d\mu. \tag{4.26}$$

This gives the required smallness, since θ can be chosen is such a way that $2\theta - 2 > -1$.

Let us now reformulate Theorem 4.2 together with supplements obtained.

Theorem 4.7. *Let assumption* (4.1) *hold. Then there is only a finite number of eigenvalues of the operator H on the interval δ. These eigenvalues have finite multiplicities. The set of eigenvalues coincides with the set $\mathcal{N}$ intervening in the formulation of Theorem 4.1. The operator H does not have the singular continuous spectrum on δ. The whole interval δ is covered by the absolutely continuous spectrum of the operator H.*

5. The wave operators. The scattering matrix

1. Results of §4 allow us to develop *the scattering theory* for the pair H_0, H on the interval δ following a standard scheme. In our presentation, proofs are essentially reduced to the appropriate references.

Denote by $\mathcal{E}(\cdot)$ the spectral measure and by P_a the projector on the absolutely continuous subspace of the operator H. Existence and completeness of *wave operators* are consequences of the limiting absorption principle (Theorem 4.1).

Theorem 5.1. *Suppose that Assumption 2.2 and the condition* (4.1) *are fulfilled. Let $\delta = \delta(\lambda, \varepsilon)$ be the interval defined in p. 2.3. Then the wave operators*

$$W_\pm(\delta) = W_\pm(H, H_0;\delta) = s - \lim_{t\to\pm\infty} \exp(iHt)\exp(-iH_0t)\mathcal{E}_0(\delta) \tag{5.1}$$

exist and are complete.

Recall that $\operatorname{Ran} W_\pm(\delta) = \mathcal{E}(\delta)P_a\mathfrak{H}$ if the limits (5.1) exist. Completeness of the operators (5.1) mean that

$$\operatorname{Ran} W_\pm(\delta) = \mathcal{E}(\delta)P_a\mathfrak{H}. \tag{5.2}$$

Theorem 5.1 can be deduced from Theorem 4.1 with a help of general tools of mathematical scattering theory (see e.g. [Ya1]). No specific features of periodic operators are used here.

2. *The scattering operator* $\mathcal{S}$ for the pair H_0, H and the interval δ is defined by the equality

$$\mathcal{S} = \mathcal{S}(\delta) = W_+^*(\delta) W_-(\delta). \tag{5.3}$$

According to (5.2), the operator (5.3) is unitary on the subspace $\mathcal{E}_0(\delta)\mathfrak{H}$. The operator $\mathcal{S}$ *commutes* with H_0 and hence it is diagonal in the direct integral constructed in p. 2.3. More precisely, the operator $\mathcal{U}\mathcal{S}\mathcal{U}^*$ acts in $L_2(\delta; \mathfrak{N})$ as multiplication by a unitary operator–valued function

$$S(\mu) : \mathfrak{N} \to \mathfrak{N}, \quad a.e.\ \mu \in \delta. \tag{5.4}$$

The operator–function (5.4) is called *a scattering matrix* for the pair H_0, H on the interval δ.

These results rely on the existence of limits (5.1) and the relation (5.2) only. An additional information about resolvents $R_0(z)$, $R_0(z)$ contained in Theorems 4.1 and 4.7 allows us to give an "explicit" expression for S–matrix (5.4). Recall that operators $J(\mu)$ and Ψ are defined by (2.16) and (2.10), respectively.

Theorem 5.2. *Under the assumptions of Theorem 5.1 the representation*

$$\Theta(\mu) := (2\pi i)^{-1}(S(\mu) - I_{\mathfrak{N}}) = -J(\mu)\Psi(V - VR(\mu+i0)V)\Psi^* J(\mu)^*, \mu \in \delta\backslash\mathcal{N}, \tag{5.5}$$

holds for all $\mu \in \delta\backslash\mathcal{N}$ *(for all* $\mu \in \delta$ *which are not eigenvalues of* H*). The operator–function* (5.5) *is Hölder continuous outside of any neighbourhood of the set* $\mathcal{N}$.

Remark 5.3. The expression in the right–hand side of (5.5) needs to be correctly defined. This will be clarified in the next part on the basis of a suitable factorization of each term. We emphasize that representations of the type (5.5) are quite general and, on a formal level, were known long ago. They were first given a precise sense in [F] (in the framework of the Friedrichs–Faddeev model) and in [BE1,2] (in the framework of the trace class scattering theory). Similar problems were considered in [K] for the perturbation of elliptic operators with constant symbols. A modern state of theory is exposed in [Ya1]. Results presented there give automatically Theorem 5.2 once Theorems 4.1, 4.7 are proven and a direct integral (which determines the mapping (5.4) uniquely) is constructed.

3. Our next goal is to study spectral properties of the operator $S(\mu)$, $\mu \in \delta\backslash\mathcal{N}$. For simplicity of notation we set $\mu = \lambda$. (Then, for example, *the operator* $J(\mu)$ *becomes the restriction* $\mathcal{I}$ *on the surface* G). Thus we accept:

Assumption 5.4. *The point* λ *is not an eigenvalue of the operator* H*, i.e.* $\lambda \notin \mathcal{N}$.

This assumption does not reduce a generality since the role of λ can be played by any point $\mu \in \delta\backslash\mathcal{N}$.

Let us now formulate estimates which, in particular, allow us to give a precise sense to the representation (5.5). Set

$$\kappa := (d-1)/(\rho - 1). \tag{5.6}$$

In p. 6.4 we shall establish

Lemma 5.5. *There is an inclusion*

$$\mathcal{I}\Psi X_r \in \Sigma_{2\kappa}(\mathfrak{H}, \mathfrak{N}), \quad 2r = \rho > 1. \tag{5.7}$$

Let us factorize V as $V = X_r \mathcal{V}_0 X_r$ where $\mathcal{V}_0 \in L_\infty$ and $2r = \rho$. By virtue of (5.7), the operator

$$Z = \mathcal{I}\Psi V(\mathcal{I}\Psi)^* \tag{5.8}$$

satisfies

$$Z = (\mathcal{I}\Psi X_r)\mathcal{V}_0(\mathcal{I}\Psi X_r)^* \in \Sigma_\kappa(\mathfrak{N}). \tag{5.9}$$

Then the equality (5.5) can be written as

$$\Theta(\lambda) + Z = (\mathcal{I}\Psi X_r)\mathcal{V}_0(X_r R(\lambda + i0)X_r)\mathcal{V}_0(\mathcal{I}\Psi X_r)^*. \tag{5.10}$$

According to Theorem 4.1 the operator $X_r R(\lambda + i0)X_r$ is compact. Thus the following statement is true.

Lemma 5.6. *Let Assumptions 2.2, 5.4 and the condition* (4.1) *hold. Then the operator*

$$\Theta(\lambda) = (2\pi i)^{-1}(S(\lambda) - I_{\mathfrak{N}})$$

admits a representation defined by formulas (5.8), (5.10). *Furthermore, inclusions* (5.9) *and*

$$\Theta(\lambda) + Z \in \Sigma_\kappa^0(\mathfrak{N}) \tag{5.11}$$

hold.

Relations (5.9), (5.11) imply that the operator $(-Z)$ is a "principal part" of the operator $\Theta(\lambda)$.

6. The asymptotics of phases of the scattering matrix

1. According to (5.9), (5.11) the operator $\Theta(\lambda)$ is compact. Therefore the spectrum of $S(\lambda)$ (it lies on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$) is discrete and has the only accumulation point $z = 1$. Let $\exp(\mp 2i\varphi_k^\pm)$ be eigenvalues of the operator $S(\lambda)$ belonging to lower and upper semicircles, respectively. Suppose $\varphi_k^\pm \in (0, \pi/2]$, $\varphi_{k+1}^\pm \le \varphi_k^\pm$; multiplicity of eigenvalues is taken into account.

Our main goal is to find the asymptotics of phases $\varphi_k^\pm$ as $k \to \infty$. We replace now the condition (4.1) by a stronger one:

$$V \in L_\infty(\mathbb{R}^d), V(x) = |x|^{-\rho}F(x/|x|) + o(|x|^{-\rho}), |x| \to \infty, \rho > 1, F \in C^\infty(\mathbb{S}^{d-1}). \tag{6.1}$$

The asymptotics of phases was studied by the authors in [BY1]. It was assumed that H_0 is a pseudodifferential operator with a constant symbol (in particular, $H_0 = -\Delta$). Now H_0 is the operator (2.1), and *the Fourier transform* Φ should be replaced in computations by the transformation (2.10). It turns out, however, that the kernel $\overline{\psi(\xi, x)} = \exp(-i\xi x)\overline{\varphi(\xi, x)}$ of the operator Ψ does not intervene explicitly in the asymptotics of phases. It is exactly to this phenomenon that we pay the main attention below. On the contrary, we give only references to [BY1] whenever methods used there can be carried over to the case studied here without significant change.

2. Let us discuss, first of all, the asymptotics of the spectrum of the operator Z. It is (see (5.8)) a selfadjoint compact operator in the space $\mathfrak{N} = L_2(G, d\sigma)$. Let us replace temporarily (6.1) by a stronger condition:

$$V \in C^\infty(\mathbb{R}^d), \quad V(x) = |x|^{-\rho} F(x/|x|) \text{ for } |x| \geq 1. \tag{6.2}$$

Obviously, the kernel of the operator Z equals (cf. (3.4))

$$(2\pi)^{-d} \int_{\mathbb{R}^d} e^{(\tilde{\xi}-\xi)x} \overline{\varphi(\xi, x)} \varphi(\tilde{\xi}, x) V(x) dx, \quad \tilde{\xi}, \xi \in G. \tag{6.3}$$

According to (3.5), (3.7), the kernel (6.3) is the sum

$$(2\pi)^{-d/2} \sum_{m \in \mathbb{Z}^d} b_m(\xi, \tilde{\xi}) \hat{V}(\xi - \tilde{\xi} - 2\pi m), \quad \hat{V} = \Phi V. \tag{6.4}$$

Denote by Y_m the operator with kernel $\hat{V}(\xi - \tilde{\xi} - 2\pi m)$. Let Z_m be operators corresponding to different terms in the sum (6.4) so that

$$Z = \sum_m Z_m. \tag{6.5}$$

Under the condition (6.2) Remark 1.8 is applicable to the function V. Therefore $Y_m \in \Sigma_t$ for $|m| = |m_1| + \cdots + |m_d| > d$ and every $t > 0$ and

$$|Y_m|_t \leq C(N, t)(1 + |m|)^{-N}, \quad |m| > d, \ \forall N > 0, \ \forall t > 0. \tag{6.6}$$

It was already noted (see p. 3.2) that the function $\varphi(\xi, x)$ is piece-wise smooth in the variable $\xi \in G_*$. According to (3.7) the same is true (in ξ and $\tilde{\xi}$) for kernels b_m. Furthermore, Theorem 1.4 and Remark 1.6 allow us to estimate multiplicator quasi-norms of kernels b_m in classes $\sum_t$ *uniformly in* m. Thus *the estimate* (6.6) *extends to the operators* Z_m. Finally, the sum of operators $Z_m, |m| > d$, can be estimated in the class Σ_t with the help of (1.5). This yields

Proposition 6.1. *Under the assumption* (6.2)

$$\sum_{|m|>d} Z_m \in \Sigma_t, \quad \forall t > 0. \tag{6.7}$$

Let us now consider the operator Y_0 with kernel $(2\pi)^{-d/2} \hat{V}(\xi - \tilde{\xi})$. It is a pseudodifferential operator of negative order $1 - \rho$ on the smooth surface G_*. The asymptotics of eigenvalues of exactly such operators was investigated in p. 3.2, 3.3 of the paper [BY1]. Thus we can use directly a corresponding result of [BY1].

Let us introduce a necessary notation. For $\xi \in G$ denote by $n(\xi) = |\nabla E(\xi)|^{-1} \nabla E(\xi)$ the unit normal to G; the set $L_\xi = \{\eta \in \mathbb{R}^d : n(\xi)\eta = 0\}$ is the tangent space to G at a point ξ. We put

$$2\pi \Xi(\xi, \eta) = |\nabla E(\xi)|^{-1} \int_0^\pi F(n(\xi) \cos\omega + \eta \sin\omega) \sin^{\rho-2} \omega \, d\omega \tag{6.8}$$

for $\xi \in G$, $\eta \in L_\xi \cap \mathbb{S}^{d-1}$ and introduce constants

$$g_\pm = \pi (2\pi)^{1-\rho} (d-1)^{-\gamma} [\int_G dG(\xi) \int_{L_\xi \cap \mathbb{S}^{d-1}} \Xi(\xi, \eta)_\pm^\kappa d\eta]^\gamma, \gamma\kappa = 1. \tag{6.9}$$

Recall that κ is defined by (5.6) and dG is the Euclidean measure on G.

Proposition 6.2. *Under the assumption* (6.1) *the asymptotics holds:*

$$\lim_{k\to\infty} k^{\gamma}\nu_k^{\pm}(Y_0) = \pi^{-1}g_{\pm}, \quad \gamma\kappa = 1. \tag{6.10}$$

Remark 6.3. In [BY1] the asymptotics (6.8)–(6.10) was, first, verified under the condition (6.2). To extend it to the case (6.1) one needs to estimate (with an arbitrary small asymptotic coefficient) the contribution of the term $o(|x|^{-\rho})$. This is done again with the help of the result obtained for the case (6.2). A similar "self-improvement " of a result we use below for the operator Z.

Remark 6.4. Constants (6.9) are defined invariantly on G. Their calculation does not require that G be replaced by G_*. These constants are constructed in terms of the function $E(\cdot)$ only. They do not depend on the corresponding eigenfunction ψ.

3. The next step is to obtain the asymptotics of the spectrum of the operator Z_0. It is convenient to accept first the condition (6.2).

Let us describe briefly a scheme of calculation in [BY1] of spectral asymptotics for operators Y_0:

1. A surface G is split up into sufficiently small piece-wise smooth parts G^j admitting a one-to-one orthogonal projection on a domain of $\mathbb{R}^{d-1}$. Thus

$$Y_0 = \sum_{i,j} Y_0^{i,j},$$

where $Y_0^{i,j}\colon L_2(G^i, d\sigma) \to L_2(G^j, d\sigma)$.

2. The operator $Y_0^{i,i}$ reduces to a pseudodifferential operator defined by some smooth *amplitude* $A_j(\bar{x};\xi,\tilde{\xi})$ where $\bar{x} \in \mathbb{R}^{d-1}$. The leading term of smoothness expansion of this pseudodifferential operator is determined by *the principal symbol* $A_j(\bar{x};\xi,\xi)$; the leading term of the spectral asymptotics depends on it only. The asymptotics of spectrum is calculated in terms of the principal symbol according to the main result of [BS3].

3. It is verified that

$$Y_0' := \sum_{i\neq j} Y_0^{i,j} \in \Sigma_\kappa^0. \tag{6.11}$$

Therefore Y_0' does not contribute to the leading term of the spectral asymtotics.

4. Asymptotics of distribution functions (of numbers $\nu_k^{\pm}$) of operators $Y_0^{j,j}$ are summed up over all j. This yields the asymptotics (6.10).

Let us apply the described scheme to the operator Z_0. Its kernel differs from that of Y_0 by a factor $b_0(\xi,\tilde{\xi})$. Since b_0 is a multiplicator, the relation (6.11) remains true for Z_0:

$$Z_0' := \sum_{i,j} Z_0^{i,j} \in \Sigma_\kappa^0.$$

The amplitude of the operator $Z_0^{j,j}$ equals $b_0(\xi,\tilde{\xi})A_j(\bar{x};\xi,\tilde{\xi})$. Taking into account that, according to (3.7), (2.8), (2.9), $b_0(\xi,\xi) = 1$, we find that its principal symbol is $A_j(\bar{x};\xi,\xi)$. Thus the asymptotics (6.10) can be extended to Z_0.

Proposition 6.5. *Under the assumption* (6.2) *the asymptotics* (6.10) *holds, where* Y_0 *is replaced by* Z_0*.*

It remains to estimate the operators Z_m for $0 < |m| \le d$. The corresponding operators Y_m satisfy $Y_m \in \sum_\kappa^0$. This can be obtained similarly to the inclusion $Y_0^{i,j} \in \Sigma_\kappa^0$ for "neighbouring" G^i, G^j (see p. 3.3 of [BY1] for more details). Taking into account that b_m are multiplicators in Σ_κ^0, we obtain

Proposition 6.6. *Under the assumption* (6.2)

$$Z_m \in \Sigma_\kappa^0, \quad 0 < |m| \le d. \tag{6.12}$$

4. In the case (6.2) the inclusion $Z - Z_0 \in \Sigma_\kappa^0$ follows from (6.5), (6.7) and (6.12). Therefore, by Proposition 6.5, the asymptotics (6.10) is fulfilled for Z. Finally, the asymptotics obtained can be extended (see Remark 6.3 and p. 3.3 of [BY1]) to the general case (6.1). Thus we arrive at

Theorem 6.7. *Under the assumption* (6.1)

$$\lim_{k\to\infty} k^\gamma \nu_k^\pm(Z) = \pi^{-1} g_\pm, \quad \gamma\kappa = 1, \tag{6.13}$$

where numbers $g_\pm$ *are defined by* (6.8), (6.9).

In particular, the asymptotics (6.13) implies that $Z \in \Sigma_\kappa$. This is equivalent to (5.7) if $V(x) = (1+|x|^2)^{-\rho/2}$. So Lemma 5.5 is proven.

It remains to add the following simple assertion of general nature.

Lemma 6.8. *Let* S *be a unitary operator,* $\Theta = (2\pi i)^{-1}(S - I)$*,* $Z = Z^*$ *and* $\Theta + Z \in \Sigma_\kappa^0$*. Suppose that the operator* Z *satisfies the asymptotics* (6.13) *for some* $\gamma > 0$*. Then the phases* $\varphi_k^\pm$ *of the operator* S *have the asymptotics*

$$\lim_{k\to\infty} k^\gamma \varphi_k^\pm = g_\pm, \quad \gamma\kappa = 1. \tag{6.14}$$

This assertion is practically the same as Lemma 1 in [BY1]. Quite an elementary device for proofs of results of this type is given in [Ya2].

Let us now combine Theorem 6.7 with Lemma 6.8 and the estimate (5.11). This leads to our main result.

Theorem 6.9. *Let Assumptions 2.2, 5.4 and the condition* (6.1) *be fulfilled. Let* $S(\lambda)$ *be the scattering matrix for the pair* H_0, H*. Then the scattering phases* $\varphi_k^\pm$ *of the operator* $S(\lambda)$ *have the asymptotics* (6.14) *where* κ *and* $g_\pm$ *are defined by* (5.6), (6.8), (6.9).

5. In conclusion we discuss how Assumption 2.2 can be relaxed. Let us consider $\lambda \in \mathbb{R}$ which is *an interior point of several bands* Λ_j, $j = l, \dots, l+s$, and is separated from all other bands. Suppose that corresponding surfaces

$$G_j = \{\xi \in \mathbb{T}^d : E_j(\xi) = \lambda\}, \quad j = l, \dots, l+s, \tag{6.15}$$

do not have critical points. If these surfaces do not interset with each other, then the above scheme remains true and only some minor modifications in formulas are required. In particular, contributions of different surfaces G_j should be summed up in the expression (6.9) for $g_\pm$: $\int_G \ldots$ is replaced by $\sum_j \int_{G_j} \ldots$. If some surfaces (6.15) do intersect, then new essential restrictions are demanded. We need to assume smoothness of all functions E_j and piece-wise smoothness of the corresponding eigenfunctions Ψ_j in a neighbourhood of G_j. Under such assumptions the above scheme still works. Actually this more general situation has many features in common with the case of matrix pseudodifferential operators considered in §4 of the paper [BY1].

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M. Sh. Birman, State University of St. Petersburg, 199164 St. Petersburg, Russia

D. R. Yafaev, IRMAR, Department of Mathematics, University of Rennes-1, 35042, Rennes, France