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THE STRUCTURE OF THE FRÉCHET DERIVATIVE IN BANACH SPACES

EVA MATOUŠKOVÁ AND CHARLES STEGALL

ABSTRACT. Our analysis of Fréchet differentiable functions obtains results of the following type. An operator is factorizable if it factors through an Asplund space or satisfies any of the conditions of [S2]. Suppose that X is a Banach space, W is an open subset of the Banach space Z and $\beta : W \to Z$ is Fréchet differentiable at every point. The set

$$\{w \in W : \beta'(w) \text{ factorizble}\}$$

is a dense subset of W if and only if for any continuous and convex function $\phi : X \to \Re$ the composed function $\phi\beta$ is Fréchet differentiable on a dense G_δ subset of W. We observe that if a continuous function is Fréchet differentiable everywhere on an open set then the derivative is in the first Baire class.

A Banach space whose dual has the Radon-Nikodým property is sometimes called an Asplund space; a Banach space whose dual has the Radon-Nikodým property is characterized by the property that each separable and linear subspace of it has a norm separable dual (examples are finite dimensional spaces or reflexive spaces). We shall say that an operator (bounded and linear function) $T : Z \to X$ is factorizable if and only if the operator factors linearly through an Asplund space; equivalently, T is factorizable if and only if $T(B_Z(0,1))$ is equimeasurable. Negatively stated, an operator $T : Z \to X$ is not factorizable if and only if there exists a weak* compact subset K of X^* and $\epsilon > 0$ so that for any weak* open subset U of K we have that the norm diameter of $T^*(U)$ is greater than ϵ; in fact, we may assume that K has a countable dense set ([S1] and [S2]). Trivially, if Z or X is an Asplund space then all operators are factorizable. If $K \subseteq B_{X^*}(0,\rho)$, $T_1 : Z \to X$ is an operator, and $\|T_1 - T\| < \epsilon/4(\rho+1)$ then T_1 is also not factorizable. It follows that the factorizable operators form a closed subset of of the Banach space of linear operators from Z to X. The factorizable operators form a vector space (see *eg* [S2]) that is a closed subspace of the space of all operators. If we define the convex function

$$\phi_K(x) = \sup\{x^*(x) : x^* \in K\}$$

it follows that the function $\phi_K T$ is not Fréchet differentiable at any point. We show even more: for any fixed $y \in X$, $\phi_K(y + T(z))$ is not differentiable at any point. Fix $w \in Z$ and choose any $z^* \in Z^*$. By assumption, the image of

$$\{x^* \in K : x^*(y + T(w)) > \phi_K(y + T(w)) - \frac{1}{n^2}\} \tag{1}$$

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under T^* has diameter more than ϵ. Choose $x_n^* \in K$ so that

$$\begin{aligned} &x^*(y+T(w)) > \phi_K(y+T(w)) - \frac{1}{n^2} \text{ and} \\ &\|T^*(x_n) - z^*\| > \frac{\epsilon}{3}. \end{aligned} \tag{2}$$

Choose $u_n \in Z$, $\|u_n\| = 1$, so that

$$(T^*(x_n) - z^*)(u_n) > \epsilon/3. \tag{3}$$

Computing the difference quotient

$$\begin{aligned} &\frac{\phi_K(y+T(w+\frac{1}{n}u_n)) - \phi_K(y+T(w)))}{1/n} - z^*(u_n) \\ &\geq \frac{x_n^*(y+T(w+\frac{1}{n}u_n)) - \phi_K(y+T(w))}{1/n}) - z^*(u_n) \\ &\geq \frac{x_n^*(y+T(w)) - \phi_K(y+T(w))}{1/n} + (T^*(x_n) - z^*)(u_n) \\ &\geq -(\frac{1}{n}) + \frac{\epsilon}{3}. \end{aligned} \tag{4}$$

This proves that z^* is not the derivative of $\phi_K(y+T(z))$ at $\phi_K(y+T(w))$; as z^* was arbitrary, there is no derivative.

Suppose that $\beta : W \to X$ is continuous where W is an open subset of the Banach space Z. Suppose that β is Fréchet differentiable at w and $\beta'(w)$ is not factorizable. Then there exists K as above so that $\phi_K\beta'(w)$ has no point of differentiability. If we write

$$\beta(v) = \beta(w) + \beta'(w)(v-w) + o(v-w)$$

then

$$\phi_K(\beta(v)) = \phi_K(\beta(w) - \beta'(w)(w) + \beta'(w)(v) + o(v-w))$$

is not Fréchet differentiable at w because

$$\begin{aligned} &|\beta(v) - \phi_K(\beta(w) - \beta'(w)(w) + \beta'(w)(v))| \\ &\quad = |\phi_K(\beta(w) - \beta'(w)(w) + \beta'(w)(v) + o(v-w)) \\ &\quad - \phi_K(\beta(w) - \beta'(w)(w) + \beta'(w)(v))| \leq |\phi_K(o(v-w))| \leq \rho\|o(v-w)\| \end{aligned} \tag{5}$$

Since

$$\frac{\|o(v-w)\|}{\|v-w\|} \to 0$$

it follows that $\phi\beta$ is differentiable at w if and only if, with $y = \beta(w) - \beta'(w)(w)$,

$$\phi_K(\beta(w) - \beta'(w)(w) + \beta'(w)(v)) = \phi_K(y + \beta'(w)(v)) \tag{6}$$

is differentiable at w, and the latter function is not differentiable at w. Similarly, if β were continuously differentiable at w and $\beta'(w)$ were not factorizable then there would exist a continuous convex function ϕ and an open neighborhood V of w such that $\phi\beta$ was not differentiable at any point of V.

Suppose that $f : M \to Z$ is a function from the complete metric space M to the Banach space Z (neither M nor Z is necessarily assumed to be separable). Then f is a function of the first Baire class (pointwise limit of a sequence of continuous functions) if and only if $f|C$ has a point of continuity for any closed (or, compact) set $C \subseteq M$ if and only if $f^{-1}(E)$ is a G_δ set for any closed set $E \subseteq Z$. This is a special case of results that appear in [S6] and other places, the basic result for separable spaces going back to Baire himself. Recall that a G_δ subset of a complete metric space has an equivalent metric that makes it complete. The following is trivial in the real case, but, surprisingly, not so easy in the general case, where the proof is not constructive. The proof does reduce to the separable case (results mentioned above) but does not seem to be easier by specifically restricting oneself to the separable case. We were unable to locate the following result in the literature, although there are a number of related results (see below). There is an extensive literature concerning real valued functions which are derivatives ([Br]).

Theorem 1. *Suppose that X is a Banach space, W is an open subset of the Banach space Z and $\beta : W \to Z$ is a continuous function that is Fréchet differentiable at every point. Then $\beta' : W \to L(Z, X)$, the bounded linear operators from Z to X, is a function of the first Baire class.*

Proof. For each pair $w,\ v$ in W we define

$$\beta(v) = \beta(w) + \beta'(w)(v - w) + o_w(v - w).$$

Fix $C \subseteq W$ that is closed and not empty. For each pair of integers n and m define

$$A_{n,m} = \{w \in C : ||o_w(v - w)|| \leq \frac{1}{n}||v - w|| \text{ for all } v \text{ such that } ||v - w|| \leq \frac{1}{m}\}.$$

Since $C = \cup_m A_{n,m}$, it follows from the Baire category theorem that $\cup_m \overset{\circ}{\overline{A}}_{n,m}$ is an open dense subset of C. Fix n, m and $w_0 \in \overset{\circ}{\overline{A}}_{n,m}$. Fix $\delta > 0$ so small that $\delta < 1/2nm$, $B(w_0, \delta) \cap C \subseteq \overset{\circ}{\overline{A}}_{n,m}$ and $||o_{w_0}(v - w_0)|| \leq \frac{1}{n}||v - w_0||$ if $||v - w_0|| \leq \delta$. Choose $\eta > 0$ so that

$$B(w_0, \eta) \subseteq \{w : (1 + ||\beta'||)\,(||w - w_0|| + ||\beta(w) - \beta(w_0)||) \leq (\delta/2)^2\}$$

Let $w \in B(w_0, \eta) \cap A_{n,m}$ and choose any v so that $||v - w|| = \frac{\delta}{2}$. We have the following easy inequalities:

$$\begin{aligned}
&1 - \frac{\delta}{2} \leq \frac{||v - w_0||}{\delta/2} = \frac{||v - w_0||}{||v - w||} \leq 1 + \frac{\delta}{2},\\
(7)\qquad &||o_{w_0}(v - w_0)|| \leq \frac{1}{n}||v - w_0|| \text{ and}\\
&||o_w(v - w)|| \leq \frac{1}{n}||v - w||.
\end{aligned}$$

Continuing, we estimate

$$
\begin{aligned}
&\| \left(\beta'(w_0) - \beta'(w)\right) \frac{(v-w)}{\delta/2} \| \\
&\quad \le \|\beta'(w_0)\frac{(w_0-w)}{\delta/2}\| + \|\beta'(w_0)\frac{(v-w_0)}{\delta/2} - \beta'(w)\frac{(v-w)}{\delta/2}\| \\
&\quad \le \frac{\delta}{2} + \|\beta'(w_0)\frac{(v-w_0)}{\delta/2} - \beta'(w)\frac{(v-w)}{\delta/2}\| \\
&\quad = \frac{\delta}{2} + \|\frac{\beta(v)-\beta(w_0)}{\delta/2} \\
&\qquad - \left(\frac{o_{w_0}(v-w_0)}{\|v-w_0\|}\right)\left(\frac{\|v-w_0\|}{\delta/2}\right) - \frac{\beta(v)-\beta(w)}{\delta/2} + \frac{o_w(v-w)}{\delta/2}\| \qquad (8)\\
&\quad \le \frac{\delta}{2} + \frac{\|\beta(v)-\beta(w_0)-\beta(v)+\beta(w)\|}{\delta/2} \\
&\qquad + (1+\frac{\delta}{2})\frac{\|o_{w_0}(v-w_0)\|}{\|v-w_0\|} + \frac{\|o_w(v-w)\|}{\delta/2} \\
&\quad \le \frac{\delta}{2} + \frac{\|\beta(w)-\beta(w_0)\|}{\delta/2} + \frac{1}{n}\left(1+\frac{\delta}{2}\right) + \frac{1}{n} \le \frac{3\delta}{2} + \frac{1}{n} + \frac{1}{n} \le \frac{3}{n}.
\end{aligned}
$$

This proves that $\|\beta'(w_0)-\beta'(w)\| \le 3/n$. Now, choose any other $w_1 \in B(w_0,\eta)\cap C$. Applying, the same techniques, we know that there exists a ball $B(w_1,\tau)$, and we assume that $B(w_1,\tau) \subseteq B(w_0,\eta)$, so that if $w \in B(w_1,\tau)\cap A_{n,m}$ then $\|\beta'(w_1) - \beta'(w)\| \le 3/n$. This proves that $\|\beta'(w_0) - \beta'(w_1)\| \le 6/n$ and the variation of β' over $B(w_0,\eta)\cap C$ is no more than $6/n$. This proves that β' restricted to C is continuous at each point of $\cap_n \cup_m \overset{\circ}{\overline{A}}_{n,m}$, which is a dense G_δ subset of C. Since C was an arbitrary closed subset of W, this proves that β' is in the first Baire class.

If $\phi : X \to \Re$ is a continuous and convex function, then $\partial\phi(x)$ denotes the subgradient of ϕ at x;

$$\partial\phi(x) = \{x^* \in X^* : x^*(y-x) \le \phi(y) - \phi(x) \text{ for all } y\} \tag{9}$$

In [S3] (see also [S4] and [S5]) we introduced a class of topological spaces, which, in turn define a class of Banach spaces more general than that of the Asplund spaces ([S2]). There are many equivalent ways of defining these spaces *loc. cit.* For more recent results see [Z] and its references.

Definition 2. *Let $\mathcal{C}$ denote the class of completely regular spaces with the properties that $T \in \mathcal{C}$ if and only if given any complete metric space M, any upper semicontinuous and compact valued mapping $\Phi : M \to \wp(T)$ then there exists a dense G_δ subset M_0 of M and a continuous function $\beta : M_0 \to T$ such that $\beta(m) \in \Phi(m)$ for each $m \in M_0$. The class $\mathcal{X}$ of Banach spaces is defined such that $X \in \mathcal{X}$ if and only if the unit ball of X^*, in the weak* topology, is in $\mathcal{C}$.*

Proposition 3. *For any pair of Banach spaces, denote by $L_{\mathcal{C}}(Z,X)$ the linear operators such that $T \in L_{\mathcal{C}}(Z,X)$ if and only if $T^*(B_{Z^*}(0,1)) \in \mathcal{C}$ (T transforms bounded sets into sets which belong to the class $\mathcal{C}$). Then $L_{\mathcal{C}}(Z,X)$ is a closed vector subspace of $L(Z,X)$ containing the factorizable operators.*

Proof. We shall give complete details in the Gâteaux differentiable case. Suppose that there exist a dense subset D of W, $w \in D$, and a function $\psi : D \to X^*$ such that $\psi(z) \in \partial\phi(\beta(z))$ and $(\beta'(w))^*\psi$ is weak* continuous at w. Fix any v, $\|v\| = 1$, and $t_n \neq 0$ and $t_n \to 0$. Define

$$o(t) = \beta(w + tv) - \beta(v) - \beta'(w)(tv) \tag{10}$$

and choose $v_n \to v$, $\|v_n\| = 1$, and $s_n \neq 0$ so that $s_n \to 0$, $w + s_n v_n \in D$,

$$\begin{aligned} &\|\frac{o(t_n)}{t_n} - \frac{\beta(w + s_n v_n) - \beta(w)}{s_n} + \beta'(w)v\| < \frac{1}{n} \text{ and} \\ &\|\frac{\phi\beta(w + t_n v) - \phi\beta(v)}{t_n} - \frac{\phi\beta(w + s_n v_n) - \phi\beta(v)}{s_n}\| < \frac{1}{n}. \end{aligned} \tag{11}$$

Then,

(12)

$$\begin{aligned} 0 &\leq \frac{\phi\beta(w + t_n v) - \phi\beta(v)}{t_n} - \psi(w)(\beta'(w)v) \\ &\leq \frac{1}{n} + \frac{\phi\beta(w + s_n v_n) - \phi\beta(v)}{s_n} - \psi(w)(\beta'(w)v) \\ &\leq \frac{1}{n} + \psi(w + s_n v_n)\left(\frac{\phi\beta(w + s_n v_n) - \phi\beta(v)}{s_n}\right) - \psi(w)(\beta'(w)v) \\ &\leq \frac{1}{n} + \psi(w + s_n v_n)\left(\frac{\phi\beta(w + s_n v_n) - \phi\beta(v)}{s_n} - \beta'(w)v\right) \\ &\quad + \left(\psi(w + s_n v_n) - \psi(w)\right)(\beta'(w)v) \\ &\leq \frac{1}{n} + \|\psi(w + s_n v_n)\|\left(\frac{\|o(t_n)\|}{|t_n|} + \frac{1}{n}\right) + \left(\psi(w + s_n v_n) - \psi(w)\right)(\beta'(w)v). \end{aligned}$$

We know that

$$\begin{aligned} &\left(\psi(w + s_n v_n) - \psi(w)\right)(\beta'(w)v) \to 0, \\ &\sup_n \|\psi(w + s_n v_n)\| < \infty \text{ and} \\ &\frac{\|o(t_n)\|}{|t_n|} \to 0. \end{aligned} \tag{13}$$

The converse is proved by using the same inequalities.

The following proposition and theorem appear in [S3] and [S4].

Proposition 4. *With the same hypothesis as above on W, Z and X, suppose that β is continuous and Gâteaux differentiable (respectively, Fréchet differentiable) at every point and let $\phi : X \to \Re$ be a continuous and convex function. Then $\phi\beta$ is Gâteaux differentiable (respectively, Fréchet differentiable) at $w \in W$ if and only if there exist a dense subset D of W, $w \in D$, and a function $\psi : D \to X^*$ such that $\psi(z) \in \partial\phi(\beta(z))$ and $(\beta'(w))^*\psi$ is weak* continuous at w (respectively, $(\beta'(w))^*\psi$ is norm continuous at w).*

Theorem 5. *With the same hypothesis as above on W and Z, suppose that $X \in \mathcal{X}$ (respectively, X is an Asplund space), and suppose that β is everywhere Gâteaux differentiable (respectively, Frechét differentiable), then for any continuous and convex function $\phi : X \to \Re$ the composed function $\phi\beta$ is Gâteaux differentiable (respectively, Frechét differentiable) on a dense G_δ subset of W.*

The main result here is the following, and in the Frechét case generalizes the result above.

Theorem 6. *Suppose that X is a Banach space, W is an open subset of the Banach space Z and $\beta : W \to Z$ is a continuous function that is Fréchet differentiable at every point. If the set*

$$\begin{aligned} &\{w \in W : \beta'(w) \in L_{\mathcal{C}}(X, Y)\} \\ &(\textit{respectively, } \{w \in W : \beta'(w) \textit{ is representable}\}) \end{aligned} \tag{14}$$

is a dense subset of W then for any continuous and convex function $\phi : X \to \Re$ the composed function $\phi\beta$ is Gâteaux differentiable (respectively, Frechét differentiable) on a dense G_δ subset of W.

A converse of this, in the Gâteaux case, is the following: suppose that

$$\{w \in W : \beta'(w) \in L_{\mathcal{C}}(X, Y)\} \tag{15}$$

is not dense in W. Then there exists $w \in W$, an open neighborhood V of w, a complete metric space M, an upper semicontinuous and compact valued mapping $\Phi : M \to \wp(X)$ so that for all $v \in V$, $\beta'(v)\Phi : M \to \wp(Z)$ has no selection as in the definition of $\mathcal{C}$.

Proposition 7. *With the same hypothesis as above on W, Z and X, suppose that β is a continuous function that is everywhere Frechét differentiable at every point, and assume that*

$$\begin{aligned} &\{w : \beta'(w) \in L_{\mathcal{C}}(Z, X)\}, \\ &(\textit{respectively, } \{w : \beta'(w) \textit{ is factorizable}\}) \end{aligned} \tag{16}$$

is dense in W. Let $\phi : X \to \Re$ be a continuous and convex function. Then there exists a dense G_δ set $G \subseteq W$ and a function $\lambda : G \to Z^*$ such that $\lambda(z) \in (\beta'(z))^* \partial\phi(\beta(z))$ that is weak* continuous (respectively, norm continuous).

Proof. We give the details in the Frechét case that $\{w : \beta'(w) \text{ is factorizable}\}$ is dense. Since β' is in the first Baire class and the space of factorizable operators is a Banach space, $\{w : \beta'(w) \text{ factorizable}\}$ is G_δ and it is dense by hypothesis. Also, the set of points where β' is continuous contains is a G_δ set. Let H be a dense G_δ such that β' is continuous at each point w of H and $\beta'(w)$ is factorizable. Define $\zeta(z) = (\beta'(z))^* \partial\phi(\beta(z))$ for each $z \in H$. Clearly, $\zeta(z)$ is weak* compact and we shall show that not only is it weak* upper semicontinuous but satisfies the hypothesis of Proposition C, page 192 of [S5]. It is also possible to argue as in the proof of Theorem 3 of [S6]. To see that ζ is weak* upper semicontinuous suppose that $z_n \to z_0$, F is a weak* closed subset of Z^* and $\zeta(z_n) \cap F \neq \emptyset$. To show that $\zeta(z_0) \cap F \neq \emptyset$, we need only show that $\zeta(z_0) \cap (F + B_{Z^*}(0, \epsilon)) \neq \emptyset$ for any $\epsilon > 0$. Observe, that since $(\beta'(z_0))^*$ is weak* continuous, the mapping $(\beta'(z_0))^* \partial\phi(\beta(z))$ is upper semicontinuous. Since β' is continuous at z_0, $\|\beta'(z_n) - \beta'(z)\| \to 0$ which means that for large n,

$$(\beta'(z_0))^* \partial\phi(\beta(z_n)) \cap (F + B_{Z^*}(0, \epsilon)) \neq \emptyset.$$

Since $\beta(z_n) \to \beta(z_0)$ and $(\beta'(z_0))^* \partial\phi(\beta(z))$ is upper semicontinuous, we have that

$$(\beta'(z_0))^* \partial\phi(\beta(z_0)) \cap (F + B_{Z^*}(0, \epsilon)) = \zeta(z_0) \cap (F + B_{Z^*}(0, \epsilon)) \neq \emptyset.$$

and, by compactness, $\zeta(z_0) \cap F \neq \emptyset$. Let $w \in H$ and $\epsilon > 0$. Since $\partial\phi(x)$ is locally bounded there exist $\rho > 0$ and $\eta > 0$ so that if $\|x - \beta(w)\| < \eta$ then $\partial\phi(x) \subseteq B_{X^*}(0, \rho)$. Let $v \in H$ such that $\|\beta(v) - \beta(w)\| < \eta$ and $\|\beta'(w) - \beta'(v)\| < \epsilon/4\rho$. Then,

$$(\beta'(v))^* \partial\phi(\beta(v)) \subseteq (\beta'(w))^* B_{X^*}(0, \rho) + B_{Z^*}(0, \epsilon)$$

and since $(\beta'(w))^* B_{X^*}(0,\rho)$ is a weak* compact set with the Radon-Nikodým property, the hypothesis of [S5] *loc. cit.* is satisfied; there exists a dense G_δ subset G of H (hence, G is a dense G_δ subset W) and $\lambda : G \to Z^*$ as required.

The case that β is an operator and X or Z is an Asplund space appears in [S2], for arbitrary β and X an Asplund space in [S3] and [S4], and another nonlinear version appears in [MS]. It is rather surprising that it is true in the following generality.

Theorem 8. *Suppose that X is a Banach space, W is an open subset of the Banach space Z, and $\beta : W \to Z$ is a continuous function that is Fréchet differentiable at every point. The set*

$$\{w \in W : \beta'(w) \text{ is factorizable}\}$$

is a dense subset of W if and only if for any continuous and convex function $\phi : X \to \Re$ the composed function $\phi\beta$ is Fréchet differentiable on a dense G_δ subset of W.

Proof. The case that $\{w \in W : \beta'(w) \text{ is factorizable}\}$ is not dense is essentially treated above. Choose w where β' is continuous at w and $\beta'(u)$ is not factorizable for u in some neighborhood of w. Proceed as above. Suppose that $\{w \in W : \beta'(w) \text{ is factorizable}\}$ is dense. Let $\lambda : G \to Z^*$ where λ and G are defined as in the proposition above. Choose arbitrarily $\psi(z) \in \partial\phi\beta(z)$ so that $(\beta'(z))^*(\psi(z)) = \lambda(z)$. Fix any $w_0 \in G$ and any sequence $\{w_n\} \subseteq G$ such that $w_n \to w_0$. We need to show that

$$\|(\beta'(w_0))^* \left(\psi(w_n) - \psi(w_0)\right)\| \to 0.$$

We may assume that $\{\partial\phi(\beta(w_n)) : n \geq 0\}$ is a bounded subset of Z^*. We have the following inequalities

$$\begin{aligned}
&\|(\beta'(w_0))^* \left(\psi(w_n) - \psi(w_0)\right)\| \\
&\qquad \leq \|\left(\beta'(w_0) - \beta'(w_n)\right)^* \left(\psi(w_n) - \psi(w_0)\right)\| + \|(\beta'(w_n))^* \left(\psi(w_n) - \psi(w_0)\right)\| \\
&\qquad \leq \|\left(\beta'(w_0) - \beta'(w_n)\right)^* \left(\psi(w_n) - \psi(w_0)\right)\| \\
&\qquad\qquad + \|\lambda(w_n) - \lambda(w_0)\| + \|\lambda(w_0) - (\beta'(w_n))^*(\psi(w_0))\|.
\end{aligned} \tag{17}$$

We know that $\lambda(w_n) \to \lambda(w_0)$,

$$\lambda(w_0) - (\beta'(w_n))^*(\psi(w_0)) = \left(\beta'(w_0) - \beta'(w_n)\right)^* (\psi(w_0)) \text{ and}$$

$(\beta'(w_0) - \beta'(w_n))^*$ converges uniformly to the origin on the bounded set

$$\{\psi(w_0), \psi(w_1), \ldots\}.$$

By the proposition above, $\phi\beta$ is Fréchet differentiable at each point of G.

Corollary 9. *With the same hypothesis as above on W, Z and β let $\phi : X \to \Re$ be a continuous and convex function. Suppose that $U \subseteq W$ is an open set and $\phi\beta$ is not differentiable at any point of U. Then there exists an open set $V \subseteq U$ and a weak* compact $K \subseteq X^*$ such that $\phi_K\beta$ is not differentiable at any point of V.*

Proof. ¿From the theorem above,

$$F = \{z \in U : \beta'(z) \text{ factorizable}\} \tag{18}$$

cannot be dense in U. Choose $w \in U \setminus \overline{F}$ so that β' is continuous at w. Choose a weak* compact $K \subseteq B_{X^*}(0,\rho)$ and $\epsilon > 0$ corresponding to $\beta'(w)$. Choose $\delta > 0$ so that if $\|v - w\| < \delta$ then $\|\beta'(v) - \beta'(w)\| < \epsilon/4(\rho+1)$. Then, as in the arguments above, $\phi_K \beta$ is not differentiable at any point of $B_Z(w, \delta)$.

Corollary 10. *With the same hypothesis as above on W, Z and β let $\phi : X \to \Re$ be a continuous and convex function. Assume that $X = C(K_0)$, the Banach space of continuous functions on the compact Hausdorff space K_0. Suppose that $U \subseteq W$ is an open set and $\phi\beta$ is not differentiable at any point of U. Then there exists an open set $V \subseteq U$ and a compact $K \subseteq K_0$ such that $\phi_K \beta$ is not differentiable at any point of V.*

Proof. A linear operator $T : Z \to C(K_0)$ is not factorizable if and only we may find $K \subseteq K_0$ and $\epsilon > 0$ so that the norm variation of T^* over any open subset of K is greater than ϵ; this is proved in [S2]. We may now follow the corollary above.

A consequence of the Bishop–Phelps theorem is that if X has a Frechét differentiable norm, then X is an Asplund space. More generally, if $\phi(x) = \|x\|$, and there is a selection $\lambda(x) \in \partial\phi(x)$ that is in some (countable) Baire class (in the norm topology) then X is an Asplund space. A separable Banach space X has an equivalent norm so that the dual norm is strictly convex, hence this norm on X is Gâteaux differentiable everywhere (except the origin). See [D] for an introduction to renorming theorems. Thus, there cannot be an analogous result for Gâteaux smooth functions, even for norms.

Preiss [P2] has shown that a Lipschitz function that is Gâteaux differentiable everywhere on an Asplund space is Fréchet differentiable on a dense set and he has an example [P1] of a continuous function on the separable Hilbert space that is Gâteaux differentiable everywhere but the set of points of Fréchet differentiability is of the first category.

If a function is Gâteaux differentiable on a neighborhood of a point, and the Gâteaux derivative is continuous at that point, then the function is Fréchet differentiable at that point (an easy consequence of the "vector valued mean value theorem" [Di]). If a continuous function is Gâteaux differentiable everywhere then it is locally Lipschitz on a dense open set [BS]; in fact, the function need only be measurable with respect to the Baire property sets (inverse images of Borel sets have the Baire property) [Ska]. Without measurability, the function can be even discontinuous everywhere. Shkarin shows in [Ska] that on every weakly compactly generated Banach space X (in particular, on any separable Banach space) there is an everywhere Gâteaux differentiable function β which is continuous at no point of X. He does not know if this is the case for an arbitrary Banach space.

Bogachev and Shkarin [BS] show that if a function is Gâteaux differentiable everywhere and Fréchet differentiable on a dense G_δ set then the derivative is continuous on a dense (necessarily G_δ) set; the converse follows from the vector mean value theorem. In particular, this implies that if a continuous function is Fréchet differentiable everywhere then its derivative is continuous at each point of a dense (necessarily G_δ) set. Our argument above shows directly that the derivative (of a continuous function that is Fréchet differentiable everywhere) is in the first Baire class, a stronger statement than that it has many points of continuity.

It should also be mentioned, in connection with the classical theory [Br], the result of Malý [Ma] that the Frechet derivative has the Darboux property.

Let F be a bounded subset of X and $x \in X$. We say that ϕ is uniformly Gâteaux differentiable in directions F if there exists a $u^* \in X^*$ such that

$$\lim_{t\to 0} \sup_{y\in F} \left| \frac{\phi(x+ty)-\phi(x)}{t} - u^*(y) \right| = 0.$$

Lemma 11. *Let $T : Z \to X$ be an operator, $\phi : X \to \Re$ a continuous and convex function and $x \in X$. Then ϕ is uniformly Gâteaux differentiable in directions $\{T(z) : \|z\| \leq 1\}$ at x if and only if there exists a function $\psi : Z \to X^*$ such that $\psi(z) \in \partial\phi(x+T(z))$ for all z and $T^*\psi$ is norm to norm continuous at 0.*

Proof. Suppose that ϕ is uniformly Gâteaux differentiable in directions $F = \{T(z) : \|z\| \leq 1\}$ and let u^* be the derivative. Make any selection $\psi(z) \in \partial\phi(x+T(z))$. We shall show that $T^*\psi(0) = T^*(u^*)$ and $T^*\psi$ is norm to norm continuous at 0. Let $t < 0$, then

$$\frac{\psi(tz)(-tTz)}{|t|} - u^*(Tz) \leq \frac{-\left(\phi(x+tTz)-\phi(x)\right)}{|t|} - u^*(Tz) \to 0$$

we have that $\|T^*\psi(tz) - T^*\psi(0)\| \to 0$. This convergence is uniform in $\{z : \|z\| \leq 1\}$ as $t \to 0^-$ but since $\{T(z) : \|z\| \leq 1\}$ is symmetric we have convergence as $t \to 0$. Also, for $t > 0$,

$$\left(\psi(0) - u^*\right)(Tz) \leq \frac{\phi(x+tTz)-\phi(x)}{t} - u^*(Tz)$$

which proves that $T^*\psi(0) = T^*(u^*)$. Conversely, suppose that we have a selection $\psi : Z \to X^*$ such that $\psi(z) \in \partial\phi(x+T(z))$ for all $z \in Z$ and $T^*\psi$ is norm to norm continuous at 0. Then, for $t > 0$,

(19)

$$0 \leq \frac{\phi(x+tTz)-\phi(x)}{t} - \psi(0)(Tz) \leq \left(\psi(tz)-\psi(0)\right)(Tz) \leq \|T^*\psi(tz) - T^*\psi(0)\|.$$

Theorem 12. *A bounded subset E of a Banach space X is equimeasurable if and only if: given any continuous and convex function $\phi : X \to \Re$ there exists at least one point in X where ϕ is uniformly Gâteaux differentiable in directions E (equivalently, there exists a dense G_δ subset of such points).*

Proof. Let E be a bounded subset of X and let $T : \ell_1(E) \to X$ be the canonical operator. The E is equimeasurable if and only if T is factorizable [S2]. The result follows from the elementary discussions at the beginning.

There exist bounded subsets of Banach spaces that have the property that any sequence in the set has a pointwise Cauchy sequence ("weakly precompact sets", by which is meant that for any sequence $\{x_n\}$ in the set there exist $y^{**} \in X^{**}$ and a sequence $\{y_m\} \subseteq \{x_n\}$ such that $\lim_m x^*(y_m) = x^{**}(x^*)$ for all $x^* \in X^*$) but need not be equimeasurable; the canonical example is the Haar system in the continuous functions on the Cantor space. It is quite easy to see that equimeasurable sets are weakly precompact (see [S2]). A pointwise Cauchy sequence is equimeasurable. Suppose that $F \subseteq X$ is bounded and weakly precompact; then any sequence $\{x_n\} \subseteq F$ has a pointwise Cauchy subsequence, say $\{y_m\}$, which is equimeasurable. Thus, for any continuous and convex $\phi : X \to \Re$, ϕ is uniformly Gâteaux differentiable in the directions $\{y_m\}$ at each point of a dense G_δ subset of X. Our results are considerable generalizations of the results of the type in, for example, [M] and [BL] and indicate that the proper setting for such results is not in the context of weakly precompact sets but that of equimeasurable sets.

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