

## A Geometrical Version of Hardy's Inequality

**M. Hoffmann–Ostenhof**  
**T. Hoffmann–Ostenhof**  
**A. Laptev**

Vienna, Preprint ESI 1017 (2001)

March 30, 2001

Supported by Federal Ministry of Science Education and Culture, Austria  
Available via <http://www.esi.ac.at>

# A GEOMETRICAL VERSION OF HARDY'S INEQUALITY

M. HOFFMANN-OSTENHOF<sup>1</sup>, T. HOFFMANN-OSTENHOF<sup>2,3</sup> AND A.LAPTEV<sup>4</sup>

ABSTRACT. We proof a version of Hardy's type inequality in a domain  $\Omega \subset \mathbb{R}^n$  which involves the distance to the boundary and the volume of  $\Omega$ . In partucular, we obtain a result which gives a positive answer to a question asked by H.Brezis and M.Marcus.

## 0. INTRODUCTION

Let  $\Omega$  be a domain in  $\mathbb{R}^n$  with Lipschitz boundary. It is known that the following extension of Hardy's inequality is valid

$$(0.1) \quad \int_{\Omega} |\nabla u(x)|^2 dx \geq \mu \int_{\Omega} \frac{|u(x)|^2}{\delta^2(x)} dx, \quad \forall u \in H_0^1(\Omega),$$

where  $\mu$  is a positive constant and  $\delta(x) = \text{dist}(x, \partial\Omega)$ . The best constant  $\mu = \mu(\Omega)$  in (0.1) depends on the domain  $\Omega$ . It is also known that for convex domains  $\mu(\Omega) = 1/4$ , but there are smooth domains such that  $\mu(\Omega) < 1/4$  (see[5], [6]).

H.Brezis and M.Marcus [2], Theorem I, have shown that for every domain  $\Omega$  of class  $C^2$  there exists a constant  $\lambda = \lambda(\Omega) \in \mathbb{R}$  such that

$$(0.2) \quad \int_{\Omega} |\nabla u(x)|^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{|u(x)|^2}{\delta^2(x)} dx + \lambda \int_{\Omega} |u(x)|^2 dx, \\ \forall u \in H_0^1(\Omega).$$

Note that there are examples ([5], [6]) which confirm that there are smooth domains with  $\lambda \leq 0$ . However, if  $\Omega$  is convex then it is proved in [2] (see Theorem II) that

$$(0.3) \quad \lambda(\Omega) \geq \frac{1}{4 \text{diam}^2(\Omega)}.$$

In this paper Brezis and Marcus have asked whether the diameter of  $\Omega$ , in (0.3) can be replaced by an expression depending on  $|\Omega| := \text{vol } \Omega$ , namely, whether  $\lambda \geq c |\Omega|^{-2/n}$  with some  $c = c(n) > 0$ .

---

1991 *Mathematics Subject Classification*. Primary 35P99; Secondary 35P20, 47A75, 47B25.

The aim of this short article is to prove that this is the case indeed and that for convex domains

$$(0.4) \quad \lambda \geq \frac{c(n)}{|\Omega|^{2/n}}, \quad c(n) = \frac{n^{(n-2)/n} s_{n-1}^{2/n}}{4},$$

where  $s_{n-1} := |\mathbb{S}^{n-1}|$ . In particular, if  $n = 2$  then  $c(2) = \pi/2$ .

The proof of this result is based on a one-dimensional version of Hardy's inequality which is obtained in Section 1. In Section 2 we extend the one-dimensional result to the many-dimensional case using arguments of E.B.Davies [3], Ch.5.3. In Section 3 we prove (0.4) and consider some other generalizations of this result.

## 1. ONE DIMENSIONAL RESULTS

We start with a simple statement which is just a corollary of the Cauchy-Schwarz inequality and partial integration. Let  $f$  be a function defined on  $(0, b)$ ,  $b > 0$ , and whose derivative is finite on  $(0, b)$ . We say that  $f$  belongs to the class  $\Phi(0, b)$  if  $f$  is real valued and there is a constant  $C = C(f)$  such that

$$(1.1) \quad \sup_{0 < t \leq b} (t|f(t)| + t^2|f'(t)|) \leq C.$$

**Lemma 1.1.** *Let  $u \in C^1(0, b)$ ,  $b > 0$ ,  $u(0) = 0$  and let  $f \in \Phi(0, b)$ . Then*

$$(1.2) \quad \int_0^b \left| \frac{du}{dt} \right|^2 dt \geq \frac{1}{4} \frac{(\int_0^b f'(t)|u|^2 dt)^2}{\int_0^b (f(t) - f(b))^2 |u|^2 dt}.$$

**Proof.** For any constant  $c$  taking into account (1.1) we have

$$\begin{aligned} \left( (f(b) - c)|u(b)|^2 - \int_0^b f'(t)|u|^2 dt \right)^2 &= \left( \int_0^b (f(t) - c)(|u|^2)' dt \right)^2 \\ &= \left( \int_0^b (f(t) - c)(u'\bar{u} + u\bar{u}') dt \right)^2 \\ &\leq 4 \left( \int_0^b |u'|^2 dt \right) \left( \int_0^b (f(t) - c)^2 |u|^2 dt \right). \end{aligned}$$

We complete the proof by substituting  $c = f(b)$ .

The next result shows that (1.2) is often sharp unlikely many other Hardy's type inequalities.

**Lemma 1.2.** *Let us assume that  $f \in \Phi(0, b)$  and*

- $\int_t^b f(s)ds \rightarrow +\infty$ , as  $t \rightarrow 0$ .
- *There is a constant  $a$ ,  $a > 0$  such that*

$$f(t)e^{-a \int_t^b f(s)ds} \in L^2(0, b).$$

*Then Lemma 1.1 is sharp.*

**Proof.** The Cauchy-Schwarz inequality which has been used in the proof of Lemma 1.1 is sharp if there exists a real valued function  $u$  such that  $u'(t)$  and  $u(t)(f(t) - f(b))$  are linearly dependent

$$(1.3) \quad u'(t) = a u(t)(f(t) - f(b)).$$

Solving (1.3) we obtain

$$u(t) = C e^{-a \int_t^b f(s)ds - a f(b)t}.$$

If now  $a > 0$ , then the first assumption implies  $u(0) = 0$ . The second one provides the inclusion  $du/dt \in L^2(0, b)$ .

**Example.**

If  $f(t) = 1/t$  and  $a > 1/2$ , then substituting in (1.2)

$$u(t) = t^a e^{-at/b}$$

we find that the left and the right hand sides of this inequality are the same. Moreover,  $u(0) = 0$ ,  $u \in C^1(0, b)$  and therefore the inequality (1.1) has an extremizer.

Although in many cases Lemma 1.1 gives sharp results, the right hand side in the inequality (1.2) is not linear with respect to  $u$ . We would now like to give the following linearized version of this inequality.

By using (1.2) we obviously have

$$\begin{aligned} & \int_0^b |u'|^2 dt \\ & \geq \frac{1}{4} \frac{(\int_0^b f'(t) |u|^2 dt)^2}{\left( \int_0^b f'(t) |u|^2 dt - \int_0^b (f'(t) - f^2(t) + 2f(t)f(b) - f^2(b)) |u|^2 dt \right)} \\ (1.4) \quad & \geq \frac{1}{4} \int_0^b \left( 2f'(t) - f^2(t) + 2f(t)f(b) - f^2(b) \right) |u|^2 dt. \end{aligned}$$

If we rewrite this inequality for the interval  $[b, 2b]$  with  $u \in C^1(b, 2b)$ ,  $u(2b) = 0$ , then

$$(1.5) \quad \int_b^{2b} |u'|^2 dt \geq \frac{1}{4} \int_b^{2b} \left( 2f'(2b-t) - f^2(2b-t) + 2f(2b-t)f(b) - f^2(b) \right) |u|^2 dx.$$

Adding up (1.4) and (1.5) and by using standard density arguments we can finally state our main one-dimensional result.

**Lemma 1.3.** *Let  $u \in H_0^1(0, 2b)$ ,  $b > 0$  and let  $f \in \Phi(0, b)$ . Then*

$$(1.6) \quad \int_0^{2b} |u'(t)|^2 dt \geq \frac{1}{4} \int_0^{2b} \left( 2f'(\rho(t)) - f^2(\rho(t)) + 2f(\rho(t))f(b) - f^2(b) \right) |u(t)|^2 dt,$$

where

$$\rho(t) = \min(t, 2b - t).$$

## 2. A RESULT FOR HIGHER DIMENSIONS

Let  $\Omega$  be a domain in  $\mathbb{R}^n$ . In order to formulate the main result of this section we need some notations. Denote by  $\tau_\nu(x)$  the distance between  $x \in \Omega$  and its nearest point belonging to the boundary  $\partial\Omega$  in the direction  $\nu \in \mathbb{S}^{n-1}$ ,

$$(2.1) \quad \tau_\nu(x) = \min\{s > 0 : x + s\nu \notin \Omega\}.$$

Let us also introduce the “distance” to the boundary  $\rho_\nu$  and the “diameter”  $D_\nu$  along the line defined by  $\nu$  via:

$$(2.2) \quad \rho_\nu(x) = \min(\tau_\nu(x), \tau_{-\nu}(x))$$

$$(2.3) \quad D_\nu(x) = \tau_\nu(x) + \tau_{-\nu}(x).$$

By  $d\omega(\nu)$  we denote the normalized measure on the unit sphere  $\mathbb{S}^{n-1}$ ,  $\int_{\mathbb{S}^{n-1}} d\omega(\nu) = 1$ .

**Theorem 2.1.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$ ,  $D \in (0, \infty]$  be its diameter and  $f \in \Phi(0, D/2)$ . Then for any  $u \in H_0^1(\Omega)$  we have*

$$(2.4) \quad \int_\Omega |\nabla u|^2 dx \geq \frac{n}{4} \int_\Omega \left( \int_{\mathbb{S}^{n-1}} (2f'(\rho_\nu(x)) - f^2(\rho_\nu(x)) + 2f(\rho_\nu(x))f(D_\nu(x)/2) - f^2(D_\nu(x)/2)) d\omega(\nu) \right) |u(x)|^2 dx,$$

**Proof.** We proceed by using E.B.Davies' arguments (see [3]). Let  $\partial_\nu$  denote partial differentiation in the direction  $\nu \in \mathbb{S}^{n-1}$ . Then Lemma 1.3 implies

$$\int_{\Omega} |\partial_\nu u|^2 dx \geq \frac{1}{4} \int_{\Omega} \left( 2f'(\rho_\nu(x)) - f^2(\rho_\nu(x)) + 2f(\rho_\nu(x))f(D_\nu(x)/2) - f^2(D_\nu(x)/2) \right) |u|^2 dx,$$

where the function  $\rho_\nu$  and  $D_\nu$  are defined in (2.2) and (2.3). Let us introduce an orthonormal basis  $\{\bar{e}_j\}_{j=1}^n$  in  $\mathbb{R}^n$ . Then

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{4} \sum_{j=1}^n \int_{\Omega} \left( 2f'(\rho_{\bar{e}_j}(x)) - f^2(\rho_{\bar{e}_j}(x)) + 2f(\rho_{\bar{e}_j}(x))f(D_{\bar{e}_j}/2) - f^2(D_{\bar{e}_j}/2) \right) |u|^2 dx.$$

Averaging both sides of the last inequality over orthonormal bases using the group  $O(n)$  we complete the proof.

### 3. APPLICATIONS OF THEOREM 2.1

**3.1. On a question of Brezis and Marcus.** Let

$$(3.1) \quad f(t) = -1/t, \quad t > 0.$$

Then the integral over  $\mathbb{S}^{n-1}$  in the right hand side of (2.4) becomes equal to

$$(3.2) \quad \begin{aligned} & \int_{\mathbb{S}^{n-1}} \left( 2f'(\rho_\nu) - f^2(\rho_\nu) + 2f(\rho_\nu)f(D_\nu/2) - f^2(D_\nu/2) \right) d\omega(\nu) \\ &= \int_{\mathbb{S}^{n-1}} \left( \frac{1}{\rho_\nu^2} + \frac{4}{\rho_\nu D_\nu} - \frac{4}{D_\nu^2} \right) d\omega(\nu). \end{aligned}$$

Let us consider the last two terms. It is clear that  $\rho_\nu(x) \leq \tau_\nu(x)$ ,  $x \in \Omega$ , where the functions  $\rho$  and  $\tau$  are defined in (2.2) and (2.1). Since  $D_\nu = \tau_\nu + \tau_{-\nu}$  (see (2.3)) we obtain

$$\begin{aligned} \frac{1}{\rho_\nu D_\nu} - \frac{1}{D_\nu^2} &\geq \frac{1}{\tau_\nu(\tau_\nu + \tau_{-\nu})} - \frac{1}{(\tau_\nu + \tau_{-\nu})^2} \\ &= \frac{\tau_{-\nu}}{\tau_\nu(\tau_\nu + \tau_{-\nu})^2}. \end{aligned}$$

This implies

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \left( \frac{1}{\rho_\nu(x) D_\nu(x)} - \frac{1}{D_\nu^2(x)} \right) d\omega(\nu) &\geq \int_{\mathbb{S}^{n-1}} \frac{\tau_{-\nu}}{\tau_\nu(\tau_\nu + \tau_{-\nu})^2} d\omega(\nu) \\ &= \frac{1}{2} \int_{\mathbb{S}^{n-1}} \left( \frac{\tau_{-\nu}}{\tau_\nu(\tau_\nu + \tau_{-\nu})^2} + \frac{\tau_\nu}{\tau_{-\nu}(\tau_\nu + \tau_{-\nu})^2} \right) d\omega(\nu) \\ &\geq \frac{1}{4} \int_{\mathbb{S}^{n-1}} \frac{1}{\tau_\nu \tau_{-\nu}} d\omega(\nu). \end{aligned}$$

In order to estimate the latter integral we apply the Cauchy-Schwarz inequality twice and obtain

$$\begin{aligned} 1 &\leq \int_{\mathbb{S}^{n-1}} \tau_\nu \tau_{-\nu} d\omega(\nu) \int_{\mathbb{S}^{n-1}} \frac{1}{\tau_\nu \tau_{-\nu}} d\omega(\nu) \\ &\leq \int_{\mathbb{S}^{n-1}} \tau_\nu^2 d\omega(\nu) \int_{\mathbb{S}^{n-1}} \frac{1}{\tau_\nu \tau_{-\nu}} d\omega(\nu), \end{aligned}$$

where we have used that  $\int_{\mathbb{S}^{n-1}} \tau_\nu^2 d\omega(\nu) = \int_{\mathbb{S}^{n-1}} \tau_{-\nu}^2 d\omega(\nu)$ .

When now applying Hölder's inequality we recall that  $\int_{\mathbb{S}^{n-1}} d\omega(\nu) = 1$ . Therefore

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \frac{1}{\tau_\nu(x) \tau_{-\nu}(x)} d\omega(\nu) &\geq \left( \int_{\mathbb{S}^{n-1}} \tau_\nu^2(x) d\omega(\nu) \right)^{-1} \\ &\geq \left( \int_{\mathbb{S}^{n-1}} \tau_\nu^n(x) d\omega(\nu) \right)^{-2/n}. \end{aligned}$$

Let us introduce the domain  $\Omega_x \subset \Omega$  defined as a part of  $\Omega$  which can be “seen” from point  $x$

$$(3.3) \quad \Omega_x := \{y \in \Omega : x + t(x - y)|x - y|^{-1} \in \Omega, \forall t \in [0, 1]\}.$$

Then

$$\int_{\mathbb{S}^{n-1}} \tau_\nu^n(x) d\omega(\nu) = \frac{n}{s_{n-1}} |\Omega_x|,$$

which finally gives us

$$(3.4) \quad \int_{\mathbb{S}^{n-1}} \left( \frac{4}{\rho_\nu(x) D_\nu(x)} - \frac{4}{D_\nu^2(x)} \right) d\omega(\nu) \geq \left( \frac{s_{n-1}}{n} \right)^{2/n} \frac{1}{|\Omega_x|^{2/n}}.$$

Now (2.4), (3.2) and (3.4) imply the following reformulation of Theorem 2.1 in the case when the function  $f$  is defined by (3.1).

**Theorem 3.1.** *For any  $\Omega \subset \mathbb{R}^n$  and any  $u \in H_0^1(\Omega)$  we have*

$$(3.5) \quad \begin{aligned} \int_{\Omega} |\nabla u|^2 dx &\geq \frac{n}{4} \int_{\Omega} \int_{\mathbb{S}^{n-1}} \frac{1}{\rho_\nu^2(x)} d\omega(\nu) |u(x)|^2 dx \\ &\quad + \frac{n^{(n-2)/n} s_{n-1}^{2/n}}{4} \int_{\Omega} \frac{|u(x)|^2}{|\Omega_x|^{2/n}} dx. \end{aligned}$$

Clearly in (3.5) the value  $|\Omega_x|$  can always be replaced by  $|\Omega|$ . If  $\Omega$  is convex then it is known (see for example [3] Exercise 5.7 and [4]) that

$$(3.6) \quad \frac{n}{4} \int_{\mathbb{S}^{n-1}} \rho_\nu^{-2}(x) d\omega(\nu) \geq \frac{1}{4} \frac{1}{\delta^2(x)},$$

Moreover, in this case  $\Omega_x = \Omega$ ,  $x \in \Omega$ , and we obtain

**Theorem 3.2.** *For any convex domain  $\Omega \subset \mathbb{R}^n$  and any  $u \in H_0^1(\Omega)$*

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{|u(x)|^2}{\delta^2(x)} dx + \frac{n^{(n-2)/n} s_{n-1}^{2/n}}{4 |\Omega|^{2/n}} \int_{\Omega} |u(x)|^2 dx.$$

Note that the domain  $\Omega$  in Theorem 3.1 can be unbounded and it is valid for a variety of domains with fractal boundaries, for example, such as the Koch snowflake in  $\mathbb{R}^2$ . So often the inequality (3.6) might hold true with a constant  $\mu < 1/4$  instead of  $1/4$  in the right hand side, whereas the second integral of the inequality (3.5) is very stable.

In particular, following E.B.Davies, Lemma 3 from [4] we can obtain:

**Corollary 3.1.** *Suppose that there is a constant  $\kappa$  such that for each  $y \in \partial\Omega$  and each  $a > 0$  there exists a disjoint from  $\Omega$  ball  $B$  with centre  $z$  and radius  $\beta \geq a\kappa$ , where  $|z - y| = a$ . Then there exists a constant  $\mu \leq 1/4$  such that*

$$\frac{n}{4} \int_{\mathbb{S}^{n-1}} \rho_\nu^{-2}(x) d\omega(\nu) \geq \mu \frac{1}{\delta^2(x)}$$

and hence

$$\int_{\Omega} |\nabla u|^2 dx \geq \mu \int_{\Omega} \frac{|u(x)|^2}{\delta^2(x)} dx + \frac{n^{(n-2)/n} s_{n-1}^{2/n}}{4} \int_{\Omega} \frac{|u(x)|^2}{|\Omega_x|^{2/n}} dx.$$

**3.2. Some refined inequalities.** The next application of Theorem 2.1 concerns the function

$$(3.7) \quad f(t) = -\frac{1}{t} + \frac{1}{t(1 - \ln(\alpha t/D))}, \quad 0 < t < D/2,$$

where  $D = \text{diam } \Omega$  and  $0 < \alpha \leq 2$ . In this case the expression appearing in the right hand side of (2.4) is equal to

$$\begin{aligned} & 2f'(\rho_\nu) - f^2(\rho_\nu) + 2f(\rho_\nu)f(D_\nu/2) - f^2(D_\nu/2) \\ &= \frac{1}{\rho_\nu^2} + \frac{1}{\rho_\nu^2 (1 - \ln(\alpha \rho_\nu/D))^2} \\ &+ 4 \left( \frac{\ln(\alpha \rho_\nu/D) \ln(\alpha D_\nu/2D)}{\rho_\nu D_\nu (1 - \ln(\alpha \rho_\nu/D))(1 - \ln(\alpha D_\nu/2D))} - \frac{\ln^2(\alpha D_\nu/2D)}{D_\nu^2 (1 - \ln(\alpha D_\nu/2D))^2} \right) \\ &\geq \frac{1}{\rho_\nu^2} + \frac{1}{\rho_\nu^2 (1 - \ln(\alpha \rho_\nu/D))^2} + 4 \left( \frac{1}{\rho_\nu D_\nu} - \frac{1}{D_\nu^2} \right) \frac{\ln^2(\alpha/2)}{(1 - \ln(\alpha/2))^2}. \end{aligned}$$



Theorem 2.1 therefore gives

$$\begin{aligned} & \int_{\Omega} |\nabla u|^2 dx \\ & \geq \frac{n}{4} \int_{\Omega} \int_{\mathbb{S}^{n-1}} \left( \frac{1}{\rho_{\nu}^2(x)} + \frac{1}{\rho_{\nu}^2 (1 - \ln(\alpha \rho_{\nu}/D))^2} \right) d\omega(\nu) |u(x)|^2 dx \\ & \quad + \frac{n \ln^2(\alpha/2)}{(1 - \ln(\alpha/2))^2} \int_{\Omega} \int_{\mathbb{S}^{n-1}} \left( \frac{1}{\rho_{\nu} D_{\nu}} - \frac{1}{D_{\nu}^2} \right) d\omega(\nu) |u(x)|^2 dx. \end{aligned}$$

Application of (3.4) leads to a more refined version of Theorem 3.1.

**Theorem 3.3.** *Let  $0 < \alpha \leq 2$ . Then for any  $\Omega \subset \mathbb{R}^n$  and any  $u \in H_0^1(\Omega)$  we have*

$$\begin{aligned} & \int_{\Omega} |\nabla u|^2 dx \\ & \geq \frac{n}{4} \int_{\Omega} \int_{\mathbb{S}^{n-1}} \left( \frac{1}{\rho_{\nu}^2(x)} + \frac{1}{\rho_{\nu}^2 (1 - \ln(\alpha \rho_{\nu}/D))^2} \right) d\omega(\nu) |u(x)|^2 dx \\ & \quad + \frac{n^{(n-2)/n} s_{n-1}^{2/n} \ln^2(\alpha/2)}{4(1 - \ln(\alpha/2))^2} \int_{\Omega} \frac{|u(x)|^2}{|\Omega_x|^{2/n}} dx. \end{aligned}$$

**Remark.** Theorem 3.3 is a stronger result than Theorem 3.1. Indeed, we obtain Theorem 3.1 from Theorem 3.3 if we let  $\alpha \rightarrow 0$ .

For  $\Omega$  convex we obtain via (3.6) that

$$\begin{aligned} & \frac{n}{4} \int_{\mathbb{S}^{n-1}} \frac{1}{\rho_{\nu}^2(x)} \left( 1 + \frac{1}{(1 - \ln(\alpha \rho_{\nu}(x)/D))^2} \right) d\omega(\nu) \\ & \geq \frac{1}{4} \frac{1}{\delta^2(x)} \left( 1 + \frac{1}{(1 - \ln(\alpha \delta(x)/D))^2} \right). \end{aligned}$$

The latter inequality and Theorem 3.3 implies a version of Theorem 3.2:

**Theorem 3.4.** *Let  $0 < \alpha \leq 2$ . For any convex domain  $\Omega \subset \mathbb{R}^n$  and any  $u \in H_0^1(\Omega)$*

$$\begin{aligned} & \int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{|u(x)|^2}{\delta^2(x)} \left( 1 + \frac{1}{(1 - \ln(\alpha \delta(x)/D))^2} \right) dx \\ (3.8) \quad & \quad + \frac{n^{(n-2)/n} s_{n-1}^{2/n} \ln^2(\alpha/2)}{4(1 - \ln(\alpha/2))^2} \frac{1}{|\Omega|^{2/n}} \int_{\Omega} |u(x)|^2 dx. \end{aligned}$$

**Remark.** The last statement is an improvement of Theorem 5.1 from [2], where for convex domains  $\Omega$  and  $u \in H_0^1(\Omega)$  the authors obtain the inequality

$$(3.9) \quad \int_{\Omega} |\nabla u(x)|^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{|u(x)|^2}{\delta^2(x)} \left( 1 + \frac{1}{(1 - \ln(\delta(x)/D))^2} \right) dx.$$

Indeed, if we choose  $\alpha = 1$  in Theorem 3.4, then the first integral in the right hand side of (3.8) coincides with the right hand side of (3.9). However,  $\alpha = 1$  still allows to have an additional non-zero term in (3.8).

**3.3. Acknowledgements.** The first and the second authors have been supported by BBWK der Republik Österreich and by the Austrian Science Foundation, grantnumber P12864-MAT. The third author has been supported by the Swedish Natural Sciences Research Council, Grant M-AA/MA 09364-320. Th.Hoffmann-Ostenhof thanks the Göran Gustafssons Stiftelse in Sweden. A.Laptev is grateful to the Erwin Schrödinger Institute for its warm hospitality. Partial financial support from the European Union through the TMR network FMRX-CT 96-0001 is gratefully acknowledged.

#### REFERENCES

- [1] H.Brezis and M. Marcus, Extremal functions for Hardy's inequality with weight. *J. Funct. Anal.* **171**, no 1, (2000), 177–191.
- [2] H.Brezis and M. Marcus, Hardy's inequalities revisited. Dedicated to Ennio De Giorgi. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **25** (1997), no. 1-2 (1998), 217–237.
- [3] E.B.Davies, *Spectral theory and differential operators*. Cambridge Studies in Advanced Mathematics, 42. Cambridge University Press, Cambridge, 1995. x+182 pp.
- [4] E.B.Davies, A review of Hardy inequalities. The Maz'ya anniversary collection, Vol. 2, *Oper. Theory Adv. Appl.* Birkhäuser, Basel, **110** (1999) 55-67.
- [5] M.Markus, V.J.Mizel and Y.Pinchover, On the best constant for Hardy's inequality in  $\mathbb{R}^n$ , *Trans. Amer. Math. Soc.*, **350** (1998) 3237–325. **28** (1997), 1601–1610
- [6] T.Matskewich and P.E.Sobolevskii, The best possible constant in a generalized Hardy's inequality for convex domains in  $\mathbb{R}^n$ ., *Nonlinear Anal.* **28** (1997), 1601–1610.
- [7] V.G.Maz'ya, *Sobolev spaces*. Springer-Verlag, 1985.
- [8] B.Opic and A.Kufner, *Hardy-type inequalities*. Pitman Research Notes in Math., vol. 219, Longman 1990.

Institut für Mathematik, Universität  
Wien<sup>1</sup>  
Institut für Theoretische Chemie,  
Universität Wien<sup>2</sup>

International Erwin Schrödinger  
Institute for Mathematical Physics<sup>3</sup>  
Department of Mathematics, KTH, 100  
44 Stockholm, Sweden<sup>4</sup>