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Vienna, Preprint ESI 1029 (2001) May 14, 2001

Supported by Federal Ministry of Science Education and Culture, Austria
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S-matrix for s-wave gravitational scattering

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(May 15, 2001)

In the s-wave approximation the 4D Einstein gravity with (scalar) matter fields can be reduced to an effective 2D dilaton gravity coupled nonminimally to the matter fields. We study the leading order (tree level) vertices. The 4-particle matrix element is calculated explicitly. It is interpreted as scattering with formation of a virtual black hole state. As one novel feature we predict the gravitational decay of s-waves.

PACS numbers: 04.60.Kz; 04.60.Gw; 11.10.Lm; 11.80.Et

a. Introduction Einstein gravity with a massless Klein-Gordon field already at the classical level for spherical symmetry exhibits many interesting properties which are by far more accessible than the general $d = 4$ case, because corresponding processes are described by an effective two dimensional theory.

When the line element ($x := (x^0, x^1)$)

$$(ds)^2_{(4)} = g_{\alpha\beta}(x)dx^\alpha dx^\beta - X(x)d\Omega^2_{S^2} \tag{1}$$

only depends on the metric on the unit 2-sphere, $d\Omega^2_{S^2}$, the "dilaton field" X and a two dimensional metric $g_{\alpha\beta}$ with signature $(+,-)$, the Hilbert-Einstein action, supplemented by an action in which the scalar field S is coupled minimally (in $d = 4$) to gravity, becomes equivalent to a two dimensional dilaton theory [1]

$$L_{\text{dil}} = \frac{4\pi}{\kappa}\int d^2x\sqrt{-g}\left[XR + \frac{(\nabla X)^2}{2X} - 2 + \kappa X\,(\nabla S)^2\right]. \tag{2}$$

R denotes the $2d$ curvature, $(\nabla X)^2 = g^{\mu\nu}\partial_\mu X\partial_\nu X$, and $\kappa = 8\pi G_N$.

The action (2) is locally and globally equivalent to a first order one [2] depending on Cartan 1-forms $e^\pm$ (we denote light-cone indices with $\pm$) and ω (the abelian gauge structure of the two dimensional spin connection $\omega^a{}_b = \varepsilon^a{}_b\omega$ is used explicitly), the dilaton field X, the auxiliary fields $X^\pm$ and the s-wave Klein-Gordon field S:

$$L_{\text{FO}} = -\frac{8\pi}{\kappa}\int\big[X^+(d-\omega)\wedge e^- + X^-(d+\omega)\wedge e^+ + \\ +Xd\wedge\omega - e^-\wedge e^+\mathcal{V} - \frac{\kappa}{2}XdS\wedge *dS\big]. \tag{3}$$

Actually, this equivalence holds for general dilaton theories. In the present spherically reduced case the "potential" in (3) becomes $\mathcal{V} = -2 - \frac{X^+X^-}{2X}$. In the following we shall take $\kappa = 1$ and drop the overall factor. Only in the final result the full κ-dependence will be restored.

b. Path integral quantization of geometry Although in the present paper we consider only classical (tree level) processes, the path integral seems to be the most adequate language to derive the scattering amplitudes. As in other well-known examples (e.g. the Klein-Nishina formula for relativistic Compton scattering [3]) this formalism is much superior to a purely classical computation, because it directly focuses on the physical observable, the S-matrix element, which leads to an immediate interpretation. In a series of papers [4–6] it has been shown that the path integral quantization, developed from the action (3), allows the exact treatment of the geometric part for the choice of a temporal gauge of the Cartan variables

$$\omega_0 = 0, \quad e_0^- = 1, \quad e_0^+ = 0. \tag{4}$$

Previous work was restricted to minimally coupled scalars in (3), i.e. the dilaton factor X in front of the matter action was omitted.

The Hamiltonian analysis in terms of the remaining field variables and associated conjugate momenta

$$q_i = (\omega_1, e_1^-, e_1^+)\,, \qquad p_i = (X, X^+, X^-)\,, \tag{5}$$

together with the introduction of the path integral in phase space, suitably extended by ghosts works here as in [4–6]. Though, as a consequence of the dilaton factor X in (3), the structure functions of the constraint algebra acquire additional terms, but the nilpotent BRST charge also here resembles the one in Yang-Mills theories. For details of the Hamiltonian analysis and path integral quantization we refer to [7]. Having integrated ghost fields and other canonical variables, the effective action including sources for q_i, p_i and S differs only slightly from the one in [5,6]:

$$L = \int\Big[-\dot{p}_iq_i + q_1p_2 - q_3\mathcal{V} + \frac{p_1}{2}\left(\partial_1 S\partial_0 S - q_2(\partial_0 S)^2\right) \\ +j_iq_i + J_ip_i + QS\Big]. \tag{6}$$

The generating functional for the Green functions reads

$$Z\,[j,J,Q] = \int(\mathcal{D}q)\,(\mathcal{D}p)\,(\mathcal{D}S)\exp{(iL)}. \tag{7}$$

After the (exact) q- and p-integrals only the integration of scalars remains. Thus, the usual perturbation theory is restricted to the incorporation of matter fields. Separating terms of $\mathcal{O}\left(S^{2n}\right)$, $n > 2$, the Gaussian path integral of the terms up to $\mathcal{O}\left(S^2\right)$ yields a typical propagator contribution, apart from terms of $\mathcal{O}(\hbar)$, like a generalized Polyakov action and a contribution from the measure. As in ref. [5] we concentrate on the (highly nontrivial) vertex $\mathcal{O}\left(S^4\right)$ in the perturbation expansion. It allows the calculation of scattering of s-wave scalars. This vertex can be extracted formally from the final effective action. However, it contains complicated multiple integrals. Hence, we use again the simple short cut introduced in [5], the idea of which we will outline briefly: It is sufficient to assume the second order combinations of the scalar field to be localized at a single point*

$$S_0 := \frac{1}{2}\left(\partial_0 S\right)^2 = c_0 \delta(x-y), \tag{8}$$

$$S_1 := \frac{1}{2}\left(\partial_0 S\right)\left(\partial_1 S\right) = c_1 \delta(x-y), \tag{9}$$

and to solve the classical equations of motion (EOM) following from the gauge fixed action (6) up to linear order in the "sources" c_0 or c_1. Then the solutions have to be substituted back into the interaction terms in (6). Higher orders in c_0, c_1 would yield either loop contributions or vertices with at least 6 outer legs. We emphasize again that we are using perturbative methods in the matter sector only. Thus no a priori split into background- and fluctuation-metric occurs in our approach.

c. Classical EOM The solution of the classical EOM in the presence of matter from (6) with vanishing sources

$$\begin{aligned} \partial_0 p_1 &= p_2, & \partial_0 q_1 &= \frac{q_3 p_2 p_3}{2p_1^2} + S_1 - q_2 S_0, \\ \partial_0 p_2 &= p_1 S_0, & \partial_0 q_2 &= -q_1 - \frac{q_3 p_3}{2p_1}, \\ \partial_0 p_3 &= 2 + \frac{p_2 p_3}{2p_1}, & \partial_0 q_3 &= -\frac{q_3 p_2}{2p_1}, \end{aligned} \tag{10}$$

to linear order in c_0 and c_1 is found easily:

$$p_1(x) = x_0 + (x_0 - y_0) c_0 y_0 h(x,y), \tag{11}$$

$$p_2(x) = 1 + c_0 y_0 h(x,y), \tag{12}$$

$$q_2(x) = 4\sqrt{p_1} + \left(2c_0 y_0^{3/2} - c_1 y_0 + (c_1 - 6c_0 y_0^{1/2}) p_1\right) h(x,y), \tag{13}$$

$$q_3(x) = \frac{1}{\sqrt{p_1}}. \tag{14}$$

Here $h(x,y) := \theta(y_0 - x_0)\delta(x_1 - y_1)$, corresponds to one of the prescriptions introduced in [6] for the boundary values at $x_0 \to \infty$. It turns out that the vertices below are *independent* of *any* such choice. The matching conditions at $x_0 = y_0$ follow from continuity properties: p_1, q_2 and q_3 are C^0 and $\partial_0 q_2(y_0+0) - \partial_0 q_2(y_0 - 0) = (c_1 - q_2(y_0)c_0)\,\delta(x_1 - y_1)$. Integration constants which would produce an asymptotic (i.e. for $x_0 \to \infty$) Schwarzschild term and a Rindler term have been fixed to zero. Thus, a black hole may appear only at an intermediate stage (the "virtual black hole", see below), but should not act asymptotically. Due to the infinite range of gravity this is necessary for a proper S-matrix element, if we want to use as asymptotic states spherical waves for the incoming and outgoing scalar particles.

d. Line element The matter dependent solutions in our gauge (4) from (5), (13) and (14) define an effective line element

$$(ds)^2 = 2drdu + K(r,u)(du)^2, \tag{15}$$

if we identify† $u = 2\sqrt{2}x_1$ and $r = \sqrt{p_1(x_0)/2}$. It then appears in outgoing Sachs-Bondi form. The Killing norm

$$K(r,u)|_{x_0<y_0} = \left(1 - \frac{2m}{r} - ar\right)(1 + \mathcal{O}(c_0)), \tag{16}$$

with $m = \delta(x_1 - y_1)(-c_1 y_0 - 2c_0 y_0^{3/2})/2^{7/2}$ and $a = \delta(x_1 - y_1)(c_1 - 6c_0 y_0^{1/2})/2^{5/2}$, has two zeros located approximately at $r = 2m$ and $r = 1/a$ corresponding for positive m and a to a Schwarzschild horizon and a Rindler type one. In the asymptotic region the Killing norm is constant by fixing $K(r,u)|_{x_0>y_0} = 1$.

e. Virtual black hole (VBH) As in [6] we turn next to the conserved quantity, which exists in all two dimensional generalized dilaton theories [8], even in the presence of matter [9]. For SRG its geometric part reads

$$\mathcal{C}^{(g)} = \frac{p_2 p_3}{\sqrt{p_1}} - 4\sqrt{p_1} \tag{17}$$

and by assumption it vanishes in the asymptotic region $x_0 > y_0$. A simple argument shows that $\mathcal{C}^{(g)}$ is discontinuous: p_1 and p_3 are continuous, but p_2 jumps at $x_0 = y_0$. This phenomenon has been called "virtual black hole" (VBH) in [6]. It is generic rather than an artifact of our special choice of asymptotic conditions.

The solutions (11) and (12) establish

$$\left.\mathcal{C}^{(g)}\right|_{x_0<y_0} = 4c_0 y_0^{3/2} \propto -m_{VBH}. \tag{18}$$

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*Actually, the sources should be localized at different points, but for the lowest order tree graphs – which are our main goal – this makes no difference.

†Note the somewhat unusual rôle of the indices 0 and 1: x_0 is asymptotically proportional to r^2, thus our Hamiltonian evolution is with respect to a "radius" as "time"-parameter.

Thus, c_1 only enters the Rindler term in the Killing norm, but not the VBH mass (18).

f. The S^4 vertex All integration constants have been fixed by the arguments in the preceding paragraphs. The fourth order vertex of quantum field theory is extracted by collecting the terms quadratic in c_0 and c_1 replacing each by S_0 and S_1, respectively.

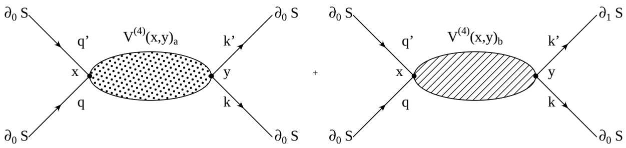

FIG. 1. Total $V^{(4)}$-vertex with outer legs

The tree graphs we obtain in that way (cf. fig. 1) contain the nonlocal vertices

$$V_a^{(4)} = \int_x \int_y S_0(x) S_0(y) \left(\frac{dq_2}{dc_0} p_1 + q_2 \frac{dp_1}{dc_0} \right) \Big|_{c_i=0}$$
$$= \int_x \int_y S_0(x) S_0(y) \, |\sqrt{y_0} - \sqrt{x_0}| \sqrt{x_0 y_0}$$
$$(3x_0 + 3y_0 + 2\sqrt{x_0 y_0}) \, \delta(x_1 - y_1), \tag{19}$$

and

$$V_b^{(4)} = \int_x \int_y \left(S_0(y) S_1(x) \frac{dp_1}{dc_0} - S_0(x) S_1(y) \frac{dq_2}{dc_1} p_1 \right) \Big|_{c_i=0}$$
$$= \int_x \int_y S_0(x) S_1(y) \, |x_0 - y_0| \, x_0 \delta(x_1 - y_1), \tag{20}$$

with $\int_x := \int_0^\infty dx^0 \int_{-\infty}^{\infty} dx^1$.

g. Asymptotics With $t := r + u$ the scalar field satisfies asymptotically the spherical wave equation. For proper s-waves only the spherical Bessel function

$$R_{k\,0}(r) = \frac{\sin(kr)}{kr} \tag{21}$$

survives in the mode decomposition ($Dk := 4\pi k^2 dk$):

$$S(r,t) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty \frac{Dk}{\sqrt{2k}} R_{k\,0} \left[a_k^+ e^{ikt} + a_k^- e^{-ikt} \right]. \tag{22}$$

With $a^\pm$ obeying the commutation relation $[a_k^-, a_{k'}^+] = \delta(k - k')/(4\pi k^2)$, they will be used to define asymptotic states and to build the Fock space. The normalization factor is chosen such that the Hamiltonian reads

$$H = \frac{1}{2} \int_0^\infty Dr \left[(\partial_t S)^2 + (\partial_r S)^2 \right] = \int_0^\infty Dk a_k^+ a_k^- k. \tag{23}$$

h. Scattering amplitude In [6] we had arrived at a trivial result in the massless case for (in $d = 2$) minimally coupled scalars: Either the S-matrix was divergent or – if the VBH was "plugged" by suitable boundary conditions on S at $r = 0$ – it vanished. Only for massive scalars we found some finite nonvanishing scattering amplitude.

In the present physical case of s-waves from $d = 4$ General Relativity at a first glance it may be surprising that the simple additional factor X in front of the matter Lagrangian induces fundamental changes in the qualitative behavior. In fact, it causes the partial differential equations (10) to become coupled, giving rise to an additional vertex ($V_b^{(4)}$).

After a long and tedious calculation (for details see [7,10]) for the S-matrix element with ingoing modes q, q' and outgoing ones k, k'

$$T(q,q';k,k') = \frac{1}{2} \left\langle 0 \left| a_k^- a_{k'}^- \left(V_a^{(4)} + V_b^{(4)} \right) a_q^+ a_{q'}^+ \right| 0 \right\rangle \tag{24}$$

having restored the full κ-dependence we arrive at

$$T(q,q';k,k') = -\frac{i\kappa\delta\left(k + k' - q - q'\right)}{2(4\pi)^4 |kk'qq'|^{3/2}} E^3 \tilde{T} \tag{25}$$

with $E = q + q'$,

$$\tilde{T}(q,q';k,k') := \frac{1}{E^3} \Bigg[\Pi \ln \frac{\Pi^2}{E^6} + \frac{1}{\Pi} \sum_{p \in \{k,k',q,q'\}} p^2 \ln \frac{p^2}{E^2}$$
$$\cdot \left(3kk'qq' - \frac{1}{2} \sum_{r \neq p} \sum_{s \neq r,p} (r^2 s^2) \right) \Bigg], \tag{26}$$

and $\Pi = (k+k')(k-q)(k'-q)$. The interesting part of the scattering amplitude is encoded in the scale independent factor $\tilde{T}$.

i. Discussion The simplicity of (26) is quite surprising, in view of the fact that the two individual contributions (cf. figure 1) are not only vastly more complicated, but also divergent. This precise cancellation urgently asks for some deeper explanation. The fact that a particular subset of graphs to a given order in perturbation theory may be gauge dependent and even divergent, while the sum over all such subsets should yield some finite, gauge-independent S-matrix is well known from gauge theory in particle physics (cf. e.g. [11]). However, it seems that only in the temporal gauge (4) one is able to integrate out the geometric degrees of freedom successfully. Also that gauge is free from coordinate singularities which we believe to be a prerequisite for a dynamical study extending across the horizon‡.

‡Other gauges of this class, e.g. the Painlevé-Gullstrand gauge [12] seem to be too complicated to allow an application of our present approach.

The only possible singularities occur if an outgoing momentum equals an ingoing one (forward scattering). Near such a pole we obtain with $k = q + \varepsilon$ and $q \neq q'$:

$$\tilde{T}(q, q'; \varepsilon) = \frac{2(qq')^2}{\varepsilon} \ln\left(\frac{q}{q'}\right) + \mathcal{O}(1). \tag{27}$$

The nonlocality of the vertex prevents the calculation of the usual s-wave cross section. However, an analogous quantity can be defined by squaring (25) and dividing by the spacetime integral over the product of the densities of the incoming waves $(\rho = (2\pi)^{-3} \sin^2(qr)/(qr)^2)$: $I = \int Drdt\rho(q)\rho(q')$, $\sigma = I^{-1} \int_0^\infty DkDk'|T|^2$. Together with the introduction of dimensionless kinematic variables $k = E\alpha, k' = E(1-\alpha), q = E\beta, q' = E(1-\beta), \alpha, \beta \in [0,1]$ this yields

$$\frac{d\sigma}{d\alpha} = \frac{1}{4(4\pi)^3} \frac{\kappa^2 E^2 |\tilde{T}(\alpha,\beta)|^2}{(1 - |2\beta - 1|)(1-\alpha)(1-\beta)\alpha\beta}. \tag{28}$$

Our result also allows the definition of a decay rate $d^3\Gamma/(DqDkDk')$ of an s-wave with ingoing momentum q decaying (!) into three outgoing ones with momenta $k, k', -q'$. Clearly, lifetimes calculated in this manner will crucially depend on assumed distributions for the momenta.

Finally, we stress that in the more general four dimensional setup of gravitational particle scattering combinations of non-spherical modes could contribute to the s-wave matrix element. Hence, our result (25) does not include the full ($4d$) classical information. Nonetheless, as the previous discussion shows, its physical content is highly nontrivial. We emphasize especially the decay of s-waves, which is a new phenomenon. Note that it cannot be triggered by graviton interaction, since there are no spherically symmetric gravitons. Still, it is caused by gravity, i.e. by gravitational self interaction encoded in our non local vertices.

Our methods are useful also for other applications, such as spherically symmetric collapse [13] or the polarized Gowdy model [14].

ACKNOWLEDGMENTS

This paper has been supported by projects P-12815-TPH and P-14650-TPH of the Austrian Science Foundation (FWF). One of the authors (D.V.) is grateful to the Deutsche Forschungsgemeinschaft (project Bo 1112/11-1) and to the Erwin Schrödinger Institute for Mathematical Physics. We thank J. Wabnig for checking parts of the calculation of the scattering amplitude.

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