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Anosov Actions on Non–Commutative Algebras

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Abstract

We construct an axiomatic framework for a quantum mechanical extension to the theory of Anosov systems, and show that this retains some of the characteristic features of its classical counterpart, e.g. positive Lyapunov exponents, a vectorial K–property, and exponential clustering. We then investigate the effects of quantisation on two prototype examples of Anosov systems, namely the iterations of an automorphism of the torus (the 'Arnold Cat' model) and the free dynamics of a particle on a surface of negative curvature. It emerges that the Anosov property survives quantisation in the case of the former model, but not of the latter one. Finally, we show that the modular dynamics of a relativistic quantum field on the Rindler wedge of Minkowski space is that of an Anosov system.

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1 Introduction

In classical mechanics, the theory of Anosov systems [An,AA] provides a paradigm for the unstable dynamics governing the mixing process, which is the basis of statistical mechanics. In particular, this theory was a forerunner to Sinai's [Si] classical works on the ergodic properties of systems of colliding particles.

In quantum mechanics, on the other hand, there is as yet no corresponding general theory of a mechanism that gives rise to similary unstable dynamics, though there is at least one quantum model that appears, *prima facie*, to be a good candidate for Anosov–type properties: this is the quantum version of the 'Arnold Cat' [BNS,Nar]. In view of this situation, it seems worthwhile to investigate the feasability of a general theory of unstable quantum dynamical systems, that is at least roughly parallel to the classical Anosov theory.

Our aim in this article is to provide a framework for a non–commutative extension of the theory of Anosov systems, that is applicable to unstable, or chaotic, quantum dynamics. This evidently requires some non–commutative generalisation of the differential structure, that is at the centre of the classical theory. We provide this in the form of derivations of the (non–abelian) algebra of observables, the natural generalisation of classical vector fields on a manifold (cf. [Co]).

In this way, we are able to construct an axiomatic formulation of quantum Anosov systems, and to show that, like their classical counterparts, these have an unstable dynamics, characterised by non–zero Lyapunov exponents and a certain vectorial, as distinct from algebraic, K–mixing property. Furthermore, we show that there are models of physical interest that satisfy our axioms, e.g. the quantised Arnold Cat and the modular dynamics of a relativistic quantum field on the Rindler wedge of Minkowski space. On the other hand, the canonical quantisation of one of the prototype classical Anosov systems, namely the free motion of a particle on a surface of constant negative curvature, destroys its Anosov property. We shall discuss this point further in section 7.

We shall organise our material as follows. In section 2, we shall present a brief review of the classical theory of Anosov systems. In section 3, we shall set out our axioms, which generalise this to the non–commutative regime, and we shall show there that the resultant generalised Anosov conditions imply an unstable dynamics, with positive Lyapunov exponents and vectorial K–mixing properties. In section 4, we shall provide a treatment of the quantum Arnold Cat, showing that it is indeed an Anosov system, and inferring therefrom certain cluster properties, stronger than those obtained in [BNS]. In section 5, we shall formulate the quantum dynamics of a free particle on a surface of constant negative curvature, and shall show that this is neither Anosov nor quasi–free, in the standard sense of algebraic quantum theory. In section 6, we shall show that the modular dynamics of a relativistic quantum field on a Rindler wedge is an Anosov system. Finally, in section 7, we shall briefly summarise our conclusions.

2 Classical Anosov Systems

Classical Anosov systems are characterized by the so–called "condition C" specified in [AA]. The phase–space of these systems is assumed to be a compact, connected Riemann manifold M. The dynamics is an action

$$T : (t,m) \in \mathbf{R} \times M \mapsto T(t)[m] \in M \tag{2.1}$$

which admits two foliations $\mathcal{F}^+$ and $\mathcal{F}^-$, stable with respect to T, and such that for each $m \in M$ the following two conditions are satisfied:

1. the two leaves $F^\pm_m$ of $\mathcal{F}^\pm$ through m intersect transversally; with $E^\pm_m$ denoting the tangent space of $F^\pm_m$ at m, $\dim E^\pm_m \equiv k^\pm > 0$ is independent of m, and $k^+ + k^- + 1 = n + 1 \equiv \dim M$; one also assumes that the orbit $T(\,\cdot\,)[m]$ intersects both $F^\pm_m$ transversally and that its tangent vector never vanishes;
2. there exist strictly positive constants a, b, λ such that for every $X^\pm_m \in E^\pm_m$

$$\|T(t)^*[X^\pm_m]\| \leq a\; e^{\mp\lambda t}\; \|X^\pm_m\| \quad \forall\, t \in \mathbf{R}^\pm \tag{2.2.a}$$

$$\|T(t)^*[X^\pm_m]\| \geq b\; e^{\pm\lambda t}\; \|X^\pm_m\| \quad \forall\, t \in \mathbf{R}^\mp. \tag{2.2.b}$$

The archetype [Had] for this structure is the geodesic flow on a compact, connected, two–dimensional, Riemannian manifold $M_c = \Gamma \setminus SL(2,\mathbf{R})/K$ of constant negative curvature, where $SL(2,\mathbf{R})$ acts on the Poincaré half–plane $(\overline{M}_c, g)$; specifically

$$\overline{M}_c = \{z \in \mathbf{C} \mid \frac{z - z^*}{i} > 0\}; \qquad g = -4(z - z^*)^{-2}\, dz\, dz^* \tag{2.3}$$

and $SL(2,\mathbf{R})$ acts by fractional transformations

$$(m,z) \in SL(2,\mathbf{R}) \times \overline{M}_c \mapsto m[z] = \frac{az+b}{cz+d} \in \overline{M}_c\;; \tag{2.4}$$

$K = S^1 \equiv \{m \in SL(2,\mathbf{R}) \mid m[i] = i\}$; $\overline{M}_c$ can be identified with $SL(2,\mathbf{R})/R$: $SL(2,\mathbf{R}) \ni m \to (m(i) \in \overline{M}_c$ and Γ is a discrete, co–compact subgroup of $SL(2,\mathbf{R})$. The "phase–space" of the model is the unit tangent bundle $M = T_1\, M_c$ which can thus be identified with $\Gamma \setminus SL(2,\mathbf{R})$. The geodesic flow on M is the action

$$T : (t,m) \in \mathbf{R} \times M \mapsto T(t)[m] \in M \tag{2.5}$$

given by

$$T(t)[m] = m \cdot \gamma(t); \qquad \gamma(t) = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}. \tag{2.6}$$

Two horocyclic actions are defined

$$S^\pm : (s,m) \in \mathbf{R} \times M \mapsto S^\pm(s)[m] \in M \tag{2.7}$$

and

$$S^{\pm}(s)[m] = m \cdot \gamma^{\pm}(s); \qquad \gamma^{+}(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}; \qquad \gamma^{-}(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}. \tag{2.8}$$

One now verifies straightforwardly that the orbits of the horocyclic actions $S^{\pm}$ are the leaves $F^{\pm}$ of two foliations $\mathcal{F}^{\pm}$ that equip the dynamical system (M, μ, T) — where μ is obtained from the right Haar measure on $SL(2, \mathbf{R})$ — with the structure of an Anosov system. Note, moreover, that

$$T(t)\, S^{\pm}(s)\, T(-t) = S^{\pm}(e^{-\lambda_{\pm} t}\, s) \quad \forall\, (s,t) \in \mathbf{R}^2 \tag{2.9}$$

and $\lambda_{\pm} = \pm 1$. This is a stronger version of (2.2), and, in higher dimensions, would subsume the Frobenius theorem on involutive distributions.

We now seek to generalize the relation (2.9) in a way that lifts the actions from the points m of the manifold M to a (sufficiently large) algebra $\mathcal{A}$ of functions f on M, constituating the observables of the classical system. Once that is achieved we extend the formalism to the non–commutative algebras of quantum systems. The first step is immediately achieved by defining the actions

$$\tau : (t, f) \in \mathbf{R} \times \mathcal{A} \mapsto f \circ T(-t) \in \mathcal{A} \tag{2.10}$$

$$\sigma^{\pm} : (s, f) \in \mathbf{R} \times \mathcal{A} \mapsto f \circ S^{\pm}(-s) \in \mathcal{A} \tag{2.11}$$

satisfying

$$\tau(t)\, \sigma^{\pm}(s)\, \tau(-t) = \sigma^{\pm}(e^{-\lambda_{\pm} t}\, s). \tag{2.12}$$

3 Non–commutative Anosov Systems

In this section we introduce first one of the most convenient extensions of the Anosov structures to non–commutative systems. Some refinements of our initial definitions are discussed at the end of the section (Remarks 3.7).

Definition 3.1 Let $\mathcal{A}$ be a von Neumann algebra, ϕ be a faithful normal state on $\mathcal{A}$, τ an action — the dynamics — of $\mathbf{R}$ on $\mathcal{A}$ leaving ϕ invariant, i.e. a map

$$\tau : (t, A) \in \mathbf{R} \times \mathcal{A} \mapsto \tau(t)[A] \in \mathcal{A} \tag{3.1}$$

such that for all t and t' in $\mathbf{R}$: $\tau(t) \in \text{Aut}\,(\mathcal{A})$; $\tau(t)\tau(t') = \tau(t+t')$; τ continuous in t; and $\phi \circ \tau(t) = \phi$.

We say that the dynamical system $(\mathcal{A}, \phi, \tau)$ admits an 'integrable Anosov structure' if there exists a collection

$$\{\, \sigma_j \mid j = 1, \ldots, k; k+1, \ldots, n \,\} \tag{3.2}$$

of 'horocyclic' actions

$$\sigma_j : (s, A) \in \mathbf{R} \times \mathcal{A} \mapsto \sigma_j(s)[A] \in \mathcal{A} \tag{3.3}$$

leaving ϕ invariant and satisfying

$$\tau(t)\sigma_j(s)\tau(-t) = \sigma_j(e^{-\lambda_j t}s) \quad \forall \quad (s,t) \in \mathbf{R}^2 \tag{3.4}$$

with $\lambda_j \in \mathbf{R}$ and

$$\lambda_1 \leq \ldots \leq \lambda_k < 0 < \lambda_{k+1} \leq \ldots \leq \lambda_n. \tag{3.5}$$

Without loss of generality, we can assume (by GNS construction) that: $\mathcal{A}$ acts on a Hilbert space $\mathcal{H}$; ϕ is a vector state, i.e. $\langle\phi; A\rangle = (\Phi, A\Phi)\,\forall\, A \in \mathcal{A}$, with $\Phi \in \mathcal{H}$ cyclic and separating for $\mathcal{A}$; τ [resp. σ_j] is implemented by a strongly continuous one-parameter unitary group V [resp. U_j] , i.e. $\forall\,(s,t) \in \mathbf{R}^2$ and $\forall\, A \in \mathcal{A}$

$$\tau(t)[A] = V(t)\,A\,V(-t) \qquad \sigma_j(s)[A] = U_j(s)\,A\,U_j(-s). \tag{3.6}$$

We have then, as a consequence of (3.4)

$$V(t)\,U_j(s)\,V(-t) = U_j(e^{-\lambda_j t}s) \quad \forall \quad (s,t) \in \mathbf{R}^2. \tag{3.7}$$

Theorem 3.2 (Lyapunov exponents) Let δ_j be the derivation generating the horocyclic action $\sigma_j(\mathbf{R})$ and $\mathcal{D}(\delta_j)$ be its domain. Then:

(1) $\tau(t)\,\delta_j\,\tau(-t) = e^{-\lambda_j t}\delta_j \quad$ on $\mathcal{D}(\delta_j) \quad \forall \quad t \in \mathbf{R}$

(2) for every $A \in \mathcal{D}(\delta_j)$ with $\delta_j[A] \neq 0$

$$\lambda_j = \lim_{t\to\infty} \frac{1}{t} \ln \|\, \delta_j \tau(t)[A]\, \|$$

(3) with $\mathcal{D}$ denoting the space of derivations of $\mathcal{A}$, g any positive bilinear map

$$g : (\delta, \delta') \in \mathcal{D} \times \mathcal{D} \mapsto g(\delta, \delta') \in \mathcal{A}\,,$$

and, for every $\delta \in \mathcal{D}$: $\|\delta\|_g \equiv g(\delta,\delta)^{1/2}$; if we assume that for every horocyclic δ_j: $\|\delta_j\|_g \neq 0$, we have:

$$\lambda_j = \lim_{t\to\infty} \frac{1}{t} \ln \|\, \tau(-t)\delta_j\tau(t)\|_g$$

(4) for every $A \in \mathcal{D}(\delta_j)$ with $\delta_j[A] \neq 0$, $f(A,s) \equiv \|\, \sigma_j(s)[A] - A\,\|$ satisfies

$$\lambda_j = \frac{1}{t} \ln \lim_{s\to 0} \left\{ \frac{f(\tau(t)[A], s)}{f(A,s)} \right\}.$$

Proof: Eqn. (3.4) implies that $\mathcal{D}(\delta_j)$ is stable under $\tau(\mathbf{R})$, and in fact, for any $A \in \mathcal{D}(\delta_j)$

$$\delta_j\, \tau(t)[A] = e^{\lambda_j t} \tau(t)[\delta_j\, A]$$

which is (1). We have then

$$\ln \|\, \delta_j\, \tau(t)[A]\, \| = \lambda_j t + \ln \|\delta_j[A]\|$$

from which (2) follows; (3) follows similarly from (1). Finally:

$$\lim_{s\to 0} \frac{f(\tau(t)[A], s)}{f(A,s)} \;=\; e^{\lambda_j t} \lim_{s\to 0} \frac{(se^{\lambda_j t})^{-1} f(A, e^{\lambda_j t} s)}{s^{-1} f(A,s)} \;=\; e^{\lambda_j t}$$

which proves (4). q.e.d.

Comments on Thm 3.2

1. Eqn. (3.7) implies that δ_j are unbounded derivations, hence $\|\tau(-t)\delta_j\tau(t)\|$ would be of no use in the above computations.
2. The conclusion (3) involves a non–commutative extension of the classical definition of a metric
$$g : (X, X') \in \mathcal{X}^\infty(M) \times \mathcal{X}^\infty(M) \mapsto g(X, X') \in \mathcal{C}^\infty(M)$$
where $\mathcal{X}^\infty(M)$ denotes the space of smooth vector fields on M. Hence (3) generalizes the classical result, according to which the Lyapunov exponents do not depend on the metric (see e.g. [Wal]).
3. $f(A,s)$ provides a measure of how far A moves along an orbit of the horocyclic action σ_j, so that (4) compares this to the similar expression for the time–translate $\tau(t)[A]$ of A.
4. $f(A,s)$ can be replaced in 3.2.4 by $f(\Phi, A, s) = \|(\sigma_j(s)[A] - A)\Phi\|$.

Theorem 3.3 (spectral properties) Let $V(\mathbf{R})$ and $U(\mathbf{R})$ be two strongly continuous one–parameter groups of unitary operators acting on some Hilbert space $\mathcal{H}$ and satisfying for some $\lambda \neq 0$

$$V(t)\, U(s)\, V(-t) = U(e^{-\lambda t} s) \quad \forall \quad (s,t) \in \mathbf{R}^2. \tag{3.8}$$

With $\{E(-\infty, k] \mid k \in \mathbf{R}\}$ denoting the spectral family of the generator K of $U(\mathbf{R})$, let

$$\mathcal{H}^o \equiv \{\Psi \in \mathcal{H} \mid U(s)\Psi = \Psi \quad \forall \quad s \in \mathbf{R}\} \tag{3.9}$$

$$\mathcal{H}^- \equiv \{\Psi \in \mathcal{H} \mid (\Psi, \Psi') = 0 \quad \forall \quad \Psi' \in \mathcal{H}^o\} \tag{3.10}$$

$$\mathcal{H}^- \equiv E(-\infty, 0]\, \mathcal{H}^- \quad ; \quad \mathcal{H}^+ \equiv E[0, \infty)\, \mathcal{H}^-. \tag{3.11}$$

Then:

(1) the decomposition $\mathcal{H}^- \oplus \mathcal{H}^o \oplus \mathcal{H}^+$ is stable under $V(\mathbf{R})$ and $U(\mathbf{R})$;

(2) with $H^\pm$ and $K^\pm$ denoting the self–adjoint generators of the restrictions $V^\pm(\mathbf{R})$ and $U^\pm(\mathbf{R})$ of $V(\mathbf{R})$ and $U(\mathbf{R})$ to $\mathcal{H}^\pm$, the four operators $H^\pm$ and $\Lambda^\pm \equiv \ln\{\pm K^\pm\}$ have homogeneous, absolutely continuous Lebesgue spectrum covering $\mathbf{R}$.

Proof: Since $\mathcal{H}^i$ $(i=-,o,+)$ are defined in terms of the spectral family of $U(\mathbf{R})$, these spaces are stable under $U(s)$ for all $s\in\mathbf{R}$. Moreover Eqn. (3.4.1) is equivalent to the relation

$$V(t)\,E(\Delta)\,V(-t)=E(e^{\lambda t}\Delta) \tag{3.12}$$

holding for every $s\in\mathbf{R}$ and every Borel subset $\Delta\subset\mathbf{R}$. This relation implies that $\mathcal{H}^i$ are also stable under $V(s)$ for all $s\in\mathbf{R}$. This proves (1).

On $\mathcal{H}^\pm$ the operators $\pm K^\pm$ are strictly positive so that $\Lambda^\pm=\ln(\pm K^\pm)$ are well defined self–adjoint operators with spectral families $F^\pm$ such that, for every $\Delta\subset\mathbf{R}$ $F^\pm(\Delta)\equiv E^-(\exp[\pm\Delta])$ where E^- denotes the restriction of E to $\mathcal{H}^\pm$.

With $W^\pm(t)\equiv V^\pm(\frac{t}{\lambda})$ Eqn. (3.4.2) reads

$$W^\pm(t)\,F^\pm(\Delta)\,W^\pm(-t)\;=\;F^\pm(\Delta+t) \tag{3.13}$$

for every $t\in\mathbf{R}$ and every Borel set $\Delta\subset\mathbf{R}$. Since Eqns. (3.13) are systems of imprimitivity on $\mathbf{R}$, the conclusion (2) of the theorem is a straightforward consequence of the Mackey–von Neumann uniqueness theorem. q.e.d.

As a consequence of Eqn. (3.4) [see (3.6) – (3.7)] the above spectral properties are satisfied for every horocyclic action σ_j $(j=1,\ldots,n)$ of a non–commutative Anosov system.

In the sequel, we shall need to control the behaviour of $E(-\epsilon,\epsilon)A\Phi$ for small ϵ. With this in mind, we introduce the following notation, in which the index j is omitted whenever it is not explicitly required. For every $A\in\mathcal{A}$ and every $f\in L^1(\mathbf{R},ds)$ we introduce

$$A(f)\equiv\int ds\,f(s)\sigma(s)[A]\,. \tag{3.14}$$

We denote with $\hat{f}$ the Fourier transform of f, and we introduce

$$\mathcal{F}^o\equiv\{f\in L^1(\mathbf{R},dx)\ \mid\ \hat{f}(0)=0\,;\ \hat{f}\ \text{ continuous around the origin}\} \tag{3.15}$$

and for every Borel set $\Delta\subset\mathbf{R}$

$$\mathcal{F}^o(\Delta)\equiv\{f\in\mathcal{F}^o\ \mid\ \text{ess–supp}\ \hat{f}\cap\Delta=\emptyset\}. \tag{3.16}$$

We define

$$\mathcal{A}^o\equiv\{A(f)\ \mid\ A\in\mathcal{A};\,f\in\mathcal{F}^o\} \tag{3.17}$$

and

$$\mathcal{A}^o(\Delta)\equiv\{A\in\mathcal{A}^o\mid A(f)=0\quad\forall\,f\in\mathcal{F}^o(\Delta)\} \tag{3.18}$$

such that $\Delta\subset\overline{\Delta}$ implies $\mathcal{A}^o(\Delta)\subset\mathcal{A}^o(\overline{\Delta})$.

Corollary 3.4 (vectorial K–filtering) Let $(\mathcal{A}, \phi, \tau)$ be a non–commutative Anosov system, σ be one of its horocyclic actions, and $\mathcal{A}^o(\Delta)$ be defined as in Eqn. (3.18) with $\Delta = (-\infty, -a) \cup (a, \infty)$, $a \in \mathbf{R}^+$. Then

1. $\mathcal{A}^o(\Delta)$ is a subspace of $\mathcal{A}^o$
2. $\tau(t)[\mathcal{A}^o(\Delta)] = \mathcal{A}^o(e^{-\lambda t}\Delta)$
3. for all $\Delta = (-\infty, -a] \cup [b, \infty)$, $a, b \in \mathbf{R}^+$ and for all t such that $\lambda t > 0$: $\mathcal{A}^o(\Delta) \subset \tau(t)\,[\mathcal{A}^o(\Delta)]$.

Proof: (1) is straightforward. For (2), note that for every $A \in \mathcal{A} : (\tau(t)[A])(f) = \tau(t)[A(f_t)]$ with $f_t(s) = e^{-\lambda t} f(e^{-\lambda t} s)$ and thus $\hat{f}_t(k) = \hat{f}(e^{\lambda t} k)$; together with (3.18) this indeed implies (2). Finally, (3) follows straightforwardly from (2). q.e.d.

Lemma 3.5 Let $V(\mathbf{R})$ and $U(\mathbf{R})$ be as in Theorem 3.3; K be the infinitesinal generator of $U(\mathbf{R})$, and E be its spectral family. Let further Ψ_1 and Ψ_2 be two vectors in $\mathcal{H}$, with $E(-a, a)\Psi_1 = 0$ for some $a > 0$, and $\Psi_2 \in \mathcal{D}(K^r)$ for some $r > 0$. Then for every $t \in \mathbf{R}$:

$$|(\Psi_2, V(t)\Psi_1)| \leq e^{-\lambda t r}\, a^{-r}\, \|\Psi_1\|\, \|K^r\, \Psi_2\|. \tag{3.19}$$

Proof: With $\Delta = (-\infty, -a] \cup [a, \infty)$ we have as a consequence of Eqn. (3.12)

$$|(\Psi_2, V(t)\Psi_1)| = |(E(e^{\lambda t}\Delta)\Psi_2, V(t)\Psi_1)| \leq \|\Psi_1\|\, (\Psi_2, E(e^{\lambda t}\Delta)\Psi_2)^{1/2} \tag{3.20}$$

from which the conclusion of the lemma follows upon using the classical inequality, holding for every $r > 0$:

$$\chi_{(-\infty,-1]\cup[1,\infty)}(x) \leq x^{2r} \quad \forall \quad x \in \mathbf{R} \tag{3.21}$$

where χ_Δ denotes the characteristic function of the subset Δ; for Δ as above, we have:

$$\chi_{\exp(\lambda t)\Delta}(x) \leq (a \exp(\lambda t))^{-2r} x^{2r} \quad \forall \quad (x, t) \in \mathbf{R}^2 \tag{3.22}$$

so that

$$(\Psi_2, E(e^{\lambda t}\Delta)\Psi_2)^{1/2} \leq (a \exp(\lambda t))^{-r}\, (\Psi_2, K^{2r}\Psi_2)^{1/2}\,. \tag{3.23}$$

q.e.d.

Note that the corollary can be strengthened in case Ψ_2 satisfies $E(\Delta_2)\Psi_2 = \Psi_2$ for Δ_2 compact; indeed there exists then $T \geq 0$ such that for all $t > T$: $e^{\lambda t}\Delta \cap \Delta_2 = \emptyset$ and thus

$$(\Psi_2, V(t)\Psi_1) = 0 \quad \forall \quad t > T. \tag{3.24}$$

Theorem 3.6 (exponential clustering) Let $(\mathcal{A}, \phi, \tau)$ be a non–commutative Anosov system; σ be one of its horocyclic actions, for which $\mathcal{A}^o$ is defined as in (3.17); δ be the derivation generating σ; $\mathcal{D}(\delta^r)$ be the domain of δ^r, with $r > 0$.

Then for every $A \in \mathcal{A}^o$ and $\epsilon > 0$, there exists $a > 0$ such that for all $t \in \mathbf{R}$, all $r > 0$ and all $B \in \mathcal{D}(\delta^r)$:

$$\begin{aligned} |\langle\phi; A^*\tau(t)[B]\rangle| \;\leq\; & e^{-\lambda t r} a^{-r} \langle\phi; A^*A\rangle^{1/2} \langle\phi; \delta^r[B]^*\delta^r[B]\rangle^{1/2} \\ & + \epsilon \langle\phi; A^*A\rangle^{1/2} \langle\phi; B^*B\rangle^{1/2} \end{aligned} \qquad (3.25)$$

with $\langle\phi; A\rangle \, \langle\phi; B\rangle = 0$, since by (3.15) and (3.17) $\langle\phi; A\rangle = 0$.

Proof: Given $A \in \mathcal{A}^o$ we have $A = A_o(f)$ for some $A_o \in \mathcal{A}$ and some $f \in \mathcal{F}^o$, hence $\langle\phi; A\rangle = 0$. Without loss of generality, we can assume $\|A\Phi\| \neq 0$; for every $a > 0$, we have then:

$$\|E(-a,a)\, A_o(f)\Phi\| = \left(\int_{-a}^{a} dk |\hat{f}(k)\Phi_{A_o}(k)|^2\right)^{1/2} \leq \sup_{[-a,a]} |\hat{f}(k)| \frac{\|A_o\Phi\|}{\|A\Phi\|} \|A\Phi\| \qquad (3.26)$$

where $\{\Phi_{A_o}(k) \mid k \in \mathbf{R}\}$ denotes the representation of $A_o\Phi$ with respect to the spectral family of K — the generator of the unitary group $U(\mathbf{R})$ implementing $\sigma(\mathbf{R})$. Since $f \in \mathcal{F}^o$, given $\epsilon > 0$ there exists $a > 0$ such that

$$\|E(-a,a)A\Phi\| \leq \epsilon \|A\Phi\|. \qquad (3.27)$$

We now write, for A and B as stated in the theorem:

$$|\langle\phi; A^*\tau[B]\rangle| \leq |(A\Phi, V(t)B\Phi)| \leq |(\Psi_2, V(t)\Psi_1)| + |(\Psi_2^c, V(t)\Psi_1)| \qquad (3.28)$$

with $\Psi_2 \equiv E(\Delta)A\Phi$, $\Psi_2^c \equiv E(\Delta^c)A\Phi$, $\Delta \equiv (-\infty, -a] \cup [a, \infty)$, $\Delta^c \equiv (-a, a)$ and $\Psi_1 \equiv B\Phi$.

The theorem then follows from (3.28) upon applying Lemma 3.5 to $|(\Psi_2, V(t)\Psi_1)|$, and using Schwartz inequality for $|(\Psi_2^c, V(t)\Psi_1)|$ together with the inequality (3.27). q.e.d.

Remarks 3.7

1. In the definition of τ, one may want to allow for $\mathbf{R}$ to be replaced by Z; this modification requires some adjustments that will be illustrated in section 4. Although we set as a separate assumption the σ–invariance of ϕ, this property alternatively can be viewed as a consequence of the following three assumptions: (i) σ is unitarily implemented; (ii) Eqn. (3.7) holds with $(s,t) \in \mathbf{R} \times \mathbf{R}$ or $\mathbf{R} \times \mathbf{Z}$; and (iii) ϕ is τ–invariant.
2. To emphasize the fact that classical Anosov systems are structures appearing in differential geometry (i.e. with $\mathcal{A}_o = C^\infty(M)$), rather than probability theory (i.e. with $\mathcal{A} = L^\infty(M, d\mu)$), and the fact that derivations provide a natural non–commutative extension of the classical concept of vector field, one may want to consider the conclusion (1) in Theorem 3.2 as the starting point of the theory rather than the relation (3.4) used in definition 3.1.

3. Although it has played no role in this section, one might ultimately want to impose that: (i) there exists a dense domain $\mathcal{D} \subset \mathcal{A}$ (compare with $C^\infty(M) \subset L^\infty(M)$ for compact M), stable under every $\delta_j (j = 1, \ldots, n)$; and (ii) that these derivations form a n–dimensional Lie algebra over $\mathcal{D}$.

4. In the same vein, it may be interesting to pursue the fact that there exists non–commutative Anosov systems (see e.g. section 4) for which the generators of the horocyclic actions form a basis in the space of outer derivations.

5. We did *not* assume, in the present section, that the dynamics $\tau(\mathbf{R})$ was the modular group for ϕ, as this would have precluded the existence of horocyclic actions $\sigma_j(\mathbf{R}) \subset$ Aut $(\mathcal{A})$. Nevertheless, see section 6 (e.g. Remark 6.3.2, Lemma 6.6 and Theorem 6.7) for a possibility to have ϕ KMS for $\tau(\mathbf{R})$ and still keep the main results of the present section.

6. The consequence (3.7) of the defining relation (3.4) rather than this relation itself was the operative condition for the present section. Hence, once (3.7) is obtained, W*–character of $\mathcal{A}$ is inessential. After that, $\mathcal{A}$ could equally well be C*–algebra, a normed *–algbra or just a *–algebra.

7. It is also possible to weaken the assumptions without damaging the essential structures and not require that ϕ be a faithful state on $\mathcal{A}$, or even dispense entirely with the introduction of a state.

8. In addition to the fact that the subspace $\mathcal{H}^o$ in theorem 3.3 is stable under $V(\mathbf{R})$ and contains the cyclic and separating vector Φ, one might wish to demand that $\dim(\mathcal{H}^o_j) = 1$ i.e. that $\mathcal{H}^o_j = C\Phi$ for each j separately; this would imply that the state ϕ is *extremal* σ_j*–invariant* for each j. This might be too much to ask. Nevertheless, a less stringent assumption will ensure an ergodic behaviour under the dynamics $\tau(\mathbf{R})$. Specifically, to obtain that $\mathcal{H}^o_V \equiv \{\Psi \mid V(t)\Psi = \Psi \quad \forall \quad t \in \mathbf{R}\}$ is one–dimensional, and thus (since we assumed $\mathcal{A}$ to be in standard form with respect to ϕ) ϕ is *extremal* τ*–invariant*, it is sufficient, by virtue of the absolute continuity of H over $\mathcal{H}^-$, to assume that
$$\dim\left(\bigcap_{j\in S} \mathcal{H}^o_j\right) = 1 \tag{3.30}$$
where S is any subset of the index set $j = 1, \ldots, n$; the cases where $S = \{1, \ldots, k\}$ [resp. $S = \{k+1, \ldots, n\}$] could be interpreted as a non–commutative analog of the standard feature of classical Anosov flows, namely that the stable [resp. unstable] manifolds are dense in M.

4 The Quantized Arnold Cat as Anosov System

Recall (see e.g. [AA]) that the phase space (M, ω) of the classical Arnold cat map is the torus $M = \Gamma \setminus \overline{M}$, with $\Gamma = \{\zeta = (\xi, \eta) \in \mathbf{Z}^2\}$ and $\overline{M} = \{z = (p, q) \in \mathbf{R}^2\}$, equipped with the symplectic form $\omega = dp \wedge dq$. The **discrete** "dynamics" of the model is given by the iterates $\{T^n \mid n \in \mathbf{Z}\}$ of the natural action on M of $T \in SL(2,\mathbf{Z})$ with tr $T > 2$. Note that the eigenvalues ε_j $(j = 1,2)$ of T satisfy $0 < \varepsilon_1 < 1 < \varepsilon_2 < \infty$; and that the principal directions $X_j = (u_j, v_j)$ $(j = 1,2)$ of T, defined by

$$T\, X_j = \varepsilon_j\, X_j \quad \text{and} \quad \|X_j\| = 1, \tag{4.1}$$

have irrational slope (v_j/u_j), so that each of the integral curves of X_j is dense in M.

To quantize this model, we introduce first the Weyl algebra $\{W(\zeta) \mid \zeta = (\xi,\eta) \in \mathbf{R}^2\}$ for $\overline{M}$, satisfying the defining relations

$$\begin{aligned} &W(\zeta)^* = W(-\zeta); \qquad \|W(\zeta)\| = 1 \\ &W(\zeta_1)\, W(\zeta_2) = e^{i2\pi\theta\sigma(\zeta_1,\zeta_2)}\, W(\zeta_1 + \zeta_2) \end{aligned} \tag{4.2}$$

where $\sigma(\zeta_1,\zeta_2) = \frac{1}{2}(\xi_1\,\eta_2 - \xi_2\,\eta_1)$ and the "deformation parameter" $\theta \neq 0$ plays here the role of the Planck constant or the magnetic field, see [Bel].

The Weyl algebra for $M = \Gamma \setminus \overline{M}$ is then obtained by restricting the domain of W to those ζ that satisfy the "periodic boundary conditions"

$$W(\eta)\, W(\zeta)\, W(\eta)^* = W(\zeta) \quad \forall\, \eta \in \frac{1}{\theta}\, \mathbf{Z}^2. \tag{4.3}$$

Hence W will now be restricted to $\zeta \in \mathbf{Z}^2$. We denote by $\mathcal{C}$ the C*–algebra obtained as the norm–closure of the linear span $\mathcal{D}$ of $W = \{W(\zeta) \mid \zeta \in \mathbf{Z}^2\}$. Note that

$$\phi : W(\zeta) \in W \mapsto \delta_{0,\zeta} \tag{4.4}$$

extends to a faithful tracial state ϕ over $\mathcal{C}$. Let π be the faithful representation of $\mathcal{C}$ obtained by GNS from ϕ, and let $\mathcal{A} = \pi(\mathcal{C})''$. Clearly $\mathcal{A}$ is a non–abelian von Neumann algebra; in particular, for irrational values of θ, $\mathcal{A}$ is the hyperfinite type II$_1$–factor. We henceforth identify $W(\zeta)$ and $\pi(W(\zeta))$; similarly $\mathcal{D}$ and $\pi(\mathcal{D})$. Note finally that

$$\tau : W(\zeta) \in W \mapsto W(\widehat{T}\zeta) \in W \tag{4.5}$$

(where $\widehat{T}$ denotes the transposed of T) extends to a continuous action

$$\tau : (n, A) \in \mathbf{Z} \times \mathcal{A} \mapsto \tau(n)[A] \in \mathcal{A} \tag{4.6}$$

satisfying

$$\phi \circ \tau(n) = \phi \quad \forall\, n \in \mathbf{Z}. \tag{4.7}$$

Eqn. (4.5) expresses the fact that the dynamics τ is quasi–free.

Definition 4.1 [BNS,Nar]: The Quantized Arnold Cat (for $\theta \neq 0$) is the non–commutative dynamical system $(\mathcal{A}, \phi, \tau)$ constructed above.

In the sense of definition 3.1, *except* that the time runs here over $\mathbf{Z}$ rather than over $\mathbf{R}$, we immediately obtain the following result.

Proposition 4.2 The Quantized Arnold Cat is a non–commutative Anosov system.

Proof: For $j = 1, 2$ and X_j as in (4.1), define for every $s \in \mathbf{R}$:

$$\sigma_j(s) : W(\zeta) \in W \mapsto e^{-is(X_j,\zeta)}\, W(\zeta) \in \mathcal{D} \tag{4.8}$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the usual scalar product in $\mathbf{R}^2$. Then σ_j extends to a weakly–continuous action of $\mathbf{R}$ on $\mathcal{A}$:

$$\sigma_j : (s, A) \in \mathbf{R} \times \mathcal{A} \mapsto \sigma_j(s)[A] \in \mathcal{A} \tag{4.9}$$

satisfying

$$\phi \circ \sigma_j(s) = \phi \quad \forall\, s \in \mathbf{R} \tag{4.10}$$

and the discrete–time version of (3.4), namely

$$\tau(n)\, \sigma_j(s)\, \tau(-n) = \sigma_j(e^{-\lambda_j n}\, s) \quad \forall\, (s, n) \in \mathbf{R} \times \mathbf{Z} \tag{4.11}$$

where $\lambda_j = \ln \varepsilon_j$ and ε_j as in (4.1), and thus $-\infty < \lambda_1 < 0 < \lambda_2 < \infty$. q.e.d.

As indicated in Rem. 3.7.1, the fact that the domain of the time–parameter is $\mathbf{Z}$ rather than $\mathbf{R}$ requires some minor modifications to the theory presented in section 3. The following result is at the root of most of these modifications.

Theorem 4.3 With $(\mathcal{A}, \phi, \tau)$ and $\sigma_j(\mathbf{R})$ as in Proposition 4.2

(i)

$$U_j(s) : W(\zeta)\Phi \in \mathcal{H} \mapsto \sigma_j(s)[W(\zeta)]\Phi \in \mathcal{H} \tag{4.12}$$

extends to a weakly–continuous, one–parameter unitary group $U_j(\mathbf{R})$; the spectrum of the generator K_j of $U_j(\mathbf{R})$ is a discrete, but dense, subgroup of $\mathbf{R}$, and it is simple.

(ii)

$$V : W(\zeta)\Phi \in \mathcal{H} \mapsto \tau(1)[W(\zeta)]\Phi \in \mathcal{H} \tag{4.13}$$

extends to a unitary operator V on $\mathcal{H}$; and V has homogeneous Lebesgue spectrum (in the sense of Kolmogorov).

Proof: Since $\mathcal{C}$ is a deformation ($\theta \neq 0$) of the classical ($\theta = 0$) algebra of functions on T^2, and since the quantum dynamics is quasi–free (4.5), the classical proof, which depends effectively only of the vector space structure of the algebra, can be extended straightforwardly to the present case. We first notice (see (4.4)) that

$$\{\Phi_\zeta \equiv W(\zeta)\Phi \mid \zeta \in \mathbf{Z}^2\} \tag{4.14}$$

is an orthonormal basis in $\mathcal{H}$. From Eqns. (4.12) and (4.8):

$$K_j\, \Phi_\zeta = (X_j, \zeta)\Phi_\zeta \quad \forall\, \zeta \in \mathbf{Z}^2 \tag{4.15}$$

and thus, with $X_j = (u_j, v_j)$ and $\zeta = (\xi, \eta)$:

$$\mathrm{Sp}\;(K_j) = \{u_j\;\xi + v_j\;\eta \mid (\xi,\eta) \in \mathbf{Z}^2\}. \tag{4.16}$$

Hence Sp (K_j) is a discrete subgroup of $\mathbf{R}$; since $m \equiv v_j/u_j$ is irrational, Sp (K_j) is dense in $\mathbf{R}$, and is simple. This proves (i). Note in particular that

$$\mathcal{H}^0 \equiv \{\psi \in \mathcal{H} \mid U_j(s)\;\psi = \psi \;\forall\; s \in \mathbf{R}\} = \mathbf{C}\Phi. \tag{4.17}$$

To prove (ii), we parametrize $\mathbf{Z}^2 \setminus \{0\}$, and hence

$$\mathcal{H}^- \equiv \{\psi \in \mathcal{H} \mid (\psi, \Phi) = 0\} \tag{4.18}$$

as follows. Since the principal directions $\widehat{X}_j$ of $\widehat{T}$ (also) have irrational slopes: $(\mathbf{R}X_j) \cap \mathbf{Z}^2 = \{0\}$, and the points of $\mathbf{Z}^2 \setminus \{0\}$ can be re–indexed as $\zeta = (\widehat{\xi}, \widehat{\eta})$ with $\widehat{\xi} \in \mathbf{Z} \setminus \{0\}$ indexing the orbits of $\widehat{T}$ (except the trivial orbit $\{0\}$), and $\widehat{\eta} \in \mathbf{Z}$ indexing the successive points of the orbit $\widehat{\xi}$. We thus have:

$$\mathcal{H}^- = \overline{\mathrm{span}}\;\{\Phi_{\widehat{\xi},\widehat{\eta}} \mid \widehat{\xi} \in \mathbf{Z} \setminus \{0\}, \widehat{\eta} \in \mathbf{Z}\} \tag{4.19}$$

$$V\;\Phi_{\widehat{\xi},\widehat{\eta}} = \Phi_{\widehat{\xi},\widehat{\eta}+1} \quad \forall \quad \widehat{\xi} \in \mathbf{Z} \setminus \{0\},\; \widehat{\eta} \in \mathbf{Z} \tag{4.20}$$

$$\{\psi \in \mathcal{H} \mid V\;\psi = \psi\} = \mathcal{H}^0 = \mathbf{C}\;\Phi. \tag{4.21}$$

Eqns. (4.20 – 4.21), together with (4.17 – 4.19), are the conclusion (ii) of the theorem.

q.e.d.

Remarks 4.4

1. The other results of section 3 (Lyapunov exponents and exponential clustering) hold for the present model. Note also that conditions (3.17) and (3.18) in Cor. 3.4 and Thm 3.6 can be replaced by the following (perhaps more visualizable) condition: for all $a > 0$, let
$$\mathcal{D}_a \equiv \{\zeta \in \mathbf{Z}^2 \mid (X, \zeta)^2 \geq a\} \tag{4.22}$$
where X is one of the principal directions of T; and
$$\mathcal{A}_a \equiv \mathrm{Span}\;\{W(\zeta) \mid \zeta \in \mathcal{D}_a\}; \tag{4.23}$$
in this case, we can even write $\varepsilon = 0$ in (3.25).

2. In connection with Rem. 3.7.4, note that in the present model, the differential version of (4.12), namely
$$\tau(n)\;\delta_j\;\tau(-n) = e^{-\lambda_j n}\;\delta_j \quad \forall\; n \in \mathbf{Z} \tag{4.24}$$
holds on the dense domain $\mathcal{D} = \mathrm{Span}\;\{W(\zeta) \mid \zeta \in \mathbf{Z}^2\}$, where
$$\delta_j[W(\zeta)] = (X_j, \zeta)\;W(\zeta) \quad \forall\; \zeta \in \mathbf{Z}^2. \tag{4.25}$$
Hence [Nar,BEJ,Bre,CR] the generators δ_j of the horocyclic actions (4.9) form a basis in the space of outer derivations for the differential structure attached to $\mathcal{D}$, i.e. every derivation $\delta : \mathcal{D} \to \mathcal{D}$ can be written as
$$\delta = c_1\;\delta_1 + c_2\;\delta_2 + \delta_3 \tag{4.26}$$
with δ_3 (approximatively) inner.

5 Free Quantum Particle on the Poincaré Half–Plane

The free motion of a classical particle on a compact manifold of constant negative curvature, such as a principal domain of the Poincaré half–plane, provides a prototype example of an Anosov flow (see section 2 above). The question therefore naturally arises as to whether its Anosov property survives quantization. We shall now show that, by contrast with the dynamics of the Arnold Cat, it does not.

The Model

As in section 2, let $\overline{M}_c \equiv \{z = (x, y) \mid x \in \mathbf{R}, y \in \mathbf{R}^+\}$ be the Poincaré half–plane; the measure $d\mu_c = y^{-2}dxdy$ is invariant under the free and transitive action on $\overline{M}_c$ obtained by restricting (2.4) to the subgroup H of upper–diagonal matrices in $SL(2, \mathbf{R})$. We parametrize H with its two subgroups

$$H_1 \equiv \{\gamma^+(v) \mid v \in \mathbf{R}\} \quad ; \quad H_2 \equiv \{\gamma(u) \mid u \in \mathbf{R}^+\} \tag{5.1a}$$

with

$$\gamma^+(v) = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \quad ; \quad \gamma(u) = \begin{pmatrix} u^{-1/2} & 0 \\ 0 & u^{1/2} \end{pmatrix}. \tag{5.1b}$$

We define $\mathcal{H}$ to be the Hilbert space $L^2(M, d\mu)$ and U to be the unitary representation of H in $\mathcal{H}$, given by

$$(U(h)f)(z) \equiv f(h^{-1}[z]) \quad \forall\, (z, h) \in \overline{M}_c \times H. \tag{5.2}$$

Thus, the infinitesimal generators of $U(H_1)$, $U(H_2)$ are $-iw_1$, $-2iw_2$, respectively, where

$$w_1 \equiv \frac{\partial}{\partial x} \; ; \quad w_2 \equiv x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{5.3}$$

We assume the algebra of observables, $\mathcal{A}$, for a quantum particle on $\overline{M}_c$ to be (cf. [Ma;Em3]) $\{U(H), C(\overline{M}_c)\}''$, the elements of $C(\overline{M}_c)$ acting multiplicatively on $\mathcal{H}$. Thus, $\mathcal{A} = \mathcal{B}(\mathcal{H})$. Further, [Em2,3], $\mathcal{A}$ is the $W^\star-$algebra generated by an extension of U to a faithful unitary representation in $\mathcal{H}$ of the Weyl group, W, of the CCR for a particle in $\overline{M}_c$. To be specific, the Lie algebra $\mathcal{W}$ of $U(W)$ has as basis the operators $(w_1, w_2, w_3, w_4, w_5)$, where w_1, w_2 are as defined by (5.3) and

$$w_3 = xy^{-1} \; ; \quad w_4 = -(1 + y^{-1}) \; ; \quad w_5 = 1 \tag{5.4}$$

all acting multiplicatively on $\mathcal{H}$. Thus, the structure of $\mathcal{W}$ is given by

$$[w_1, w_2] = w_1 \; ; \quad [w_1, w_3] = w_4 + w_5 \; ; \quad [w_2, w_4] = w_4 + w_5 \tag{5.5}$$

all other commutators between the w_k's vanishing. The group W is then [Em3] a central extension by $\mathbf{R}$ of a group G, which is itself a central extension of H by $\mathbf{R}^2$.

In a standard way, we term an automorphism, α, of $\mathcal{A}$, quasi–free if $U(W)$ is stable under α, i.e., if there is an automorphism, $\tilde{\alpha}$, of W, such that $\alpha U(F) \equiv U(\tilde{\alpha} F)$.

The free dynamics of a particle on $\overline{M}_c$ is governed by the one–parameter group $\tau(\mathbf{R})$ of automorphisms of $\mathcal{A}$, given by

$$\tau(t)[A] \equiv V(t) A V(-t) \tag{5.6}$$

where

$$V(t) = \exp(-i\Delta t) \tag{5.7}$$

and Δ is the Laplace–Beltrami operator for $\overline{M}_c$, i.e.

$$\Delta \equiv y^2 (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}). \tag{5.8}$$

The model of a quantum particle on $\overline{M}_c$ is thus given by $(\mathcal{A}, \tau)$. One defines similarly the dynamics on a fundamental domain $M_c = \Gamma \setminus \overline{M}_c$ (see section 2 and [Em3]), with $\mathcal{H}_c = L^2(M_c, d\mu_c)$, $\mathcal{A}_c = \mathcal{B}(\mathcal{H}_c)$ and Δ_c the Laplace–Beltrami operator for M_c.

Violation of Anosov and Quasi–Free Conditions

In the case of the Arnold Cat, the Anosov property of τ stems from the fact that it is quasi–free and implemented by classical Anosov automorphisms of its Weyl group. Since, the dynamics of the present models are also obtained by quantisation of free Anosov dynamics, it is reasonable to ask whether the same situation also prevails here. However, the following Propositions show that it does not.

Proposition 5.1 For any normal, τ_Γ−invariant state, ϕ_Γ, on $\mathcal{A}_\Gamma$, the dynamical system $(\mathcal{A}_\Gamma, \tau_\Gamma, \phi_\Gamma)$ is not Anosov.

Proof: Since [BGM] the spectrum of Δ_Γ is discrete, it follows from Theorem 3.3 that the model cannot be Anosov. q.e.d.

Proposition 5.2 The model $(\mathcal{A}, \tau)$ is neither quasi–free nor Anosov.

Proof: Since the spectrum of Δ is positive, it follows again from Theorem 3.3 that the model cannot satisfy the Anosov condition.

The proof that τ is not quasi–free, i.e., that $U(W)$ is not stable under $\tau(\mathbf{R})$, is an immediate consequence of Lemmas 5.3 and 5.4 below. We include them for completeness but the result is to be expected since here even classically the Lie–algebra does not close if one adds the Δ to the w's. q.e.d.

Lemma 5.3 The restriction of $\mathcal{W}$ to $\mathcal{D}$ is not stable under $Ad(\Delta)$.

Lemma 5.4 If $i\xi$ is the infinitesimal generator of a one–parameter subgroup, $S(\mathbf{R})$, of $U(W)$, such that $\tau(\mathbf{R}) : S(\mathbf{R}) \rightarrow U(W)$, then $Ad(\Delta)[\xi]$, as defined on $\mathcal{D}$, belongs to $\mathcal{W}$.

Proof of Lemma 5.3 It follows from our definitions that

$$[\Delta, w_3] = 2(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - w_3) \quad \text{on } \mathcal{D}.$$

The required result follows from the fact that, by eqns. (5.3) and (5.4), the r.h.s. of this equation manifestly does not belong to $\mathcal{W}$. q.e.d.

Proof of Lemma 5.4 Let $S_t(s) \equiv \tau(t)[S(s)] \equiv V(t)S(s)V(-t)$, $\forall s, t \in \mathbf{R}$, and let $i\xi(t)$ be the infinitesimal generator of $S_t(\mathbf{R})$. Then the assumption that $\tau(\mathbf{R}) : S(\mathbf{R}) \rightarrow U(W)$ implies that $\xi(t)$ lies in $\mathcal{W}$. Hence, $\xi(t)$ is a complex linear combination of the $w_j's$, i.e.,

$$\xi(t) = \sum\nolimits_{k=1}^{5} c_k(t) w_k. \tag{5.9}$$

Further, by the above definition of S_t,

$$(V(t)f, S_t(s)g) = (S(-s)f, V(-t)g) \quad \forall f, g \in \mathcal{H}. \tag{5.10}$$

Hence, as $\mathcal{D}$ lies in the domains of all elements of $\mathcal{W}$,

$$(V(t)f, \xi(t)g) = (\xi f, V(-t)g) \quad \forall f, g \in \mathcal{D} \tag{5.11}$$

i.e., by (5.11),

$$\sum\nolimits_k c_k(t)(V(t)f, w_k g) = (\xi f, V(-t)g) \quad \forall f, g \in \mathcal{D}. \tag{5.12}$$

Consequently, if $f_1, .. , f_5, g \in \mathcal{D}$ and

$$L_{jk}(t) \equiv (V(t)f_j, w_k g) \quad \text{and} \quad d_j(t) \equiv (\xi f_j, V(-t)g); \tag{5.13}$$

then

$$\sum\nolimits_k L_{jk}(t) c_k(t) = d_j(t). \tag{5.14}$$

We note here that, since $\mathcal{D}$ lies in the domains of the polynomials in the w_j's and Δ, it follows that L_{jk} and d_j are C^∞ functions on $\mathbf{R}$.

In order to ensure that the matrix $[L_{jk}(t)]$ is invertible, at least for t in a neighbourhood of the origin, we choose g so that the vectors $\{w_j g \mid j = 1, \dots 5\}$ are linearly independent; and then we choose each f_j to be orthogonal to $\{w_k g \mid k \neq j\}$, but not to $w_j g$. Thus, $[L_{jk}(0)]$ is a diagonal matrix, whose determinant is non–zero. Hence, it follows from (5.13) and the C^∞ character of $[L_{jk}(t)]$ that, for t in a neighbourhood of $\{0\}$, this matrix is invertible and its inverse, $[L_{jk}^{-1}(t)]$ is C^∞. Consequently, by (5.14) and the C^∞ property of the d_j's, the functions $c_j(t)$ are C^∞, for t in this neighbourhood of $\{0\}$.

Reverting now to the situation where f, g are arbitrary elements of $\mathcal{D}$, we find, on differentiating eqn. (5.14) w.r.t. t at $t = 0$, that

$$\sum\nolimits_k [\dot{c}_k(0)(f, w_k g) + i c_k(0)(\Delta f, w_k g)] = i(\xi f, \Delta g) \quad \forall f, g \in \mathcal{D},$$

i.e, as $\xi = \sum_k c_k(0) w_k$, by (5.11),

$$[\Delta, \xi] = i \sum_k \dot{c}_k(0) w_k, \quad \text{on} \;\; \mathcal{D}.$$

Since the r.h.s. of this equation lies in $\mathcal{W}$, this establishes the required result. q.e.d.

6 Quantized Field on the Rindler Wedge

Let $(M^{1,1}, g)$ denote the 2–dimensional Minkowski space, with null, future–directed, coordinates (u, v); and with metric $g = 4 \; dudv$. In the Poincaré group $P^{1,1}$, we consider the following one–parameter subgroups: the Lorentz boosts

$$\Lambda : (t; (u, v)) \in \mathbf{R} \times M^{1,1} \mapsto (e^t \; u, e^{-t} \; v) \in M^{1,1} \tag{6.1}$$

and the two null–translations

$$N_1 : (s, (u, v)) \in \mathbf{R} \times M^{1,1} \mapsto (u + s, v) \in M^{1,1} \qquad \forall \; s \in \mathbf{R} \tag{6.2}$$

$$N_2 : (s, (u, v)) \in \mathbf{R} \times M^{1,1} \mapsto (u, v + s) \in M^{1,1} \qquad \forall \; s \in \mathbf{R}. \tag{6.3}$$

For any weakly continuous, unitary representation U of $P^{1,1}$, we write (with $j = 1, 2$):

$$V(t) = U(\Lambda(t)) \quad \text{and} \quad U_j(s) = U(N_j(s)) \quad \forall \; s, t \in \mathbf{R}. \tag{6.4}$$

Note that these operators satisfy the Anosov property:

$$V(t) \; U_j(s) \; V(-t) = U_j(e^{-\lambda_j t} \; s) \quad \forall \; (s, t) \in \mathbf{R}^2 \tag{6.5}$$

with $\lambda_1 = -1$ and $\lambda_2 = +1$.

This relation is therefore satisfied, in particular, for every relativistic QFT on $M^{1,1}$ with $U(P^{1,1})$–invariant vacuum Φ. ¿From the Reeh–Schlieder theorem [RS], Φ is cyclic and separating for the von Neumann algebra $\mathcal{A}$ of the observables relative to the wedge

$$W = \{(u, v) \in M^{1,1} \;\mid\; u \geq 0, v \leq 0\} \tag{6.6}$$

which is, moreover, stable under the action (6.1). Let finally

$$\phi : A \in \mathcal{A} \mapsto (\Phi, A \; \Phi) \in \mathbf{C} \tag{6.7}$$

$$\tau : (t, A) \in \mathbf{R} \times \mathcal{A} \mapsto V(t) \; A \; V(-t) \in \mathcal{A}. \tag{6.8}$$

Definition 6.1 [Rin]: The quantized Rindler wedge (in Minkowski space) is the non–commutative dynamical system $(\mathcal{A}, \phi, \tau)$ constructed above.

Proposition 6.2 On the quantized Rindler wedge $(\mathcal{A}, \phi, \tau)$ the semi–groups of endomorphisms, defined for $j = 1, 1$, by

$$\sigma_j : (s, A) \in \mathbf{R}_j \times \mathcal{A} \mapsto U_j(s)\; A\; U_j(-s) \in \mathcal{A} \tag{6.9}$$

(where $\mathbf{R}_1 \equiv \mathbf{R}^+$, and $\mathbf{R}_2 \equiv \mathbf{R}^-$) satisfy the Anosov condition

$$\tau(t)\; \sigma_j(s)\; \tau(-t) = \sigma_j(e^{-\lambda_j t}\; s) \quad \forall\, (s, t) \in \mathbf{R}_j \times \mathbf{R} \tag{6.10}$$

with $\lambda_1 = -1$ and $\lambda_2 = +1$.

Proof: Immediate from (6.5, 6.8, 6.9) and from the fact that the wedge (6.6) is stable under $N_j(\mathbf{R}_j)$. q.e.d.

Remarks 6.3

1. The inclusion $\sigma_j(s)\; [\mathcal{A}] \subset \mathcal{A}$ for $s \in \mathbf{R}_j \setminus 0$ is strict; this follows from the fact that $\sigma_j(s)\; [\mathcal{A}]$ is the algebra of observables w.r.t. the wedge $N_j(s)\; [\mathcal{W}] \subset \mathcal{W}$.
2. The fact that the $\sigma_j(s)$ for $s \in \mathbf{R}_j \setminus 0$ are *–algebraic maps that are injective but not surjective reflects the fact, first pointed out by Bisognano and Wichmann [BW] that $\tau(\mathbf{R})$, as defined by (6.8), (6.4) and (6.1), is the modular group for ϕ, so that (6.10) is incompatible with $\phi \circ \sigma_j(s) = \phi$ and $\sigma_j(s) \in \mathrm{Aut}\; (\mathcal{A})$, since the latter would imply [Tak] that $\sigma_j(s)$ would commute with $\tau(\mathbf{R})$.
3. The modular automorphisms, τ, represent the timme–translations of relativistic quantum fields, as percieved by a uniformly accelerated observer, and are directly relevant to the Unruh and Hawking effects [Se].

Corollary 6.4 For every $s \in \mathbf{R}$ (not just $\mathbf{R}_j$), let

$$\sigma_j(s) : A \in \mathcal{A} \mapsto U_j(s)\; A\; U_j(s)^* \in \mathcal{B}(\mathcal{H}). \tag{6.11}$$

Then

(1) $\sigma_j(s)[\mathcal{A}] \subset \mathcal{A} \quad \forall\, s \in \mathbf{R}_j$

(2) $\bigcap_{s \in \mathbf{R}} \sigma_j(s)[\mathcal{A}] = \mathbf{C}\; I$

(3) $\bigvee_{s \in \mathbf{R}} \sigma_j(s)[\mathcal{A}] = \mathcal{B}(\mathcal{H})$

(4) For each $s \in \mathbf{R}$ $\sigma_j(s)[\mathcal{A}]\Phi$ is dense in $\mathcal{H}$.

(5) With K_j denoting the generator of the unitary group $U_j(\mathbf{R})$ implementing $\sigma_j(\mathbf{R})$: Sp $(K_j) = \mathbf{R}^+$ and the eigenvalue 0 of K_j is simple.

Proof: (1) – (4) are immediate consequences of the argument in Remark 6.3.1. From $\phi \circ \sigma_j(s) = \phi$ and (6.5) and by Theorem 3.3, we know that Sp (K_j) contains at least $\mathbf{R}^+$ or $\mathbf{R}^-$; $\mathbf{R}^- \setminus 0$ however is ruled out by: the spectral condition of the energy–momentum $P = (P^0, P^1)$ of a relativistic QFT; and $K_j = P_0 + (-1)^{j+1}\ P^1$. $\{0\}$ simple follows similarly. q.e.d.

Remark 6.5 Thus, the quantum field on the Rindler wedge induces an *algebraic* K–structure [Em3,NT] on the dynamical systems $(\mathcal{B}(\mathcal{H}), \phi, \sigma_j)$, attached to the whole Minkowski space $M^{1,1}$, *in addition* to the fact that the dynamical system $(\mathcal{A}, \phi, \tau)$, attached to the wedge $\mathcal{W}$, satisfies the *vectorial* K–filtering of Corollary 3.4. Notice moreover that the generator H of $V(\mathbf{R})$ has homogeneous Lebesgue spectrum over the full real line $\mathbf{R}$, in contrast to the above conclusion (5), namely Sp $(K_j) = \mathbf{R}^+$ only.

The following result, essentially due to Borchers [Bor], shows the sense in which the above properties of the wedge are generic.

Lemma 6.6 Let ϕ be a faithful, non–tracial, normal state on a von Neumann algebra $\mathcal{A}$ (presented in the standard form w.r.t. ϕ on $\mathcal{H}$). Let σ_j $(j = 1, 2)$ be a weakly continuous action

$$\sigma_j : (s, A) \in \mathbf{R}_j \times \mathcal{A} \mapsto \sigma_j(s)[A] \in \mathcal{A} \tag{6.12}$$

of the additive semi–group $\mathbf{R}_j = \mathbf{R}^+$ [resp. $\mathbf{R}^-$] for $j = 1$ [resp. $j = 2$]; and assume that for each $s \in \mathbf{R}_j$

$$\sigma_j(s) \in \text{End } (\mathcal{A}) \quad ; \quad \sigma_j(s)[\mathcal{A}]\Phi \ \text{ dense in } \mathcal{H}. \tag{6.13}$$

Then, there exists a weakly continuous, one–parameter group of unitaries $U_j(\mathbf{R})$ $(j = 1, 2)$, acting on $\mathcal{H}$, such that

$$\sigma_j(s)[A] = U_j(s)\ A\ U_j(-s) \quad \forall\ (s, A) \in \mathbf{R}_j \times \mathcal{A}. \tag{6.14}$$

Moreover, if the generator K_j of $U_j(\mathbf{R})$ has positive spectrum, then $\sigma_j(\mathbf{R})$ satisfy the Anosov condition

$$\tau(t)\ \sigma_j(s)\ \tau(-t) = \sigma_j(e^{-\lambda_j t}\ s) \quad \forall\ (s, t) \in \mathbf{R} \times \mathbf{R} \tag{6.15}$$

for $\lambda_1 = -1$, $\lambda_2 = +1$, and for the modular action

$$\tau : (t, A) \in \mathbf{R} \times \mathcal{A} \mapsto \Delta^{it/2\pi}\ A\ e^{-it/2\pi} \in \mathcal{A} \tag{6.16}$$

relative to ϕ ; and

$$\phi \circ \sigma_j(s) = \phi. \tag{6.17}$$

Proof: (6.13) imply that for every $s \in \mathbf{R}_j$

$$U_j(s) : A\Phi \in \mathcal{D}_o \mapsto \sigma_j(s)[A]\Phi \in \mathcal{D}_s \tag{6.18}$$

are isometries, with dense domain $\mathcal{D}_o = \mathcal{A}\Phi$ and dense range $\mathcal{D}_s = \sigma_j(s)[\mathcal{A}]\Phi$; they can therefore be extended uniquely to unitaries acting on $\mathcal{H}$; $U_j(\cdot)$ is then extended from $s \in \mathbf{R}_j$ to $s \in \mathbf{R}$ by $U_j(-s) \equiv U_j(s)^*$. This proves (6.14). From [Bor], we know that $V(t) = \Delta^{it/2\pi}$ and $U_j(s)$ satisfy the Anosov condition (6.5), from which (6.15 – 6.16) follow immediately; so does (6.17) by virtue of the second part of Remark 3.7.1.

Theorem 6.7 The quantum field on the Rindler wedge, and more generally any dynamical system described as in Lemma 6.6, satisfy the essential properties (Lyapunov exponents, and exponential clustering) ascribed to the non–commutative Anosov systems of section 3.

Proof: These properties are consequence of (3.7) $\equiv$ (6.5) which is satisfied for the Rindler wedge by construction, and in general from the assumption of Lemma 6.6. q.e.d.

Remarks 6.8

1. For the quantum field on the Rindler wedge, the exponential clustering w.r.t. Lorentz boosts could have been obtained directly from the Lehmann–Källen representation [Le,Ka].
2. Theorem 6.7 opens applications beyond the Rindler wedge since Lemma 6.6 holds independently of whether the actions σ_j commute with one another (see e.g. the discussion of the de Sitter universe in [Thi]).

7 Concluding Remarks

We have provided a framework for a generalisation of the classical theory of Anosov systems to the quantum regime, where the algebra of observables is non–commutative. Here, as in Connes's non–commutative geometry [Co], the required differential structure is provided by derivations of this algebra, which, in the classical case, correspond to vector fields on a Riemannian manifold. The dynamical system obtained by imposing the Anosov hyperbolicity conditions onto this structure then has the *vectorial* K–property.

Our principal results on concrete models are the following.

(1) The quantum version of the 'Arnold Cat', as given by automorphisms of the non–commutative torus, is a quasi–free, exponentially clustering, Anosov system, whose hyperbolic dynamics is inherited from that of its Weyl group.

(2) On the other hand, the dynamics of a quantum particle, moving freely, i.e. without external forces, on the Poincaré half–plane, is neither quasi–free nor Anosov. Thus, in this case, quantisation destroys the Anosov property.

(3) The modular dynamics of an arbitrary relativistic quantum field on the Rindler wedge of Minkowski space possesses the Anosov property.

Thus, of the above examples, (2) is the only one that lacks the Anosov property. It is also the only one whose algebra of observables is a type I factor. For the algebra of (1) is (cf. [BNS]) either a type II_1 factor or a tensor product of a classical algebra $L^{\infty}(T^2)$ and a finite type I factor I_n, depending on whether the non–commutativity parameter of the model is rational or irrational; while the algebra of (3) is always of type III. Hence, our results are in line with the conventional wisdom that the only systems, apart from the (essentially) classical ones, that enjoy good ergodic properties are those whose algebras of observables are of type II or III.

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