

Linearization of Regular Proper Groupoids

Alan Weinstein

Vienna, Preprint ESI 1049 (2001)

June 27, 2001

Supported by Federal Ministry of Science and Transport, Austria
Available via <http://www.esi.ac.at>

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Alan Weinstein*
Department of Mathematics
University of California
Berkeley, CA 94720 USA
(alanw@math.berkeley.edu)

June 25, 2001

Abstract

Let G be a Lie groupoid over M such that the target-source map from G to $M \times M$ is proper. We show that, if \mathcal{O} is an orbit of finite type (i.e. which admits a proper function with finitely many critical points), then the restriction $G|_{\mathcal{U}}$ of G to some neighborhood \mathcal{U} of \mathcal{O} in M is isomorphic to a similar restriction of the action groupoid for the linear action of the transitive groupoid $G|_{\mathcal{O}}$ on the normal bundle $N\mathcal{O}$. The proof uses a deformation argument based on a cohomology vanishing theorem, along with a slice theorem which is derived from a new result on submersions with a fibre of finite type.

1 Introduction

A Lie groupoid $G \rightrightarrows X$ is called **proper** if the (target,source) map $G \rightarrow X \times X$ is a proper map. Such groupoids arise, for example, as the transformation groupoids attached to smooth proper actions of groups. For proper group actions, and hence for these transformation groupoids, there is a normal form valid in the neighborhood of each orbit. (We refer to the first two chapters of [8] for a detailed treatment of the theory of smooth proper

*Research partially supported by NSF Grants DMS-96-25122 and DMS-99-71505 and the Miller Institute for Basic Research in Science.

MSC2000 Subject Classification Numbers: 58H05 (Primary), 57R99 (Secondary).

Keywords: Lie groupoid, proper action, submersion.

actions.) Transformation to the normal form depends on: (1) the existence of slices, which relate the behavior near an orbit to that near a fixed point of the isotropy subgroup; (2) linearizability of actions of compact groups near fixed points. Together, these two ingredients show that a proper group action is equivalent in a neighborhood of each orbit to a linear action on a vector bundle.

As is explained in [21], we hope eventually to establish a linearization theorem for all proper groupoids near their orbits. Its most general version would require a still unproven linearization theorem around fixed points, so in the present paper we sidestep this problem by assuming that the groupoid is **regular** in the sense that its orbits all have the same dimension.

The assumption that G is regular greatly simplifies the local problem, since the restriction G_Σ of a regular groupoid G to a slice Σ is **essentially étale** in the sense that its action on the slice factors through an étale groupoid. The étale groupoid is easily linearized near its fixed point, and a deformation argument using Crainic's cohomology vanishing theorem [5] allows us to linearize the essentially étale groupoid G_Σ ; i.e., we prove that G_Σ is locally isomorphic to the action groupoid for the linear action of the isotropy group G_x on the tangent space $T_x\Sigma$.

The second building block of our normal form is the restriction of G to an orbit, which is transitive and is therefore the gauge groupoid of a principal bundle. Combining the action groupoid over a slice with the gauge groupoid over an orbit produces the action groupoid for the (linear) action of the gauge groupoid on the normal bundle of the orbit, which we call the linear approximation to G along the orbit. Our main theorem asserts that, under a differential-topological finiteness assumption which we describe in the following paragraph, G is isomorphic to its linear approximation in some neighborhood of each orbit. We refer to such an isomorphism as a linearization of G along the orbit.

The main theorem follows from a slice theorem which asserts that a linearization of the restriction to a slice extends to a linearization along an orbit. This slice theorem holds even in the nonregular case, but it turns out to require a differential-topological assumption on the orbit itself. The necessity of this assumption appears already in the special case where the proper groupoid has trivial isotropy groups, i.e. where G is the equivalence relation whose equivalence classes are the fibres of a submersion $f : X \rightarrow Y$. Applied to such a groupoid, our normal form theorem asserts that, restricted to some open neighborhood \mathcal{U} of each fibre $f^{-1}(y)$, f is a trivial fibration onto $f(\mathcal{U})$. Already in this case, it turns out that an additional hypothesis is necessary; the most natural one seems to be that the fibre $f^{-1}(y)$ is of

finite type in the sense that it admits a proper map to \mathbb{R} with finitely many critical points. (See Appendix B for a discussion of this finiteness condition.)

The body of the paper begins with a discussion of the definition of proper groupoids, followed by some examples showing the importance of a local triviality condition. We prove our linearization theorem for groupoids associated with submersions, and then for étale groupoids. After proving a rigidity theorem for proper groupoids, we deal with the effectively étale case. Finally, we prove the slice theorem and use it to deduce the main theorem. Two appendices are devoted to background material on proper mappings and manifolds of finite type.

Proper groupoids seem to have appeared only infrequently in the literature. Moerdijk and Pronk [17] characterized orbifolds as Morita equivalence classes of étale proper groupoids on manifolds. The orbit spaces of regular proper groupoids are orbifolds as well, and the Morita equivalence classes of these groupoids may be thought of as principal bundles over orbifolds. In connection with operator algebras, proper groupoids appear in Connes' book [4] as the appropriate setting for the geometric realization of cycles in K -theory; an appendix (Section 6) on proper groupoids in the related paper of Tu [20] includes the construction of a cutoff function used by Crainic [5] in the proof of his cohomology vanishing theorem.

I would like to thank Ronnie Brown, Marius Crainic, Bob Gompf, Rob Kirby, Kirill Mackenzie, Gaël Meigniez, and Ieke Moerdijk for their helpful advice concerning groupoids on the one hand and manifolds of finite type on the other. For hospitality during the final preparation of this paper, I would like to thank Harold Rosenberg at l'Institut de Mathématiques de Jussieu, Yvette Kosmann-Schwarzbach and Claude Viterbo at l'École Polytechnique, and Peter Michor at the Erwin Schrödinger Institute.

2 Definition of proper groupoids

For basic notions about groupoids, we refer the reader to [3] or [13]. Our conventions and notation include the following. “Lie groupoid” will always mean “smooth groupoid,” not necessarily transitive as was the case in [13]. All manifolds will be Hausdorff; Appendix A shows why this assumption is reasonable in the context of proper groupoids. All neighborhoods will be open. In a groupoid $G \rightrightarrows X$, we will denote the target and source maps by α and β respectively. We will sometimes call the map $(\alpha, \beta) : G \rightarrow X \times X$ the **anchor** of the groupoid, following [13]. If A and B are subsets of X ,

we denote by G_{AB} the set $\alpha^{-1}(A) \cap \beta^{-1}(B)$. If $A = B$, we write G_A for G_{AA} . For one-point subsets, we abbreviate $G_{\{x\}\{y\}}$ by G_{xy} . Combining the two abbreviations leads to the usual notation G_x for the isotropy group of $x \in X$.

A submanifold $S \subseteq G$ for which $(\alpha, \beta)(S)$ is the graph of a diffeomorphism $\phi_S : X \rightarrow X$ is called a **bisection** of G . The bisections form a group under setwise G -multiplication which acts on G by left or right translations and on X via the homomorphism $S \mapsto \phi_S$.

Following [15], we will say that a groupoid $G \rightrightarrows M$ is “source-xxx” if the target and source maps of G (or their fibres, as will be clear from the context) each have the property “xxx.” For instance, we may refer to groupoids as being source-proper, source-connected, etc.

A groupoid G is **regular** if its orbits all have the same dimension. This condition is equivalent to constancy of rank for either the anchor of G or the anchor $\mathcal{A}(G) \rightarrow TX$ of its Lie algebroid.

We now recall two definitions concerning group actions.

Definition 2.1 *An action of a Lie group Γ on a topological space X is **proper** if the mapping $(\gamma, x) \mapsto (\gamma x, x)$ is a proper mapping from $\Gamma \times X$ to $X \times X$.*

Definition 2.2 *Given an action of a group Γ with identity element e on a set X , the corresponding **action groupoid** is the groupoid $\Gamma \times X \rightrightarrows X$ with target and source maps $(\gamma, x) \mapsto \gamma x$ and $(\gamma, x) \mapsto x$, product $(\gamma, \gamma'x)(\gamma', x) = (\gamma\gamma', x)$, unit embedding $x \mapsto (e, x)$, and inversion $(\gamma, x)^{-1} = (\gamma^{-1}, \gamma x)$.*

On the basis of these two definitions, we introduce the notion of proper groupoid.

Definition 2.3 *A **proper groupoid** is a Lie groupoid $G \rightrightarrows X$ for which the anchor mapping $(\alpha, \beta) : G \rightarrow X \times X$ is proper.*

Here is an important property of proper groupoids.

Proposition 2.4 *Each orbit of a proper groupoid is a closed submanifold.*

Proof. Let \mathcal{O} be the G -orbit through $x \in X$. The isotropy G_x is a compact group acting freely on G_{xX} by multiplication from the left, and β factors through the natural projection to give a map from the quotient G_{xX}/G_x to X which is an injective immersion with image \mathcal{O} . To show that \mathcal{O} is closed, it suffices to show that this immersion is proper.

To this end, let g_i be a sequence of elements in G_{xX} such that βg_i is convergent. Then the anchor of G applied to g_i gives the convergent sequence $(x, \beta g_i)$. Since the groupoid is proper, g_i contains a convergent sequence, hence so does the corresponding sequence $[g_i]$ in the quotient space G_{xX}/G_x .

□

It turns out that properness of a groupoid is not sufficient to imply some of the nice properties which we associate with proper actions of groups. For this reason, we will sometimes impose the additional condition of **source-local triviality**. Examples in Section 3 below will show that this condition does not follow from the properness of the anchor. For action groupoids associated with group actions, the target and source maps are globally, hence locally, trivial fibrations.

3 Groupoid actions and stability

Let $G \rightrightarrows X$ be a groupoid, and let $\mu : Y \rightarrow X$ be a surjective mapping. An **action** of G on Y is a mapping $(g, y) \mapsto gy$ to Y from the fibre product $G \times_X Y$ (using the source map from G to X) satisfying the usual conditions for associativity and action of the identities. Connes [4] notes that an action of a Lie groupoid can be thought of as a functor from the groupoid considered as a category to the category of smooth manifolds and smooth mappings.

Example 3.1 Any groupoid $G \rightrightarrows X$ acts on its base X by the rule $gx = \alpha(g)$ whenever $\beta(g) = x$. If H and A are subsets of G and X respectively, HA is defined in the usual way, as in the case of group actions.

Definition 2.2 is easily extended from group actions to groupoid actions. Given an action of $G \rightrightarrows X$ on Y , the associated **action groupoid** is the groupoid $G \times_X Y \rightrightarrows Y$ with anchor $(g, y) \mapsto (gy, y)$ and multiplication $(h, gy)(g, y) = (hg, y)$. The mapping μ is sometimes called the **moment map** of the groupoid action; in the differentiable category, we require μ to be a submersion, so that we may require the action to be differentiable, in which case the action groupoid $G \times_X Y \rightrightarrows Y$ is again a Lie groupoid. The action groupoid for the action of G on its base X is naturally isomorphic to G itself.

Motivated by the case of group actions, we make the following definition.

Definition 3.2 *A fixed point x of a topological groupoid $G \rightrightarrows X$ is **stable** if every neighborhood of x contains a G -invariant neighborhood.*

The following theorem shows that proper groupoids share an important property with proper actions of groups.

Theorem 3.3 *Every fixed point of a source-locally trivial proper topological groupoid is stable.*

Proof. Let x be a fixed point of $G \rightrightarrows X$. To show that a given neighborhood \mathcal{U} of x contains an invariant neighborhood, we may assume to begin that \mathcal{U} is small enough so that $\beta^{-1}(\mathcal{U}) \approx \mathcal{U} \times G_x$ as spaces over \mathcal{U} . Since G_x is compact, each neighborhood of $\beta^{-1}(y)$ for y in \mathcal{U} contains a neighborhood of the form $\beta^{-1}(\mathcal{V})$.

Now we define the **core** $C(\mathcal{U})$ of \mathcal{U} to be

$$\{y \in \mathcal{U} | Gy \subseteq \mathcal{U}\} = \{y \in \mathcal{U} | \beta^{-1}(y) \subseteq \alpha^{-1}(\mathcal{U})\}.$$

$C(\mathcal{U})$ is invariant: if $y \in C(\mathcal{U})$ and gy is defined, then $h(gy) = (hg)y \in \mathcal{U}$ whenever $h(gy)$ is defined, so $gy \in C(\mathcal{U})$. $C(\mathcal{U})$ is open: if $y \in C(\mathcal{U})$, then $\beta^{-1}(y)$ is contained in the open set $\alpha^{-1}(\mathcal{U})$, so y has a neighborhood \mathcal{V} such that $\beta^{-1}(\mathcal{V}) \subseteq \alpha^{-1}(\mathcal{U})$, i.e. $\mathcal{V} \subseteq C(\mathcal{U})$.

□

Examples 3.4 and 3.5 below show that Theorem 3.3 requires an assumption like source-local triviality. These examples also show that, while the condition of properness is preserved under Morita equivalence of groupoids [16], source-local triviality and stability are not.

Example 3.4 Let X_1 be the plane \mathbb{R}^2 with the origin removed. Let the groupoid G_1 be the equivalence relation on X_1 , with quotient space \mathbb{R} , consisting of all the pairs of points lying on the same vertical line. The anchor of G_1 is proper, but the source map is not locally trivial over any point on the vertical line through the origin. There are no fixed points.

Now let G_2 be the restriction of G_1 to any horizontal line X_2 *not* passing through the origin. G_2 is just the trivial groupoid, with all elements units, so it is source-locally trivial as well as proper. Every element of X_2 is a stable fixed point for G_2 . The two groupoids are Morita equivalent since the second is the restriction of the first to a closed submanifold passing through all orbits and intersecting them transversely. (See Example 2.7 in [18] for the relevant topological result; the smooth case is handled similarly).

Finally, let G_3 be the restriction of G_1 to the union X_3 of *two* horizontal lines, one of which passes through (but does not include) the origin. This

groupoid is equivalent to the first two; like G_1 it has a proper anchor map, but it is not source-locally trivial. The point of X_3 lying on the vertical line through the origin is a fixed point for G_3 (the only one), but it is not stable.

Example 3.5 (Dipole foliation) Let X be the plane with the two points $(0, 1)$ and $(0, -1)$ removed. In this plane, consider the foliation given by the level curves of the potential function produced by a unit positive charge at one deleted point and a unit negative charge at the other. This foliation is symmetric about the horizontal axis; its leaves are this axis and simple closed curves surrounding the two deleted points. Let G the equivalence relation determined by the foliation, considered as a groupoid over X . G is easily seen to be proper; for instance, it is Morita equivalent to its restriction to an open line segment joining the two charges, which is a trivial groupoid. On the other hand, G is not source-locally trivial around points of the horizontal axis. We do obtain an equivalent source-locally trivial groupoid by restricting G to the open strip between the horizontal lines through the charges. The latter groupoid is in fact isomorphic to an action groupoid for an \mathbb{R} action.

4 The main theorem

In this section, we state our main theorem. Succeeding sections are devoted to proofs of special cases, culminating in the proof of the theorem itself.

Recall that, if \mathcal{O} is an orbit of any Lie groupoid $G \rightrightarrows X$, the restricted groupoid $G_{\mathcal{O}}$ has a natural representation on the normal bundle $N\mathcal{O}$. (See, for example, Appendix B in [10].) The action groupoid $G_{\mathcal{O}} \times_{\mathcal{O}} N\mathcal{O}$ should be thought of as the linear approximation to G along \mathcal{O} , so the following theorem states that, near an orbit of finite type, a proper groupoid is isomorphic to its linear approximation.

Theorem 4.1 *Let $G \rightrightarrows X$ be a regular, proper Lie groupoid, and let \mathcal{O} be an orbit of G which is a manifold of finite type. Then there is a neighborhood \mathcal{U} of \mathcal{O} in X such that the restriction of G to \mathcal{U} is isomorphic to the restriction of the action groupoid $G_{\mathcal{O}} \times_{\mathcal{O}} N\mathcal{O}$ to a neighborhood of the zero section in $N\mathcal{O}$.*

5 Local semitriviality of submersions

The prefix “semi” in the title of this section is meant in the same sense as in “semicontinuous”.

If $f : X \rightarrow Y$ is a submersion, then the fibre product $X \times_Y X$ is an equivalence relation, as well as being a submanifold of $X \times X$. With these structures, $X \times_Y X$ becomes a proper Lie subgroupoid of $X \times X$. (In fact, these are the only proper Lie subgroupoids of $X \times X$.) The isotropy groups of $X \times_Y X$ are trivial and the orbits are the fibres of f . In this situation, our main theorem is essentially equivalent to Theorem 5.1 below, of which Example 3.5 is an illustration. The equivariant case is included for later use.

Theorem 5.1 *Let $f : X \rightarrow Y$ be a submersion. For any $y \in Y$, if $\mathcal{O} = f^{-1}(y)$ is a manifold of finite type, then there is a neighborhood \mathcal{U} of \mathcal{O} in X such that $f|_{\mathcal{U}} : \mathcal{U} \rightarrow f(\mathcal{U})$ is a trivial fibration. In other words, there is a retraction $\rho : \mathcal{U} \rightarrow \mathcal{O}$ such that $(\rho, f) : \mathcal{U} \rightarrow \mathcal{O} \times f(\mathcal{U})$ is a diffeomorphism.*

If f is equivariant with respect to actions of a compact group K on X and Y , with y a fixed point, and $f^{-1}(y)$ is of finite type as a K -manifold, then \mathcal{U} can be chosen to be K -invariant and ρ to be K -equivariant.

Proof. All which follows can be done equivariantly. Let $h : \mathcal{O} \rightarrow [0, \infty)$ be a proper function whose critical points form a compact set. Choose N large enough so that all the critical points of h lie inside $h^{-1}([0, N - 1])$. Since \mathcal{O} is a closed submanifold of X , it has a tubular neighborhood \mathcal{T} on which there is a smooth retraction $\rho_0 : \mathcal{T} \rightarrow \mathcal{O}$. By composition with this retraction, we extend the function h to \mathcal{T} and call the extension h as well.

For any $N > 0$ and any subset A of \mathcal{T} , we will denote the subset of A on which $h(x) < N$ by $A_{(N)}$ and the subset on which $h(x) \leq N$ by $A_{[N]}$.

Since $\mathcal{O}_{[N+1]}$ is compact, we can choose a small open disc \mathcal{W} around y in Y such that the restriction of (f, ρ_0) to $f^{-1}(\mathcal{W})_{[N+1]}$ is a diffeomorphism from $f^{-1}(\mathcal{W})_{[N+1]}$ to $\mathcal{W} \times \mathcal{O}_{[N+1]}$. We denote the inverse of this diffeomorphism by Ψ_0 . By choosing \mathcal{W} small enough, we can also assure that the function h has no fibrewise (for f) critical points in $f^{-1}(\mathcal{W})_{[N+1]} \setminus f^{-1}(\mathcal{W})_{(N-1)}$.

Now let \mathcal{V} be the union of $f^{-1}(\mathcal{W})_{(N+1)} \setminus f^{-1}(\mathcal{W})_{[N-1]}$ and the set of points in $f^{-1}(\mathcal{W})$ where h is not critical along the fibres of f . Choose (see Lemma A.5) a complete riemannian metric on \mathcal{V} . With the help of a compactly supported function equal to 1 on $f^{-1}(\mathcal{W})_{[N+1]}$, we may modify this metric without affecting its completeness so that its restriction to $f^{-1}(\mathcal{W})_{[N+1]}$ becomes a product metric with respect to the diffeomorphism with $\mathcal{W} \times \mathcal{O}_{[N+1]}$. As a result, the projection ρ_0 becomes a local isometry to \mathcal{O} when further restricted to each fibre of f .

Let ξ be the vector field on \mathcal{V} which is the fibrewise gradient of h . Since the metric restricted to each fibre is complete, and h is proper, ξ is a complete vector field. Denote the flow of ξ by $t \mapsto \phi_t$, and let \mathcal{U} be the union of all

the trajectories of ξ which intersect $f^{-1}(\mathcal{W})_{(N+1)}$. Since \mathcal{U} is the union over all t of $\phi_t(f^{-1}(\mathcal{W})_{(N+1)})$, it is an open subset of X .

Now we define a retraction $\rho : \mathcal{U} \rightarrow \mathcal{O}$ as follows. If $h(x) < N + 1$, we let $\rho(x) = \rho_0(x)$, where $\rho_0(x)$ is the original tubular neighborhood retraction on \mathcal{T} . If $h(x) > N$, there is a unique positive number $\tau(x)$, depending smoothly on x , such that $h(\phi_{-\tau(x)}(x)) = N$. We then define $\rho(x) = \phi_{\tau(x)}(\rho_0(\phi_{-\tau(x)}(x)))$. By the product structure of the metric, the map ρ_0 commutes with the flow on $f^{-1}(\mathcal{W})_{[N+1]}$, and so these two definitions agree on the intersection of the domains. It is also clear that ρ is a retraction.

To see that the product map (ρ, f) is a diffeomorphism from \mathcal{U} to $\mathcal{W} \times \mathcal{O}$, we observe that it has an inverse Ψ defined on $\mathcal{W} \times \mathcal{O}_{(N+1)}$ by $\Psi = \Psi_0$ and on $\mathcal{W} \times (\mathcal{O} \setminus \mathcal{O}_{[n]})$ by $\Psi(y, z) = \phi_{\tau(z)}(\Psi_0(y, \phi_{-\tau(z)}(z)))$.

□

A proper submersion is always a locally trivial fibration. Compactness of all the fibres is not enough to insure local triviality (restrict Example 5.2 below to $(0, 1) \times \mathbb{N}$), but it is an easy corollary of Theorem 5.1 that compactness and connectedness of the fibres *are* enough. These results were already essentially proved by Ehresmann [9]. Meigniez [14] has recently given several sufficient conditions for submersions with noncompact fibres to be locally trivial.

Here is a very simple example which shows that the hypothesis of finite type cannot be omitted from Theorem 5.1.

Example 5.2 (Railroad to infinity) Let $X \subset (-1, 1) \times \mathbb{N}$ be the subset $\{(x, n) | nx^2 < 1\}$, $Y = (-1, 1)$, and f the projection on the first factor. Then there is no neighborhood of $f^{-1}(0)$ on which f becomes a trivial fibration, since the inverse image of any $y \neq 0$ is finite and hence cannot contain an embedded copy of $f^{-1}(0)$.

Lest the reader think that connectedness might help, we give another example.

Example 5.3 (Ladder to heaven) Let S be an infinite-holed torus with one end, and let $h : S \rightarrow [0, \infty)$ be a proper function for which the inverse image of each interval $[0, n]$ is a surface of finite genus bounded by two circles. Now let $X \subset (-1, 1) \times S$ be the subset $\{(x, s) | h(s)x^2 < 1\}$, $Y = (-1, 1)$, and f the projection on the first factor. Then there is no neighborhood of $f^{-1}(0)$ on which f becomes a trivial fibration, since the inverse image of any $y \neq 0$ has finite genus and hence cannot contain an embedded copy of $f^{-1}(0)$.

(To prove the latter fact, notice that the intersection form on $H_1(S, \mathbb{Z})$ has infinite rank, and that this then has to be true for any manifold in which S is embedded as an open subset.)

Here is a more exotic example, which we will use later on as a counterexample to an alternative version of our main theorem.

Example 5.4 (Exotic \mathbb{R}^4 's) Let R be a manifold whose underlying topological space is \mathbb{R}^4 , but which carries an exotic differentiable structure, and let $h : R \rightarrow \mathbb{R}$ be the length-squared function. Although h might not be differentiable, it is certainly continuous, so $X = \{(x, r) | h(r)x^2 < 1\}$ is an open subset of $(-1, 1) \times R$ and is therefore a smooth manifold. Let f be the projection on the first factor; it is locally trivial topologically but not in the smooth sense. In fact, if R is chosen so that the fibres of f form what Gompf and Stipsicz [11] call a radial family, their Theorem 9.4.10 implies immediately that f is not locally semitrivial, because R cannot be embedded into any of the other fibres.

To end this section, we show that Theorem 5.1 implies our main theorem for the case of groupoids with trivial isotropy.

Corollary 5.5 *Given a submersion $f : X \rightarrow Y$, let $G : X \rightrightarrows Y$ be the groupoid which is the equivalence relation $X \times_Y X$. For $y \in Y$, if $\mathcal{O} = f^{-1}(y)$ is of finite type, then there is a neighborhood \mathcal{U} of \mathcal{O} in X such that $G_{\mathcal{U}}$ is isomorphic to $\mathcal{O} \times \mathcal{O} \times f(\mathcal{U}) \rightrightarrows \mathcal{O} \times f(\mathcal{U})$, the product of the pair groupoid $\mathcal{O} \times \mathcal{O}$ with the trivial groupoid $f(\mathcal{U})$. This product groupoid is in turn isomorphic to the action groupoid for the action of $G_{\mathcal{O}}$ on a neighborhood of the zero section in the normal bundle $N\mathcal{O}$.*

Proof. The diffeomorphism $(\rho, f) : \mathcal{U} \rightarrow \mathcal{O} \times f(\mathcal{U})$ of Theorem 5.1 gives an isomorphism from $G_{\mathcal{U}}$ to $(\mathcal{O} \times f(\mathcal{U})) \times_{f(\mathcal{U})} (\mathcal{O} \times f(\mathcal{U}))$, which is isomorphic to the product groupoid in the statement of the corollary.

Furthermore, the derivative Tf induces an isomorphism between $N\mathcal{O}$ and $\mathcal{O} \times T_y Y$. With respect to this isomorphism, the action of $G_{\mathcal{O}} = \mathcal{O} \times \mathcal{O}$ on $N\mathcal{O}$ is just given by its action on \mathcal{O} , with the trivial action on $T_y Y$, so that the action groupoid is $\mathcal{O} \times \mathcal{O} \times T_y Y$. Finally, after shrinking \mathcal{U} if necessary, we may identify $f(\mathcal{U})$ with a neighborhood of zero in $T_y Y$, which gives the required isomorphism of $G_{\mathcal{U}}$ with the action groupoid.

□

6 Étale groupoids

Recall that a Lie groupoid $G \rightrightarrows X$ is **étale** if its target (or source) map is a local diffeomorphism. The following theorem is the specialization (slightly strengthened) of Theorem 4.1 to the case where the groupoid is étale and the orbit consists of a single point.

Theorem 6.1 *Let $G \rightrightarrows X$ be an étale groupoid with fixed point $x \in X$. Then there is a neighborhood \mathcal{U} of x in X such that the restriction of G to \mathcal{U} is isomorphic to the restriction of the action groupoid $G_x \times T_x X \rightrightarrows T_x X$ to a neighborhood of zero in $T_x X$. If the groupoid is source-locally trivial, the neighborhood \mathcal{U} can be taken to be G -invariant.*

Proof. Since G is proper and étale, G_x is a finite group. Since x is a fixed point, G_x is also the fibre $\alpha^{-1}(x)$. Hence, by (a very simple case of) Theorem 5.1, there is a neighborhood \mathcal{V} of G_x in G such that the restriction of α to \mathcal{V} is a trivial fibration. \mathcal{V} is then a disjoint union of finitely many open subsets of G which are mapped diffeomorphically by α to a neighborhood \mathcal{U}_1 of x . We may assume that \mathcal{U}_1 is connected.

Let r be the unique continuous retraction from \mathcal{V} to G_x . This map is a “homomorphism” (in quotes because \mathcal{V} is not necessarily a subgroupoid of G) in the sense that

$$r(g^{-1}h) = (r(g))^{-1}r(h)$$

for all (g, h) in the fibre product $\mathcal{V} \times_{\mathcal{U}_1} \mathcal{V}$. To prove this, we note that each side of the displayed equation is a continuous function of (g, h) taking values in the discrete space G_x , and that any pair (g, h) in the fibre product can be connected by a path to a pair in $G_x \times G_x$, where the equation is obviously satisfied.

Via the action of G on X (see Example 3.1), the components of \mathcal{V} define an action of G_x by local diffeomorphisms of X fixing x . By the Bochner linearization theorem [1] for actions of compact groups, this action is equivalent in a neighborhood of x to the linearized action of G_x on $T_x X$. In particular, we can find a disc \mathcal{U} about x which is invariant under the action, and hence is invariant for $G_{\mathcal{U}_1}$ (though not necessarily for G itself). The restriction $G_{\mathcal{U}}$ is then isomorphic to the action groupoid for G_x acting on a neighborhood of the origin in $T_x X$.

If G is source-locally trivial, it follows from Theorem 3.3 that \mathcal{U} contains an invariant neighborhood of x ; the isomorphism with a transformation groupoid still holds there.

□

Remark 6.2 We note for later use (in the proof of Theorem 9.1) that the groupoid morphism $r : G_{\mathcal{U}} \rightarrow G_x$ associated to the isomorphism between $G_{\mathcal{U}}$ and an action groupoid is a covering morphism in the sense of [2]; i.e. (r, β) is a diffeomorphism from $G_{\mathcal{U}}$ to $G_x \times \mathcal{U}$.

Remark 6.3 The groupoid G_3 in Example 3.4 is proper and étale but is not isomorphic to an action groupoid on any invariant neighborhood of its fixed point.

Remark 6.4 We conjecture that Theorem 6.1 extends to the non-étale case. See the discussion in Section 4 of [21].

7 Deformation of proper groupoids

In this section, we prove that (target,source)-preserving deformations of regular proper groupoids are trivial. The result will be used to extend the linearization theorem from étale to effectively étale groupoids. As is usual in such deformation problems, we use cohomology.

Theorem 7.1 *Let $\{m_t\}$ be a smooth family of proper, regular, groupoid structures on G over X , defined for $t \in [0, 1]$, having a fixed map $(\alpha, \beta) : G \rightarrow X \times X$ as anchor and a fixed $\epsilon : X \rightarrow G$ as identity section. Then there is a family $\{A_t\}$ of diffeomorphisms of G such that A_0 is the identity, $A_t \circ \epsilon = \epsilon$, and A_t is a groupoid isomorphism from (G, m_t) to (G, m_0) .*

Proof. Since the anchor is fixed, the submanifold $G^{(2)} \subseteq G \times G$ of composable pairs is independent of t . For $(g, h) \in G^{(2)}$, we denote $m_t(g, h)$ by $g *_t h$. The derivative Y_t of m_t with respect to t is a mapping from $G^{(2)}$ to TG which lifts m_t . In fact, for each composable pair (g, h) , $Y_t(g, h)$ lies in the subspace of $T_{g *_t h} G$ which is annihilated by $(T\alpha, T\beta)$. Differentiating the associativity law for the products m_t with respect to t , we obtain the identity

$$Y_t(g, h) *_t k + Y_t(g *_t h, k) = g *_t Y_t(h, k) + Y_t(g, h *_t k), \quad (1)$$

where the operation $*_t$ applied to a tangent vector and an element of G denotes the derivative of right [or left] translation for the multiplication m_t ; this derivative acts on vectors tangent to the fibres of β [or α].

Equation (1) is actually a cocycle condition. To see this, we right-translate the vectors $Y_t(g, h)$ back to the identity section, i.e. we define $c_t(g, h) = Y_t(g, h) *_t (g *_t h)^{-1}$. Since the values of Y_t are tangent to the fibres of both α and β , the same is true of the values of c_t ; i.e. $c_t(g, h)$ belongs to the fibre at $\alpha(g *_t h) = \alpha(g)$ of the isotropy subalgebroid \mathfrak{b} of the Lie algebroid \mathfrak{g} of G . As a vector bundle, \mathfrak{b} is independent of t , and for each t there is an adjoint action of G on \mathfrak{b} defined by $g \bullet_t v = g *_t v *_t g^{-1}$. (See also Appendix B of [10]).

The infinitesimal associativity law (1) for Y_t becomes the following identity for c_t :

$$c_t(g, h) + c_t(g *_t h, k) = g \bullet_t c_t(h, k) + c_t(g, h *_t k).$$

This identity says precisely that c_t is a 2-cocycle on G with values in the representation bundle \mathfrak{b} .

We now appeal to Proposition 1 of [5], which establishes the triviality of the higher cohomology of any proper groupoid with coefficients in any representation. This provides us with a family $\{b_t\}$ of 1-cochains whose coboundaries are c_t ; i.e. $b_t : G \rightarrow \mathfrak{b}$ with $b_t(g) \in \mathfrak{b}_{\alpha(g)}$ and

$$b_t(g) + g \bullet_t b_t(h) - b_t(g *_t h) = c_t(g, h). \quad (2)$$

Although it is not stated explicitly in [5], it follows from the proof of Proposition 1 that the “primitive” b_t can be chosen to depend smoothly on t . (Alternatively, one can apply Proposition 1 to the single groupoid $G \times [0, 1] \rightrightarrows X \times [0, 1]$ obtained by combining all the (G, m_t) .) Furthermore, the fact that all m_t agree along the identity section implies that $Y_t(g, g) = 0$ whenever g is an identity element, and then the construction of b_t shows that $b_t(g) = 0$ as well.

Reversing the translation procedure which led from Y_t to c_t , we construct from b_t the vector field X_t on G defined by $X_t(g) = b_t(g) *_t g$. Since these vector fields vanish on the identities, the family $\{X_t\}$ may be integrated at least locally to a smooth family $\{A_t\}$ of diffeomorphisms of G which fix the identities and which commute with the anchor (α, β) . Since the anchor is proper, the integration can be done globally. Finally, the coboundary relation (2) is just the differentiated version of the statement that A_t is, for all t , a groupoid homomorphism from (G, m_t) to (G, m_0) .

□

Example 7.2 When X is a point, Theorem 7.1 implies the stability of group structures on compact Lie groups and, with a “multiparameter t ,” the

local triviality of smooth bundles of Lie groups. In particular, the isotropy subgroupoid of a regular proper groupoid is such a bundle, and hence all the isotropy groups over a connected component of the base are isomorphic.

Similarly, if we have a locally trivial bundle of compact groups over a base manifold Y acting freely on a locally trivial fibration $X \rightarrow Y$, the action groupoids form a locally trivial bundle of proper regular groupoids over Y , and the orbit spaces form a locally trivial bundle over Y .

Remark 7.3 It seems likely that Theorem 7.1 remains valid even if the proper groupoid G is not regular. In this case, the cohomology problem lives in a family of vector spaces which is not a smooth bundle, but this should not be a serious difficulty.

Finally, we raise the question of deformability of proper groupoids, regular or not, without the restriction that the anchor be fixed. It seems possible that a rigidity theorem like Theorem 7.1 might still hold; it would be related to the Reeb stability theorem.

8 Effectively étale groupoids

A groupoid G over X is étale when its maximal source-connected subgroupoid is trivial. We say that G is **effectively étale** when its maximal source-connected subgroupoid acts trivially on X ; equivalently, G is effectively étale when the anchor $\mathcal{A}(G) \rightarrow TX$ of its Lie algebroid is identically zero.

If G is effectively étale, then its maximal source-connected subgroupoid is a bundle of groups B which is a normal subgroupoid of G , and the quotient G/B is étale. In other words, an effectively étale groupoid is an extension of an étale groupoid by a bundle of groups. When the groupoid is proper, the bundle of groups is locally trivial (see Example 7.2).

Theorem 8.1 *Let $G \rightrightarrows X$ be an effectively étale groupoid with fixed point $x \in X$. Then there is a neighborhood \mathcal{U} of x in X such that the restriction of G to \mathcal{U} is isomorphic to the restriction of the action groupoid $G_x \times T_x X \rightrightarrows T_x X$ to a neighborhood of zero in $T_x X$. If the groupoid is source-locally trivial, the neighborhood \mathcal{U} can be taken to be G -invariant.*

Proof. Let B be the maximal source-connected subgroupoid of G . Since G is proper, so is G/B , and hence we may apply Theorem 6.1 to find a neighborhood \mathcal{U} of X such that $(G/B)_{\mathcal{U}}$ can be identified with the restriction of the action groupoid $(G/B)_x \times T_x X \rightrightarrows T_x X$ to a neighborhood \mathcal{V} of 0 in

$T_x X$. Note that $(G/B)_x \cong G_x/B_x$. By choosing a representative of each coset of B_x , we also obtain a diffeomorphism (though generally not a group isomorphism) between G_x and $B_x \times G_x/B_x$.

By Theorem 5.1, B is a locally trivial bundle of groups; its typical fibre is B_x . Hence, as a manifold, $G_{\mathcal{U}}$ may be identified with $B_x \times G_x/B_x \times \mathcal{V}$, and the target and source maps of its groupoid structure factor through those of the action groupoid $G_x/B_x \times \mathcal{V} \rightrightarrows \mathcal{V}$. Recalling that the groupoid multiplication on the latter has the form $(g, hy)(h, y) = (gh, y)$, we find that the multiplication on $B_x \times G_x/B_x \times \mathcal{V}$ coming from the groupoid structure on G must have the form $m((a, g, hy), (b, h, y)) = (M(a, b, g, h, y), gh, y)$, where $M : B_x \times B_x \times G_x/B_x \times G_x/B_x \times \mathcal{U} \rightarrow B_x$ is a smooth map. When $y = 0$ (corresponding to the fixed point x in \mathcal{U}), we have $(M(a, b, g, h, 0), gh) = (a, g) \cdot (b, h)$, the product in the isotropy group G_x .

We may now construct for $t \in [0, 1]$ the smooth 1-parameter family of multiplications m_t all having the anchor $(\alpha, \beta)(b, h, y) = (hy, y)$ by the formula

$$m_t((a, g, hy), (b, h, y)) = (M(a, b, g, h, ty), gh, y).$$

These multiplications are all associative. To see this, we note first that $t(hy) = h(ty)$ since the action of G_x on $T_x X$ is linear, so

$$m((a, g, thy), (b, h, ty)) = (M(a, b, g, h, ty), gh, ty),$$

which shows that, for $t \neq 0$, m_t is the pullback of the associative operation m by the diffeomorphism $(a, g, y) \mapsto (a, g, ty)$. On the other hand, for $t = 0$, we have

$$m_0((a, g, hy), (b, h, y)) = (M(a, b, g, h, 0), gh, y) = ((a, g) \cdot (b, h), y)$$

which is the product in the action groupoid $G_x \times \mathcal{V} \rightrightarrows \mathcal{V}$. We are thus in a position to apply Theorem 7.1, which gives an isomorphism between this action groupoid and the groupoid with multiplication m_1 , which is just $G_{\mathcal{U}}$ itself.

If G is assumed source-locally trivial, then, by Theorem 3.3, x is a stable fixed point from G , so the neighborhood \mathcal{U} contains a G -invariant neighborhood on which we still have an isomorphism with the linear approximation.

□

9 The slice theorem

A **slice** through a point x on an orbit \mathcal{O} of a groupoid $G \rightrightarrows X$ will be defined simply as a submanifold Σ of X which meets \mathcal{O} only at x , with

$T_x X = T_x \Sigma \oplus T_x \mathcal{O}$. Only a small neighborhood of x in a slice will be of interest, and we can choose the neighborhood small enough so that it is everywhere transverse to the orbits of G , so that the restriction G_x is again a Lie groupoid. Since, by Proposition 2.4, \mathcal{O} is a closed submanifold, we can also suppose that Σ intersects \mathcal{O} only at x , so that G_Σ has x as a fixed point.

In this section, we will show that a proper groupoid can be linearized around the orbit \mathcal{O} if its restriction to a slice can be linearized around $x \in \mathcal{O}$, and if \mathcal{O} is of finite type. Combined with Theorem 8.1, this slice theorem will immediately imply our main theorem.

Theorem 9.1 *Let $G \rightrightarrows X$ be a proper groupoid, and let \mathcal{O} be an orbit of G which is a manifold of finite type. Suppose that the restriction of G to a slice through $x \in \mathcal{O}$ is isomorphic to the restriction of the action groupoid $G_x \times N_x \mathcal{O} \rightrightarrows N_x \mathcal{O}$ to a neighborhood of zero. Then there is a neighborhood \mathcal{U} of \mathcal{O} in X such that the restriction of G to \mathcal{U} is isomorphic to the restriction of the action groupoid $G_{\mathcal{O}} \times_{\mathcal{O}} N \mathcal{O} \rightrightarrows N \mathcal{O}$ to a neighborhood of the zero section.*

Proof. The proof involves several steps, the basic idea being to apply Theorem 5.1 to the restriction of α to $G_{\Sigma X}$, where Σ is a slice.

Step 1. Let Σ be a slice as in the statement of the theorem, assumed small enough so that it is everywhere transverse to the orbits of G and intersects \mathcal{O} only at x , so that x is a fixed point of G_Σ . G_x is the isotropy group of x in both G and G_Σ , and the natural identification of $N_x \mathcal{O}$ with $T_x \mathcal{O}$ is G_x -equivariant.

By assumption, the restriction G_Σ is isomorphic, via a retraction of groupoids $r : G_\Sigma \rightarrow G_x$ and an open embedding $i : \Sigma \rightarrow T_x \Sigma$, to the restriction of the action groupoid $G_x \times T_x \Sigma \rightrightarrows T_x$ to a neighborhood of zero. We may assume that this neighborhood is invariant for the action groupoid. We then have an action of G_x on Σ as well as an embedding of G_x into the group of bisections of G_Σ .

Step 2. Since the target map α is a submersion, $G_{\Sigma X} = \alpha^{-1}(\Sigma)$ is a closed submanifold of G . The compact group G_x acts on Σ as mentioned in Step 1, and it also acts freely on $G_{\Sigma X}$ by left translations via the embedding of G_x into the group of bisections of G_Σ . The restricted submersion $\alpha : G_{\Sigma X} \rightarrow \Sigma$ is equivariant with respect to these actions. Applying Theorem 5.1, we find a G_x -equivariant local trivialization of α on a neighborhood \mathcal{V} of $G_{xX} = \alpha^{-1}(x)$ in $G_{\Sigma X}$. That is, there is a G_x -equivariant retraction $\rho : \mathcal{V} \rightarrow G_{xX}$ such that $(\rho, \alpha) : \mathcal{V} \rightarrow G_{xX} \times \Sigma$ is a diffeomorphism. We will denote the inverse of this diffeomorphism by Φ .

The G_x orbits in \mathcal{V} are just the fibres of the source map β , so if we let \mathcal{U} be $\beta(\mathcal{V})$, we obtain by equivariance a retraction, also to be denoted by ρ , from \mathcal{U} to $\beta(G_{xX}) = \mathcal{O}$.

Note that $\beta : \mathcal{V} \rightarrow \mathcal{U}$ and $\beta : G_{xX} \rightarrow \mathcal{O}$ are principal bundles with structure group G_x (contrary to the usual conventions, the structure group is acting on the left).

Step 3. In this step, we will construct a retraction of groupoids $R : G_{\mathcal{U}} \rightarrow G_{\mathcal{O}}$.

Let $p \in G_{\mathcal{U}}$. Since $\alpha(p)$ and $\beta(p)$ both belong to $\mathcal{U} = \beta(\mathcal{V})$, we can find h and k in \mathcal{V} such that $\beta(h) = \beta(p)$ and $\beta(k) = \alpha(p)$. The product kph^{-1} is then defined and, since $\alpha(kph^{-1}) = \alpha(k)$ and $\beta(kph^{-1}) = \beta(h^{-1}) = \alpha(h)$, kph^{-1} lies in G_{Σ} ; hence, we have the element $r(kph^{-1})$ of G_x . Now we define $R(p)$ to be $\rho(k)^{-1}r(kph^{-1})\rho(h)$. The target and source of $R(p)$ are then $\alpha(R(p)) = \beta(\rho(k)) = \rho(\beta(k)) = \rho(\alpha(p))$ and $\beta(R(p)) = \beta(\rho(h)) = \rho(\beta(h)) = \rho(\beta(p))$ respectively; in particular, $R(p)$ belongs to $G_{\mathcal{O}}$.

We must show that $R(p)$ is independent of the choices we made. If h and k are replaced by h' and k' , we have $h' = bh$ and $k' = ck$ for b and c in G_{Σ} . Now we have $k'ph'^{-1} = c(kph^{-1})b^{-1}$, so $r(k'ph'^{-1}) = r(c)r(kph^{-1})r(b)^{-1}$, so $\rho(k')^{-1}r(k'ph'^{-1})\rho(h') = \rho(ck)^{-1}r(c)r(kph^{-1})r(b)^{-1}\rho(bh)$. By the equivariance of ρ , this becomes

$$\rho(k)^{-1}r(c)^{-1}r(c)r(kph^{-1})r(b)^{-1}r(b)\rho(h) = \rho(k)^{-1}r(kph^{-1})\rho(h)$$

as before, so $R(p)$ is well defined.

To see that R is a retraction, if we assume that p belongs to $G_{\mathcal{O}}$, we may choose h and k in G_{xX} , the image of the retraction ρ , so that $kph^{-1} \in G_X$, and hence $R(p) = \rho(k)^{-1}r(kph^{-1})\rho(h) = k^{-1}kph^{-1}h = p$.

Finally, we will show that R is a groupoid homomorphism, with the map on objects being $\rho : \mathcal{U} \rightarrow \mathcal{O}$. We have already seen that R and ρ are compatible with the source and target maps. If p and q are in \mathcal{U} with $\beta(p) = \alpha(q)$, and we choose h and k as above to compute $R(p)$, then we may use some g and the same h to compute $R(q)$, while g and k may be used to compute $R(pq)$. With $a = kph^{-1}$ and $b = hqg^{-1}$, we find $ab = kpqg^{-1}$, and so $R(p)R(q) = \rho(k)^{-1}r(a)\rho(h)\rho(h)^{-1}r(b)\rho(g) = \rho(k)^{-1}r(ab)\rho(g) = R(pq)$.

Step 4. To identify $G_{\mathcal{U}}$ with an action groupoid, we will show that the homomorphism R is a covering morphism in the sense of [2]. This means that we must show that the map (R, β) is a diffeomorphism from $G_{\mathcal{U}}$ to $G_{\mathcal{O}} \times_{\mathcal{O}} \mathcal{U}$, the fibre product with respect to the pair (β, ρ) . To do so, we will construct an inverse map Ψ .

Let $(m, z) \in G_{\mathcal{O}} \times_{\mathcal{O}} \mathcal{U}$. Since $\mathcal{U} = \beta(\mathcal{V})$, we may write $z = \beta(h)$ for some $h \in \mathcal{V}$; in particular, $\alpha(h) \in \Sigma$. Then $\beta(\rho(h)) = R(\beta(h)) = R(z) =$

$\beta(m)$, since (m, z) is in the fibre product, so the product $u = \rho(h)m^{-1}$ is defined, with $\alpha(u) = x$ and $\beta(u) = \alpha(m) \in \mathcal{O}$. Recalling that $(\rho, \alpha) : \mathcal{V} \rightarrow G_{xX} \times \Sigma$ is a diffeomorphism with (equivariant) inverse Φ , we set $\Psi(m, z) = \Phi(u, \alpha(h))^{-1}h$, i.e. $\Psi(m, z) = \Phi(\rho(h)m^{-1}, \alpha(h))^{-1}h$.

To see that Ψ is well defined, we replace h by $h' = ah$, where $a \in G_\Sigma$. Carrying out the construction of the previous paragraph with this new choice, we have $u' = \rho(ah)m^{-1} = r(a)\rho(h)m^{-1}$ and

$$\begin{aligned}\Phi(u', \alpha(h')) &= \Phi(r(a)\rho(h)m^{-1}, \alpha(ah)) = \Phi(r(a)\rho(h)m^{-1}, a\alpha(h)) \\ &= a\Phi(\rho(h)m^{-1}, \alpha(h)) = a\Phi(u, \alpha(h)).\end{aligned}$$

Thus $\Phi(u', \alpha(h'))^{-1}h' = \Phi(u, \alpha(h))^{-1}a^{-1}ah = \Phi(u, \alpha(h))^{-1}h$, so Ψ is well-defined.

Now we show that Ψ is indeed an inverse to (R, β) . For $p \in G\mathcal{U}$, we choose h and k as in Step 3 and set $a = kph^{-1}$; then $R(p) = \rho(k)^{-1}r(a)\rho(h)$. Now, since $\beta(h) = \beta(p)$,

$$\begin{aligned}\Psi(R(p), \beta(p)) &= \Phi(\rho(h)R(p)^{-1}, \alpha(h))^{-1}h \\ &= (\Phi(\rho(h)\rho(h)^{-1}r(a)^{-1}\rho(k), \alpha(h))^{-1}h \\ &= (\Phi(r(a)^{-1}\rho(k), \alpha(h))^{-1}h(\Phi(r(a)^{-1}\rho(k), a^{-1}a\alpha(h))^{-1}h \\ &= (\Phi(\rho(k), \alpha(ah))^{-1}ah = (\Phi(\rho(k), \alpha(kp))^{-1}kp = k^{-1}kp = p.\end{aligned}$$

In the other direction, for $(m, z) \in G_{\mathcal{O}} \times_{\mathcal{O}} \mathcal{U}$, with $z = \beta(h)$, we have

$$\beta(R(\Psi(m, z))) = \beta(\Phi(\rho(h)m^{-1}, \alpha(h))^{-1}h) = \beta(h) = z,$$

while

$$R(\Psi(m, z)) = R(\Phi(\rho(h)m^{-1}, \alpha(h))^{-1}h).$$

To show that the last expression is equal to m , thus completing the proof, we may choose for the h in the definition of R the h which we used to define $\Psi(m, z)$. For the k in the definition we take $\Phi(u, \alpha(h)) = \Phi(\rho(h)m^{-1}, \alpha(h))$. Then

$$\begin{aligned}R(\Psi(m, z)) &= \rho(k)^{-1}r(k\Psi(m, z)h^{-1})\rho(h) \\ &= \rho\Phi(u, \alpha(h))^{-1}r(\Phi(u, \alpha(h))\Psi(m, z)h^{-1})\rho(h) \\ &= u^{-1}r(hh^{-1})\rho(h) = u^{-1}\rho(h) = m.\end{aligned}$$

□

10 Proof of the main theorem

For convenience, we restate the main theorem.

Theorem 4.1 *Let $G \rightrightarrows X$ be a regular, proper Lie groupoid, and let \mathcal{O} be an orbit of G which is a manifold of finite type. Then there is a neighborhood \mathcal{U} of \mathcal{O} in X such that the restriction of G to \mathcal{U} is isomorphic to the restriction of the action groupoid $G_{\mathcal{O}} \times_{\mathcal{O}} N\mathcal{O} \rightrightarrows N\mathcal{O}$ to a neighborhood of the zero section in $N\mathcal{O}$.*

Proof.

Let Σ be a slice through $x \in \mathcal{O}$. If Σ is chosen small enough, then x is a fixed point of G_{Σ} , so the anchor of the Lie algebroid $\mathcal{A}(G_{\Sigma})$ is zero at x . Since G is regular, so is G_x , and hence the anchor of $\mathcal{A}(G_{\Sigma})$ is identically zero; i.e. G_{Σ} is effectively étale. By Theorem 6.1, we can choose Σ small enough so that G_{Σ} is locally isomorphic to its linearization at x . By Theorem 9.1, G is isomorphic to its linearization along \mathcal{O} .

□

We close with some remarks.

Example 10.1 It was tempting to substitute an assumption of source-local triviality of $G \rightrightarrows X$ for the hypothesis in the main theorem that \mathcal{O} be of finite type. The following example shows that this is not possible.

We begin with the submersion $f : X \rightarrow (-1, 1)$ of Example 5.4. Let $G \rightrightarrows X$ be the groupoid which is the product of the equivalence relation $X \times_{(-1,1)} X \rightrightarrows X$ and the group $SU(2)$, where the latter is considered as a groupoid over a one-point base. The fibre over (x, r) of the source map of G is $f^{-1}(r) \times SU(2)$, which is topologically the product $\mathbb{R}^4 \times SU(2)$, and the source map is locally trivial as a topological fibration. By Theorem 2 of [14], the source map is locally trivial in the differentiable sense as well; i.e. $G \rightrightarrows X$ is source-locally trivial.

On the other hand, $G \rightrightarrows X$ is not isomorphic to its linearization around $\mathcal{O} = f^{-1}(0)$ in any neighborhood of \mathcal{O} , since if it were, the restriction of f to such a neighborhood would be locally trivial, and we saw in Example 5.4 that this is not the case.

Finally, we would also like to mention an alternative approach to understanding proper regular groupoids, due in part to I. Moerdijk (private communication). Instead of first restricting $G \rightrightarrows X$ to a slice to get an

effectively étale groupoid and then dividing by the identity component of the isotropy, we may first divide G itself by the identity component C of its entire isotropy, which turns out to be a smooth bundle of compact groups. The quotient G/C is then a **foliation groupoid** over X , i.e. it is a groupoid for which the Lie algebroid anchor is injective. (Note that the orbits, or “leaves,” of $G/C \rightrightarrows X$ are not necessarily connected.) The foliation groupoid can be analyzed via the slice theorem in terms of its restriction to a slice, which is the same étale groupoid as was obtained in the first approach. Finally, the original groupoid $G \rightrightarrows X$ may be seen as an extension of a foliation groupoid by a bundle of compact groups. The bundle is locally trivial (Example 7.2), and the extension is then classified by a degree 2 cohomology class of the foliation groupoid with values in the bundle of groups. If we restrict to a neighborhood of an orbit, we can again use a deformation argument, somewhat more complicated than that in Theorem 8.1, to recover the main linearization theorem.

A Appendix: Proper mappings

Perhaps the most common definition of properness for mappings between topological spaces is:

Definition A.1 *A mapping $f : X \rightarrow Y$ between topological spaces is proper if it is continuous and if $f^{-1}(A)$ is compact in X for every compact subset A of Y .*

There are two other definitions of properness which are equivalent to this one when X and Y are Hausdorff spaces, but which differ in general. Since many interesting groupoids, such as holonomy groupoids of foliations, may not be Hausdorff, we mention these other definitions.

James [12] defines properness of a map in the following way.

Definition A.2 *A mapping $f : X \rightarrow Y$ between topological spaces is proper if it is continuous and if, for every topological space Z , the product mapping $f \times 1_Z : X \times Z \rightarrow Y \times Z$ is closed, in the sense that it maps closed sets to closed sets.*

Actually, James calls such mappings “compact,” but we will not use this term. James actually goes on to define a topological space X to be compact if the map from X to a point is proper, and then he proves that this definition is equivalent to the usual one in terms of open coverings. He also proves:

Proposition A.3 *A continuous mapping is proper if and only if it is closed and the inverse image of each point is compact.*

Another definition of properness is given by Crainic and Moerdijk in [6].

Definition A.4 *A continuous mapping $f : X \rightarrow Y$ is proper if: (i) the image of the diagonal $X \rightarrow X \times_Y X$ is closed; and (ii) $f^{-1}(A)$ is compact whenever A is a compact subset of a Hausdorff open subset of Y .*

This definition is adapted to the study of non-Hausdorff groupoids and their associated operator algebras. It leads easily to the conclusion that, if $G \rightrightarrows X$ is a proper groupoid, and X is Hausdorff, then G is Hausdorff as well. It is not clear how this definition relates to the one which is expressed in terms of products.

For the reader's convenience, we also include here a proof of a standard fact used in Section 5.

Lemma A.5 *If K is a compact group, any K -manifold M admits a complete, invariant riemannian metric.*

Proof. It is clearly sufficient to prove the lemma under the assumption that M is connected. This implies that M admits a partition of unity by compactly supported functions $M \rightarrow [0, 1]$ which can be enumerated π_1, π_2, \dots . The sum $\sum_{n=1}^{\infty} n\pi_n$ (or the corresponding finite sum if M is compact) is then a proper, nonnegative function on M . By averaging with respect to K we obtain an invariant function λ . Using the compactness of K , one shows easily that the averaged function is again proper.

Now let $\langle \cdot, \cdot \rangle_0$ be any invariant riemannian metric on M , and define the new metric $\langle \cdot, \cdot \rangle$ to be $(1 + \langle \nabla_0 \lambda, \nabla_0 \lambda \rangle) \langle \cdot, \cdot \rangle_0$. Then the gradient $\nabla \lambda$ of λ with respect to $\langle \cdot, \cdot \rangle$ has length everywhere less than 1. It follows that $\langle \cdot, \cdot \rangle$ is complete, since the proper function λ is bounded on any curve of finite length, which implies that closed bounded subsets of M are compact.

□

B Appendix: Manifolds of finite type

There does not seem to be a standard name for the following concept.

Definition B.1 *Let K be a compact group. A K -manifold of finite type is a K -manifold M which admits a proper K -invariant function whose critical points form a compact set. When K is the trivial group, we simply say that M is a manifold of finite type.*

Here are some elementary observations about manifolds of finite type.

By squaring a given proper function, we can arrange that the proper function in the definition above take values in $[0, \infty)$. If K is finite, we can arrange that the function have finitely many critical points, and even that they be nondegenerate if K is trivial. We also note that M is a K -manifold of finite type if and only if it is equivariantly diffeomorphic to the interior of a compact K -manifold with boundary.

If M is of finite type, then so is any K -equivariant bundle over M with compact fibres, e.g. a finite covering. If K is a compact group acting freely on M , then M is a K -manifold of finite type if and only if M/K is a manifold of finite type. But note the following example.

Example B.2 There exist many manifolds which are homeomorphic to \mathbb{R}^4 and which are not of finite type. (See [11]. In fact, Gompf and Stipsicz remark on page 366 that the existence of an exotic \mathbb{R}^4 of finite type would lead to a counterexample to the differentiable Poincaré conjecture in either dimension 3 or dimension 4.) Let Q be such a manifold of infinite type, and let $M = Q \times SU(2)$. Then M is simply connected and is simply connected at infinity, so it is of finite type, according to Siebenmann [19]. Let $SU(2)$ act on M by left translation on the second factor. Then $M/SU(2)$ is Q , which is not of finite type.

Remark B.3 It would be interesting to know whether a finite group can act freely on a manifold of finite type such that the quotient manifold is of infinite type. An example might be used to extend Example 10.1 to cover the étale case.

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