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Vienna, Preprint ESI 105 (1994) May 31, 1994

Supported by Federal Ministry of Science and Research, Austria
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UWThPh-1994-18
June 1, 1994

Quasifree Second Quantization and Its Relation to Noncommutative Geometry

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Abstract

Schwinger terms of current algebra can be identified with nontrivial cyclic cocycles of a Fredholm module. We discuss its temperature dependence. Similar anomalies may occur also in spin systems. In simple examples already an operator–valued cocycle shows up.

Lectures given at the XXX–th Karpacz Winter School in Theoretical Physics, Poland, 1994.

1 Introduction

A long time ago, we became familiar with the occurrence of nontrivial cocycles, while studying group representations. They occur already if we consider chiral transformations within the simplest models of quantum field theory. We shall discuss external field problems since they provide examples of many body systems, which exhibit features like fractional charges [1], geometric phases [2] and anomalous commutation relations. Here we shall focus on the temperature dependence of the latter [3].

We shall follow the algebraic approach and treat in part two the most general quasifree states over the C* algebra generated by operators fulfilling the canonical anticommutation rules (CAR). The GNS construction provides us with a Hilbert space, an algebra representation and a cyclic vector. Next we shall try to represent unitary operators from the one–particle Hilbert space. If the second quantized implementer exists, we say that the symmetry is of the Wigner type, if it does not exist, the symmetry is said to be spontaneously broken. In the first case Schwinger terms occur in the algebra of charges, which are cyclic cocycles of a two–summable Fredholm module [4] (chapter four). The implementers provide us with a multiplier representation.

If the state is not pure, the GNS representation is reducible and the implementers of unitary transformations are defined up to elements of the commutant. There are various ways to proceed. In part three we discuss a definition of charges in temperature states, such that the cocycles remain temperature independent [5]. We mention a different procedure which allows to kill the anomaly [6]. Part five contains a short discussion of an anomaly in spin systems [7]. Both procedures are compared and give quite different answers.

2 Quantization of External Field Problems

Let $\mathcal{H}$ be the Hilbert space of a relativistic fermi field in two dimensions. To any $f \in \mathcal{H}$ we assign creation and annihilation operators $a^\dagger(f)$ and $a(f)$ fulfilling the CAR

$$\{a(f), a^\dagger(g)\} = \langle f|g\rangle, \qquad \{a(f), a(g)\} = 0. \tag{1}$$

To each positive operator $0 \leq T \leq 1$ there corresponds a quasifree state ω_T over the CAR with vanishing truncated n–point functions, and a two–point function, which is given by

$$\omega_T(a(f)a^\dagger(g)) = \langle f|Tg\rangle. \tag{2}$$

Denote by $\mathcal{H}_T$, Π_T and Ω_T the Hilbert space, the representation and the cyclic vector associated with ω_T via the GNS representation. The spectral properties of T imply properties of the GNS triple [8]. If T has discrete spectrum, ω_T is a factor state which is pure, if and only if T is a projector. Π_T is then irreducible and called a Fock representation.

Typical examples are given by taking T to be the projector of the positive (or negative) part P_+ (or P_-) of a Dirac operator. This corresponds to filling the Dirac sea, respectively the Fermi ball, $P_+ - P_- = F$ determines the polarization.

Two representations, Π_{T_1} and Π_{T_2} are equivalent, in the sense that the generated von Neumann algebras are isomorphic, if and only if $\sqrt{T_1} - \sqrt{T_2}$ and $\sqrt{1-T_1} - \sqrt{1-T_2}$ have finite Hilbert–Schmidt norm. For Fock representations the simplified Shale Stinespring criterion results: Two Fock representations are equivalent if and only if $T_1 - T_2$ is Hilbert–Schmidt.

We may apply the above results to clarify the question of implementability of unitary transformations U of the one particle space. Given U we obtain an automorphism of the CAR by defining

$$\tau_U \; a(f) = a(Uf). \tag{3}$$

The existence of the second quantized operator $\Gamma_T(U)$ implementing the Bogoliubov transformation (3) through the relation

$$\Gamma_T(U)\Pi_T(a(f))\Gamma_T^\dagger(U) = \Pi_T(\tau_U(a(f))), \tag{4}$$

is equivalent to the condition of unitary equivalence of the states ω_T and $\omega_{UTU^\dagger}$ and equivalent to requiring that [8]

$$\|[F,U]\|_{HS} < \infty. \tag{5}$$

Example: Let $T = \Theta(-H_0)$ and H_0 be the free massive Dirac operator. If we take for $U = \exp(i\gamma_5\Lambda_5(x))$ axial gauge transformations, (5) implies the asymptotic conditions that $\Lambda_5(\pm\infty) = N_\pm\pi$ with $N_\pm \in Z$. The difference $|N_+ - N_-|$ equals the charge of the new vacuum.

Implementation of unitary groups $U_t = \exp(tA)$ requires for the generators that $\|P_\pm U P_\mp\|_{HS} < \infty$, and $\Lambda_5(+\infty) = \Lambda_5(-\infty)$ for the above example.

The second quantized form of the generators $d\Gamma_P(A) = \,:Q(A):$ have to be defined in its normal ordered form. If we work first with trace–class operators we easily derive the algebra of charges

$$[:Q(A):, :Q(B):] = \,:Q([A,B]): + S(A,B). \tag{6}$$

(6) can be extended to A and B's being in the Hilbert–Schmidt class. The Schwinger term

$$S(A,B) = 2i \; \mathrm{Im} \; \mathrm{Tr} \; (P_- A P_+ B P_-) \tag{7}$$

depends on the polarization and cannot be removed by redefining the smeared charges. Special examples are the current algebra cocycles, Kac–Moody algebra extensions as well as the Virasoro algebra extension.

By irreducibility of the Fock representation the implementable automorphisms extend to inner automorphisms of $\Pi_T''(\mathrm{CAR})$ and $d\Gamma_P$ are automatically affiliated and can be obtained as strong limits from local quantities. The Schwinger term (7) turns out to be identical to the curvature of the determinant line bundle of an infinite dimensional Grassmannian and occurs even in a classical setting as an obstruction for lifting a group action [9].

3 Current Algebra at Finite Temperature

At finite temperature we take $T = (e^{\beta H} + 1)^{-1}$ and Π_T becomes reducible. From the Tomita Takesaki theory it follows that $\Pi_T''(\mathrm{CAR})$ is antiisomorphic to its commutant, since the KMS property of ω_T implies that Ω_T is cyclic and separating. Therefore the arbitrary phase factors of the zero temperature implementers become now arbitrary unitary elements from the commutant.

The degree of arbitrariness in choosing the implementers is best seen by applying the doubling trick [8]. Define

$$P = \begin{pmatrix} T & (T(1-T))^{1/2} \\ (T(1-T))^{1/2} & 1-T \end{pmatrix}. \tag{8}$$

P is a projector on $\mathcal{H} \oplus \mathcal{H}$ such that the Fock state ω_P over CAR($\mathcal{H} \oplus \mathcal{H}$) restricts to ω_T over CAR($\mathcal{H} \oplus 0$) $\simeq$ CAR($\mathcal{H}$). The temperature representation corresponds therefore to a Fock representation of the doubled system. All results on symmetries stated before apply to thermal representations of the doubled system, provided we give a prescription how the Bogoliubov automorphism of CAR($\mathcal{H}$) is lifted to a Bogoliubov automorphism of CAR($\mathcal{H} \oplus \mathcal{H}$). Let ϕ be a homomorphism of the set of unitary operators on $\mathcal{H}$ which are implementable; a possible lift is given by $U \mapsto U \oplus \phi(U)$. The most interesting cases are:

$\phi = id$. Then $U \to U \oplus U$ induces a lift $X \mapsto X \oplus X$ in the Lie algebra. An easy calculation confirms that the Schwinger term vanishes identically for this diagonal embedding; the anomaly melts. A drawback is that the charges $d\Gamma_P(X \oplus X)$ are not affiliated to $\Pi_P''(\mathrm{CAR}(\mathcal{H} \oplus 0))$ and do not correspond to observable quantities.

$\phi \equiv \mathbf{1}$. Then $U \mapsto U \oplus \mathbf{1}$ leads to $X \mapsto X \oplus 0$ and the charges obey (6) with S given by

$$S(A, B) = 2i \ \mathrm{Im} \ \mathrm{Tr} \ TA(1-T)B. \tag{9}$$

In this case the charges are affiliated to $\Pi_P''(\mathrm{CAR}(\mathcal{H} \oplus 0))$ and are physical observables.

4 Connection to Non–Commutative Geometry

A p–summable Fredholm module $(\mathcal{H}, F, \Gamma, \mathcal{A}, \Pi)$ [10] consists of an algebra which is represented through Π as bounded operators on a Hilbert space $\mathcal{H}$. Γ defines the grading of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\Gamma\mathcal{H}_\pm = \pm\mathcal{H}_\pm$. F is an odd operator such that $F^2 = 1$. The set of a's $\in \mathcal{A}$, such that $[F, \Pi(a)] \in \mathcal{L}^p(\mathcal{H})$, where $\mathcal{L}^p$ denotes the p–th Schatten class, form two–sided ideals within $\mathcal{B}(\mathcal{H})$. $[\cdot,\cdot]$ denotes the graded commutator. One easily shows that $da = i[F, \Pi(a)]$ defines an anti–derivation of $\Pi(\mathcal{A})$ and $d^2 = 0$. We define $\Omega^0(\mathcal{A}) = \mathcal{A}$, $\Omega^1(\mathcal{A}) = \{a_0 da_1 | a_i \in \mathcal{A}\} \dots \Omega^k(\mathcal{A}) = \{a_0 da_1 \dots da_k | a_j \in \mathcal{A}\}$ and obtain a graded differential complex $\bigoplus_k \Omega^k(\mathcal{A})$.

An element of $\bigotimes^{p+1} \mathcal{A}$ is called a p–chain in this setting. They form sets C_p. On C_p the Hochschild homology can be defined. Dual to the complex of chains there is the complex of cochains, which are multilinear functions from C_p to $\mathbf{C}$. Let $s(a_0, \dots, a_n)$ be a multilinear function of $n+1$ elements $a_i \in \mathcal{A}$. The Hochschild coboundary operator δ assigns to s a function δs of $n+2$ elements from $\mathcal{A}$:

$$(\delta s)(a_0, \dots, a_{n+1}) := \sum_{j=0}^{n} s(a_0, \dots, (a_j a_{j+1}), \dots, a_{n+1}) + (-1)^{n+1} s((a_{n+1} a_0), a_1, \dots, a_n). \tag{10}$$

It is easy to show that $\delta(\delta s) = 0$ for all s. A cyclic n–cocycle is a multilinear function $s(a_0, \dots, a_n)$ which fulfills $\delta s = 0$ and

$$s(a_0, \dots, a_n) = (-1)^n s(a_n, a_0, \dots, a_{n-1}). \tag{11}$$

A cochain s on n elements is called a coboundary, if there exists a cochain r on $n-1$ elements with $s = \delta r$. The quotient of cyclic cocycles modulo coboundaries defines the cyclic cohomology.

Take for example $n = 1$. Then $s(a_0, a_1) = -s(a_1, a_0)$ has to be antisymmetric and $s(a_0 a_1, a_2) - s(a_0, a_1 a_2) + s(a_2 a_0, a_1) = 0$ is the cocycle property. A coboundary is a linear function of the commutator $s(a_0, a_1) = (\delta r)(a_0, a_1) = f([a_0, a_1])$.

We consider now the map

$$a_0 da_1 \dots da_n \overset{\rho}{\mapsto} \Pi(a) d\Pi(a_1) \dots d\Pi(a_n), \tag{12}$$

which has a kernel $J_0 = \ker(\rho)$. Then $J_0 \oplus \delta J_0$ is a differential ideal. Factorization with respect to this ideal allows to obtain a differential complex.

We define now cyclic cocycles on a p–summable Fredholm module [10] as

$$s(a_0, \dots, a_{p-1}) := \mathrm{Tr}\; F[F, \Pi(a_0)][F, \Pi(a_1)] \dots [F, \Pi(a_{p-1})]. \tag{13}$$

For even p,s $\equiv 0$. One can prove that (13) fulfills all properties mentioned [4]. For odd p, the grading is absent.

Looking back to chapter 2, we observe from eq. (5) that a $p = 2$ summable Fredholm module was constructed. A simple calculation shows that (7) is proprtional to (13) with $p = 2$, $A = \Pi(a_0)$ and $B = \Pi(a_1)$.

In our attempts to compare cocycles in various representations we realized that they are rather rigid. There is, for example the following [11]

Theorem: Let P_1 and P_2 be projection operators on $\mathcal{H} = L^2(S^1) \otimes \mathbf{C}^2$ and α and β be maps from S^1 to $\mathbf{R}$ such that $\exp(it\alpha)$ and $\exp(it\beta)$ are implemented strongly continuously in Π_{P_1} and Π_{P_2}. If $P_1 \alpha P_1 - P_2 \alpha P_2$ and $P_1 \beta P_1 - P_2 \beta P_2$ are trace–class then $S_{P_1}(\alpha, \beta)$ is cohomologous to $S_{P_2}(\alpha, \beta)$ and they differ at most by a function of the commutator.

We worked out [12] conditions on T, such that for Dirac operators $h_m = \alpha p + \beta m$ (with $\alpha = \sigma^3$) on $\widetilde{\mathcal{H}} = L^2(\mathbf{R}, dp) \otimes \mathbf{C}^2$ identical cocycles arise:

Theorem: Let T be a matrix multiplication operator on $\widetilde{\mathcal{H}}$ with $0 \le T(p) \le 1$. Denote by Π_T the GNS representation of the quasifree state ω_T. If α, β are strongly continuously implementable maps in Π_T then $\lim_{p \to \pm\infty} T_{11} = \ell_1^{\pm}$ and $\lim_{p \to \pm\infty} T_{22} = \ell_2^{\pm}$ exist and $\ell_{1,2}^{\pm} \in \{0, 1\}$. $S_T(\alpha, \beta)$ depends then only on $\ell_1^{\pm}$ and $\ell_2^{\pm}$; for example for $\ell_1^+ = \ell_2^- = 0$ but $\ell_1^- = \ell_2^+ = 1$ the charge–axial charge cocycle

$$S_T(\alpha, \beta) = \int_{-\infty}^{\infty} \frac{dx}{2\pi}\, \alpha(x) \beta'(x) \tag{14}$$

results.

Typical examples are free massless or massive fermions in a zero temperature or a temperature state. Although all these representations are inequivalent identical Schwinger terms show up in the charge algebra. In these examples we clearly have chosen the implementation with $\phi \equiv \mathbf{1}$.

5 Anomalies in Lattice Spin Systems

Even for noninteracting spins in a constant magnetic field B, it is easy to obtain nontrivial cocycles and various temperature dependences can be studied.

As for the C* algebra $\mathcal{A}$ we take the inductive limit of products of two times two matrix algebras $\mathcal{A}(\Lambda) = \bigotimes_{k \in \Lambda} M_2(\mathbf{C})_k$. Let $A_k := \dots \mathbf{1} \otimes A \otimes \mathbf{1} \dots$, where A is on the k–th place. The group $\mathbf{Z}$ of translations on the lattice is then represented on $\mathcal{A}$ by *–automorphisms τ_n, defined by the assignment $A_k \mapsto \tau_n(A_k) := A_{k+n}$. The time evolution is given by the one parameter group of *–automorphisms

$$A_k \mapsto \alpha_t(A_k) := e^{it\,\frac{B}{2}\,\sigma_k^3} A_k e^{-it\,\frac{B}{2}\,\sigma_k^3}. \tag{15}$$

Since the external field is translation invariant, we observe that $\tau_n \circ \alpha_t = \alpha_t \circ \tau_n$. The state is chosen to be kink–like:

$$\prod_{\ell=1}^{N} A_{k_\ell} \mapsto \omega_\beta\left(\prod_{\ell=1}^{N} A_{k_\ell}\right) := \prod_{\{\ell : k_\ell < 0\}} \mathrm{Tr}\ \sigma^1 \rho_\beta \sigma^1 A_{k_\ell} \prod_{\{m : k_m \geq 0\}} \mathrm{Tr}\ \rho_\beta A_{k_m} \tag{16}$$

where Tr denotes the trace in $M_2(\mathbf{C})$ and ρ_β is given by

$$\rho_\beta = \begin{pmatrix} (1+e^{\beta B})^{-1} & 0 \\ 0 & (1+e^{-\beta B})^{-1} \end{pmatrix}. \tag{17}$$

Note that ω_β is invariant under the time evolution automorphism group α_t. But nontranslation invariance of the kink state implies that the implementers of translations $V_\infty(n)$ and time evolution $U_\infty(t)$ in the state ω_∞ do not commute, but

$$V_\infty(n) U_\infty(t) = e^{-itBn} U_\infty(t) V_\infty(n) \tag{18}$$

an anomaly shows up [6].

At finite temperature α_t is still implemented by a unitary group $U_\beta(t)$ which is, up to a phase, unique, since ω_β is invariant under α_t. In contrast the implementers $V_\beta(n)$ of translations are left undetermined up to a unitary element from the commutant $\Pi'_\beta(\mathcal{A})$. There are again various choices possible. One of them, proposed in [6] uses the modular theory to construct implementers acting on Ω_β as

$$\widetilde{V}_\beta(n)\Omega_\beta = \Pi_\beta\left(\prod_{j=0}^{n-1} \sigma_j^1\right) \Pi'_\beta\left(\prod_{j=0}^{n-1} \sigma_j^1\right) \Omega_\beta, \qquad \text{if } n \geq 1, \tag{19}$$

where $\Pi'_\beta(\cdot) = J\Pi_\beta(\cdot)J$ and J denotes the modular conjugation. Due to the antilinearity of Π'_β the anomaly vanishes now. One price to be paid is that the expectation value of the soliton ground state of H_β^S vanishes, and does not tend to the zero temperature expectation value for $T \searrow 0$:

$$\langle \widetilde{V}_\beta(n)\Omega_\beta | H_\beta^S \widetilde{V}_\beta(n)\Omega_\beta \rangle = 0 \stackrel{T\searrow 0}{\not\to} \langle V_\infty(n)\Omega_\infty | H_\infty^S V_\infty(n)\Omega_\infty \rangle = |n| M \ \forall\, \beta, \tag{20}$$

although translating the ground state changes the magnetization proportional to $n \tanh(\beta B/2)$ [7].

We propose a different implementation $V_\beta(n)$, which is a straightforward generalization of the zero temperature case [7]:

$$V_\beta(n)\Omega_\beta = \Pi_\beta\left(\prod_{j=0}^{n-1} \sigma_j^1\right) \Omega_\beta, \qquad n > 0. \tag{21}$$

The expectation value of H_β^S turns out to be

$$\langle V_\beta(n)\Omega_\beta | H_\beta^S V_\beta(n)\Omega_\beta \rangle = |n| B \tanh \frac{\beta B}{2}, \tag{22}$$

which tends to the $T = 0$ value $|n|M$ for $T \searrow 0$.

Although our choice of $V_\beta(n)$ yields reasonable zero temperature limits, here too $V_\beta(n) \notin \Pi''_\beta(\mathcal{A})$. In addition we obtain an example of an operator–valued cocycle:

Theorem [7]: The group commutator between translations and time evolution takes values in $\Pi'_\beta(\mathcal{A})$:

$$V_\beta(n)U_\beta(t)V_\beta^\dagger(n)U_\beta^\dagger(t) = \begin{cases} \Pi'_\beta\left(\prod_{j=0}^{n-1} e^{-itB\sigma_j^1}\right) & \text{for } n > 0 \\ \mathbf{1} & \text{for } n = 0 \\ \Pi'_\beta\left(\prod_{j=-|n|}^{-1} e^{-iB\sigma_j^3}\right) & \text{for } n < 0 \end{cases} . \qquad (23)$$

Here we obtained an example of an operator–valued cohomology class in the sense of [13].

In summarizing we note that the two anomaly melt–down mechanisms for systems at finite temperature work quite differently. The first, of part three, uses the ambiguity of lifting the Bogoliubov automorphisms to the doubled algebra at finite temperature. Through a diagonal embedding the Schwinger terms are killed at the price that generators of automorphisms are not affiliated.

The second mechanism is based on the Tomika–Takesaki theory and uses antilinearity of the modular conjugation to kill the anomalous phase factor in the projective representation of Wigner symmetries. Moreover for automorphisms which commute with time evolution, the corresponding implementers cannot be affiliated.

A certain choice of an implementer can be selected in particular situations on physical grounds. Work in progress concerns the temperature dependence of cocycles in interacting systems.

Acknowledgements: These notes summarize our common work with C. Reitberger and W. Maderner. I would like to thank them for their enthusiasm, many discussions and the fruitful collaboration.

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