

**On the Absence of Eigenvalues of a
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T.A. Suslina

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ON THE ABSENCE OF EIGENVALUES OF A MATRIX PERIODIC SCHRÖDINGER OPERATOR IN A LAYER

T. A. SUSLINA

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Dedicated to Michael Semenovich Agranovich on his seventieth birthday

ABSTRACT. A matrix Schrödinger operator $-\Delta + \mathcal{V}(\mathbf{x})$ in a layer $\mathbb{R}^{d-1} \times (0, T)$, $d \geq 2$, is considered. The potential \mathcal{V} is assumed to be periodic along the layer. On the boundary we set appropriate boundary conditions. The Dirichlet and Neumann boundary conditions, third type condition with periodic coefficients and quasiperiodic conditions are allowed. It is shown that the spectrum of the corresponding operators is free of eigenvalues. In the selfadjoint case the spectrum is absolutely continuous.

§0. INTRODUCTION

1. In the original paper [T] by L. Thomas it was shown that the spectrum of the Schrödinger operator $-\Delta + V(\mathbf{x})$ in $L_2(\mathbb{R}^d)$ is absolutely continuous if a potential V is real-valued and periodic. Thomas suggested the method of the proof of absolute continuity of the spectrum which was used nearly in all further investigations. The detailed exposition of the Thomas method can be found in the books [RSi], [Ku], and also in the paper [BSu]. In [Ku], [KuL], the Thomas method has been applied in non-selfadjoint case to prove the absence of eigenvalues. In particular, in [KuL], it was shown that the matrix Schrödinger operator $-\Delta + \mathcal{V}(\mathbf{x})$ in $L_2(\mathbb{R}^d; \mathbb{C}^M)$ has no eigenvalues if an $(M \times M)$ -matrix-valued function $\mathcal{V}(\mathbf{x})$ is periodic. Absolute continuity of the spectrum of the periodic magnetic Schrödinger operator, the Schrödinger operator with a variable periodic metric, the periodic Dirac and Maxwell operators was studied by a number of authors. The detailed survey on absolute continuity of the spectrum of periodic operators can be found in [BSu], [Su].

We turn our attention to the results for the Schrödinger operator with singular potentials. In [BShSu], [ShSu1], the periodic magnetic Schrödinger operator with a singular potential of the form $V(\mathbf{x}) + \sigma(\mathbf{x})\delta_\Sigma(\mathbf{x})$ was studied. Here $V(\mathbf{x})$ is an "ordinary" part of potential, Σ is a periodic system of hypersurfaces, $\delta_\Sigma(\mathbf{x})$ is the delta-function supported on Σ , and $\sigma(\mathbf{x})$ is a periodic function on Σ . The two-dimensional case was studied in [BShSu], and the case $d \geq 3$ was studied in

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[ShSu1]. Under wide conditions on the problem data, the absolute continuity of the spectrum (in the selfadjoint case) was proved.

Most papers are devoted to periodic problems in the whole space \mathbb{R}^d . Absolute continuity of the spectrum of the Schrödinger operator in *periodic waveguides* was studied only in the two-dimensional case. Rather complete results for the magnetic Schrödinger operator with a variable metric in a two-dimensional periodic waveguide were obtained in the papers [SoW], [ShaSo], [ShSu2]. The Dirichlet and Neumann problems, as well as the third boundary value problem with a periodic coefficient in the boundary condition were studied.

2. In the present paper, we study a matrix Schrödinger operator

$$\mathcal{H} = -\Delta + \mathcal{V}(\mathbf{x}), \quad (0.1)$$

in a layer $\Pi = \mathbb{R}^{d-1} \times (0, T)$ for $d \geq 2$. It is assumed that the $(M \times M)$ -matrix-valued function $\mathcal{V}(\mathbf{x}) = \mathcal{V}(\mathbf{x}', x_d)$ is periodic in $\mathbf{x}' \in \mathbb{R}^{d-1}$ with respect to a lattice $\Gamma \subset \mathbb{R}^{d-1}$. An elementary cell of the lattice Γ is denoted by Ω' ; we put $\Omega := \Omega' \times (0, T)$. For $d \geq 3$ local conditions on $\mathcal{V}(\mathbf{x})$ (see condition (1.17) below) are formulated in terms of the Lorentz classes $L_{p,\infty}^0(\Omega)$; for $d = 2$ we assume that $\mathcal{V} \in L_p(\Omega)$, $p > 1$. (For the problem in \mathbb{R}^d , the same local conditions on potential have been posed in [BSu]). For $d = 2, 3, 4$ conditions (1.16), (1.17) are optimal in the Lorentz scale, while for $d \geq 5$ they are non-optimal.

We put appropriate boundary conditions. Namely, the components of a vector \mathbf{u} are subject to the Dirichlet or Neumann conditions, or to the third type condition $\partial u / \partial x_d + \sigma(\mathbf{x}')u = 0$ with a periodic coefficient $\sigma(\mathbf{x}')$ or to the quasiperiodic conditions. It is assumed that $\sigma \in L_q(\Omega')$, $q > 1$, for $d = 2$, and that $\sigma \in L_{2,\infty}^0(\Omega')$ for $d = 3$ (see conditions (1.19), (1.20) below). These conditions are optimal in the Lorentz scale. For $d \geq 4$ we impose the (non-optimal) condition $\sigma \in L_{2d-2}(\Omega')$. Mention that conditions (1.16), (1.19) for $d = 2$ can be relaxed in the spirit of the papers [Sh1,2], [ShSu2]. The precise definition of the operator is given in terms of the corresponding quadratic form (see §1). The main result of the paper is Theorem 1.3 about the absence of eigenvalues of operator (0.1); in the selfadjoint case the spectrum of operator (0.1) is absolutely continuous.

3. As usual, we lean upon the Thomas method. Periodic operator (0.1) is decomposed into the direct integral. The operator \mathcal{H} is associated with the operator family $\mathcal{H}(\mathbf{k}')$ acting in $L_2(\Omega)$ and depending on the parameter $\mathbf{k}' \in \mathbb{R}^{d-1}$ (the quasimomentum). The Thomas approach is based on analytic extension of the operator family $\mathcal{H}(\mathbf{k}')$ to the complex values of \mathbf{k}' . Further considerations reduce to estimating the resolvent of the operators $\mathcal{H}(\mathbf{k}')$ for large imaginary values of the quasimomentum. In each particular case, such estimates are the most difficult part of the proof. The estimates required in our case are given in Theorem 2.2.

For the proof of Theorem 2.2, we need to study the symbol of the "free" operator carefully. Part of technical results is borrowed from [BSu], [BShSu], [ShSu1] in readiness for use. In many aspects, the third boundary value problem in a layer is similar to the problem in \mathbb{R}^d for the Schrödinger operator with a singular potential of the form $\sigma\delta_\Sigma$. The latter problem was studied in [BShSu] in the case $d = 2$ and in [ShSu1] in the case $d \geq 3$. However, for $d \geq 3$, the estimates from [ShSu1] are not sufficient for our goal. For the problem in a layer, the quasimomentum is tangential to the boundary in all points. In this case considerations of [ShSu1] are

unapplicable. The key estimates necessary for the study of the problem in a layer for $d \geq 3$ are obtained in Proposition 3.3.

4. The result of the paper can be applied to the study of the periodic Maxwell operator in a layer with boundary conditions of ideal conductivity. The periodic Maxwell operator in \mathbb{R}^3 was studied in [Mo], [Su]. In [Mo], it was shown that the spectrum of the Maxwell operator is absolutely continuous when the dielectric permeability $\varepsilon(\mathbf{x})$ and the magnetic permeability $\mu(\mathbf{x})$ are scalar positive periodic functions in \mathbb{R}^3 (under some smoothness conditions on ε and μ). Later, in the paper [Su], the proof of [Mo] has been simplified significantly. Namely, it turned out that, on the basis of some identities from [Mo], the question of absolute continuity of the spectrum of the *selfadjoint* periodic Maxwell operator in \mathbb{R}^3 is reduced to the question of absence of eigenvalues of certain *matrix non-selfadjoint* Schrödinger operator. The same method allows us to reduce the problem of absolute continuity of the spectrum of the periodic Maxwell operator in a layer (with conditions of ideal conductivity) to the question of absence of eigenvalues for the matrix Schrödinger operator with appropriate boundary conditions. This reduction will be published elsewhere.

5. The main definitions and results are presented in §1. §2 contains the necessary material concerning the Thomas approach, together with the formulation of the basic Theorem 2.2 about estimates. In §3–5 we obtain estimates necessary for the proof of Theorem 2.2. In §6, we establish Theorem 2.2.

Mention that the set of the problem and the corresponding results can be generalized and intensified in various directions. In particular, the class of admissible boundary conditions can be extended significantly. However, such generalizations would require serious technical complications.

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§ 1. DEFINITION OF THE OPERATOR. THE MAIN RESULT

1. Notation. If \mathfrak{H} , \mathfrak{G} are two separable Hilbert spaces, and A is a continuous linear operator from \mathfrak{H} to \mathfrak{G} , then its operator norm is denoted by $\|A\|_{\mathfrak{H} \rightarrow \mathfrak{G}}$. Often we write simply $\|A\|$ if this does not lead to confusion.

Let $n \in \mathbb{N}$. The symbols $\langle \cdot, \cdot \rangle_n$, $|\cdot|_n$ stand for the standard inner product and the norm in \mathbb{C}^n ; often we omit the index n . By $\mathbf{1}_n$ we denote the unit $(n \times n)$ -matrix. For a $(n \times n)$ -matrix a , its operator norm in \mathbb{C}^n is denoted by $\|a\|$, and its modulus is denoted by $|a| := (a^*a)^{1/2}$. Symbols C , c (with or without indices) denote various constants in estimates.

Let $\Pi = \Pi_T$, $T > 0$, be a layer in \mathbb{R}^d defined by

$$\Pi := \mathbb{R}^{d-1} \times (0, T), \quad d \geq 2. \quad (1.1)$$

We fix an orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_d$ in \mathbb{R}^d so that the vector \mathbf{e}_d is normal to the boundary of the layer, i. e. $\Pi = \{\mathbf{x} = (\mathbf{x}', x_d) : \mathbf{x}' = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}, x_d \in (0, T)\}$. We use the notation $\nabla = \text{grad} = (\partial_1, \dots, \partial_d)$, $\mathbf{D} = -i\nabla = (D_1, \dots, D_d)$. The boundary of the layer is the union of two hypersurfaces: $\partial\Pi = \Sigma_0 \cup \Sigma_1$,

$$\Sigma_0 := \{\mathbf{x} = (\mathbf{x}', 0), \mathbf{x}' \in \mathbb{R}^{d-1}\}, \quad \Sigma_1 := \{\mathbf{x} = (\mathbf{x}', T), \mathbf{x}' \in \mathbb{R}^{d-1}\}.$$

Let vectors $\mathbf{a}_1, \dots, \mathbf{a}_{d-1} \in \mathbb{R}^d$ such that $\mathbf{a}_r \perp \mathbf{e}_d$, $r = 1, \dots, d-1$, be a basis of a $(d-1)$ -dimensional lattice Γ . The set

$$\Omega' := \{\mathbf{x}' \in \mathbb{R}^{d-1} : \mathbf{x}' = t_1 \mathbf{a}_1 + \dots + t_{d-1} \mathbf{a}_{d-1}, 0 < t_r < 1, r = 1, \dots, d-1\}$$

is an elementary cell of the lattice Γ ; we put $\Omega := \Omega' \times (0, T)$. We use the notation

$$S_0 := \{\mathbf{x} = (\mathbf{x}', 0), \mathbf{x}' \in \Omega'\}, \quad S_1 := \{\mathbf{x} = (\mathbf{x}', T), \mathbf{x}' \in \Omega'\}.$$

Let \mathcal{G} be a domain in \mathbb{R}^d ; $n \in \mathbb{N}$. By $L_p(\mathcal{G}; \mathbb{C}^n)$, $1 \leq p < \infty$, we denote the space of all \mathbb{C}^n -valued functions in \mathcal{G} integrable with the power p ; $L_\infty(\mathcal{G}; \mathbb{C}^n)$ is a class of all measurable bounded \mathbb{C}^n -valued functions. The norm in $L_p(\mathcal{G}; \mathbb{C}^n)$ is denoted by $\|\cdot\|_{p, \mathcal{G}}$. Sometimes we write simply $L_p(\mathcal{G})$ if this does not lead to confusion. The Sobolev classes of order l are denoted by $H^l(\mathcal{G}; \mathbb{C}^n)$; for $n = 1$ we abbreviate this to $H^l(\mathcal{G})$.

We fix $M \in \mathbb{N}$. Let $L \in \mathbb{Z}$, $0 \leq L \leq M$, and let $\mathcal{I} = \{i_1, \dots, i_L\}$ be an ordered system of indices:

$$1 \leq i_1 < i_2 < \dots < i_L \leq M.$$

If $L = 0$, then we put $\mathcal{I} = \emptyset$. Similarly, $\mathcal{J} = \{j_1, \dots, j_K\}$, $\mathcal{L} = \{l_1, \dots, l_N\}$ are ordered systems of indices:

$$\begin{aligned} 1 \leq j_1 < j_2 < \dots < j_K \leq M, \quad 0 \leq K \leq M, \\ 1 \leq l_1 < l_2 < \dots < l_N \leq M, \quad 0 \leq N \leq M. \end{aligned}$$

Assume that

$$\mathcal{L} \cap (\mathcal{I} \cup \mathcal{J}) = \emptyset.$$

Let $\Xi = \{\xi_1, \xi_2, \dots, \xi_N\}$ be a system of numbers such that $0 \leq \xi_s < 2\pi$, $s = 1, \dots, N$. We introduce the notation

$$\begin{aligned} H^1(\Pi; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi) := \{\mathbf{u} = (u_1, \dots, u_M) \in H^1(\Pi; \mathbb{C}^M) : \\ u_s|_{\Sigma_0} = 0 \text{ for } s \in \mathcal{I}, \quad u_s|_{\Sigma_1} = 0 \text{ for } s \in \mathcal{J}, \quad u_s|_{\Sigma_0} = e^{i\xi_s} u_s|_{\Sigma_1} \text{ for } s \in \mathcal{L}\}. \end{aligned} \quad (1.2)$$

Here the traces $u_s|_{\Sigma_0}$, $u_s|_{\Sigma_1}$ are understood in the sense of the trace embedding theorem; the equality $u_s|_{\Sigma_0} = e^{i\xi_s} u_s|_{\Sigma_1}$ means that

$$u_s(\mathbf{x}', 0) = e^{i\xi_s} u_s(\mathbf{x}', T), \quad \mathbf{x}' \in \mathbb{R}^{d-1}, \quad s \in \mathcal{L}. \quad (1.3)$$

In the case $\mathcal{I} = \mathcal{J} = \{1, \dots, M\}$, $\mathcal{L} = \emptyset$, we use the notation $\mathring{H}^1(\Pi; \mathbb{C}^M)$ for the space (1.2). By $\tilde{H}^l(\Omega; \mathbb{C}^M)$ we denote the subspace formed by the functions in $H^l(\Omega; \mathbb{C}^M)$ the Γ -periodic extension (in \mathbf{x}') of which belongs to $H_{\text{loc}}^l(\Pi; \mathbb{C}^M)$. We introduce the notation

$$\begin{aligned} \tilde{H}^1(\Omega; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi) := \{\mathbf{u} = (u_1, \dots, u_M) \in \tilde{H}^1(\Omega; \mathbb{C}^M) : \\ u_s|_{S_0} = 0 \text{ for } s \in \mathcal{I}, \quad u_s|_{S_1} = 0 \text{ for } s \in \mathcal{J}, \quad u_s|_{S_0} = e^{i\xi_s} u_s|_{S_1} \text{ for } s \in \mathcal{L}\}. \end{aligned} \quad (1.4)$$

2. Weak L_p -classes. Let $\mathcal{G} \subseteq \mathbb{R}^d$ be a domain. We recall the definition of the classes $L_{p, \infty}(\mathcal{G})$ of the Lorentz scale. For a measurable function f we put

$$\rho_f(t) = \text{meas} \{\mathbf{x} \in \mathcal{G} : |f(\mathbf{x})| > t\}, \quad t > 0.$$

The class $L_{p,\infty}(\mathcal{G})$, $0 < p < \infty$, consists of all functions f such that the *quasinorm*

$$\|f\|_{p,\infty,\mathcal{G}} := \sup_{t>0} t(\rho_f(t))^{1/p} \quad (1.5)$$

is finite. The class $L_{p,\infty}(\mathcal{G})$ is complete with respect to the quasinorm (1.5), and it is non-separable. We distinguish the separable subspace $L_{p,\infty}^0(\mathcal{G})$ of $L_{p,\infty}(\mathcal{G})$ formed by the functions $f \in L_{p,\infty}(\mathcal{G})$ such that

$$\rho_f(t) = o(t^{-p}), \quad t \rightarrow 0, \quad t \rightarrow \infty. \quad (1.6)$$

The class $L_p(\mathcal{G})$ (and, therefore, the class $C_0^\infty(\mathcal{G})$) is dense in $L_{p,\infty}^0(\mathcal{G})$. If $\text{meas}\mathcal{G} < \infty$, then as $t \rightarrow 0$ condition (1.6) is fulfilled automatically.

The classes $l_{p,\infty}(\Gamma')$, $0 < p < \infty$, where Γ' is some countable set, are defined in a similar way. Let $\mathbf{g} = \{g_{\mathbf{q}}\}$, $\mathbf{q} \in \Gamma'$; we put $\varrho_{\mathbf{g}}(t) = \#\{\mathbf{q} \in \Gamma' : |g_{\mathbf{q}}| > t\}$, $t > 0$. The class $l_{p,\infty}(\Gamma')$ is formed by all the sequences \mathbf{g} such that the quasinorm

$$\|\mathbf{g}\|_{p,\infty,\Gamma'} := \sup_{t>0} t(\varrho_{\mathbf{g}}(t))^{1/p}$$

is finite. The separable subspace $l_{p,\infty}^0(\Gamma')$ consists of all $\mathbf{g} \in l_{p,\infty}(\Gamma')$ that satisfy $\varrho_{\mathbf{g}}(t) = o(t^{-p})$ as $t \rightarrow 0$.

3. Here we present some auxiliary statements related to embedding theorems.

Proposition 1.1. *1°. Let ψ be a function such that*

$$\psi \in L_\rho(\Omega), \quad \rho > 1, \quad d = 2; \quad \psi \in L_{d/2,\infty}(\Omega), \quad d \geq 3. \quad (1.7)$$

Then the form $\int_\Omega |\psi||u|^2 d\mathbf{x}$ is continuous in $H^1(\Omega)$, and

$$\begin{aligned} \int_\Omega |\psi||u|^2 d\mathbf{x} &\leq C(\psi) \int_\Omega (|\nabla u|^2 + |u|^2) d\mathbf{x}, \quad u \in H^1(\Omega), \\ C(\psi) &= c\|\psi\|_{\rho,\Omega}, \quad d = 2; \quad C(\psi) = c\|\psi\|_{d/2,\infty,\Omega}, \quad d \geq 3. \end{aligned} \quad (1.8)$$

If

$$\psi \in L_\rho(\Omega), \quad \rho > 1, \quad d = 2; \quad \psi \in L_{d/2,\infty}^0(\Omega), \quad d \geq 3, \quad (1.9)$$

then the form $\int_\Omega |\psi||u|^2 d\mathbf{x}$ is compact in $H^1(\Omega)$ and, consequently,

$$\int_\Omega |\psi||u|^2 d\mathbf{x} \leq \varepsilon \int_\Omega |\nabla u|^2 d\mathbf{x} + C(\varepsilon, \psi) \int_\Omega |u|^2 d\mathbf{x}, \quad u \in H^1(\Omega), \quad 0 < \varepsilon \leq 1. \quad (1.10)$$

2°. Suppose that a function $\psi(\mathbf{x}) = \psi(\mathbf{x}', x_d)$ in Π is Γ -periodic in \mathbf{x}' . Under the assumption (1.7), we have

$$\int_\Pi |\psi||u|^2 d\mathbf{x} \leq C(\psi) \int_\Pi (|\nabla u|^2 + |u|^2) d\mathbf{x}, \quad u \in H^1(\Pi).$$

If condition (1.9) is fulfilled, then

$$\int_\Pi |\psi||u|^2 d\mathbf{x} \leq \varepsilon \int_\Pi |\nabla u|^2 d\mathbf{x} + C(\varepsilon, \psi) \int_\Pi |u|^2 d\mathbf{x}, \quad u \in H^1(\Pi), \quad 0 < \varepsilon \leq 1.$$

The statement 1° follows from the embedding theorems (see, for example, [BS, §9]). Writing down inequalities (1.8), (1.10) for the shifted cells $\Omega + \mathbf{l}$ and summing over $\mathbf{l} \in \Gamma$, we obtain statement 2°.

Proposition 1.2. 1°. *If a function σ satisfies condition*

$$\sigma \in L_q(\Omega'), \quad q > 1, \quad d = 2; \quad \sigma \in L_{d-1,\infty}(\Omega'), \quad d \geq 3, \quad (1.11)$$

then the form $\int_{\Omega'} |\sigma(\mathbf{x}')| |u(\mathbf{x}', b)|^2 d\mathbf{x}'$, with $b \in [0, T]$, is continuous in $H^1(\Omega)$, and

$$\int_{\Omega'} |\sigma(\mathbf{x}')| |u(\mathbf{x}', b)|^2 d\mathbf{x}' \leq C(\sigma) \int_{\Omega} (|\nabla u|^2 + |u|^2) d\mathbf{x}, \quad u \in H^1(\Omega), \quad (1.12)$$

$$C(\sigma) = c \|\sigma\|_{q,\Omega'}, \quad d = 2; \quad C(\sigma) = c \|\sigma\|_{d-1,\infty,\Omega'}, \quad d \geq 3.$$

Under the condition

$$\sigma \in L_q(\Omega'), \quad q > 1, \quad d = 2; \quad \sigma \in L_{d-1,\infty}^0(\Omega'), \quad d \geq 3, \quad (1.13)$$

the form $\int_{\Omega'} |\sigma(\mathbf{x}')| |u(\mathbf{x}', b)|^2 d\mathbf{x}'$ is compact in $H^1(\Omega)$ and, consequently,

$$\int_{\Omega'} |\sigma(\mathbf{x}')| |u(\mathbf{x}', b)|^2 d\mathbf{x}' \leq \varepsilon \int_{\Omega} |\nabla u|^2 d\mathbf{x} + C(\varepsilon, \sigma) \int_{\Omega} |u|^2 d\mathbf{x}, \quad u \in H^1(\Omega), \quad 0 < \varepsilon \leq 1. \quad (1.14)$$

2°. *Let $\sigma(\mathbf{x}')$ be a Γ -periodic function in \mathbb{R}^{d-1} and let $b \in [0, T]$. Under the condition (1.11), we have*

$$\int_{\mathbb{R}^{d-1}} |\sigma(\mathbf{x}')| |u(\mathbf{x}', b)|^2 d\mathbf{x}' \leq C(\sigma) \int_{\Pi} (|\nabla u|^2 + |u|^2) d\mathbf{x}, \quad u \in H^1(\Pi).$$

If condition (1.13) is fulfilled, then

$$\int_{\mathbb{R}^{d-1}} |\sigma(\mathbf{x}')| |u(\mathbf{x}', b)|^2 d\mathbf{x}' \leq \varepsilon \int_{\Pi} |\nabla u|^2 d\mathbf{x} + C(\varepsilon, \sigma) \int_{\Pi} |u|^2 d\mathbf{x},$$

$$u \in H^1(\Pi), \quad 0 < \varepsilon \leq 1.$$

In 1° (correspondingly, in 2°) the traces of a function $u \in H^1(\Omega)$ (correspondingly, $u \in H^1(\Pi)$) on the section $x_d = b$ are understood in the sense of the trace embedding theorem. For $d \geq 3$, statement 1° is contained, for example, in Proposition 1.4 of the paper [ShSu1]. For $d = 2$, statement 1° is a direct consequence of the Hölder inequality and standard embedding theorems. Statement 2° can be proved by summing inequalities (1.12), (1.14) over the cells of the lattice.

4. We pass to the precise definition of the Schrödinger operator in a layer. The operator is defined in the sense of the corresponding quadratic form. Here we deal with *sectorial forms and m -sectorial operators*. These objects were systematically studied in the book [K] by T. Kato.

Let $\mathcal{V}(\mathbf{x})$ be an $(M \times M)$ -matrix-valued function (generally speaking, with complex-valued entries) in Π . Assume that $\mathcal{V}(\mathbf{x})$ is Γ -periodic in \mathbf{x}' :

$$\mathcal{V}(\mathbf{x} + \mathbf{a}_r) = \mathcal{V}(\mathbf{x}), \quad \mathbf{x} \in \Pi, \quad r = 1, \dots, d-1. \quad (1.15)$$

Also, we assume that

$$\mathcal{V} \in L_p(\Omega), \quad p > 1, \quad d = 2; \quad (1.16)$$

$$\mathcal{V} \in L_{p,\infty}^0(\Omega), \quad 2p = d, \quad d = 3, 4; \quad p = d - 2, \quad d \geq 5. \quad (1.17)$$

We introduce the operator

$$\begin{aligned} \Theta : L_2(\Pi; \mathbb{C}^M) &\rightarrow L_2(\mathbb{R}^{d-1}; \mathbb{C}^{2M}), \quad \text{Dom } \Theta = H^1(\Pi; \mathbb{C}^M), \\ (\Theta \mathbf{u})(\mathbf{x}') &:= (\mathbf{u}(\mathbf{x}', 0), \mathbf{u}(\mathbf{x}', T)). \end{aligned}$$

Let $\sigma(\mathbf{x}')$ be a Γ -periodic $(2M \times 2M)$ -matrix-valued function (generally speaking, with complex-valued entries) in \mathbb{R}^{d-1} :

$$\sigma(\mathbf{x}' + \mathbf{a}_r) = \sigma(\mathbf{x}'), \quad \mathbf{x}' \in \mathbb{R}^{d-1}, \quad r = 1, \dots, d-1. \quad (1.18)$$

We assume that

$$\sigma \in L_q(\Omega'), \quad q > 1, \quad d = 2, \quad (1.19)$$

$$\sigma \in L_{2,\infty}^0(\Omega'), \quad d = 3, \quad (1.20)$$

$$\sigma \in L_{2d-2}(\Omega'), \quad d \geq 4. \quad (1.21)$$

Let a be a positive $((d-1) \times (d-1))$ -matrix with real-valued entries, and let \hat{a} be a block $(d \times d)$ -matrix of the form

$$\hat{a} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}. \quad (1.22)$$

In the Hilbert space $L_2(\Pi; \mathbb{C}^M)$, we consider the following quadratic form

$$\begin{aligned} h[\mathbf{u}, \mathbf{u}] &= h(\mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi; \mathcal{V}, \sigma)[\mathbf{u}, \mathbf{u}] := \int_{\Pi} \left(\sum_{s=1}^M \langle \hat{a} \mathbf{D}u_s, \mathbf{D}u_s \rangle_d + \langle \mathcal{V}(\mathbf{x}) \mathbf{u}, \mathbf{u} \rangle_M \right) d\mathbf{x} + \\ &\int_{\mathbb{R}^{d-1}} \langle \sigma(\mathbf{x}') \Theta \mathbf{u}, \Theta \mathbf{u} \rangle_{2M} d\mathbf{x}', \quad \mathbf{u} \in \text{Dom } h = H^1(\Pi; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi). \end{aligned} \quad (1.23)$$

By Propositions 1.1, 1.2, under conditions (1.15)–(1.21) the form (1.23) (see [K, Theorem VI.3.9]) is *closed and sectorial*. This form gives rise to an *m-sectorial operator* $\mathcal{H} = \mathcal{H}(\mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi; \mathcal{V}, \sigma)$ in $L_2(\Pi; \mathbb{C}^M)$. If matrices \mathcal{V} and σ are Hermitian, then the operator \mathcal{H} is selfadjoint.

The operator \mathcal{H} is interpreted as a (matrix) Schrödinger operator $(-\text{div } \hat{a} \nabla) \mathbf{1}_M + \mathcal{V}(\mathbf{x})$ in a layer Π with appropriate boundary conditions. Namely, the component u_s of a vector \mathbf{u} satisfies the Dirichlet condition on Σ_0 for $s \in \mathcal{I}$ and on Σ_1 for $s \in \mathcal{J}$. The other components satisfy *natural* (in variational sense) boundary conditions which arise under varying the form (1.23). When varying the form, one should take into account that u_s satisfies condition (1.3) when $s \in \mathcal{L}$. For example, when $\mathcal{I} = \mathcal{J} = \mathcal{L} = \emptyset$, the natural boundary conditions are of the form

$$\text{col}(-\partial_d \mathbf{u}|_{\Sigma_0}, \partial_d \mathbf{u}|_{\Sigma_1}) = \sigma \Theta \mathbf{u}. \quad (1.24)$$

When $\sigma = 0$, condition (1.24) turns into the Neumann condition on $\partial\Pi$. We illustrate the other possibilities for the "scalar" case $M = 1$. If $\mathcal{I} = \mathcal{J} = \{1\}$, $\mathcal{L} = \emptyset$, then $\text{Dom } h = \dot{H}^1(\Pi)$, and the corresponding operator \mathcal{H} is associated with the Dirichlet problem (then the integral over \mathbb{R}^{d-1} in (1.23) is automatically equal to zero). If $\mathcal{I} = \{1\}$, $\mathcal{J} = \emptyset$, $\mathcal{L} = \emptyset$, then u satisfies the Dirichlet condition $u|_{\Sigma_0} = 0$ on Σ_0 and the (natural) third boundary condition $(\partial_d u + \sigma_{22}(\mathbf{x}')u)|_{\Sigma_1} = 0$ on Σ_1 . The case $\mathcal{I} = \emptyset$, $\mathcal{J} = \{1\}$, $\mathcal{L} = \emptyset$ can be considered similarly. Finally, if $\mathcal{I} = \mathcal{J} = \emptyset$, $\mathcal{L} = \{1\}$, then the condition $u|_{\Sigma_0} = e^{i\xi}u|_{\Sigma_1}$ is accompanied by the natural condition

$$(\partial_d u - (\sigma_{11}(\mathbf{x}') + \sigma_{21}(\mathbf{x}')e^{i\xi})u)|_{\Sigma_0} = e^{i\xi}(\partial_d u + (\sigma_{12}(\mathbf{x}')e^{-i\xi} + \sigma_{22}(\mathbf{x}'))u)|_{\Sigma_1}.$$

If, besides, $\sigma = 0$, then the operator \mathcal{H} corresponds to the problem with the quasiperiodic boundary conditions.

One should take into account that the natural boundary conditions can be understood in the sense of the trace embedding theorem only under additional restrictions on σ . In general case, these conditions are understood in the "weak" sense.

We introduce the notation $\mathcal{H}^0 = \mathcal{H}^0(\mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi) := \mathcal{H}(\mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi; 0, 0)$ for the "free" operator which corresponds to $\mathcal{V}(\mathbf{x}) \equiv 0$, $\sigma(\mathbf{x}') \equiv 0$. Obviously, this operator is selfadjoint.

5. The main result of the paper is the following theorem.

Theorem 1.3. *Let $d \geq 2$, $M \in \mathbb{N}$; and let Π be a layer in \mathbb{R}^d defined by (1.1). Suppose that an $(M \times M)$ -matrix-valued function $\mathcal{V}(\mathbf{x})$ in Π satisfies conditions (1.15)–(1.17). Let $\sigma(\mathbf{x}')$ be a $(2M \times 2M)$ -matrix-valued function in \mathbb{R}^{d-1} satisfying conditions (1.18)–(1.21). Suppose that a constant positive $(d \times d)$ -matrix \hat{a} with real-valued entries is of the form (1.22). Let $H^1(\Pi; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi)$ be the space defined by (1.2). Let $\mathcal{H}(\mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi; \mathcal{V}, \sigma)$ be the m -sectorial operator in $L_2(\Pi; \mathbb{C}^M)$ corresponding to the form (1.23). Then the spectrum of the operator $\mathcal{H}(\mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi; \mathcal{V}, \sigma)$ does not contain eigenvalues. If matrices $\mathcal{V}(\mathbf{x})$, $\sigma(\mathbf{x}')$ are Hermitian for all values of arguments, then the operator $\mathcal{H}(\mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi; \mathcal{V}, \sigma)$ is selfadjoint, and its spectrum is absolutely continuous.*

§ 2. DIRECT INTEGRAL DECOMPOSITION. THE THOMAS APPROACH

1. Direct integral. Let $\tilde{\Gamma}$ be the $(d-1)$ -dimensional lattice dual to the lattice Γ . By $\mathbf{b}_1, \dots, \mathbf{b}_{d-1}$ we denote the basis of the lattice $\tilde{\Gamma}$ such that $\mathbf{b}_j \mathbf{a}_l = 2\pi\delta_{jl}$, $j, l = 1, \dots, d-1$. The elementary cell of the lattice $\tilde{\Gamma}$ is denoted by $\tilde{\Omega}$; we have

$$\tilde{\Omega} = \{\mathbf{k}' \in \mathbb{R}^{d-1} : \mathbf{k}' = \tau_1 \mathbf{b}_1 + \dots + \tau_{d-1} \mathbf{b}_{d-1}; 0 < \tau_r < 1, r = 1, \dots, d-1\}.$$

The cell $\tilde{\Omega}$ is dual to Ω' .

Consider the operator

$$\begin{aligned} \Theta_0 : L_2(\Omega; \mathbb{C}^M) &\rightarrow L_2(\Omega'; \mathbb{C}^{2M}), \quad \text{Dom } \Theta_0 = H^1(\Omega; \mathbb{C}^M), \\ (\Theta_0 \mathbf{u})(\mathbf{x}') &:= (\mathbf{u}(\mathbf{x}', 0), \mathbf{u}(\mathbf{x}', T)). \end{aligned}$$

For every $\mathbf{k}' \in \mathbb{R}^{d-1}$ (the parameter \mathbf{k}' is called the *quasimomentum*) we put

$\mathbf{k} = (\mathbf{k}', 0) \in \mathbb{R}^d$ and consider the following quadratic form in $L_2(\Omega; \mathbb{C}^M)$:

$$\begin{aligned} h(\mathbf{k}')[\mathbf{u}, \mathbf{u}] &= h(\mathbf{k}'; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi; \mathcal{V}, \sigma)[\mathbf{u}, \mathbf{u}] := \sum_{s=1}^M \int_{\Omega} \langle \widehat{a}(\mathbf{D} + \mathbf{k})u_s, (\mathbf{D} + \mathbf{k})u_s \rangle_d d\mathbf{x} + \\ &\int_{\Omega} \langle \mathcal{V}(\mathbf{x})\mathbf{u}, \mathbf{u} \rangle_M d\mathbf{x} + \int_{\Omega'} \langle \sigma(\mathbf{x}')\Theta_0\mathbf{u}, \Theta_0\mathbf{u} \rangle_{2M} d\mathbf{x}', \\ \mathbf{u} \in \text{Dom } h(\mathbf{k}') &= \widetilde{H}^1(\Omega; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi). \end{aligned} \quad (2.1)$$

Here we use the notation (1.4). The domain of the form (2.1) does not depend on \mathbf{k}' . Propositions 1.1, 1.2 together with Theorem VI.3.9 from [K] show that, under conditions (1.16), (1.17), (1.19)–(1.21), the form (2.1) is closed and sectorial in $L_2(\Omega; \mathbb{C}^M)$ (see estimates (2.12) below). The form (2.1) gives rise to an m -sectorial operator $\mathcal{H}(\mathbf{k}') = \mathcal{H}(\mathbf{k}'; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi; \mathcal{V}, \sigma)$ in $L_2(\Omega; \mathbb{C}^M)$. Since the embedding of $\widetilde{H}^1(\Omega; \mathbb{C}^M)$ in $L_2(\Omega; \mathbb{C}^M)$ is compact, *the resolvent of the operator $\mathcal{H}(\mathbf{k}')$ is compact*. The "free" operator corresponding to $\mathcal{V}(\mathbf{x}) \equiv 0$, $\sigma(\mathbf{x}') \equiv 0$ is denoted by $\mathcal{H}^0(\mathbf{k}') = \mathcal{H}^0(\mathbf{k}'; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi) := \mathcal{H}(\mathbf{k}'; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi; 0, 0)$.

The direct integral decomposition for periodic operators is constructed with the help of the *Gelfand transformation* \mathcal{U} . Consider the Hilbert space

$$\mathcal{K} := \int_{\widetilde{\Omega}} \oplus L_2(\Omega; \mathbb{C}^M) d\mathbf{k}'. \quad (2.2)$$

First, we define the mapping $\mathcal{U} : L_2(\Pi; \mathbb{C}^M) \rightarrow \mathcal{K}$ on the functions $\mathbf{f} \in C^\infty(\overline{\Pi}; \mathbb{C}^M)$ rapidly decreasing as $|\mathbf{x}'| \rightarrow \infty$ by the formula

$$(\mathcal{U}\mathbf{f})(\mathbf{k}', \mathbf{x}', x_d) = (\text{meas } \widetilde{\Omega})^{-1/2} e^{-i\langle \mathbf{k}', \mathbf{x}' \rangle} \sum_{\mathbf{q} \in \Gamma} e^{-i\langle \mathbf{k}', \mathbf{q} \rangle} \mathbf{f}(\mathbf{x}' + \mathbf{q}, x_d).$$

Then \mathcal{U} is extended by continuity to a *unitary* operator acting from $L_2(\Pi; \mathbb{C}^M)$ onto \mathcal{K} . Moreover, it turns out that if

$$\mathbf{f} \in H^1(\Pi; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi), \quad (2.3)$$

then

$$(\mathcal{U}\mathbf{f})(\mathbf{k}', \cdot) \in \widetilde{H}^1(\Omega; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi), \quad \text{for a. e. } \mathbf{k}' \in \widetilde{\Omega}, \quad (2.4)$$

$$h[\mathbf{f}, \mathbf{f}] = \int_{\widetilde{\Omega}} h(\mathbf{k}') [(\mathcal{U}\mathbf{f})(\mathbf{k}', \cdot), (\mathcal{U}\mathbf{f})(\mathbf{k}', \cdot)] d\mathbf{k}'. \quad (2.5)$$

Conversely, if $\mathbf{f} \in L_2(\Pi; \mathbb{C}^M)$ satisfies (2.4) and the integral (2.5) is finite, then (2.3) is true. From (2.4), (2.5) it is clear that in the direct integral (2.2) the action of the operator \mathcal{H} reduces to multiplication by the operator-valued function $\mathcal{H}(\mathbf{k}')$:

$$(\mathcal{U}\mathbf{f})(\mathbf{k}', \cdot) \in \text{Dom } \mathcal{H}(\mathbf{k}') \quad \text{for a. e. } \mathbf{k}' \in \widetilde{\Omega}, \quad \text{if } \mathbf{f} \in \text{Dom } \mathcal{H}, \quad (2.6)$$

$$(\mathcal{U}\mathcal{H}\mathbf{f})(\mathbf{k}', \cdot) = \mathcal{H}(\mathbf{k}')(\mathcal{U}\mathbf{f})(\mathbf{k}', \cdot), \quad \text{for a. e. } \mathbf{k}' \in \widetilde{\Omega}. \quad (2.7)$$

Relations (2.6), (2.7) can be written as

$$\mathcal{U}\mathcal{H}\mathcal{U}^{-1} = \int_{\widetilde{\Omega}} \oplus \mathcal{H}(\mathbf{k}') d\mathbf{k}'.$$

Remark 2.1. *By a linear change of variables \mathbf{x}' in \mathbb{R}^{d-1} , the lattice Γ can be transformed into a cubic lattice. With that, the matrix a converts into another matrix of the same type. Thus, in the proofs we may assume, without loss of generality, that*

$$\Gamma = \mathbb{Z}^{d-1}, \quad \Omega' = (0, 1)^{d-1},$$

so that $\tilde{\Omega} = (0, 2\pi)^{d-1}$.

We assume this in what follows.

2. Complexification. The Thomas method involves extension of the forms (2.1) and the corresponding operators $\mathcal{H}(\mathbf{k}')$ to the complex values $\mathbf{k}' \in \mathbb{C}^{d-1}$ of the quasimomentum. We rewrite the form (2.1) as follows:

$$h(\mathbf{k}')[\mathbf{u}, \mathbf{u}] = \sum_{s=1}^M \int_{\Omega} \langle \hat{a} \mathbf{D} u_s, \mathbf{D} u_s \rangle_d d\mathbf{x} + w_1(\mathbf{k}')[\mathbf{u}, \mathbf{u}] + w_2[\mathbf{u}, \mathbf{u}] + w_3[\mathbf{u}, \mathbf{u}], \quad (2.8)$$

$$\mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi),$$

$$w_1(\mathbf{k}')[\mathbf{u}, \mathbf{u}] := \sum_{s=1}^M \int_{\Omega} (\langle \hat{a} \mathbf{D} u_s, \bar{\mathbf{k}} u_s \rangle_d + \langle \hat{a} \mathbf{k} u_s, \mathbf{D} u_s \rangle_d) d\mathbf{x} + \int_{\Omega} \langle \hat{a} \mathbf{k}, \bar{\mathbf{k}} \rangle_d |\mathbf{u}|^2 d\mathbf{x}, \quad \mathbf{k} = (\mathbf{k}', 0), \quad (2.9)$$

$$w_2[\mathbf{u}, \mathbf{u}] := \int_{\Omega} \langle \mathcal{V}(\mathbf{x}) \mathbf{u}, \mathbf{u} \rangle_M d\mathbf{x}, \quad (2.10)$$

$$w_3[\mathbf{u}, \mathbf{u}] := \int_{\Omega'} \langle \sigma(\mathbf{x}') \Theta_0 \mathbf{u}, \Theta_0 \mathbf{u} \rangle_{2M} d\mathbf{x}'. \quad (2.11)$$

The conditions imposed on \mathcal{V} , σ and Propositions 1.1, 1.2 ensure the estimates

$$\begin{aligned} |w_1(\mathbf{k}')[\mathbf{u}, \mathbf{u}]| &\leq \varepsilon \sum_{s=1}^M \int_{\Omega} \langle \hat{a} \mathbf{D} u_s, \mathbf{D} u_s \rangle_d d\mathbf{x} + C(\varepsilon, \mathbf{k}', a) \int_{\Omega} |\mathbf{u}|^2 d\mathbf{x}, \\ |w_2[\mathbf{u}, \mathbf{u}]| &\leq \varepsilon \sum_{s=1}^M \int_{\Omega} \langle \hat{a} \mathbf{D} u_s, \mathbf{D} u_s \rangle_d d\mathbf{x} + C(\varepsilon, a, \mathcal{V}) \int_{\Omega} |\mathbf{u}|^2 d\mathbf{x}, \\ |w_3[\mathbf{u}, \mathbf{u}]| &\leq \varepsilon \sum_{s=1}^M \int_{\Omega} \langle \hat{a} \mathbf{D} u_s, \mathbf{D} u_s \rangle_d d\mathbf{x} + C(\varepsilon, a, \sigma) \int_{\Omega} |\mathbf{u}|^2 d\mathbf{x}, \\ &\mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi), \quad 0 < \varepsilon \leq 1. \end{aligned} \quad (2.12)$$

Formula (2.9) allows us to extend $w_1(\mathbf{k}')$ to arbitrary $\mathbf{k}' \in \mathbb{C}^{d-1}$ analytically. At the same time, (2.8) yields an analytic extension of $h(\mathbf{k}')$. The first estimate in (2.12) remains valid for all $\mathbf{k}' \in \mathbb{C}^{d-1}$. This implies (see [K, Theorem VI.3.9]) that the form $h(\mathbf{k}')$ is closed and sectorial for all $\mathbf{k}' \in \mathbb{C}^{d-1}$. Such a form gives rise to an m -sectorial operator in $L_2(\Omega; \mathbb{C}^M)$, still denoted by $\mathcal{H}(\mathbf{k}')$, $\mathbf{k}' \in \mathbb{C}^{d-1}$. The resolvent of the operator $\mathcal{H}(\mathbf{k}')$ is compact for all $\mathbf{k}' \in \mathbb{C}^{d-1}$.

3. We want to deal with the dependence of the family $\mathcal{H}(\mathbf{k}')$ on a one-dimensional parameter; for this, we single out a direction by fixing some vector $\mathbf{m} \in \mathbb{Z}^{d-1}$ and putting

$$\boldsymbol{\alpha} := \langle a^{-1}\mathbf{m}, \mathbf{m} \rangle^{-1/2} a^{-1}\mathbf{m}. \quad (2.13)$$

Next, we fix some vector

$$\tilde{\mathbf{k}} \in \mathbb{R}^{d-1}, \quad \tilde{\mathbf{k}} \perp \mathbf{m}, \quad (2.14)$$

and put

$$\mathbf{k}' = z\boldsymbol{\alpha} + \tilde{\mathbf{k}}, \quad z \in \mathbb{C}. \quad (2.15)$$

Relative to parameter $z \in \mathbb{C}$, the operators $\mathcal{H}(z\boldsymbol{\alpha} + \tilde{\mathbf{k}})$ form an *analytic family of type (B) with compact resolvent*. Recall that a family of type (B) arises when we define operators in terms of sectorial forms with common domain (see [K, §VII.4]). As applied to the operator families of type (B), the Thomas approach (in the selfadjoint case) was described in detail in [BSu]. As applied to non-selfadjoint operators, the Thomas method was presented in [Ku], [KuL]. The arguments of [KuL, §2] are applicable to any analytic family with compact resolvent. The proof of the absence of eigenvalues of the operator \mathcal{H} reduces to the proof of the fact that $\|(\mathcal{H}((\mu + iy)\boldsymbol{\alpha} + \tilde{\mathbf{k}}))^{-1}\| \rightarrow 0$ as $|y| \rightarrow \infty$ for an appropriate μ and all $\tilde{\mathbf{k}}$ satisfying (2.14). In the selfadjoint case, this implies absolute continuity of the spectrum of \mathcal{H} . We put

$$z = \mu + iy, \quad \mu := \pi \langle a^{-1}\mathbf{m}, \mathbf{m} \rangle^{-1/2}, \quad \mathbf{k}^0 := \mu\boldsymbol{\alpha} + \tilde{\mathbf{k}}. \quad (2.16)$$

In this case, we agree to write $\mathcal{H}(y)$ in place of $\mathcal{H}((\mu + iy)\boldsymbol{\alpha} + \tilde{\mathbf{k}})$; and similarly for the other operators and forms.

Thus, Theorem 1.3 is a consequence of the following theorem, which is our main technical result.

Theorem 2.2. *Suppose that the conditions of Theorem 1.3 are satisfied. Let $\mathbf{m} \in \mathbb{Z}^{d-1}$ and let $\tilde{\mathbf{k}}$ satisfy (2.14). Suppose that $\boldsymbol{\alpha}, \mu$ are defined by (2.13), (2.16). Let $\mathcal{H}(y) = \mathcal{H}((\mu + iy)\boldsymbol{\alpha} + \tilde{\mathbf{k}})$ be the operator in $L_2(\Omega; \mathbb{C}^M)$ generated by the quadratic form (2.8). Then there exist constants*

$$y_0 = y_0(a, T, \mathcal{V}, \sigma, \mathbf{m}, \tilde{\mathbf{k}}), \quad C_0 = C_0(a, \mathbf{m})$$

such that the operator $\mathcal{H}(y)$ is invertible for $|y| \geq y_0$, and

$$\|(\mathcal{H}(y))^{-1}\| \leq C_0(1 + |y|)^{-1}, \quad |y| \geq y_0. \quad (2.17)$$

If $\tilde{\mathbf{k}}$ runs through a bounded set of the plane $\Pi_{\mathbf{m}} = \{\mathbf{k}' \in \mathbb{R}^{d-1} : \mathbf{k}' \perp \mathbf{m}\}$, then y_0 can be chosen independent of $\tilde{\mathbf{k}}$.

§ 3. ESTIMATES OF THE FREE OPERATOR

1. Let $\mathcal{I}, \mathcal{J}, \mathcal{L}$ and Ξ be fixed systems of numbers introduced in Subsection 1.1. For each $s \in \{1, \dots, M\}$, we put

$$\Gamma_s = \begin{cases} \mathbb{Z}, & \text{for } s \in \mathcal{L}, \\ \mathbb{N}, & \text{for } s \in \mathcal{I} \cap \mathcal{J}, \\ \mathbb{Z}_+, & \text{in other cases.} \end{cases}$$

Let $\{\varphi_n^{(s)}(x_d)\}$, $n \in \Gamma_s$, $s = 1, \dots, M$, be a sequence of functions on $(0, T)$, defined as follows:

$$\varphi_n^{(s)}(x_d) = \begin{cases} T^{-1/2} \exp\{iT^{-1}(2\pi n - \xi_s)x_d\}, & \text{if } s \in \mathcal{L}, \\ \sqrt{2}T^{-1/2} \sin(\pi T^{-1}nx_d), & \text{if } s \in \mathcal{I} \cap \mathcal{J}, \\ \sqrt{2}T^{-1/2} \cos(\pi T^{-1}nx_d), & \text{if } s \notin \mathcal{I} \cup \mathcal{J} \cup \mathcal{L}, \quad n \geq 1, \\ T^{-1/2}, & \text{if } s \notin \mathcal{I} \cup \mathcal{J} \cup \mathcal{L}, \quad n = 0, \\ \sqrt{2}T^{-1/2} \sin(\pi(2T)^{-1}(2n+1)x_d), & \text{if } s \in \mathcal{I} \setminus \mathcal{J}, \\ \sqrt{2}T^{-1/2} \cos(\pi(2T)^{-1}(2n+1)x_d), & \text{if } s \in \mathcal{J} \setminus \mathcal{I}. \end{cases} \quad (3.1)$$

We also introduce a sequence of numbers $\{\lambda_n^{(s)}\}$, $n \in \Gamma_s$, $s = 1, \dots, M$:

$$\lambda_n^{(s)} := \begin{cases} T^{-2}(2\pi n - \xi_s)^2, & \text{if } s \in \mathcal{L}, \\ \pi^2 T^{-2} n^2, & \text{if } s \in \mathcal{I} \cap \mathcal{J} \text{ or } s \notin \mathcal{I} \cup \mathcal{J} \cup \mathcal{L}, \\ \pi^2 (2T)^{-2} (2n+1)^2, & \text{if } s \in (\mathcal{I} \setminus \mathcal{J}) \cup (\mathcal{J} \setminus \mathcal{I}). \end{cases} \quad (3.2)$$

Let $\mathbf{u} = (u_1, \dots, u_M) \in L_2(\Omega; \mathbb{C}^M)$. We decompose $u_s \in L_2(\Omega)$ into the Fourier series

$$u_s(\mathbf{x}', x_d) = \sum_{\mathbf{n}=\{\mathbf{n}', n\} \in \mathbb{Z}^{d-1} \times \Gamma_s} \hat{u}_{\mathbf{n}}^{(s)} e^{2\pi i(\mathbf{n}', \mathbf{x}')} \varphi_n^{(s)}(x_d), \quad s = 1, \dots, M. \quad (3.3)$$

In accordance with (3.3), let

$$\mathcal{F}_s : L_2(\Omega) \rightarrow l_2(\mathbb{Z}^{d-1} \times \Gamma_s), \quad \mathcal{F}_s u_s := \{\hat{u}_{\mathbf{n}}^{(s)}\}, \quad \mathbf{n} \in \mathbb{Z}^{d-1} \times \Gamma_s, \quad s = 1, \dots, M, \quad (3.4)$$

be the discrete (unitary) Fourier transformation. Suppose that the quasimomentum is chosen according to (2.13)–(2.16). Then the "free" operator $\mathcal{H}^0(y) = \mathcal{H}^0(y; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi)$ is represented in the form

$$\mathcal{H}^0(y) = \mathcal{H}^{(1)}(y) \oplus \dots \oplus \mathcal{H}^{(M)}(y), \quad (3.5)$$

$$\mathcal{H}^{(s)}(y) = \mathcal{F}_s^* \mathbf{h}^{(s)}(y) \mathcal{F}_s, \quad s = 1, \dots, M, \quad (3.6)$$

where the symbols $\mathbf{h}^{(s)}(y) = \{h_{\mathbf{n}}^{(s)}(y)\}$, $\mathbf{n} = \{\mathbf{n}', n\} \in \mathbb{Z}^{d-1} \times \Gamma_s$, $s = 1, \dots, M$, are defined by the formulas

$$h_{\mathbf{n}}^{(s)}(y) = \langle a(2\pi \mathbf{n}' + \mathbf{k}^0), 2\pi \mathbf{n}' + \mathbf{k}^0 \rangle + \lambda_n^{(s)} - y^2 + 2iy \langle a(2\pi \mathbf{n}' + \mathbf{k}^0), \boldsymbol{\alpha} \rangle, \quad (3.7)$$

$$\mathbf{n} = \{\mathbf{n}', n\} \in \mathbb{Z}^{d-1} \times \Gamma_s, \quad s = 1, \dots, M.$$

Relations (2.13), (2.14), (2.16) imply the inequality

$$|\langle a(2\pi \mathbf{n}' + \mathbf{k}^0), \boldsymbol{\alpha} \rangle| \geq \mu, \quad \mathbf{n}' \in \mathbb{Z}^{d-1}. \quad (3.8)$$

Hence,

$$|\operatorname{Im} h_{\mathbf{n}}^{(s)}(y)| = 2|y| |\langle a(2\pi \mathbf{n}' + \mathbf{k}^0), \boldsymbol{\alpha} \rangle| \geq 2\mu|y|, \quad y \in \mathbb{R}, \quad \mathbf{n} \in \mathbb{Z}^{d-1} \times \Gamma_s, \quad s = 1, \dots, M. \quad (3.9)$$

From (2.13), (3.8) it follows that

$$|a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)| \geq \mu, \quad \mathbf{n}' \in \mathbb{Z}^{d-1}. \quad (3.10)$$

In the case $2|y| \leq \mu$, by (3.7), (3.10), we have

$$\begin{aligned} \operatorname{Re} h_{\mathbf{n}}^{(s)}(y) &= |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 + \lambda_n^{(s)} - y^2 \geq \lambda_n^{(s)} + \mu^2/2, \\ 2|y| &\leq \mu, \quad \mathbf{n} \in \mathbb{Z}^{d-1} \times \Gamma_s, \quad s = 1, \dots, M. \end{aligned} \quad (3.11)$$

Relations (3.9), (3.11) yield the estimate

$$|h_{\mathbf{n}}^{(s)}(y)| \geq c_0^{-1}(1 + |y|), \quad y \in \mathbb{R}, \quad \mathbf{n} \in \mathbb{Z}^{d-1} \times \Gamma_s, \quad s = 1, \dots, M, \quad c_0 = c_0(a, \mathbf{m}). \quad (3.12)$$

From (3.5), (3.6), (3.12) it is clear that the operator $\mathcal{H}^0(y)$ is invertible for all $y \in \mathbb{R}$, and

$$\|(\mathcal{H}^0(y))^{-1}\| \leq c_0(1 + |y|)^{-1}, \quad y \in \mathbb{R}, \quad c_0 = c_0(a, \mathbf{m}).$$

2. We will rely on the following statement which is contained in [BKaS, Proposition 4.2].

Proposition 3.1. *Let f be a function and \mathbf{g} a sequence such that*

$$f \in L_{\theta, \infty}(\Omega), \quad \mathbf{g} \in l_{\theta, \infty}(\mathbb{Z}^{d-1} \times \Gamma_s), \quad \theta > 2.$$

Let \mathcal{F}_s , $s = 1, \dots, M$, be the operators defined by (3.3), (3.4). Then the operators

$$f\mathcal{F}_s^* \mathbf{g} : l_2(\mathbb{Z}^{d-1} \times \Gamma_s) \rightarrow L_2(\Omega), \quad s = 1, \dots, M, \quad (3.13)$$

are bounded, and

$$\|f\mathcal{F}_s^* \mathbf{g}\| \leq c_1 \|f\|_{\theta, \infty, \Omega} \|\mathbf{g}\|_{\theta, \infty, \mathbb{Z}^{d-1} \times \Gamma_s}, \quad s = 1, \dots, M, \quad c_1 = c_1(d, \theta, T).$$

If either $f \in L_{\theta, \infty}^0(\Omega)$, or $\mathbf{g} \in l_{\theta, \infty}^0(\mathbb{Z}^{d-1} \times \Gamma_s)$, then the operators (3.13) are compact.

We need the estimate of the symbol $\mathbf{h}^{(s)}(y)$ in the Lorentz classes.

Proposition 3.2. *Let $d \geq 2$ and $2\theta \geq d$. Let $\boldsymbol{\alpha}$ and \mathbf{k}^0 be as in (2.13), (2.14), (2.16). Then the sequence*

$$(\mathbf{h}^{(s)}(y))^{-1} = \{(h_{\mathbf{n}}^{(s)}(y))^{-1}\}, \quad \mathbf{n} \in \mathbb{Z}^{d-1} \times \Gamma_s, \quad s = 1, \dots, M,$$

defined by (3.7) satisfies the estimate

$$\begin{aligned} \|(\mathbf{h}^{(s)}(y))^{-1}\|_{\theta, \infty, \mathbb{Z}^{d-1} \times \Gamma_s} &\leq c_2(1 + |y|)^{-\tau}, \quad y \in \mathbb{R}, \quad s = 1, \dots, M, \\ \tau &= 1 - (d-2)\theta^{-1} \quad \text{for } \theta \geq 2, \quad \tau = 2 - d\theta^{-1} \quad \text{for } \theta < 2. \end{aligned} \quad (3.14)$$

The constant $c_2 = c_2(a, \mathbf{m}, T, \theta)$ does not depend on y and $\tilde{\mathbf{k}}$.

Actually, the proof of Proposition 3.2 follows from [BSu, Theorem 3.1].

3. The following statement is crucial for us.

Proposition 3.3. *Let $d \geq 3$. Let $\boldsymbol{\alpha}$ and \mathbf{k}^0 be as in (2.13), (2.14), (2.16). Then the sequence $\mathbf{h}^{(s)}(y)$ defined by (3.7) satisfies the estimate*

$$\sup_{\mathbf{n}' \in \mathbb{Z}^{d-1}} \sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1} \leq c_3(1 + |y|)^{-1/2}, \quad y \in \mathbb{R}, \quad s = 1, \dots, M, \quad c_3 = c_3(a, \mathbf{m}, T). \quad (3.15)$$

Proof.¹ We start with the case $2|y| \leq \mu$. Relations (3.2), (3.11) imply the inequality

$$\begin{aligned} \sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1} &\leq \sum_{n \in \Gamma_s} \left(\lambda_n^{(s)} + \mu^2/2 \right)^{-1} \leq c_4 \leq c_4(1 + \mu)^{1/2}(1 + |y|)^{-1/2}, \\ 2|y| &\leq \mu, \quad \mathbf{n}' \in \mathbb{Z}^{d-1}, \quad s = 1, \dots, M, \quad c_4 = c_4(a, \mathbf{m}, T), \end{aligned}$$

which yields (3.15) for $2|y| \leq \mu$.

Now, let $2|y| > \mu$ and $|a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 \geq y^2$. Then, by (3.7), (3.9),

$$\begin{aligned} \sqrt{2}|h_{\mathbf{n}}^{(s)}(y)| &\geq |\operatorname{Re} h_{\mathbf{n}}^{(s)}(y)| + |\operatorname{Im} h_{\mathbf{n}}^{(s)}(y)| \geq \lambda_n^{(s)} + 2\mu|y|, \\ 2|y| &> \mu, \quad |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 \geq y^2, \quad s = 1, \dots, M. \end{aligned} \quad (3.16)$$

From (3.2), (3.16) it follows that

$$\begin{aligned} \sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1} &\leq \sqrt{2} \sum_{n \in \Gamma_s} \left(\lambda_n^{(s)} + 2\mu|y| \right)^{-1} \leq c_5(1 + |y|)^{-1/2}, \\ 2|y| &> \mu, \quad |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 \geq y^2, \quad s = 1, \dots, M, \quad c_5 = c_5(a, \mathbf{m}, T). \end{aligned} \quad (3.17)$$

It remains to consider the case where $2|y| > \mu$ and $|a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 < y^2$. In this case, we introduce the notation $b(\mathbf{n}', y) := (y^2 - |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2)^{1/2}$,

$$\begin{aligned} \chi^{(s)}(n; \mathbf{n}', y) &:= \begin{cases} 1, & \text{if } \lambda_n^{(s)} > b^2(\mathbf{n}', y), \\ 0, & \text{if } \lambda_n^{(s)} \leq b^2(\mathbf{n}', y), \end{cases} \\ \tilde{\chi}^{(s)}(n; \mathbf{n}', y) &:= 1 - \chi^{(s)}(n; \mathbf{n}', y), \\ n \in \Gamma_s, \quad 2|y| &> \mu, \quad |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 < y^2, \quad s = 1, \dots, M. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1} &= \Lambda^{(s)}(\mathbf{n}', y) + \tilde{\Lambda}^{(s)}(\mathbf{n}', y), \\ 2|y| &> \mu, \quad |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 < y^2, \quad s = 1, \dots, M, \end{aligned} \quad (3.18)$$

where

$$\begin{aligned} \Lambda^{(s)}(\mathbf{n}', y) &:= \sum_{n \in \Gamma_s} \chi^{(s)}(n; \mathbf{n}', y) |h_{\mathbf{n}}^{(s)}(y)|^{-1}, \\ \tilde{\Lambda}^{(s)}(\mathbf{n}', y) &:= \sum_{n \in \Gamma_s} \tilde{\chi}^{(s)}(n; \mathbf{n}', y) |h_{\mathbf{n}}^{(s)}(y)|^{-1}. \end{aligned}$$

¹The proof was obtained jointly with F. V. Petrov.

For definiteness, assume that $s \in \mathcal{I} \cap \mathcal{J}$ (see (3.2)). The other cases can be treated similarly.

For $\lambda_n^{(s)} = \pi^2 T^{-2} n^2 > b^2(\mathbf{n}', y)$, by (3.7), (3.9), we have $\sqrt{2}|h_{\mathbf{n}}^{(s)}(y)| \geq \lambda_n^{(s)} - b^2(\mathbf{n}', y) + 2\mu|y|$. Since $n \in \mathbb{N}$, then $n \geq [b(\mathbf{n}', y)T\pi^{-1}] + 1$, and

$$\begin{aligned} \lambda_n^{(s)} - b^2(\mathbf{n}', y) &= \pi^2 T^{-2} (n^2 - b^2(\mathbf{n}', y)T^2 \pi^{-2}) \geq \pi^2 T^{-2} r^2, \\ r &:= n - [b(\mathbf{n}', y)T\pi^{-1}] - 1 \in \mathbb{Z}_+. \end{aligned}$$

Hence, $\sqrt{2}|h_{\mathbf{n}}^{(s)}(y)| \geq \pi^2 T^{-2} r^2 + 2\mu|y|$. Thus,

$$\begin{aligned} \Lambda^{(s)}(\mathbf{n}', y) &\leq \sqrt{2} \sum_{r \in \mathbb{Z}_+} (\pi^2 T^{-2} r^2 + 2\mu|y|)^{-1} \leq c_5 (1 + |y|)^{-1/2}, \\ 2|y| &> \mu, \quad |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 < y^2, \quad c_5 = c_5(a, \mathbf{m}, T). \end{aligned} \quad (3.19)$$

For $\lambda_n^{(s)} = \pi^2 T^{-2} n^2 \leq b^2(\mathbf{n}', y)$ we have $n \leq [b(\mathbf{n}', y)T\pi^{-1}]$, and, by (3.7), (3.9), $\sqrt{2}|h_{\mathbf{n}}^{(s)}(y)| \geq b^2(\mathbf{n}', y) - \pi^2 T^{-2} n^2 + 2\mu|y|$. Then

$$\begin{aligned} \tilde{\Lambda}^{(s)}(\mathbf{n}', y) &\leq \sqrt{2} \sum_{1 \leq n \leq [b(\mathbf{n}', y)T\pi^{-1}]} (b^2(\mathbf{n}', y) - \pi^2 T^{-2} n^2 + 2\mu|y|)^{-1}, \\ 2|y| &> \mu, \quad |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 < y^2. \end{aligned} \quad (3.20)$$

The sum (3.20) is estimated via the corresponding integral:

$$\begin{aligned} \tilde{\Lambda}^{(s)}(\mathbf{n}', y) &\leq \sqrt{2}(2\mu|y|)^{-1} + \sqrt{2} \int_0^{[b(\mathbf{n}', y)T\pi^{-1}]} (b^2(\mathbf{n}', y) - \pi^2 T^{-2} t^2 + 2\mu|y|)^{-1} dt \leq \\ &\sqrt{2}(2\mu|y|)^{-1} + \sqrt{2}T\pi^{-1} \int_0^{b(\mathbf{n}', y)} (b^2(\mathbf{n}', y) - \tau^2 + 2\mu|y|)^{-1} d\tau = \sqrt{2}(2\mu|y|)^{-1} + \\ &2^{-1/2}T\pi^{-1}(b^2(\mathbf{n}', y) + 2\mu|y|)^{-1/2} \ln \left| \frac{b(\mathbf{n}', y) + (b^2(\mathbf{n}', y) + 2\mu|y|)^{1/2}}{b(\mathbf{n}', y) - (b^2(\mathbf{n}', y) + 2\mu|y|)^{1/2}} \right|. \end{aligned}$$

Using the elementary identity

$$\ln \left| \frac{(1 + \rho)^{1/2} + 1}{(1 + \rho)^{1/2} - 1} \right| \leq 5(1 + \rho^{-1})^{1/2}, \quad \rho > 0,$$

we obtain the estimate

$$\begin{aligned} \tilde{\Lambda}^{(s)}(\mathbf{n}', y) &\leq \sqrt{2}(2\mu|y|)^{-1} + 5T(2\pi)^{-1}(\mu|y|)^{-1/2} \leq c_6(1 + |y|)^{-1/2}, \\ 2|y| &> \mu, \quad |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 < y^2, \quad c_6 = c_6(a, \mathbf{m}, T). \end{aligned} \quad (3.21)$$

Now, relations (3.18), (3.19), (3.21) imply that

$$\sum_{n \in \mathbb{N}} |h_{\mathbf{n}}^{(s)}(y)|^{-1} \leq (c_5 + c_6)(1 + |y|)^{-1/2}, \quad 2|y| > \mu, \quad |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 < y^2. \quad (3.22)$$

Finally, (3.17), (3.22) yield inequality (3.15) for $2|y| > \mu$. •

Remark 3.4. 1) It is easily seen that the estimate (3.15) is of sharp order, i. e., we have the following estimate from below:

$$\sup_{\mathbf{n}' \in \mathbb{Z}^{d-1}} \sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1} \geq c_7(1 + |y|)^{-1/2},$$

$$y \in \mathbb{R}, \quad s = 1, \dots, M, \quad d \geq 3, \quad c_7 = c_7(a, \mathbf{m}, T) > 0.$$

2) In the case $d = 2$ the analogue of (3.15) is the estimate

$$\sup_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1} \leq c(1 + |y|)^{-1} \ln(2 + |y|),$$

$$y \in \mathbb{R}, \quad s = 1, \dots, M, \quad d = 2, \quad c = c(a, \mathbf{m}, T).$$

However, we will not use the latter estimate.

4. Let $P^{(s)}(y)$, $s = 1, \dots, M$, be the operator in $L_2(\Omega)$ defined by the formula

$$P^{(s)}(y) := \mathcal{F}_s^* \mathbf{P}^{(s)}(y) \mathcal{F}_s, \quad s = 1, \dots, M, \quad (3.23)$$

$$\mathbf{P}^{(s)}(y) := \{|h_{\mathbf{n}}^{(s)}(y)|^{-1/2}\}, \quad \mathbf{n} \in \mathbb{Z}^{d-1} \times \Gamma_s.$$

We put

$$P(y) = P^{(1)}(y) \oplus \dots \oplus P^{(M)}(y). \quad (3.24)$$

Then, by (3.12), the operator $P(y)$ is bounded in $L_2(\Omega; \mathbb{C}^M)$, and

$$\|P(y)\| \leq c_0^{1/2}(1 + |y|)^{-1/2}, \quad y \in \mathbb{R}. \quad (3.25)$$

Besides, by (3.14), (3.23), we have

$$\|\mathbf{P}^{(s)}(y)\|_{2p, \infty, \mathbb{Z} \times \Gamma_s} \leq c_2^{1/2}(1 + |y|)^{-\tau/2}, \quad \tau = 1 \text{ for } p \geq 2, \quad (3.26)$$

$$\tau = 2 - 2p^{-1} \text{ for } 1 < p < 2, \quad s = 1, \dots, M, \quad d = 2, \quad c_2 = c_2(a, \mathbf{m}, T, p),$$

$$\|\mathbf{P}^{(s)}(y)\|_{\varkappa, \infty, \mathbb{Z}^{d-1} \times \Gamma_s} \leq c_2^{1/2}, \quad \varkappa = \max\{2(d-2), d\}, \quad (3.27)$$

$$s = 1, \dots, M, \quad d \geq 2, \quad c_2 = c_2(a, \mathbf{m}, T, \varkappa/2).$$

We put

$$c_* = c_*(a, \mathbf{m}, \tilde{\mathbf{k}}) := \max\{\pi^{-1} |a^{-1}|^{1/2}, (2\pi)^{-1} |\mathbf{k}^0|\}.$$

In accordance with (3.3), we define the projectors $\mathcal{Z}^{(s)}(y)$, $\tilde{\mathcal{Z}}^{(s)}(y)$, $s = 1, \dots, M$, in $L_2(\Omega)$ by the formulas

$$(\mathcal{Z}^{(s)}(y)u_s)(\mathbf{x}', x_d) = \sum_{\mathbf{n} \in \mathbb{Z}^{d-1} \times \Gamma_s, |\mathbf{n}'| \leq c_*(1+|y|)} \hat{u}_{\mathbf{n}}^{(s)} e^{2\pi i(\mathbf{n}', \mathbf{x}')} \varphi_n^{(s)}(x_d), \quad (3.28)$$

$$\tilde{\mathcal{Z}}^{(s)}(y) = I - \mathcal{Z}^{(s)}(y), \quad s = 1, \dots, M. \quad (3.29)$$

Let $\mathcal{Z}(y)$, $\tilde{\mathcal{Z}}(y)$ be the projectors in $L_2(\Omega; \mathbb{C}^M)$ defined by

$$\mathcal{Z}(y) := \mathcal{Z}^{(1)}(y) \oplus \dots \oplus \mathcal{Z}^{(M)}(y), \quad (3.30)$$

$$\tilde{\mathcal{Z}}(y) := I - \mathcal{Z}(y) = \tilde{\mathcal{Z}}^{(1)}(y) \oplus \dots \oplus \tilde{\mathcal{Z}}^{(M)}(y).$$

Proposition 3.5. *Let $d \geq 2$. Let $\boldsymbol{\alpha}$ and \mathbf{k}^0 be as in (2.13), (2.14), (2.16). Let $P^{(s)}(y)$ be the operators defined by (3.7), (3.23), and let $\tilde{\mathcal{Z}}^{(s)}(y)$ be the operators defined by (3.28), (3.29). Then the operators $\mathbf{D}\tilde{\mathcal{Z}}^{(s)}(y)P^{(s)}(y) : L_2(\Omega) \rightarrow L_2(\Omega; \mathbb{C}^d)$, $s = 1, \dots, M$, are bounded for all $y \in \mathbb{R}$, and*

$$\|\mathbf{D}\tilde{\mathcal{Z}}^{(s)}(y)P^{(s)}(y)\| \leq c_8, \quad y \in \mathbb{R}, \quad s = 1, \dots, M, \quad c_8 = c_8(a, \mathbf{m}, \tilde{\mathbf{k}}). \quad (3.31)$$

If $\tilde{\mathbf{k}}$ runs through a bounded set of $\Pi_{\mathbf{m}}$, then the constant c_8 can be chosen independent of $\tilde{\mathbf{k}}$.

Proof. According to (3.23), (3.28), (3.29), for the proof of (3.31), we have to check the estimate

$$\left(|2\pi\mathbf{n}'|^2 + \lambda_n^{(s)} \right)^{1/2} |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} \leq c_8, \quad y \in \mathbb{R}, \quad |\mathbf{n}'| > c_*(1 + |y|), \quad n \in \Gamma_s. \quad (3.32)$$

For $|\mathbf{n}'| > c_*(1 + |y|)$ we have $|a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 > 2y^2$. Therefore, (see (3.7))

$$\operatorname{Re} h_{\mathbf{n}}^{(s)}(y) > \frac{1}{2} |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 + \lambda_n^{(s)}.$$

Hence,

$$\begin{aligned} |2\pi\mathbf{n}' + \mathbf{k}^0| |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} &< \sqrt{2} \lfloor a^{-1/2} \rfloor, \\ (\lambda_n^{(s)})^{1/2} |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} &< 1 \quad \text{for } |\mathbf{n}'| > c_*(1 + |y|). \end{aligned}$$

Combining this with (3.12), we arrive at (3.32). •

§ 4. ESTIMATES OF PERTURBATION

1. The matrix $\mathcal{V}(\mathbf{x})$ admits the following decomposition:

$$\begin{aligned} \mathcal{V}(\mathbf{x}) &= \mathcal{W}_1(\mathbf{x})U(\mathbf{x})\mathcal{W}_2(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ \mathcal{W}_1(\mathbf{x}) &:= |\mathcal{V}(\mathbf{x})^*|^{1/2}, \quad \mathcal{W}_2(\mathbf{x}) := |\mathcal{V}(\mathbf{x})|^{1/2}, \quad \mathbf{U}(\mathbf{x}) = 1. \end{aligned} \quad (4.1)$$

Proposition 4.1. *Let $d \geq 2$. Suppose that conditions (1.16), (1.17) are valid. Let $\mathcal{W}_j(\mathbf{x})$, $j = 1, 2$, be defined by (4.1). Let $\boldsymbol{\alpha}$ and \mathbf{k}^0 be as in (2.13), (2.14), (2.16). Let $P(y)$ be the operator defined by (3.7), (3.23), (3.24). Then the operators $\mathcal{W}_j P(y) : L_2(\Omega; \mathbb{C}^M) \rightarrow L_2(\Omega; \mathbb{C}^M)$, $j = 1, 2$, are bounded for all $y \in \mathbb{R}$, and*

$$\lim_{|y| \rightarrow \infty} \|\mathcal{W}_j P(y)\| = 0, \quad j = 1, 2. \quad (4.2)$$

The limit is uniform with respect to the parameter $\tilde{\mathbf{k}}$.

Proof. We start with the case $d = 2$. By (1.16), (4.1), $\mathcal{W}_j \in L_{2p}(\Omega)$, $p > 1$. Applying Proposition 3.1 (with $\theta = 2p > 2$) together with (3.26), we obtain

$$\begin{aligned} \|\mathcal{W}_j P(y)\| &\leq c_1 c_2^{1/2} (1 + |y|)^{-\tau/2} \|\mathcal{W}_j\|_{2p, \Omega}, \quad j = 1, 2, \\ \tau = 1 &\text{ for } p \geq 2, \quad \tau = 2 - 2p^{-1} \text{ for } 1 < p < 2, \quad d = 2. \end{aligned}$$

This yields (4.2) for $d = 2$.

Now, let $d \geq 3$. By (1.17), (4.1), $\mathcal{W}_j \in L_{\varkappa, \infty}^0(\Omega)$, $\varkappa = \max\{2(d-2), d\}$. For a given $\varepsilon > 0$, we choose an $(M \times M)$ -matrix-valued function $\widetilde{\mathcal{W}}_j(\mathbf{x})$ such that

$$\widetilde{\mathcal{W}}_j \in L_\infty(\Omega), \quad \|\mathcal{W}_j - \widetilde{\mathcal{W}}_j\|_{\varkappa, \infty, \Omega} \leq \varepsilon, \quad j = 1, 2, \quad \varkappa = \max\{2(d-2), d\}.$$

We have

$$\mathcal{W}_j P(y) = \widetilde{\mathcal{W}}_j P(y) + (\mathcal{W}_j - \widetilde{\mathcal{W}}_j) P(y), \quad j = 1, 2. \quad (4.3)$$

For estimating the second summand in the right-hand side of (4.3), we apply Proposition 3.1 (with $\theta = \varkappa > 2$) together with (3.27). The first summand is estimated with the help of (3.25). Thus,

$$\|\mathcal{W}_j P(y)\| \leq c_0^{1/2} \|\widetilde{\mathcal{W}}_j\|_{\infty, \Omega} (1 + |y|)^{-1/2} + c_1 c_2^{1/2} \varepsilon, \quad j = 1, 2, \quad d \geq 3,$$

which implies (4.2) for $d \geq 3$. •

2. The matrix $\sigma(\mathbf{x}')$ can be decomposed as follows:

$$\begin{aligned} \sigma(\mathbf{x}') &= \psi_1(\mathbf{x}') \nu(\mathbf{x}') \psi_2(\mathbf{x}'), \\ \psi_1(\mathbf{x}') &= |\sigma(\mathbf{x}')^*|^{1/2}, \quad \psi_2(\mathbf{x}') = |\sigma(\mathbf{x}')|^{1/2}, \quad \|\nu(\mathbf{x}')\| = 1, \quad \mathbf{x}' \in \Omega'. \end{aligned} \quad (4.4)$$

We introduce the operators $G_j : L_2(\Omega; \mathbb{C}^M) \rightarrow L_2(\Omega'; \mathbb{C}^{2M})$ defined by the relations

$$G_j \mathbf{u} := \psi_j \Theta_0 \mathbf{u}, \quad \text{Dom } G_j = \widetilde{H}^1(\Omega; \mathbb{C}^M), \quad j = 1, 2. \quad (4.5)$$

By Proposition 1.2, under conditions (1.19)–(1.21), we have $G_j \mathbf{u} \in L_2(\Omega'; \mathbb{C}^{2M})$ for any $\mathbf{u} \in \widetilde{H}^1(\Omega; \mathbb{C}^M)$.

Repeating the arguments of [BSu, §4], it is easy to obtain the following statement.

Proposition 4.2. *Let $d = 2$. Suppose that condition (1.19) is satisfied. Let $\boldsymbol{\alpha}$ and \mathbf{k}^0 be as in (2.13), (2.14), (2.16). Let $P(y)$ be the operator defined by (3.7), (3.23), (3.24), and let G_j , $j = 1, 2$, be the operators defined by (4.5). Then the operators $G_j P(y) : L_2(\Omega; \mathbb{C}^M) \rightarrow L_2(\Omega'; \mathbb{C}^{2M})$ are bounded for all $y \in \mathbb{R}$, and*

$$\begin{aligned} \|G_j P(y)\| &\leq c_9 \|\sigma\|_{q, \Omega'}^{1/2} (1 + |y|)^{-\vartheta}, \quad y \in \mathbb{R}, \quad j = 1, 2, \\ 4\vartheta &= 1 - q^{-1} \quad \text{for } 1 < q \leq 3; \quad 6\vartheta = 1 \quad \text{for } q \geq 3; \quad c_9 = c_9(a, \mathbf{m}, T, q, \widetilde{\mathbf{k}}). \end{aligned} \quad (4.6)$$

If $\widetilde{\mathbf{k}}$ runs through a bounded set of $\Pi_{\mathbf{m}}$, then the constant c_9 can be chosen independent of $\widetilde{\mathbf{k}}$.

3. Now we proceed to the case $d \geq 3$.

Proposition 4.3. *Let $d \geq 3$, and assume that*

$$\sigma \in L_\infty(\Omega'). \quad (4.7)$$

Let $\boldsymbol{\alpha}$ and \mathbf{k}^0 be as in (2.13), (2.14), (2.16). Let $P(y)$ be the operator defined by (3.7), (3.23), (3.24), and let G_j , $j = 1, 2$, be the operators defined by (4.5). Then the operators $G_j P(y) : L_2(\Omega; \mathbb{C}^M) \rightarrow L_2(\Omega'; \mathbb{C}^{2M})$ are bounded for all $y \in \mathbb{R}$, and

$$\|G_j P(y)\| \leq c_{10} \|\sigma\|_{\infty, \Omega'}^{1/2} (1 + |y|)^{-1/4}, \quad y \in \mathbb{R}, \quad j = 1, 2, \quad c_{10} = c_{10}(a, \mathbf{m}, T). \quad (4.8)$$

Proof. We have

$$\begin{aligned} \|G_j P(y) \mathbf{u}\|_{2, \Omega'}^2 &= \int_{\Omega'} |\psi_j(\mathbf{x}') \Theta_0 P(y) \mathbf{u}|^2 d\mathbf{x}' \leq \\ \|\sigma\|_{\infty, \Omega'} &\left(\int_{\Omega'} |(P(y) \mathbf{u})(\mathbf{x}', 0)|^2 d\mathbf{x}' + \int_{\Omega'} |(P(y) \mathbf{u})(\mathbf{x}', T)|^2 d\mathbf{x}' \right), \quad \mathbf{u} \in L_2(\Omega; \mathbb{C}^M). \end{aligned} \quad (4.9)$$

Let us consider the first integral in the bracket in the right-hand side of (4.9). The second integral can be treated similarly. By (3.3), (3.23), (3.24), we have

$$\begin{aligned} (P(y) \mathbf{u})_s(\mathbf{x}', 0) &= (P^{(s)}(y) u_s)(\mathbf{x}', 0) = \\ &\sum_{\mathbf{n} \in \mathbb{Z}^{d-1} \times \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} \hat{u}_{\mathbf{n}}^{(s)} e^{2\pi i \langle \mathbf{n}', \mathbf{x}' \rangle} \varphi_n^{(s)}(0), \quad s = 1, \dots, M. \end{aligned}$$

Taking account of (3.1), we obtain

$$|\varphi_n^{(s)}(0)| \leq \sqrt{2} T^{-1/2}, \quad n \in \Gamma_s, \quad s = 1, \dots, M. \quad (4.10)$$

By the Parseval identity, we have

$$\begin{aligned} \int_{\Omega'} |(P(y) \mathbf{u})_s(\mathbf{x}', 0)|^2 d\mathbf{x}' &= \sum_{\mathbf{n}' \in \mathbb{Z}^{d-1}} \left| \sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} \hat{u}_{\mathbf{n}}^{(s)} \varphi_n^{(s)}(0) \right|^2 \leq \\ &2T^{-1} \sup_{\mathbf{n}' \in \mathbb{Z}^{d-1}} \left(\sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1} \right) \|u_s\|_{2, \Omega}^2. \end{aligned}$$

By Proposition 3.3,

$$\begin{aligned} \int_{\Omega'} |(P(y) \mathbf{u})(\mathbf{x}', 0)|^2 d\mathbf{x}' &= \sum_{s=1}^M \int_{\Omega'} |(P(y) \mathbf{u})_s(\mathbf{x}', 0)|^2 d\mathbf{x}' \leq \\ &2T^{-1} c_3 (1 + |y|)^{-1/2} \|\mathbf{u}\|_{2, \Omega}^2, \quad \mathbf{u} \in L_2(\Omega; \mathbb{C}^M). \end{aligned}$$

The term $\int_{\Omega'} |(P(y) \mathbf{u})(\mathbf{x}', T)|^2 d\mathbf{x}'$ satisfies a similar estimate. Combining this with (4.9), we obtain (4.8). •

4. Now, our purpose is to pass from condition (4.7) to the more free condition $\sigma \in L_{2d-2}(\Omega')$.

Proposition 4.4. *Let $d \geq 3$, and suppose that*

$$\sigma \in L_{2d-2}(\Omega'). \quad (4.11)$$

Let $\boldsymbol{\alpha}$ and \mathbf{k}^0 be as in (2.13), (2.14), (2.16). Let $P(y)$ be the operator defined by (3.7), (3.23), (3.24), and let G_j , $j = 1, 2$, be the operators defined by (4.5). Then the operators $G_j P(y) : L_2(\Omega; \mathbb{C}^M) \rightarrow L_2(\Omega'; \mathbb{C}^{2M})$ are bounded for all $y \in \mathbb{R}$, and

$$\|G_j P(y)\| \leq c_{11} \|\sigma\|_{2d-2, \Omega'}^{1/2}, \quad y \in \mathbb{R}, \quad j = 1, 2, \quad c_{11} = c_{11}(a, \mathbf{m}, T, \tilde{\mathbf{k}}). \quad (4.12)$$

If $\tilde{\mathbf{k}}$ runs through a bounded set of $\Pi_{\mathbf{m}}$, then c_{11} can be chosen independent of $\tilde{\mathbf{k}}$.

Proof. By Proposition 1.2 (inequality (1.12)) and relations (3.24), (3.25), (3.30), (3.31),

$$\|G_j \tilde{\mathcal{Z}}(y)P(y)\| \leq c_{12} \|\sigma\|_{d-1, \infty, \Omega'}^{1/2} \leq c_{12} \|\sigma\|_{2d-2, \Omega'}^{1/2}, \quad c_{12} = c_{12}(a, \mathbf{m}, \tilde{\mathbf{k}}), \quad j = 1, 2.$$

Therefore, for the proof of inequality (4.12), it is sufficient to obtain the estimate

$$\|G_j \mathcal{Z}(y)P(y)\| \leq c_{13} \|\sigma\|_{2d-2, \Omega'}^{1/2}, \quad y \in \mathbb{R}, \quad j = 1, 2, \quad c_{13} = c_{13}(a, \mathbf{m}, T, \tilde{\mathbf{k}}). \quad (4.13)$$

Let $\mathbf{u} \in L_2(\Omega; \mathbb{C}^M)$. Then

$$\|G_j \mathcal{Z}(y)P(y)\mathbf{u}\|_{2, \Omega'}^2 = \int_{\Omega'} |\psi_j(\mathbf{x}')(\Theta_0 \mathcal{Z}(y)P(y)\mathbf{u})(\mathbf{x}')|^2 d\mathbf{x}' \leq \sum_{s=1}^M (\mathfrak{A}_s[u_s] + \mathfrak{B}_s[u_s]), \quad (4.14)$$

where

$$\begin{aligned} \mathfrak{A}_s[u_s] &:= \int_{\Omega'} |\sigma(\mathbf{x}')| |v_s(\mathbf{x}')|^2 d\mathbf{x}', \quad v_s(\mathbf{x}') := (\mathcal{Z}^{(s)}(y)P^{(s)}(y)u_s)(\mathbf{x}', 0), \\ \mathfrak{B}_s[u_s] &:= \int_{\Omega'} |\sigma(\mathbf{x}')| |w_s(\mathbf{x}')|^2 d\mathbf{x}', \quad w_s(\mathbf{x}') := (\mathcal{Z}^{(s)}(y)P^{(s)}(y)u_s)(\mathbf{x}', T). \end{aligned} \quad (4.15)$$

For definiteness, consider the term $\mathfrak{A}_s[u_s]$. By (3.23), (3.28), we have

$$\begin{aligned} v_s(\mathbf{x}') &= \sum_{\mathbf{n}' \in \mathbb{Z}^{d-1}, |\mathbf{n}'| \leq c_*(1+|y|)} \hat{v}_{\mathbf{n}'}^{(s)} e^{2\pi i(\mathbf{n}', \mathbf{x}')}, \\ \hat{v}_{\mathbf{n}'}^{(s)} &= \sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} \hat{u}_{\mathbf{n}}^{(s)} \varphi_n^{(s)}(0). \end{aligned} \quad (4.16)$$

Under the assumption $\sigma \in L_\rho(\Omega')$, $\rho > 1$, applying the Hölder inequality and the Young inequality and taking account of (4.15), (4.16), we obtain

$$\mathfrak{A}_s[u_s] \leq \|\sigma\|_{\rho, \Omega'} \|v_s\|_{2\rho', \Omega'}^2 \leq c \|\sigma\|_{\rho, \Omega'} \left(\sum_{\mathbf{n}' \in \mathbb{Z}^{d-1}, |\mathbf{n}'| \leq c_*(1+|y|)} |\hat{v}_{\mathbf{n}'}^{(s)}|^r \right)^{2/r}, \quad (4.17)$$

where $\rho^{-1} + (\rho')^{-1} = 1$, $r^{-1} + (2\rho')^{-1} = 1$. Next, by (4.10), (4.16),

$$|\hat{v}_{\mathbf{n}'}^{(s)}| \leq \sqrt{2}T^{-1/2} \left(\sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1} \right)^{1/2} \left(\sum_{n \in \Gamma_s} |\hat{u}_{\mathbf{n}}^{(s)}|^2 \right)^{1/2}.$$

Hence, by the Hölder inequality,

$$\begin{aligned} & \sum_{\mathbf{n}' \in \mathbb{Z}^{d-1}, |\mathbf{n}'| \leq c_*(1+|y|)} |\hat{v}_{\mathbf{n}'}^{(s)}|^r \leq \\ & 2^{r/2} T^{-r/2} \left(\sum_{\mathbf{n} \in \mathbb{Z}^{d-1} \times \Gamma_s, |\mathbf{n}'| \leq c_*(1+|y|)} |\hat{u}_{\mathbf{n}}^{(s)}|^2 \right)^{r/2} \times \\ & \left(\sum_{\mathbf{n}' \in \mathbb{Z}^{d-1}, |\mathbf{n}'| \leq c_*(1+|y|)} \left(\sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1} \right)^{rt/2} \right)^{1/t}, \end{aligned} \quad (4.18)$$

where $t^{-1} + r/2 = 1$. Obviously, $t = \rho + 1$, $rt/2 = \rho$. From (4.17), (4.18) it follows that

$$\mathfrak{A}_s[u_s] \leq 2cT^{-1} \|\sigma\|_{\rho, \Omega'} \|u_s\|_{2, \Omega}^2 \left(\sum_{\mathbf{n}' \in \mathbb{Z}^{d-1}, |\mathbf{n}'| \leq c_*(1+|y|)} \left(\sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1} \right)^\rho \right)^{1/\rho}. \quad (4.19)$$

By (3.15),

$$\begin{aligned} & \sum_{\mathbf{n}' \in \mathbb{Z}^{d-1}, |\mathbf{n}'| \leq c_*(1+|y|)} \left(\sum_{n \in \Gamma_s} |h_{\mathbf{n}}^{(s)}(y)|^{-1} \right)^\rho \leq \\ & c_3^\rho (1+|y|)^{-\rho/2} \#\{\mathbf{n}' \in \mathbb{Z}^{d-1} : |\mathbf{n}'| \leq c_*(1+|y|)\} \\ & \leq c_{14} (1+|y|)^{d-1-\rho/2}, \quad c_{14} = c_{14}(a, \mathbf{m}, T, \tilde{\mathbf{k}}, \rho). \end{aligned} \quad (4.20)$$

When $\rho = 2d - 2$, relations (4.19), (4.20) imply that

$$\mathfrak{A}_s[u_s] \leq c_{15} \|\sigma\|_{2d-2, \Omega'} \|u_s\|_{2, \Omega}^2, \quad u_s \in L_2(\Omega'), \quad c_{15} = c_{15}(a, \mathbf{m}, T, \tilde{\mathbf{k}}), \quad s = 1, \dots, M.$$

The term $\mathfrak{B}_s[u_s]$ satisfies a similar estimate. Combining this with (4.14), we arrive at (4.13). •

Proposition 4.5. *Suppose that conditions of Proposition 4.4 are satisfied. Then*

$$\lim_{|y| \rightarrow \infty} \|G_j P(y)\| = 0, \quad j = 1, 2. \quad (4.21)$$

If $\tilde{\mathbf{k}}$ runs through a bounded set of $\Pi_{\mathbf{m}}$, then the limit is uniform with respect to the parameter $\tilde{\mathbf{k}}$.

Proof. For a given $\varepsilon > 0$, we choose a $(2M \times 2M)$ -matrix-valued function $\tilde{\sigma}(\mathbf{x}')$ such that

$$\tilde{\sigma} \in L_\infty(\Omega'), \quad \|\sigma - \tilde{\sigma}\|_{2d-2, \Omega'} \leq \varepsilon. \quad (4.22)$$

For $\mathbf{u} \in L_2(\Omega; \mathbb{C}^M)$, we have

$$\|G_j P(y)\mathbf{u}\|_{2, \Omega'}^2 \leq \int_{\Omega'} |\tilde{\sigma}(\mathbf{x}')| |\Theta_0 P(y)\mathbf{u}|^2 d\mathbf{x}' + \int_{\Omega'} |\sigma(\mathbf{x}') - \tilde{\sigma}(\mathbf{x}')| |\Theta_0 P(y)\mathbf{u}|^2 d\mathbf{x}'. \quad (4.23)$$

The second summand in the right-hand side of (4.23) can be estimated by Proposition 4.4 together with (4.22). The first summand is estimated with the help of Proposition 4.3. As a result, we obtain

$$\begin{aligned} \|G_j P(y)\mathbf{u}\|_{2, \Omega'}^2 & \leq \left(c_{10}^2 \|\tilde{\sigma}\|_{\infty, \Omega'} (1+|y|)^{-1/2} + c_{11}^2 \varepsilon \right) \|\mathbf{u}\|_{2, \Omega}^2, \\ & \mathbf{u} \in L_2(\Omega; \mathbb{C}^M), \quad j = 1, 2. \end{aligned}$$

This implies (4.21). •

Remark 4.6. 1) *On the basis of Remark 3.4(2), we can prove the analogs of Propositions 4.3–4.5 in the two-dimensional case. This yields another (relative to Proposition 4.2 which follows from considerations of [BShSu]) way of the proof of relations (4.21) in the case where $d = 2$ and $\sigma \in L_q(\Omega')$, $q > 1$.*

2) *We have proved estimate (4.20) on the basis of (3.15). There is a certain reserve for relaxation of condition (1.21) on σ for $d \geq 4$. For this, one should directly examine the expression in the left-hand side of (4.20) and improve the estimate (4.20). However, such refinement would require further technical complications.*

§ 5. ESTIMATES IN THE THREE-DIMENSIONAL CASE

1. Estimates of the free operator. In the case $d = 3$ we are able to replace condition (4.11) by the wider condition (1.20). For this, we need to analyse the symbol of the free operator carefully. In accordance with (3.3), we introduce projectors $\mathcal{X}^{(s)}(y)$, $\tilde{\mathcal{X}}^{(s)}(y)$, $s = 1, \dots, M$, in $L_2(\Omega)$ defined as follows:

$$(\mathcal{X}^{(s)}(y)u_s)(\mathbf{x}', x_3) = \sum_{\mathbf{n} \in \mathbb{Z}^2 \times \Gamma_s, \lambda_n^{(s)} \leq 2y^2} \hat{u}_{\mathbf{n}}^{(s)} e^{2\pi i \langle \mathbf{n}', \mathbf{x}' \rangle} \varphi_n^{(s)}(x_3), \quad (5.1)$$

$$\tilde{\mathcal{X}}^{(s)}(y) = I - \mathcal{X}^{(s)}(y), \quad s = 1, \dots, M. \quad (5.2)$$

Let $\mathcal{X}(y)$, $\tilde{\mathcal{X}}(y)$ be projectors in $L_2(\Omega; \mathbb{C}^M)$ defined by the formulas

$$\begin{aligned} \mathcal{X}(y) &:= \mathcal{X}^{(1)}(y) \oplus \dots \oplus \mathcal{X}^{(M)}(y), \\ \tilde{\mathcal{X}}(y) &:= I - \mathcal{X}(y) = \tilde{\mathcal{X}}^{(1)}(y) \oplus \dots \oplus \tilde{\mathcal{X}}^{(M)}(y). \end{aligned} \quad (5.3)$$

Proposition 5.1. *Let $d = 3$. Let $\boldsymbol{\alpha}$ and \mathbf{k}^0 be as in (2.13), (2.14), (2.16). Let $P^{(s)}(y)$ be the operators defined by (3.7), (3.23), and let $\tilde{\mathcal{X}}^{(s)}(y)$ be the operators defined by (5.1), (5.2). Then the operators $\mathbf{D}\tilde{\mathcal{X}}^{(s)}(y)P^{(s)}(y) : L_2(\Omega) \rightarrow L_2(\Omega; \mathbb{C}^d)$, $s = 1, \dots, M$, are bounded for all $y \in \mathbb{R}$, and*

$$\|\mathbf{D}\tilde{\mathcal{X}}^{(s)}(y)P^{(s)}(y)\| \leq c_{16}, \quad y \in \mathbb{R}, \quad s = 1, \dots, M, \quad c_{16} = c_{16}(a, \mathbf{m}, \tilde{\mathbf{k}}). \quad (5.4)$$

If $\tilde{\mathbf{k}}$ runs through a bounded set of $\Pi_{\mathbf{m}}$, then the constant c_{16} can be chosen independent of $\tilde{\mathbf{k}}$.

Proof. According to (3.23), (5.1), (5.2), for the proof of (5.4), we have to check the estimate

$$\left(|2\pi\mathbf{n}'|^2 + \lambda_n^{(s)} \right)^{1/2} |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} \leq c_{16}, \quad y \in \mathbb{R}, \quad \mathbf{n}' \in \mathbb{Z}^2, \quad n \in \Gamma_s, \quad \lambda_n^{(s)} > 2y^2. \quad (5.5)$$

By (3.7), for $\lambda_n^{(s)} > 2y^2$, we have

$$\operatorname{Re} h_{\mathbf{n}}^{(s)}(y) > |a^{1/2}(2\pi\mathbf{n}' + \mathbf{k}^0)|^2 + \lambda_n^{(s)}/2.$$

Hence,

$$|2\pi\mathbf{n}' + \mathbf{k}^0| |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} < \lfloor a^{-1/2} \rfloor, \quad (\lambda_n^{(s)})^{1/2} |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} < \sqrt{2} \quad \text{for } \lambda_n^{(s)} > 2y^2.$$

Combining this with (3.12), we arrive at (5.5). •

Let us fix $n \in \Gamma_s$, and consider a sequence

$$\mathbf{r}^{(s)}(y; n) := \left\{ \left(h_{\{\mathbf{n}', n\}}^{(s)}(y) \right)^{-1} \right\}, \quad \mathbf{n}' \in \mathbb{Z}^2. \quad (5.6)$$

We need to estimate the norm of the sequence (5.6) in $l_{2,\infty}(\mathbb{Z}^2)$ for a fixed $n \in \Gamma_s$.

Proposition 5.2. *Let $d = 3$. Let $\boldsymbol{\alpha}$ and \mathbf{k}^0 be as in (2.13), (2.14), (2.16). Let $\mathbf{r}^{(s)}(y; n)$ be the sequence defined by (3.7), (5.6). Then we have*

$$\sum_{n \in \Gamma_s, \lambda_n^{(s)} \leq 2y^2} \|\mathbf{r}^{(s)}(y; n)\|_{2, \infty, \mathbb{Z}^2} \leq c_{17}, \quad y \in \mathbb{R}, \quad s = 1, \dots, M, \quad c_{17} = c_{17}(a, \mathbf{m}, T, \tilde{\mathbf{k}}). \quad (5.7)$$

If $\tilde{\mathbf{k}}$ runs through a bounded set of $\Pi_{\mathbf{m}}$, then the constant c_{17} can be chosen independent of $\tilde{\mathbf{k}}$.

Actually, the proof of Proposition 5.2 follows from [ShSu1, Lemma 5.2].

Remark 5.3. *For $d \geq 4$, the similar sum $\sum_{n \in \Gamma_s, \lambda_n^{(s)} \leq 2y^2} \|\mathbf{r}^{(s)}(y; n)\|_{p, \infty, \mathbb{Z}^{d-1}}$ grows together with $|y|$ for any $p < \infty$.*

2. Now, we proceed to estimating the operators $G_j P(y)$. For this, we need the following analogue of Proposition 3.1.

Proposition 5.4. *Let f be a function and $\mathbf{g} = \{g_{\mathbf{n}'}\}$ be a sequence such that*

$$f \in L_{\theta, \infty}(\Omega'), \quad \mathbf{g} \in l_{\theta, \infty}(\mathbb{Z}^{d-1}), \quad \theta > 2.$$

Let $T_{f\mathbf{g}} : l_2(\mathbb{Z}^{d-1}) \rightarrow L_2(\Omega')$ be the operator which takes a sequence $\mathbf{w} = \{w_{\mathbf{n}'}\}$, $\mathbf{n}' \in \mathbb{Z}^{d-1}$, into the function

$$(T_{f\mathbf{g}}\mathbf{w})(\mathbf{x}') = f(\mathbf{x}') \sum_{\mathbf{n}' \in \mathbb{Z}^{d-1}} g_{\mathbf{n}'} w_{\mathbf{n}'} e^{2\pi i(\mathbf{n}', \mathbf{x}')}.$$

Then the operator $T_{f\mathbf{g}}$ is bounded, and

$$\|T_{f\mathbf{g}}\| \leq c_{18} \|f\|_{\theta, \infty, \Omega'} \|\mathbf{g}\|_{\theta, \infty, \mathbb{Z}^{d-1}}, \quad c_{18} = c_{18}(d, \theta).$$

If either $f \in L_{\theta, \infty}^0(\Omega')$, or $\mathbf{g} \in l_{\theta, \infty}^0(\mathbb{Z}^{d-1})$, then the operator $T_{f\mathbf{g}}$ is compact.

Proposition 5.4 follows from [BKaS, Proposition 4.2].

Proposition 5.5. *Let $d = 3$, and suppose that $\sigma \in L_{2, \infty}(\Omega')$. Let $\boldsymbol{\alpha}$ and \mathbf{k}^0 be as in (2.13), (2.14), (2.16). Let $P(y)$ be the operator defined by (3.7), (3.23), (3.24), and let G_j , $j = 1, 2$, be the operators defined by (4.5). Then the operators $G_j P(y) : L_2(\Omega; \mathbb{C}^M) \rightarrow L_2(\Omega'; \mathbb{C}^{2M})$ are bounded for all $y \in \mathbb{R}$, and*

$$\|G_j P(y)\| \leq c_{19} \|\sigma\|_{2, \infty, \Omega'}^{1/2}, \quad y \in \mathbb{R}, \quad j = 1, 2, \quad c_{19} = c_{19}(a, \mathbf{m}, T, \tilde{\mathbf{k}}).$$

If $\tilde{\mathbf{k}}$ runs through a bounded set of $\Pi_{\mathbf{m}}$, then the constant c_{19} can be chosen independent of $\tilde{\mathbf{k}}$.

Proof. The following estimate is a direct consequence of Proposition 1.2 (inequality (1.12)) and relations (3.24), (3.25), (5.3), (5.4):

$$\|G_j \tilde{\mathcal{X}}(y) P(y)\| \leq c_{20} \|\sigma\|_{2, \infty, \Omega'}^{1/2}, \quad y \in \mathbb{R}, \quad j = 1, 2, \quad c_{20} = c_{20}(a, \mathbf{m}, \tilde{\mathbf{k}}).$$

Therefore, for the proof of Proposition, it is sufficient to check the estimate

$$\|G_j \mathcal{X}(y) P(y)\| \leq c_{21} \|\sigma\|_{2, \infty, \Omega'}^{1/2}, \quad y \in \mathbb{R}, \quad j = 1, 2, \quad c_{21} = c_{21}(a, \mathbf{m}, T, \tilde{\mathbf{k}}). \quad (5.8)$$

For $\mathbf{u} \in L_2(\Omega; \mathbb{C}^M)$, we have

$$\begin{aligned} \|G_j \mathcal{X}(y) P(y) \mathbf{u}\|_{2, \Omega'}^2 &= \int_{\Omega'} |\psi_j(\mathbf{x}') (\Theta_0 \mathcal{X}(y) P(y) \mathbf{u})(\mathbf{x}')|^2 d\mathbf{x}' \leq \\ &\sum_{s=1}^M (\mathcal{A}_s[u_s] + \mathcal{B}_s[u_s]), \\ \mathcal{A}_s[u_s] &:= \int_{\Omega'} |\boldsymbol{\sigma}(\mathbf{x}')| \left| (\mathcal{X}^{(s)}(y) P^{(s)}(y) u_s)(\mathbf{x}', 0) \right|^2 d\mathbf{x}', \\ \mathcal{B}_s[u_s] &:= \int_{\Omega'} |\boldsymbol{\sigma}(\mathbf{x}')| \left| (\mathcal{X}^{(s)}(y) P^{(s)}(y) u_s)(\mathbf{x}', T) \right|^2 d\mathbf{x}'. \end{aligned} \quad (5.9)$$

By (3.23), (5.1),

$$\begin{aligned} &(\mathcal{X}^{(s)}(y) P^{(s)}(y) u_s)(\mathbf{x}', 0) = \\ &\sum_{n \in \Gamma_s, \lambda_n^{(s)} \leq 2y^2} \varphi_n^{(s)}(0) \sum_{\mathbf{n}' \in \mathbb{Z}^2} |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} \hat{u}_{\mathbf{n}}^{(s)} e^{2\pi i \langle \mathbf{n}', \mathbf{x}' \rangle}, \quad s = 1, \dots, M. \end{aligned}$$

Taking account of (4.10), we obtain

$$\begin{aligned} &(\mathcal{A}_s[u_s])^{1/2} \leq \\ &\sum_{n \in \Gamma_s, \lambda_n^{(s)} \leq 2y^2} \left(\frac{2}{T} \int_{\Omega'} |\boldsymbol{\sigma}(\mathbf{x}')| \left| \sum_{\mathbf{n}' \in \mathbb{Z}^2} |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} \hat{u}_{\mathbf{n}}^{(s)} e^{2\pi i \langle \mathbf{n}', \mathbf{x}' \rangle} \right|^2 d\mathbf{x}' \right)^{1/2}. \end{aligned} \quad (5.10)$$

Introduce the notation

$$\mathbf{g}^{(s)}(y; n) := \{|h_{\{\mathbf{n}', n\}}^{(s)}|^{-1/2}\}, \quad \mathbf{n}' \in \mathbb{Z}^2.$$

By Proposition 5.4 (applied with $\theta = 4$), taking account of (5.6), we have

$$\begin{aligned} &\left(\int_{\Omega'} |\boldsymbol{\sigma}(\mathbf{x}')| \left| \sum_{\mathbf{n}' \in \mathbb{Z}^2} |h_{\mathbf{n}}^{(s)}(y)|^{-1/2} \hat{u}_{\mathbf{n}}^{(s)} e^{2\pi i \langle \mathbf{n}', \mathbf{x}' \rangle} \right|^2 d\mathbf{x}' \right)^{1/2} \leq \\ &c_{18} \|\boldsymbol{\sigma}\|_{4, \infty, \Omega'}^{1/2} \|\mathbf{g}^{(s)}(y; n)\|_{4, \infty, \mathbb{Z}^2} \left(\sum_{\mathbf{n}' \in \mathbb{Z}^2} |\hat{u}_{\mathbf{n}}^{(s)}|^2 \right)^{1/2} = \\ &c_{18} \|\boldsymbol{\sigma}\|_{2, \infty, \Omega'}^{1/2} \|\mathbf{r}^{(s)}(y; n)\|_{2, \infty, \mathbb{Z}^2}^{1/2} \left(\sum_{\mathbf{n}' \in \mathbb{Z}^2} |\hat{u}_{\mathbf{n}}^{(s)}|^2 \right)^{1/2}. \end{aligned}$$

Combining this with (5.10), we arrive at the estimate

$$\begin{aligned} (\mathcal{A}_s[u_s])^{1/2} &\leq c_{22} \|\boldsymbol{\sigma}\|_{2, \infty, \Omega'}^{1/2} \|u_s\|_{2, \Omega} \left(\sum_{n \in \Gamma_s, \lambda_n^{(s)} \leq 2y^2} \|\mathbf{r}^{(s)}(y; n)\|_{2, \infty, \mathbb{Z}^2} \right)^{1/2}, \quad (5.11) \\ &c_{22} = c_{22}(T). \end{aligned}$$

The term $\mathcal{B}_s[u_s]$ satisfies a similar estimate. Hence, by (5.7), (5.9), we obtain (5.8).

•

Proposition 5.6. *Let $d = 3$, and suppose that condition (1.20) is valid. Let $\boldsymbol{\alpha}$ and \mathbf{k}^0 be as in (2.13), (2.14), (2.16). Let $P(y)$ be the operator defined by (3.7), (3.23), (3.24), and let G_j , $j = 1, 2$, be the operators defined by (4.5). Then the operators $G_j P(y) : L_2(\Omega; \mathbb{C}^M) \rightarrow L_2(\Omega'; \mathbb{C}^{2M})$ are bounded for all $y \in \mathbb{R}$, and*

$$\lim_{|y| \rightarrow \infty} \|G_j P(y)\| = 0. \quad (5.12)$$

If $\tilde{\mathbf{k}}$ runs through a bounded set of $\Pi_{\mathbf{m}}$, then the limit is uniform in $\tilde{\mathbf{k}}$.

Proof. For a given $\varepsilon > 0$, we choose a $(2M \times 2M)$ -matrix-valued function $\sigma_\varepsilon(\mathbf{x}')$ such that

$$\sigma_\varepsilon \in L_\infty(\Omega'), \quad \|\sigma - \sigma_\varepsilon\|_{2, \infty, \Omega'} \leq \varepsilon. \quad (5.13)$$

For $\mathbf{u} \in L_2(\Omega; \mathbb{C}^M)$, we have

$$\|G_j P(y) \mathbf{u}\|_{2, \Omega'}^2 \leq \int_{\Omega'} \|\sigma_\varepsilon(\mathbf{x}')\| |\Theta_0 P(y) \mathbf{u}|^2 d\mathbf{x}' + \int_{\Omega'} \|\sigma(\mathbf{x}') - \sigma_\varepsilon(\mathbf{x}')\| |\Theta_0 P(y) \mathbf{u}|^2 d\mathbf{x}'. \quad (5.14)$$

We apply Proposition 5.5 together with (5.13) for estimating the second summand in the right-hand side of (5.14). The first summand is estimated with the help of Proposition 4.3. Then

$$\begin{aligned} \|G_j P(y) \mathbf{u}\|_{2, \Omega'}^2 &\leq \left(c_{10}^2 \|\sigma_\varepsilon\|_{\infty, \Omega'} (1 + |y|)^{-1/2} + c_{19}^2 \varepsilon \right) \|\mathbf{u}\|_{2, \Omega}^2, \\ &\mathbf{u} \in L_2(\Omega; \mathbb{C}^M), \quad y \in \mathbb{R}, \quad j = 1, 2. \end{aligned}$$

This implies (5.12). •

§ 6. PROOF OF THEOREM 2.2

1. By (3.5), (3.6), (3.23), (3.24), we have

$$R(y) := (\mathcal{H}^0(y))^{-1} = P(y) \mathcal{T}(y) P(y) = (P(y))^2 \mathcal{T}(y), \quad (6.1)$$

where

$$\begin{aligned} \mathcal{T}(y) &:= \mathcal{T}^{(1)}(y) \oplus \cdots \oplus \mathcal{T}^{(M)}(y), \\ \mathcal{T}^{(s)}(y) &:= \mathcal{F}_s^* \mathbf{t}^{(s)}(y) \mathcal{F}_s, \quad s = 1, \dots, M, \end{aligned} \quad (6.2)$$

$$\mathbf{t}^{(s)}(y) = \{ |h_{\mathbf{n}}^{(s)}(y)| (h_{\mathbf{n}}^{(s)}(y))^{-1} \}, \quad \mathbf{n} \in \mathbb{Z}^{d-1} \times \Gamma_s. \quad (6.3)$$

Obviously, $\mathcal{T}(y)$ is a unitary operator in $L_2(\Omega; \mathbb{C}^M)$.

2. Proof of Theorem 2.2. For the proof of estimate (2.17), it is sufficient to show that for sufficiently large y_0 and $|y| \geq y_0$ the following is true. For any function $0 \neq \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi)$ there exists such function $0 \neq \mathbf{v} \in \tilde{H}^1(\Omega; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi)$ (depending also on y) that

$$|h(y; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi; \mathcal{V}, \sigma)[\mathbf{u}, \mathbf{v}]| \geq C_0^{-1} (1 + |y|) \|\mathbf{u}\|_{2, \Omega} \|\mathbf{v}\|_{2, \Omega}, \quad |y| \geq y_0. \quad (6.4)$$

Let $0 \neq \mathbf{u} \in \tilde{H}^1(\Omega; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi)$ and let (see (6.1))

$$\mathbf{v} := (R(y))^* (P(y))^{-2} \mathbf{u} = P(y) (\mathcal{T}(y))^* P(y)^{-1} \mathbf{u} = (\mathcal{T}(y))^* \mathbf{u}. \quad (6.5)$$

From (6.2), (6.3), (6.5) it is clear that

$$\mathbf{v} \in \widetilde{H}^1(\Omega; \mathbb{C}^M; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi), \quad \|\mathbf{v}\|_{2, \Omega} = \|\mathbf{u}\|_{2, \Omega}. \quad (6.6)$$

By (2.8)–(2.11),

$$\begin{aligned} h(y)[\mathbf{u}, \mathbf{v}] &= h(y; \mathcal{I}, \mathcal{J}, \mathcal{L}, \Xi; \mathcal{V}, \sigma)[\mathbf{u}, \mathbf{v}] = \sum_{s=1}^M \int_{\Omega} \langle \widehat{a}(\mathbf{D} + \mathbf{k})u_s, (\mathbf{D} + \overline{\mathbf{k}})v_s \rangle_d d\mathbf{x} + \\ &\quad \int_{\Omega} \langle \mathcal{V}(\mathbf{x})\mathbf{u}, \mathbf{v} \rangle_M d\mathbf{x} + \int_{\Omega'} \langle \sigma(\mathbf{x}')\Theta_0\mathbf{u}, \Theta_0\mathbf{v} \rangle_{2M} d\mathbf{x}', \end{aligned} \quad (6.7)$$

where the quasimomentum $\mathbf{k} = (\mathbf{k}', 0)$ is chosen according to (2.13)–(2.16). By (4.1), (4.4), (4.5), (6.5), we obtain

$$\sum_{s=1}^M \int_{\Omega} \langle \widehat{a}(\mathbf{D} + \mathbf{k})u_s, (\mathbf{D} + \overline{\mathbf{k}})v_s \rangle d\mathbf{x} = \|(P(y))^{-1}\mathbf{u}\|_{2, \Omega}^2, \quad (6.8)$$

$$\left| \int_{\Omega} \langle \mathcal{V}(\mathbf{x})\mathbf{u}, \mathbf{v} \rangle d\mathbf{x} \right| \leq \|\mathcal{W}_1 P(y)\| \|\mathcal{W}_2 P(y)\| \|(P(y))^{-1}\mathbf{u}\|_{2, \Omega}^2, \quad (6.9)$$

$$\left| \int_{\Omega'} \langle \sigma(\mathbf{x}')\Theta_0\mathbf{u}, \Theta_0\mathbf{v} \rangle d\mathbf{x}' \right| \leq \|G_1 P(y)\| \|G_2 P(y)\| \|(P(y))^{-1}\mathbf{u}\|_{2, \Omega}^2. \quad (6.10)$$

Relations (6.7)–(6.10) yield the inequality

$$\begin{aligned} |h(y)[\mathbf{u}, \mathbf{v}]| &\geq \\ (1 - \|\mathcal{W}_1 P(y)\| \|\mathcal{W}_2 P(y)\| - \|G_1 P(y)\| \|G_2 P(y)\|) \|(P(y))^{-1}\mathbf{u}\|_{2, \Omega}^2, \quad y \in \mathbb{R}. \end{aligned} \quad (6.11)$$

The bracketed terms in the right-hand side of (6.11) are estimated with the help of Proposition 4.1, and also Proposition 4.2 (for $d = 2$), Proposition 4.5 (for $d \geq 4$) and Proposition 5.6 (for $d = 3$). Then, for sufficiently large $y_0 = y_0(a, T, \mathcal{V}, \sigma, \mathbf{m}, \widetilde{\mathbf{k}})$, we have

$$|h(y)[\mathbf{u}, \mathbf{v}]| \geq \frac{1}{2} \|(P(y))^{-1}\mathbf{u}\|_{2, \Omega}^2, \quad |y| \geq y_0. \quad (6.12)$$

If $\widetilde{\mathbf{k}}$ runs through a bounded set of $\Pi_{\mathbf{m}}$, then y_0 can be chosen independent of $\widetilde{\mathbf{k}}$. By (3.25),

$$\|(P(y))^{-1}\mathbf{u}\|_{2, \Omega}^2 \geq c_0^{-1}(1 + |y|)\|\mathbf{u}\|_{2, \Omega}^2, \quad y \in \mathbb{R}, \quad c_0 = c_0(a, \mathbf{m}). \quad (6.13)$$

Relations (6.6), (6.12), (6.13) imply the estimate

$$|h(y)[\mathbf{u}, \mathbf{v}]| \geq (2c_0)^{-1}(1 + |y|)\|\mathbf{u}\|_{2, \Omega}^2 = (2c_0)^{-1}(1 + |y|)\|\mathbf{u}\|_{2, \Omega}\|\mathbf{v}\|_{2, \Omega}, \quad |y| \geq y_0,$$

which yields (6.4) with $C_0 = 2c_0$. Theorem 2.2 is proved.

REFERENCES

- [BKaS] M. Sh. Birman, G. E. Karadzhov and M. Z. Solomyak, *Boundedness conditions and spectrum estimates for the operators $b(X)a(D)$ and their analogs*, Estimates and Asymptotics for Discrete Spectra of Integral and Differential Equations, Adv. Soviet Math. **7** (1991), Amer. Math. Soc., Providence, RI, 85–106.
- [BS] M. Sh. Birman and M. Z. Solomyak, *Schrödinger operator. Estimates for number of bound states as function-theoretical problem*, Spectral theory of operators (Novgorod, 1989), Amer. Math. Soc. Transl. Ser. 2 **150** (1992), Amer. Math. Soc., Providence, RI, 1–54.
- [BSHSu] M. Sh. Birman, R. G. Shterenberg and T. A. Suslina, *Absolute continuity of the spectrum of a two-dimensional Schrödinger operator with potential supported on a periodic system of curves*, Algebra i Analiz **12** (2000), no. 6, 140–177; St. Petersburg Math. J. **12** (2001), no. 6.
- [BSu] M. Sh. Birman and T. A. Suslina, *Periodic magnetic Hamiltonian with variable metric. The problem of absolute continuity*, Algebra i Analiz **11** (1999), no. 2, 1–40; St. Petersburg Math. J. **11** (2000), no. 2, 203–232.
- [K] T. Kato, *Perturbation theory for linear operators*, Grundlehren Math. Wiss., Bd. 132, Springer-Verlag New York Inc., New York, 1966.
- [Ku] P. Kuchment, *Floquet theory for partial differential equations*, Birkhäuser Verlag, Basel, 1993.
- [KuL] P. Kuchment, S. Levendorskii, *On the spectra of periodic elliptic operators*, Preprint mp_arc 00-388 (2000).
- [Mo] A. Morame, *The absolute continuity of the spectrum of Maxwell operator in a periodic media*, Preprint mp_arc 99-308 (1999).
- [RSi] M. Reed and B. Simon, *Methods of modern mathematical physics*, vol. IV, Academic Press, New York, 1975.
- [ShaSo] E. Shargorodsky and A. V. Sobolev, *Quasi-conformal mappings and periodic spectral problems*, University of Sussex at Brighton, Preprint (2001).
- [Sh1] R. G. Shterenberg, *Absolute continuity of a two-dimensional magnetic periodic Schrödinger operator with measure derivative like potential*, Zapiski nauchnyh semin. POMI **271** (2000), 276–312; Preprint TRITA-MAT-2001-02 (2001), Royal Institute of Technology, Stockholm.
- [Sh2] R. G. Shterenberg, *Absolute continuity of spectra of two-dimensional periodic Schrödinger operators with positive electric potentials*, Algebra i Analiz **13** (2001), no. 4; St. Petersburg Math. J. **13** (2002), no. 4.
- [ShSu1] R. G. Shterenberg and T. A. Suslina, *Absolute continuity of the spectrum of a Schrödinger operator with potential supported on a periodic system of hypersurfaces*, Algebra i Analiz **13** (2001), no. 5; St. Petersburg Math. J. **13** (2002), no. 5.
- [ShSu2] R. G. Shterenberg and T. A. Suslina, *Absolute continuity of the spectrum of a magnetic Schrödinger operator with metric in a two-dimensional periodic waveguide*, in preparation.
- [SoW] A. V. Sobolev and J. Walthoe, *Absolute continuity in periodic waveguides*, University of Sussex at Brighton, Preprint 2000-19 (2000).
- [Su] T. A. Suslina, *Absolute continuity of the spectrum of periodic operators of mathematical physics*, Journées équations aux dérivées partielles, Nantes, 5-9 juin 2000, XVIII-1–XVIII-13.
- [T] L. Thomas, *Time dependent approach to scattering from impurities in a crystal*, Comm. Math. Phys. **33** (1973), 335–343.

ST. PETERSBURG STATE UNIVERSITY, DEPARTMENT OF PHYSICS, DIVISION OF MATHEMATICAL PHYSICS. ST. PETERSBURG, 198904, PETRODVORETS, UL'YANOVSKAYA, 1

E-mail address: tanya@petrov.stoic.spb.su