N–transitivity of Certain Diffeomorphism Groups

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n-TRANSITIVITY OF CERTAIN DIFFEOMORPHISM GROUPS

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Abstract. It is shown that some groups of diffeomorphisms of a manifold act n-transitively for each finite n.

In this paper we show that the following groups of diffeomorphism of a manifold M act transitively of order n for each finite n: All diffeomorphisms with compact support (this is folklore, the first trace is in [8]) and all real analytic diffeomorphisms (from [7]). Furthermore all real analytic diffeomorphisms, or smooth ones with compact support, which preserve either a volume form, or a symplectic form, or are contact diffeomorphisms. The symplectic ones can also be chosen 'globally hamiltonian'. For the smooth cases 1-transitivity is due to [3], n-transitivity to [1].

1. Proposition. *Let M be a connected smooth manifold of dimension $\dim M \geq 2$. Then the group $\operatorname{Diff}_c(M)$ of all smooth diffeomorphisms with compact support acts n-transitively on M, for each finite n. Thus for any two ordered sets of n different points $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ in M there is a smooth diffeomorphism f with compact support such that $f(x_i) = y_i$ for each i.*

This result is folklore. In order to be complete and since we shall need an argument later on we include a short proof.

Proof. Let us first choose a finite $n \in \mathbb{N}$. Let $M^{(n)}$ denote the open submanifold of all n-tuples $(x_1, \dots, x_n) \in M^n$ of pairwise distinct points. $\operatorname{Diff}_c(M)$ acts on $M^{(n)}$ by the diagonal action, and we have to show, that this action is transitive.

Let us first assume that $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are pairwise disjoint. For some $\varepsilon > 0$ let $c_i : (-\varepsilon, 1+\varepsilon) \to M$ be smooth curves with $c_i(0) = x_i$ and $c_i(1) = y_i$ which are embeddings and do not intersect each other. From a drawing it can be seen that this exists if $\dim M \geq 2$, since $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ are disjoint. We choose pairwise disjoint tubular neighborhoods U_i of $c_i(-\varepsilon, 1+\varepsilon)$, extend the velocity vector fields of the curves to them, and use a smooth bump function to

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obtain vector fields X_i with compact support in U_i which coincides with the velocity vector field $c_i' \circ c_i^{-1}$ along each curve $c_i|[0,1]$. Then the vector field $X = X_1 + \dots X_n$ on M with compact support coincides with the velocity vector field $c' \circ c^{-1}$ along each curve c_i, and the flow mapping Fl_1^X maps each x_i to y_i.

This argument shows that each $\operatorname{Diff}_c(M)$-orbit in $M^{(n)}$ is dense. We may replace in the argument the points y_i by points z_i in small open pairwise disjoint neighborhoods U_i of y_i, not meeting $\{x_1, \dots, x_n\}$. Then the argument shows that each orbit contains an open set in $M^{(n)}$, thus is open. Since the dimension of M is at least 2, $M^{(n)}$ is connected, so there is only one orbit and the result on n-transitivity follows. □

2. Lemma. *Let M be a real analytic manifold. Then the group $\operatorname{Diff}^\omega(M)$ of all real analytic diffeomorphisms is dense in the group $\operatorname{Diff}^\infty(M)$ of smooth diffeomorphisms, in the Whitney C^∞-topology.*

Proof. By [2], theorem 3, there is a real analytic embedding $i : M \to \mathbb{R}^k$ on a closed submanifold, for some k. We use the constant standard inner product on $\mathbb{R}^k$ to obtain a real analytic tubular neighborhood U of $i(M)$ with projection $p : U \to i(M)$. By [2], proposition 8, applied to each coordinate of $\mathbb{R}^k$ the space $C^\omega(M, \mathbb{R}^k)$ of real analytic $\mathbb{R}^k$-valued functions is dense in the space $C^\infty(M, \mathbb{R}^k)$ of smooth functions, in the Whitney C^∞-topology. If $f : M \to M$ is a smooth diffeomorphism we may approximate $i \circ f$ by real analytic mappings g in $C^\omega(M, U)$, then $p \circ g$ is real analytic $M \to i(M)$ and approximates $i \circ f$. Since the set of diffeomorphisms is open in the Whitney topology, this approximation becomes eventually a diffeomorphism. □

3. Lemma. *Let M be a real analytic manifold. Then for any real analytic vector bundle $E \to M$ the space $C^\omega(E)$ of real analytic sections of E is dense in the space $C^\infty(E)$ of smooth sections. In particular the space $\mathfrak{X}^\omega(M)$ of real analytic vector fields is dense in the space $\mathfrak{X}(M)$ of smooth vector fields, in the Whitney C^∞-topology.*

Proof. Either repeat the proof of lemma 2 with some changes or use [6], 7.5. □

4. Theorem. *Let M be a connected real analytic manifold of dimension $m \geq 2$. Then the group $\operatorname{Diff}^\omega(M)$ acts n-transitively on M, for each finite n.*

Proof. Let us fix a natural number n. The group $\operatorname{Diff}^\omega(M)$ acts on the open submanifold $M^{(n)}$ of all n-tuples $(x_1, \dots, x_n) \in M^n$ of pairwise distinct points by the diagonal action. Again we have to show, that this action is transitive.

First we show that each $\operatorname{Diff}^\omega(M)$-orbit in $M^{(n)}$ is dense. Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ be in $M^{(n)}$ and consider an open neighborhood of $(y_1, \dots, y_n)$ in $M^{(n)}$, which we may suppose to be of the form $\prod_i U_i$, where U_i is a neighborhood of y_i in M for each i. Then by proposition 1 there is a smooth diffeomorphism $f : M \to M$ with $f(x_i) = y_i$ for all i, and by lemma 2 there exists a real analytic diffeomorphism $g \in \operatorname{Diff}^\omega(M)$ with $g(x_i) \in U_i$ for each i. So $g \cdot (x_1, \dots, x_n) \in \prod_i U_i$.

Next we show that the orbit through $(x_1, \dots, x_n) \in M^{(n)}$ in $M^{(n)}$ contains an open neighborhood of $(x_1, \dots, x_n)$. This will finish the proof: Since each orbit is dense, each orbit meets this nonempty open subset, so all orbits coincide.

We choose again a complete Riemannian metric g on M. Then we let $(Y_{ij})_{j=1}^m$ be an orthonormal basis of $T_{x_i}M$ with respect to g, for all i. Then we choose real analytic vector fields X_k for $1 \leq k \leq N = nm$ which satisfy the following conditions:

$$\begin{aligned} &|X_k(x_i) - Y_{ij}|_g < \varepsilon && \text{for } k = (i-1)m + j, \\ &|X_k(x_i)|_g < \varepsilon && \text{for all } k \notin [(i-1)m+1, im], \\ &|X_k(x)|_g < 2 && \text{for all } x \in M \text{ and all } k. \end{aligned}$$

These fields exist by lemma 3. Since the fields are bounded with respect to a complete Riemannian metric, they have complete real analytic flows, see e.g. [4]. We consider the real analytic mapping

$$f : \mathbb{R}^N \to M^{(n)}$$

$$f((t_1, \dots, t_N) := \begin{pmatrix} (\mathrm{Fl}^{X_1}_{t_1} \circ \dots \circ \mathrm{Fl}^{X_N}_{t_N})(x_1) \\ \dots \\ (\mathrm{Fl}^{X_1}_{t_1} \circ \dots \circ \mathrm{Fl}^{X_N}_{t_N})(x_n) \end{pmatrix}$$

which has values in the $\mathrm{Diff}^\omega(M)$-orbit through $(x_1, \dots, x_n)$. To get the tangent mapping at 0 of f we consider the partial derivatives

$$\frac{\partial}{\partial t_k}|_0 f(0, \dots, 0, t_k, 0, \dots, 0) = (X_k(x_1), \dots, X_k(x_n)).$$

If $\varepsilon > 0$ is small enough, this is near an orthonormal basis of $T_{(x_1,\dots,x_n)}M^{(n)}$ with respect to the product metric $g \times \dots \times g$. So $T_0 f$ is invertible and the image of f contains thus an open subset. $\square$

5. Lemma. *Let $c : (-\varepsilon, 1+\varepsilon) \to M^m$ be a smooth embedding. Then every 1-form (respectively $(m-1)$-form) along $c([0,1])$ can be extended to an exact 1-form (respectively $(m-1)$-form) on M with compact support in a tubular neighborhood of the image of c.*

Proof. There exists a tubular neighborhood of $c(-\varepsilon, 1+\varepsilon)$, i.e. a diffeomorphism from $(-\varepsilon, 1+\varepsilon) \times \mathbb{R}^{m-1}$ to an open neighborhood U of the image of c in M which coincides with c on $(-\varepsilon, 1+\varepsilon) \times \{0\}$, and whose inverse $u : U \to (-\varepsilon, 1+\varepsilon) \times \mathbb{R}^{m-1}$ we may use as a chart with $u(c(t)) = (t, 0)$.

(i) The case of a 1-form.

A 1-form along c is given by $\omega(t) = \sum_{i=1}^m a_i(t) du^i|_{c(t)}$ for $t \in [0,1]$, where $a_i : [0,1] \to \mathbb{R}$ are smooth and we may extend them smoothly to $a_i : (-\varepsilon, 1+\varepsilon) \to \mathbb{R}$. Consider the function $f : U \to \mathbb{R}$, given by

$$f = A_1(u^1) + u^2 a_2(u^1) + \dots + u^m a_m(u^1),$$

where $A_1(t) = \int_0^t a_1(s) ds$. Then $df(c(t)) = \omega(t)$. Let $h, k : \mathbb{R} \to \mathbb{R}$ be smooth bump functions such that $\operatorname{supp} h \subset (-\delta, \delta)$, $\operatorname{supp} k \subset (-\varepsilon, 1+\varepsilon)$, $h = 1$ in a neighborhood of 0, and $k = 1$ in a neighborhood of $[0,1]$. Then

$$F := k(u^1)h(u^2) \dots h(u^m) f$$

has compact support in U, so we extend it by 0 to the whole of M, and $dF = df$ near $c([0,1])$, so dF is also an extension of ω.

(ii) The case of an $(m-1)$-form.
An $(m-1)$-form along c is given by

$$\omega(t) = \sum_{i=1}^{m} b_i(t)du^1 \wedge \dots \wedge \widehat{du^i} \wedge \dots \wedge du^n|_{c(t)}$$

where $b_i : [0,1] \to \mathbb{R}$ are smooth functions which we may extend smoothly to $(-\varepsilon, 1+\varepsilon)$. Let us write $m = 2k$ or $m = 2k+1$. Then the following $(m-2)$-form $\beta \in \Omega^{m-2}(U)$ satisfies $d\beta|_{c(t)} = \omega(t)$.

$$\begin{aligned}
\beta &= \sum_{i=1}^{k} \beta_i du^1 \wedge \dots \wedge du^{2(i-1)} \wedge du^{2i+1} \wedge \dots \wedge du^m + \bar\beta du^1 \wedge \dots \wedge du^{m-2},\\
\beta_1 &= u^2 b_1(u^1) + \int_0^{u^1} b_2(t)dt,\\
\beta_i &= u^{2i} b_{2i-1}(u^1) + u^{2i-1} b_{2i}(u^1) \quad \text{for } 2 \le i \le k,\\
\bar\beta &= \begin{cases} -u^{m-1} b_m(u^1) & \text{for } m = 2k+1.\\ 0 & \text{for } m = 2k \end{cases}
\end{aligned}$$

Then $\tilde\beta := k(u^1)h(u^2)\dots h(u^m)\beta$, where h, k are bump functions as above, has compact support in U, so it may be extended by 0 to the whole of M, and since $\tilde\beta = \beta$ near $c([0,1])$ we still have $d\tilde\beta|_{c(t)} = \omega(t)$. □

6. Theorem. *Let (M, σ) be a connected symplectic smooth manifold of dimension $m \ge 2$. Then the group $\operatorname{Diff}_c(M, \sigma)$ of all smooth diffeomorphisms with compact support which preserve the symplectic form σ acts n-transitively on M, for each finite n.*

If M is a real analytic manifold with a real analytic symplectic form σ, then also the group $\operatorname{Diff}^\omega(M, \sigma)$ of real analytic symplectomorphisms acts n-transitively on M, for each finite n.

The n-transitivity of the group of smooth symplectomorphisms is due to [1], with essentially the same method. The proof will also show that the Lie subgroup of $\operatorname{Diff}_c(M, \sigma)$ whose Lie algebra is the Lie algebra of compactly supported globally Hamiltonian vector fields acts n-transitively on M. This group has been identified as a Lie group in [11], for compact M. Also in the real analytic case the subgroup of globally Hamiltonian real analytic symplectomorphisms act n-transitively.

Proof. First the smooth case. By the argument used at the end of the proof of proposition 1 it suffices to show, that there exists $\varphi \in \operatorname{Diff}_c(M, \sigma)$ with $ph(x_i) = y_i$, for any $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ in $M^{(n)}$ which are pairwise disjoint sets in M. We take again smooth curves $c_i : (-\varepsilon, 1+\varepsilon) \to M$ with $c_i(0) = x_i$ and $c_i(1) = y_i$ which are embeddings and do not intersect. Let U_i be pairwise disjoint tubular neighborhoods of $c_i(-\varepsilon, 1+\varepsilon)$.

The velocity field of the curve c_i defines the 1-form $\alpha_i = i_{c_i'}\sigma$ along the curve c_i. Using lemma 6 we extend this form to an exact 1-form df_i on M with $\operatorname{supp} f_i \subset$

U_i. Let $f := f_1 + \dots + f_n$ and consider the (globally) Hamiltonian vector field $\operatorname{grad}^\sigma(f) = -\sigma^{-1}df$ with compact support corresponding to f. It coincides with the velocity field $c_i' \circ c_i^{-1}$ on $c_i([0,1])$. Hence the flow $\operatorname{Fl}_t^{\operatorname{grad}^\sigma(f)} \in \operatorname{Diff}_c(M,\sigma)$ and $Fl_1^{\operatorname{grad}^\sigma(f)}(x_i) = y_i$.

If M and σ are real analytic, we may approximate the smooth function f from above by a real analytic function g in the Whitney C^1-topology in such a way that:

(1) The Hamiltonian vector field $\operatorname{grad}^\sigma(g)$ is bounded with respect to some complete Riemannian metric and thus has a global real analytic flow $\operatorname{Fl}_t^{\operatorname{grad}^\sigma(g)} \in \operatorname{Diff}^\omega(M,\sigma)$.
(2) $\operatorname{Fl}_1^{\operatorname{grad}^\sigma(g)}(x_i)$ is near y_i for all i.

Thus it follows that each $\operatorname{Diff}^\omega(M,\sigma)$-orbit in $M^{(n)}$ is dense. Similarly as in the proof of theorem 4 we will show that the orbit through $(x_1,\dots,x_n) \in M^{(n)}$ is open, which finishes the proof of n-transitivity.

We choose again a complete Riemannian metric g on M. Then we let $(Y_{ij})_{j=1}^m$ be an orthonormal basis of $T_{x_i}M$ with respect to g, for all i. Then we choose real analytic functions f_k for $1 \le k \le N = nm$ whose Hamiltonian vector fields satisfy the following conditions:

$$\begin{aligned}
&|\operatorname{grad}^\sigma(f_k)(x_i) - Y_{ij}|_g < \varepsilon && \text{for } k = (i-1)m + j,\\
&|\operatorname{grad}^\sigma(f_k)(x_i)|_g < \varepsilon && \text{for all } k \notin [(i-1)m+1, im],\\
&|\operatorname{grad}^\sigma(f_k)(x)|_g < 2 && \text{for all } x \in M \text{ and all } k.
\end{aligned}$$

Since these conditions describe Whitney C^1 open subsets, such functions exist by [2], proposition 8. Now we may finish the proof as at the end of theorem 4. □

7. Contact manifolds. Let M be a smooth manifold of dimension $m = 2n+1 \ge 3$. A *contact form* on M is a 1-form $\alpha \in \Omega^1(M)$ such that $\alpha \wedge (d\alpha)^n \in \Omega^{2n+1}(M)$ is nowhere zero. This is sometimes called an *exact* contact structure. The pair (M,α) is called a *contact manifold* (see [5]). The *contact vector field* $X_\alpha \in \mathfrak{X}(M)$ is the unique vector field satisfying $i_{X_\alpha}\alpha = 1$ and $i_{X_\alpha}d\alpha = 0$.

A diffeomorphism $f \in \operatorname{Diff}(M)$ with $f^*\alpha = \lambda_f.\alpha$ for a nowhere vanishing function $\lambda_f \in C^\infty(M, \mathbb{R}\setminus 0)$ is called a *contact diffeomorphism.* Note that then $\lambda_f = i_{X_\alpha}(\lambda_f.\alpha) = i_{X_\alpha}f^*\alpha = f^*(i_{(f^{-1})^*X_\alpha}\alpha) = f^*(i_{f_*X_\alpha}\alpha)$. The group of all contact diffeomorphisms will be denoted by $\operatorname{Diff}(M,\alpha)$.

A vector field $X \in \mathfrak{X}(M)$ is called a contact vector field if $\mathcal{L}_X\alpha = \mu_X.\alpha$ for a smooth function $\mu_X \in C^\infty(M,\mathbb{R})$. The linear space of all contact vector fields will be denoted by $\mathfrak{X}_\alpha(M)$ and it is clearly a Lie algebra. Contraction with α is a linear mapping again denoted by $\alpha : \mathfrak{X}_\alpha(M) \to C^\infty(M,\mathbb{R})$. It is bijective since we may apply i_{X_α} to the equation $\mathcal{L}_X\alpha = i_X\,d\alpha + d\alpha(X) = \mu_X.\alpha$ and get $0 + i_{X_\alpha}d\alpha(X) = \mu_X$; but the equation uniquely determines X from $\alpha(X)$ and μ_X. The inverse $f \mapsto \operatorname{grad}^\alpha(f)$ of $\alpha : \mathfrak{X}_\alpha(M) \to C^\infty(M,\mathbb{R})$ is a linear differential operator of order 1.

Theorem. *Let M be a connected smooth manifold of dimension $m \ge 2$, and let α be a contact form on M. Then the group $\operatorname{Diff}_c(M,\alpha)$ of contact diffeomorphisms with compact support acts n-transitively on M for all finite n.*

If M and α are real analytic then also the group $\operatorname{Diff}^\omega(M,\alpha)$ of real analytic contact diffeomorphisms acts n-transitively on M for each finite n.

The n-transitivity of $\operatorname{Diff}_c(M,\alpha)$ is due to [1].

Proof. By the argument used at the end of the proof of proposition 1 it suffices to show, that there exists $\varphi \in \operatorname{Diff}_c(M,\mu)$ with $\varphi(x_i) = y_i$, for any $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ in $M^{(n)}$ which are pairwise disjoint sets in M. For $\varepsilon > 0$ let again $c_i : (-\varepsilon, 1+\varepsilon) \to M$ be smooth embeddings with $c_i(0) = x_i$, $c_i(1) = y_i$ which do not intersect. We choose pairwise disjoint tubular neighborhoods U_i of $c_i(-\varepsilon, 1+\varepsilon)$. Let $f_i : M \to \mathbb{R}$ be a smooth extension of $\alpha(c_i' \circ c_i^{-1}) : c_i([0,1]) \to \mathbb{R}$ with support in U_i and $f := \sum_{i=1}^n f_i \in C_c^\infty(M,\mathbb{R})$. Then the contact vector field $\operatorname{grad}^\alpha(f) \in \mathfrak{X}_\alpha(M)$ coincides with the velocity field $c_i' \circ c_i^{-1}$ on $c_i([0,1])$ for each i. Hence $\operatorname{Fl}_1^X \in \operatorname{Diff}_c(M,\alpha)$ and $\operatorname{Fl}_1^X(x_i) = y_i$ for $i = 1,\dots,n$.

If M and α are real analytic, we may approximate the smooth function f from above by a real analytic function g in the Whitney C^1-topology in such a way that:

(1) The contact vector field $\operatorname{grad}^\alpha(g)$ is bounded with respect to a complete Riemannian metric and so has a global real analytic flow $\operatorname{Fl}_t^{\operatorname{grad}^\alpha(g)} \in \operatorname{Diff}^\omega(M,\alpha)$, see [4].

(2) $\operatorname{Fl}_1^{\operatorname{grad}^\alpha(g)}(x_i)$ is near y_i for all i.

Thus it follows that each $\operatorname{Diff}^\omega(M,\alpha)$-orbit in $M^{(n)}$ is dense. Similarly as in the proof of theorem 4 we will show that the orbit through $(x_1,\dots,x_n) \in M^{(n)}$ is open, which finishes the proof.

We choose again a complete Riemannian metric g on M. Then we let $(Y_{ij})_{j=1}^m$ be an orthonormal basis of $T_{x_i}M$ with respect to g, for all i. Then we choose real analytic functions f_k for $1 \le k \le N = nm$ which satisfy the following conditions:

$$\begin{aligned}
&|\operatorname{grad}^\alpha(f_k)(x_i) - Y_{ij}|_g < \varepsilon && \text{for } k = (i-1)m + j,\\
&|\operatorname{grad}^\alpha(f_k)(x_i)|_g < \varepsilon && \text{for all } k \notin [(i-1)m+1, im],\\
&|\operatorname{grad}^\alpha(f_k)(x)|_g < 2 && \text{for all } x \in M \text{ and all } k.
\end{aligned}$$

Since these conditions describe Whitney C^1 open subsets, such functions exist by [2], proposition 8. Now we may finish the proof as at the end of theorem 4. □

8. Theorem. *Let (M,μ) be a connected smooth manifold of dimension $m \ge 2$ with a positive volume density. Then the group $\operatorname{Diff}_c(M,\mu)$ of all smooth volume preserving diffeomorphisms of M with compact support acts n-transitively on M, for each finite n.*

If M and μ are real analytic then also the group $\operatorname{Diff}^\omega(M,\mu)$ of real analytic volume preserving diffeomorphisms acts n-transitively on M, for each finite n.

Proof. First the smooth case. By the argument used at the end of the proof of proposition 1 it suffices to show, that there exists $f \in \operatorname{Diff}_c(M,\mu)$ with $f(x_i) = y_i$, for any $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ in $M^{(n)}$ which are pairwise disjoint sets in M.

Having fixed the points, we may find an orientable connected open subset U of M containing all points. Since we are going to construct a volume preserving diffeomorphism with support in U, for the smooth case we can replace M by U and without loss assume that M is orientable. But we shall need the setting $U \subset M$ later.

For some $\varepsilon > 0$ let $c_i : (-\varepsilon, 1+\varepsilon) \to M$, $i = 1, \dots, n$ be smooth embeddings with $c_i(0) = x_i$, $c_i(1) = y_i$ which do not intersect. We choose pairwise disjoint tubular neighborhoods U_i of $c_i(-\varepsilon, 1+\varepsilon)$, $i = 1, \dots, n$.

We can find a Riemannian metric g on M whose volume form is μ. Then the divergence of a vector field $X \in \operatorname{Vect}(M)$ is $\operatorname{div} X = *d*X^\flat$, where $X^\flat = g(X) \in \Omega^1(M)$ (here we view $g : TM \to T^*M$) and $*$ is the Hodge star operator. The velocity field of the curve c_i defines an $(m-1)$-form $*(c_i' \circ c_i^{-1})^\flat$ along $c_i([0,1])$. Using lemma 8 we extend it to an exact $(m-1)$-form $d\alpha_i$ on M with $\operatorname{supp} \alpha_i \subset U_i$, and we put $\alpha = \sum_{i=1}^n \alpha_i \in \Omega^{m-2}(M)$. We consider the vector field

$$X_\alpha = (-1)^{m+1}(*d\alpha)^\sharp = (-1)^{m+1} g^{-1} * d\alpha, \tag{1}$$

i.e. by the relation $d\alpha = *X_\alpha^\flat$. Then X_α is divergence free, $\operatorname{div} X_\alpha = *d*X_\alpha^\flat = *dd\alpha = 0$, and has compact support in the union of all U_i. It also coincides on $c_i([0,1])$ with the velocity field of the curve c_i. Hence $\operatorname{Fl}_1^{X_\alpha} \in \operatorname{Diff}_c(M,\mu)$ with $\operatorname{Fl}_1^{X_\alpha}(x_i) = y_i$.

We treat now the real analytic case. The Riemannian metric g with volume form μ can be chosen real analytic. We also choose a complete Riemannian metric γ.

First we assume that M is orientable. We approximate the smooth $(m-2)$-form α from above by real analytic $(m-2)$-forms β in such a way that:

(2) The real analytic vector field $X_\beta = (-1)^{m+1} g^{-1} * d\beta$ is bounded with respect to the complete Riemannian metric γ and thus has a global real analytic flow $\operatorname{Fl}_t^{X_\beta} \in \operatorname{Diff}^\omega(M,\mu)$.

(3) $\operatorname{Fl}_1^{X_\beta}(x_i)$ is near $y_i = \operatorname{Fl}_1^{X_\alpha}(x_i)$ for all i.

Since these conditions describe a Whitney C^1-open set, such real analytic forms β exist by lemma 3. Thus it follows that each $\operatorname{Diff}^\omega(M,\mu)$-orbit in $M^{(n)}$ is dense. Similarly as in the proof of theorem 4 we will show that the orbit through $(x_1, \dots, x_n) \in M^{(n)}$ is open, which finishes the proof.

We let $(Y_{ij})_{j=1}^m$ be an orthonormal basis of $T_{x_i}M$ with respect to the complete Riemannian metric γ, for all i. Then we choose real analytic $(m-2)$-forms β_k for $1 \le k \le N = nm$ whose vector fields $X_{\beta_k} = (-1)^{m+1} g^{-1} * d\beta_k$ satisfy the following conditions:

$$\begin{array}{ll} |X_{\beta_k}(x_i) - Y_{ij}|_\gamma < \varepsilon & \text{for } k = (i-1)m + j, \\ |X_{\beta_k}(x_i)|_\gamma < \varepsilon & \text{for all } k \notin [(i-1)m+1, im], \\ |X_{\beta_k}|_\gamma < 2 & \text{for all } x \in M \text{ and all } k. \end{array} \tag{4}$$

Since these conditions describe Whitney C^1 open subsets, such $(m-2)$-forms exist by lemma 3. Now we may finish the proof as at the end of theorem 4.

Now we treat the case of non-orientable M. Let $\pi : \tilde M \to M$ be the real analytic connected oriented double cover of M, and let $\varphi : \tilde M \to \tilde M$ be the real analytic involutive covering map. Recall the orientable connected open subset $U \subset M$ containing all points x_i and y_i from above. The form α from above had compact support in U. The inverse image $\pi^{-1}(U) \subset \tilde{M}$ is the disjoint union of two connected open subsets W_1 and W_2 such that $\pi|W_p : W_p \to U$ is a diffeomorphism for both $p = 1, 2$. We let $x_i^p = (\pi|W_p)^{-1}(x_i)$ and $y_i^p = (\pi|W_p)^{-1}(y_i)$, and we pull back both metrics to $\tilde{M}$, so $\tilde{g} := \pi^* g$ and $\tilde{\gamma} := \pi^* \gamma$.

We approximate the smooth $(m-2)$-form $\tilde{\alpha} := \pi^* \alpha$ by real analytic $(m-2)$-forms $\beta \in \Omega^{m-2}(\tilde{M})$ in such a way that the conditions (2) and (3) from above are satisfied now on $\tilde{M}$ for x_i^p and y_i^p.

$(\tilde{2})$ The real analytic vector field $X_\beta = (-1)^{m+1} \tilde{g}^{-1} * d\beta$ is bounded with respect to the complete Riemannian metric $\tilde{\gamma}$ and thus has a global real analytic flow $\mathrm{Fl}_t^{X_\beta} \in \mathrm{Diff}^\omega(\tilde{M}, \pi^* \mu)$.

$(\tilde{3})$ $\mathrm{Fl}_1^{X_\beta}(x_i^p)$ is near $y_i^p = \mathrm{Fl}_1^{X_{\tilde{\alpha}}}(x_i^p)$ for all i, and for $p = 1, 2$.

Since these conditions describe a Whitney C^1-open set, such real analytic forms β exist by lemma 3. Since $\tilde{\alpha} = \pi^* \alpha$ is invariant under φ^*, the real analytic vector field $\frac{1}{2}(X_\beta + \varphi_* X_\beta)$ still satisfies both (1) and (2), is divergence free, invariant under the covering transformation φ, thus it induces a real analytic vector field $Z_\beta \in \mathfrak{X}(M)$ which is bounded with respect to γ, such that $\mathrm{Fl}_t^{Z_\beta}(x_i)$ is still near y_i for each i, and Z_β is now divergence free in the sense that $\mathcal{L}_{Z_\beta} \mu = 0$. Thus it follows that each $\mathrm{Diff}^\omega(M, \mu)$-orbit in $M^{(n)}$ is dense.

Next we will show that the orbit through $(x_1, \dots, x_n) \in M^{(n)}$ is open, which finishes the proof. We choose real analytic $(m-2)$-forms $\beta_k \in \Omega^{m-2}(\tilde{M})$ for $1 \le k \le N = nm$ whose vector fields $X_{\beta_k} = (-1)^{m+1} \tilde{g}^{-1} * d\beta_k$ satisfy the following conditions, where we put $Y_{ij}^p := T_{x_{ij}^p} \pi^{-1} . Y_{ij}$ for $p = 1, 2$:

$$(\tilde{4}) \qquad \begin{array}{ll} |X_{\beta_k}(x_i^p) - Y_{ij}^p|_{\tilde{\gamma}} < \varepsilon & \text{for } k = (i-1)m + j, p = 1, 2, \\ |X_{\beta_k}(x_i^p)|_{\tilde{\gamma}} < \varepsilon & \text{for all } k \notin [(i-1)m + 1, im], p = 1, 2, \\ |X_{\beta_k}|_{\tilde{\gamma}} < 2 & \text{for all } x \in \tilde{M} \text{ and all } k. \end{array}$$

Since these conditions describe Whitney C^1 open subsets, such $(m-2)$-forms exist by lemma 3. Then the vector fields $\frac{1}{2}(X_{\beta_k} + \varphi_* X_{\beta_k})$ still satisfy the conditions $(\tilde{4})$, are still divergence free and induce divergence free vector fields $Z_{\beta_k} \in \mathfrak{X}(M)$ which satisfy the conditions (4) on M as in the oriented case, and we may finish the proof as above. $\square$

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