

Approximately Transitive Actions of Abelian Groups and Spectrum

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Approximately transitive actions of Abelian groups and spectrum

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Abstract

It is proved that any measure-preserving approximately transitive (AT) action of a locally compact separable Abelian group on a Lebesgue space has simple spectrum. The proof is based on investigation of properties of nontransient product cocycles and actions associated with them. We used our result that two nontransient cocycles with isomorphic associated (Mackey) actions are weakly equivalent.

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1 Introduction

Approximately transitive (AT) ergodic actions of locally compact separable (l.c.s.) groups were introduced by A. Connes and E.J. Woods [1] in connection with the description problem of factors which are infinite tensor products of type I_n ($n < \infty$) factors. Further investigation of AT actions were continued in the papers [2-5]. V. Ya. Golodets and N.I. Nessonov [4] considered AT group actions as a range of cocycles of an ergodic automorphism, preserving finite measure, with values in an arbitrary countable discrete group. Developing the methods of [1] they showed, in particular, that if an AT action of countable Abelian group is the range of a cocycle, then this cocycle is cohomologous to a product cocycle. But according to [6] every amenable action of l.c.s. group H is a range of a cocycle of an ergodic automorphism, preserving a measure, with values in H . Hence in view of [4] any AT actions of an Abelian discrete group is a range of a product cocycle. The same is true for nondiscrete groups [5].

Let Abelian l.c.s. group G has an ergodic Borel action on a Lebesgue space (Ω, δ) , preserving a measure δ . This action is called AT if for an arbitrary family of nonnegative functions $f_1, \dots, f_l \in L_1(\Omega, \delta)_+$, $l < \infty$, and any $\epsilon > 0$ there exist nonnegative functions $\lambda_1, \dots, \lambda_l \in L_1(G, dg)$ and $f \in L_1(\Omega, \delta)_+$, satisfying the inequality

$$\| f_j - \int_G \lambda_j(g) f(g\omega) dg \|_1 < \epsilon, \quad 1 \leq j \leq l.$$

(see [1]).

We can consider the unitary representation for G in the space $L_2(\Omega, \delta)$ given by

$$(U_g f)(\omega) = f(g\omega), \quad f \in L_2(\Omega, \delta), \quad g \in G.$$

If the representation $g \rightarrow U_g$ of an Abelian group G has a simple spectrum, then the G -action will be said to have simple spectrum.

In view of the definition of AT action it is natural to ask about the simplicity of spectrum for an AT action. In this paper we prove the following theorem.

Theorem 1 *Any measure-preserving AT action of Abelian l.c.s. group has simple spectrum.*

The theorem presents an important invariant which allows to one to define the given group action to be not AT (see Section 5 below).

Theorem 1 is a consequence of the following result.

Theorem 2 *Let a measure-preserving AT action W of a countable Abelian group G be the range of a nontransient product cocycle. Then W has simple spectrum.*

This theorem is proved in the paper below using the results of [7]. Now we shall show that it implies Theorem 1.

Indeed suppose we deal with a measure-preserving AT action W of an arbitrary Abelian l.c.s. group G . Let D be a countable dense subgroup G . The restriction

of the action W to D is AT. If it has simple spectrum then the action W of G has simple spectrum too and Theorem 1 will be true. But in view of the above remark the action W of D is a range of a product cocycle and it has simple spectrum by Theorem 2.

Note since G is an Abelian l.c.s. group, then it is sufficient to prove Theorem 1 in assumption that W is a free action of G . Hence we can assume that action W of D is also free. But in Section 1 (below) we construct a standard nontransient cocycle α such that its Mackey range W_α is isomorphic to W .

The paper consists of five sections. Section 1 contains some preliminary results on cocycles. Section 2 contains some results on simple spectrum of unitary representations of an Abelian group. In Section 3 we prove Theorem 2 for some simple case, and in Section 4 we consider the general case. Section 5 contains some examples and consequences of Theorem 1.

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2 Preliminary results. Cocycles

Let (P, m) be a Lebesgue space with a probability measure m , T an ergodic automorphism of (P, m) , preserving the measure m , G an Abelian discrete group and dg the Haar measure on G .

A 1-cocycle α on $((P, m), T, G)$ with values in a group G is a Borel map $\alpha : P \times Z \rightarrow G$ possessing the following property

$$\alpha(p, T^{n+m}) = \alpha(p, T^n) + \alpha(T^n p, T^m)$$

for almost all $p \in P$. Denote by $[T]$ the full group of automorphisms of (P, m) , generating by $T : [T] = (\theta \in \text{Aut}(P, m) : \theta p = T^{n(p)} p)$ (see [8]). A cocycle α extended naturally on $[T]$: if $\theta \in [T]$ and $\theta p = T^n p$ then $\alpha(p, \theta) = \alpha(p, T^n)$, $p \in P$.

Let $Z^1((P, m), T, G)$ be a set of 1-cocycles on $[T]$ with values in G . The cocycles α_1 and $\alpha_2 \in Z^1(P, T, G)$ are called cohomologous if there exists a Borel map $f : P \rightarrow G$ satisfying the relation

$$\alpha_2(p, \theta) = f(p) + \alpha_1(p, \theta) - f(\theta p), \quad \theta \in [T]$$

for a.a. $p \in P$.

Let us consider the Lebesgue space $(P \times G, \nu)$ where $\nu = m \times dg$, and two actions on this space

$$\tilde{T}(p, g) = (Tp, g + \alpha(p, T)), \tag{1}$$

$$\tilde{h}(p, g) = (p, g + h), \quad g, h \in G. \quad (2)$$

Let ζ be the measurable partition of $(P \times G, \nu)$ into ergodic components of \tilde{T} . We denote by (Ω, δ) the quotient space of $(P \times G, \nu)$ by ζ . Since the action (1) and (2) are permutable, it is possible to define the quotient action W_α of (2) on (Ω, δ) . $W_\alpha(g), g \in G$, is called the associated action of a group G constructed by α or Mackey action (or range of α).

Denote

$$S_\alpha = \{t \in [T] : \alpha(p, t) = 0, \quad p \in P\}.$$

S_α is called the kernel of a cocycle α , $S_\alpha = \ker(\alpha)$. A cocycle α is called transient [9] if S_α is of type I. We shall consider only nontransient cocycles. Then S_α will be of type II_1 .

Let W be a free action of an Abelian discrete group G on a Lebesgue space (Ω, δ) , preserving σ -finite measure δ . Then we can construct the standard nontransient cocycles $\alpha_0 \in Z_1((P, m), T, G)$ such that $W_{\alpha_0} \sim W$.

Let us put

$$\alpha_1(\omega, W(g)) = g, \quad \omega \in \Omega, \quad g \in G.$$

Then α_1 is a transient cocycle and $W \sim W_{\alpha_1}$. Let $[W]$ be the full group of automorphisms of (Ω, δ) generated by $W(g), g \in G$, and $[W]_X$ the restriction of $[W]$ on $X \in \Omega, \delta(X) = 1$. Then there exists an automorphism $Q \in [W]_X$ such that

$$[Q] = [W]_X \quad (\text{see [10]}).$$

Denote by α_2 the restriction α_1 on (X, Q) . It is evident that α_2 is also transient and $W_{\alpha_2} \sim W$.

Now let (Y, ν) be a Lebesgue space with a finite measure ν , $\nu(Y) = 1$, and S an ergodic automorphism of $(Y, \nu,)$ preserving ν . Consider the group of transformations of the space $(X \times Y, \mu \times \nu)$ where $\mu = \delta|_X$, generated by $Q_1 = Q \times \mathbf{1}$ and $S_2 = \mathbf{1} \times S$ and define the cocycle α_0 :

$$\alpha_0((x, y), Q_1^n S_2^m) = \alpha(x, Q^n), \quad x \in X, \quad y \in Y, \quad m, n \in Z.$$

It is evident that S_{α_0} is of type II_1 , hence α_0 is a nontransient cocycle and its associated action $W_0 = W_{\alpha_0}$ of the group G is isomorphic to W . Moreover, a Lebesgue space (Ω_0, δ_0) , in which W_0 acts, one can identify with a Borel subset of $(X \times G, \mu \times \mu_G), \mu_G$ is the Haar measure of G , which has a following form.

$$\Omega_0 = \bigcap_{g \in G} (X_g \times g)$$

where $X_e = X$, $\delta_e = \mu$; $X_g \subseteq X$, $\delta|_{X_g} = \mu|_{X_g}$.

Now denote by $[Q_1, S_2]$ the full group of $Aut(X \times Y, \mu \times \nu,)$ generated by Q_1 and S_2 , and by $N[Q_1, S_2]$ the normalizer of $[Q_1, S_2]$ (see [8]).

If $\beta \in Z_1((X \times Y, \mu \times \nu), [Q_1, S_2], G)$, β is transient and $W_\beta \sim W_0$ then there exists $\theta \in N[Q_1, S_2]$ such that the cocycles $\alpha_0((x, y), \gamma)$ and $\beta \circ \theta((x, y), \gamma)$, where

$(x, y) \in X \times Y$, $\gamma \in [Q_1, S_2]$, $\beta \circ \theta(x, y, \gamma) = \beta(\theta(x, y), \theta\gamma\theta^{-1})$, are cohomologous (see [7]). (θ preserves the measure $\mu \times \nu$).

Now we remind the definition of a product cocycle. Let $P_k = \{0, 1, \dots, N_{k-1}\}$ $\tau_k(i) = i+1 \pmod{N_k}$, $N_k(i) = 1/N_k$, $i \in P_k$, $(P, m) = \prod_{k=1}^{\infty} (P_k, m_k)$, where $N_k \in \mathbb{N}$.

Denote by Γ the full group of automorphisms of P , generated by $\{\tau_k\}_{k=1}^{\infty}$. Then there is an ergodic automorphism T of (P, m) such that $[T] = \Gamma$ (see [10]).

Let $\alpha_{pr} \in Z_1((P, m), [T], G)$ and $\alpha_{pr}(p, \tau_k^i)$ depends only on k -th coordinate of the point $p = (p_i) \in P$, where $p_i \in P_i$, $1 \leq i < \infty$. Such cocycle α_{pr} is called a product cocycle of $((P, m), \Gamma)$.

3 Unitary representation of an Abelian group and simple spectrum.

The following lemma represents something like generalization of the Baxter lemma [11].

Lemma 3 *Let $g \rightarrow U_g$ be an unitary representation of an Abelian group G in the Hilbert space H . Suppose also that there exists a sequence of orthogonal projections P_n in H converging strongly to the unit operator I , where each P_n and each $U_g, g \in G$, are permutable. If the representation $g \rightarrow U_g P_n$ in the space $P_n H$ has a simple spectrum for every n then the whole representation $g \rightarrow U_g$ has a simple spectrum too.*

Proof. Let $b \in (U_g)'$, prove that $b \in (U_g)'' = A$. Simple computations show that $b_n = P_n b P_n$ is permutable with $P_n U_g, g \in G$. Hence $b_n \in (P_n U_g)'$ and $b_n \in (P_n U_g)''$ because $g \rightarrow P_n U_g$ has a simple spectrum in $P_n H$.

Now the mapping $a \rightarrow P_n a, a \in A$, defined $*$ -algebraic homomorphism of W^* -algebra A on W^* -algebra $P_n A$, since P_n commutate with any element of A . Therefore there exists the projector $Q_n \in A$ such that $P_n(I - Q_n) = 0$ and the mapping $a \rightarrow P_n a$ defined $*$ -isomorphism of W^* -algebra $Q_n A$ onto W^* -algebra $P_n A$ (see [12]).

If $c_n \in Q_n A$ such that $c_n P_n = b_n \in P_n A$ then $\|c_n\| = \|b_n\| \leq \|b\|$.

Let us prove that $c_n \rightarrow b$ for $n \rightarrow \infty$ in the strong topology. Consider neighbourhoods

$$V(b, f_1, \dots, f_s, \epsilon) = \{a : \| (a - b) f_k \| \leq \epsilon, k = 1, \dots, s\}$$

where $f_i \in H, 1 \leq i \leq s, s < \infty$. They made a fundamental neighbourhood system of b . Let n be sufficiently large that

$$\|f_k - P_n f_k\| < 2^{-1} \|b\|^{-1} \epsilon$$

and simultaneously

$$\|(b_n - b) f_k\| = \|(P_n b P_n - b) f_k\| < 2^{-1} \epsilon, k = 1, \dots, s.$$

Since $c_n P_n = b_n$, then $c_n \in V(b_n, P_n f_1, \dots, P_n f_s, 2^{-1}\epsilon)$.

Therefore

$$\begin{aligned} & \| c_n f_k - b f_k \| = \\ & \| c_n f_k - c_n P_n f_k \| + \| c_n P_n f_k - b_n P_n f_k \| + \| (b_n - b) f_k \| < \\ & 2^{-1} \| c_n \| \| b \|^{-1} \epsilon + 2^{-1} \epsilon < \epsilon. \diamond \end{aligned}$$

Proposition 4 *Let $(g, \omega) \rightarrow g\omega$ be a Borel ergodic action of l.c.s. Abelian group G on a Lebesgue space (Ω, δ) , preserving σ -finite measure δ and $g \rightarrow U_g, g \in G$, a corresponding unitary of G in the space $L_2(\Omega, \delta)$. If for any finite set of vectors $f_i, 1 \leq i \leq l$, from $L_2(\Omega, \delta)$ there are a vector $f \in L_2(\Omega, \delta)$, finite set of numbers $\lambda_{i,k} \in \mathbf{C} (1 \leq i \leq l)$ and finite set of elements $g_{i,k} \in G (1 \leq i \leq l)$ such that*

$$\| f_i - \sum_k \lambda_{i,k} U_{g_{i,k}} f \|_2 < \epsilon, 1 \leq i \leq l, l < \infty, \quad (3)$$

then the representation $g \rightarrow U_g$ has simple spectrum.

Proof. Let $\{f_i\}_{i=1}^\infty$ be a sequence of functions from $L_2(\Omega, \delta)$ such that the closed linear span of all $\{f_i\}_{i=1}^\infty$ is $L_2(\Omega, \delta)$. According to the assumption of the Proposition for each $n \in \mathbf{N}$ there exists $f_0^n \in L_2(\Omega, \delta)$ such that

$$\| f_i - \sum \lambda_{i,k} U_{g_{i,k}} f_0^n \|_2 < n^{-1} \epsilon, 1 \leq i \leq n, \quad (4)$$

Now let H_n be the closed linear span of all vectors $U_g f_0^n, g \in G$. It is clear that H_n is G -invariant subspace of H and the orthogonal projectors P_n on H_n are permutable with every $U_g, g \in G$. The restriction of the representation $g \rightarrow U_g$ to $H_n = P_n H$ has simple spectrum due to the construction. Moreover [4] implies that P_n strongly tend to I , so we have only to apply lemma 3. \diamond

Remark. J. Hawkins and E.A. Robinson [2] defined $AT(p)$ actions, where $p \leq 1$ indicates which of the L^p -norms of the approximation take place in. In particular, $AT(1)$ coincides with the Connes-Woods AT action. They also proved that $AT(2)$ -actions has simple spectrum, using another methods.

Corollary 5 *Let $(\Omega, \delta), G$ and $g \rightarrow U_g$ be as in Proposition 4. If for any $\epsilon > 0$ and any finite set of vectors (f_i) where f_i has a form $f_i = \chi_{A_i}, A_i \in \Omega, \delta(A_i) > 0$ and χ_A is the indicator of a set A , there is $f \in L_2(\Omega, \delta)$ such that (3) takes place in, then the representation $g \rightarrow U_g$ has simple spectrum. Moreover $g \rightarrow U_g$ will have simple spectrum if all $A_i \subset X_\theta \subset \Omega$, where $\delta(X_\theta) > 0$.*

A proof is evident, the last fact follows from the ergodicity of an action of G on (Ω, δ) .

4 The Proof of Theorem 2, special case

In this section we prove Theorem 2 for the some special case. We do not use complicated estimates. Thus one has a possibility to understand an algebraic side of the problem.

Let $(P, m), T$ and G be as in Section 1, $\alpha \in Z^1(P, T, G), W_\alpha$ be the associated action of G constucted by α . Let W_α acts in the Lebesgue space (Ω, δ) .

Denote by X a Borel section of P which intersects every ergodic orbit of S_α in one point and denote by μ the projection of m on X . Suppose that (Ω, δ) is a Borel subspace of $(X \times G, \mu \times dg)$ and (Ω, δ) has a following structure

$$(\Omega, \delta) = \bigcup_g (X_g, \delta_g \times g) \in (X \times G, \mu \times dg)$$

where $X_e = X, \delta_e \sim \mu$ and $\delta_g \prec \mu, \text{supp } \delta_g = X_g \subset X$. It follows that $\delta_g = \delta_e|_{X_g}$.

Since

$$\mu(F) = \int_F p(x) d\delta_e(x), F \in X, \quad (5)$$

where $p(x)$ is the positive Borel function on X .

Lemma 6 *Let $(P, m), T, G, \alpha \in Z_1(P, T, G), W_\alpha$ and (Ω, δ) be as above. Suppose that α is a product cocycle and there is a real positive number C such that*

$$0 < \text{supvrai } p(x) < C < \infty,$$

then W_α has simple spectrum.

Proof Let $2^{-1}C^{-1}\epsilon > 0$ and $f_i \in L_1(X, \delta_e)_+, 0 \leq f_i \leq p(x), 1 \leq i \leq l, l < \infty$. It is sufficient to prove that there exist a function $f_0(x), 0 \leq f_0(x) \leq p(x)$, a finite set of elements $g_{i,k} \in G$ and a finite set of integers $\lambda_{i,k}, 1 \leq i \leq l$, such that

$$\| f_i(x) - \sum_k \lambda_{i,k} f_0(g_{i,k}x) \|_1 < \epsilon, 1 \leq i \leq l, \quad (6)$$

$$| \sum_k \lambda_{i,k} f_0(g_{i,k}x) | < p(x) \quad (7)$$

where we write $gx, g \in G, x \in X$, instead of $W_\alpha(g)x$ to simplify notations.

Indeed, it follows from (6) and (7) that

$$\| f_i(x) - \sum_k \lambda_{i,k} f_0(g_{i,k}x) \|_2 \leq$$

$$2C \| f_i(x) - \sum_k \lambda_{i,k} f_0(g_{i,k}x) \|_1 < 2C\epsilon$$

and one can conclude from Proposition 4 that the action $W_\alpha(g), g \in G$, on (Ω, δ) has simple spectrum.

Now if $E \subset P$, then

$$m(E) = \int_X \nu_x(E) d\delta_e(x)$$

where ν_x is a system of canonical conditional measures on P related with the measure m and the Borel section X [8], besides

$$0 \leq \nu_x(E) \leq p(x).$$

Since $0 \leq f_i(x) \leq p(x)$ then there are subsets $E_i \subset P$ such that

$$\nu(E_i) = f_i(x), x \in X \quad [8].$$

But $(P, m) = \prod_{n=1}^{\infty} (P_i, m_i)$ and $E_i \subset P$, $i = 1, \dots, l$, can be approximated by set $E_i^0 \subset \prod_{i=1}^N (P_i, m_i)$ such that

$$m(E_i \Delta E_i^0) < \epsilon, i = 1, \dots, l, \quad (8)$$

if $N \in \mathbf{N}$ is sufficiently large. Let P_j^N be atoms of $\prod_{i=1}^N (P_i, m_i) : P_i^N \cap P_j^N = \emptyset, i \neq j, \cup_j P_j^N = P$ and there is a periodical automorphism τ of $P, \tau \in [T]$, such that

$$\tau^j P_0^N = P_j^N \quad (9)$$

Since the cocycle α is a product cocycle we can believe that functions

$$\alpha(p, \tau^j) = g_j \quad (10)$$

do not depend on $p \in P_0^N$.

Now let $[(p, q)]$ be an ergodic component of the group

$$\gamma_\alpha(p, g) = (\gamma p, g + \alpha(p, \gamma)), \gamma \in [\Gamma] = [T],$$

containing the point $(p, g) \in P \times G$.

In view of (9) and (10) it take a place in

$$x = [(p, 0)] = [(\tau^j p, g_j)] = W_\alpha(g_j)[(\tau^j p, 0)],$$

or

$$g_j^{-1} x = g_j [(p, 0)] = [(\tau^j p, 0)], p \in P_0^N.$$

Hence

$$dm(\tau^j p) = d\nu(\tau^j p) d\delta_\epsilon(g_j x).$$

Since $dm(\tau_j p) = dm(p)$ and $d\delta_\epsilon(p) = d\delta(g_j x)$ we have

$$d\nu_x(p) = d\nu_{g_j x}(\tau_j p), p \in P_0^N.$$

Therefore in view of (10)

$$\nu_x(P_j^N) = \nu_x(\tau^j P_0^N) = \nu_{g_j^{-1} x}(P_0^N) \quad (11)$$

Put $f_0(x) = \nu_x(P_0^N)$, then in view of (11)

$$\nu_x(E_i^0) = \sum_k \nu_x(P_{i,k}^N) = \sum_k \lambda_{i,k} f_0(g_{i,k}^{-1}x)$$

and

$$0 \leq | \sum_k \lambda_{i,k} f_0(g_{i,k}^{-1}x) | = | \nu_x(E_i^0) | \leq p(x) < C. \quad (12)$$

Moreover in view of (8)

$$\begin{aligned} & \| f_i(x) - \sum_k \lambda_{i,k} f_0(g_{i,k}^{-1}x) \|_1 = \\ & \| \nu_x(E_i) - \nu_x(E_i^0) \|_1 \leq \\ & \int | \chi_{E_i}(p) - \chi_{E_i^0}(p) | dm(p) = \mu(E_i^0 \Delta E_i) < \epsilon \end{aligned} \quad (13)$$

Now if we compare (12) and (13) with (6) and (7) and note that

$$0 \leq f_0(x) = \nu_x(P_0^N) \leq p(x),$$

we can make the conclusion that Lemma is true \diamond

5 The proof of Theorem 2, general case.

Let $(P, m), T, G, \alpha \in Z_1(P, T, G)$ be as in Section 1, α is a product nontransient cocycle, W_α is the action of G , associated with α .

Define a cocycle $\alpha_0 \in Z_1((P, m)T, G)$ such that $W_0 = W_{\alpha_0} \sim W_\alpha$ and α_0 has a form, described in Section 1. This means that

$$P = X \times Y, \quad m = \mu \times \nu, \quad \mu(X) = \nu(Y) = 1,$$

$$Q_1(x, y) = (Qx, y), \quad S_2(x, y) = (x, Sy).$$

where Q (resp. S) is an ergodic automorphism of (X, μ) (resp. (Y, ν)), preserving measure μ (resp. ν), $[Q_1, S_2] = [T]$.

$$\alpha_0((x, y), Q_1^n) = \beta(x, Q^n), \quad \alpha_0((x, y), S_2) = 0,$$

where $\beta(x, Q^n)$ is a transient cocycle and $\beta \in Z_1(X, Q, G)$.

Moreover, the space (Ω_0, δ_0) , in which W_0 acts has the following structure

$$\Omega = \bigcup_g (X_g \times g)$$

where $X_e = X$ and $\delta_0|_{X_e} = \mu, X_g \subset X$.

Now, as we remarked in Section 1, there exists $\zeta \in N[T]$ such that

$$\alpha_0(\zeta p, \zeta t \zeta^{-1}) = \varphi(p) - \varphi(tp) + \alpha(p, t) \quad (14)$$

where $t \in [T]$ and the Borel function $\varphi : P \rightarrow G$ has a form

$$\varphi(p) = \sum_k \chi_{D_k}(p) g_k,$$

where D_k is a partition of P , χ_D is an indicator of D , $m(D_k) > 0$, $g_k \in G$.

Let us fix k_0 and denote

$$D' = D_{k_0}, \quad D = \zeta D' \subset X \times Y = P$$

Put

$$\nu_x(D) = \nu(D_x)$$

where $D_x \subset Y$ and $(x \times Y) \cap D = x \times D_x$. Then the function $x \rightarrow \nu_x(D)$ is Borel and

$$0 \leq \nu_x(D) \leq 1,$$

since $\nu(Y) = 1$. Hence there exists a real number θ , $0 < \theta < 1$, such that the set

$$X_\theta = \{x \in X : \theta \leq \nu_x(D) \leq 1\}$$

has a positive μ -measure, that is $\mu(X_\theta) > 0$.

Consider

$$E = (X_\theta \times Y) \cap D.$$

It is evident

$$0 < \theta \leq \nu_x(E) \leq 1, \quad x \in X_\theta.$$

Now consider any finite family of subsets $A_i \subset X_\theta$, $i = 1, \dots, l$, $\mu(A_i) > 0$. Then there are subsets $B_i \subset E$, $i = 1, \dots, l$, such that $\nu_x(B_i) = \theta \chi_{A_i}(x)$, $i = 1, \dots, l$, where χ_A is the indicator of A . Denote $B'_i = \zeta^{-1} B_i \subset P$. Since α is a product cocycle there exists a partition P_j^N of P with the properties (9) and (10), such that

$$m(B'_i \Delta (\bigcup_k P_{i(k)}^N)) < 16^{-2} (\epsilon \theta)^2$$

Put $\zeta P_i^N = F_i$, since ζ preserves the measure m , then

$$m(B_i \Delta (\bigcup_k F_{i(k)})) < 16^{-2} (\epsilon \theta)^2$$

and we can write

$$\begin{aligned} & \| \theta \chi_{A_i}(x) - \sum_k \nu_x(F_{i(k)}) \|_1 \\ & \leq \int | \chi_{B_i}(z) - \sum_k \chi_{F_{i(k)}}(z) | dm(z) \\ & = m(B_i \Delta \bigcup_k F_{i(k)}) < 16^{-2} (\epsilon \theta)^2. \end{aligned}$$

But

$$0 \leq \theta \chi_{A_i} \leq 1,$$

$$0 \leq \sum_k \nu_x(F_{i(k)}) = \nu_x\left(\bigcup_k F_{i(k)}\right) \leq 1$$

and hence

$$\| \theta \chi_{A_i}(x) - \sum_k \nu_x(F_{i(k)}) \|_2 \leq 2(16)^{-2}(\epsilon\theta)^2, \quad i = 1, \dots, l, \quad (15)$$

Now we keep in the sets $\bigcup_k F_{i(k)}$ only such subsets $F_{i(k)}$ that

$$m(F_{i(k)} \triangle B_i) < (16)^{-1}(\epsilon\theta)m(F_{i(k)}).$$

Then instead of (15) we get the following estimates

$$\| \theta \chi_{A_i}(x) - \sum_k \nu_x(F_{i(k)}) \|_2 < 4^{-1}\epsilon\theta, \quad i = 1, \dots, l. \quad (16)$$

Let us fix $F_0 = F_{i_0(k_0)}$ for which

$$m(F_0 \triangle B_{i_0}) < 16^{-1}\epsilon\theta m(F_0) \quad (17)$$

Since $B_i \subset E$ then

$$m(F_0 \cap E) > (1 - (16)^{-1}\epsilon\theta)m(F_0) \quad (18)$$

It follows from (17) and (18) that there are subsets $F_{0,i(k)} \subset F_0 \cap E$ and $F_{i(k),0} \subset F_{i(k)} \cap E$ such that

$$\begin{aligned} m((F_0 \cap E) \setminus F_{0,i(k)}) &< 8^{-1}\epsilon\theta m(F_0), \\ m((F_{i(k)} \setminus F_{i(k),0})) &< 8^{-1}\epsilon\theta m(F_0) \end{aligned} \quad (19)$$

and

$$\tau_1^j F_{0,i(k)} = F_{i(k),0}$$

where $\tau_1 = \zeta\tau\zeta^{-1} \in [T]$ and τ is defined in Section 3, see (9). We can believe that $j = i(k)$ and therefore we have the relations

$$\tau_1^{i(k)} F_{0,i(k)} = F_{i(k),0} \subset F_{i(k)} \subset E$$

But it follows from (10) and (14) that

$$\alpha(z, \tau_1^{i(k)}) = g_{i(k)}, \quad z \in F_{0,i(k)}.$$

Therefore as in (11) we get the relations

$$\nu_x(F_{i,0}) = \nu_{g_i^{-1}x}(F_{0,i}) \quad (20)$$

for some integers i and $x \in X_\theta$. Let us put

$$f_0 = \tau_x(F_0)$$

and estimate the expression:

$$\| \nu_x(F_i) - f_0(g_i^{-1}x) \|_1 =$$

$$\begin{aligned}
& \| \nu_x(F_i) - \nu_{g_i^{-1}x}(F_0) \|_1 \\
&= \| \nu_x(F_i \setminus F_{i,0}) + \nu_x(F_{i,0}) - \nu_{g_i^{-1}x}(F_0) \|_1 \\
&\leq \| \nu_x(F_i \setminus F_{i,0}) \|_1 + \| \nu_x(F_{i,0}) - \nu_{g_i^{-1}x}(F_0) \|_1
\end{aligned}$$

Using (20), (19) and G -invariance of the measure $\delta_e = \mu$ we get

$$\begin{aligned}
& \| \nu_{g_i^{-1}x}(F_0) - \nu_{g_i^{-1}x}(F_{0,i}) \|_1 \\
&= \| \nu_{g_i^{-1}x}(F_0 \setminus F_{0,i}) \|_1 = \int \nu_{g_i^{-1}x}(F_0 \setminus F_{0,i}) d\mu(x) \\
&= m(F_0 \setminus F_{0,i}) \leq 8^{-1} \epsilon \theta m(F_i).
\end{aligned}$$

Applying the same arguments we get

$$\| \nu_x(F_i \setminus F_{i,0}) \|_1 < 8^{-1} \epsilon \theta m(F_i)$$

Thus

$$\| \nu_x(F_i) - f_0(g_i^{-1}x) \|_1 < 4^{-1} \epsilon \theta m(F_i)$$

and since $0 \leq f_0(x), \nu_x(F_i) \leq 1$, then

$$\| \nu_x(F_i) - f_0(g_i^{-1}x) \|_2 < 2^{-1} \epsilon \theta m(F_i) \tag{21}$$

Now in view of (16) and (21)

$$\begin{aligned}
& \| \theta \chi_{A_i}(x) - \sum_k f_0(g_{i(k)}^{-1}x) \|_2 \\
&\leq \| \theta \chi_{A_i}(x) - \sum_k \nu_x(F_{i(k)}) \|_2 + \\
&\| \sum_k (\nu_x(F_{i(k)}) - f_0(g_{i(k)}^{-1}x)) \|_2 \leq 4^{-1} \epsilon \theta + 2^{-1} \epsilon \theta < \epsilon \theta.
\end{aligned}$$

Thus we have

$$\begin{aligned}
& \| \chi_{A_i} - \theta^{-1} \sum_k U_{g_{i(k)}}^* f_0 \|_2 = \\
& \| \chi_{A_i}(x) - \theta^{-1} \sum_k f_0(g_{i(k)}^{-1}x) \|_2 < \epsilon
\end{aligned}$$

for any $\epsilon > 0$ and any finite family of subsets $A_i \subset X_\theta, \mu(A_i) > 0$. Note that $X_\theta \subset X$ and $\mu(X_\theta) > 0, X \subset \Omega_0, \delta_0|_X = \mu$ and $\mu(X) = 1$. Therefore it follows from Corollary 6 that action $W_0(\sim W_\alpha)$ has simple spectrum \diamond

6 Corollaries and examples

As a simple consequence of Theorem 1 we get

Corollary 7 *Any measure- preserving AT action of an Abelian locally compact has zero entropy.*

Indeed,as well known [13], positive entropy implies an existence of a spectral component with AT property. This fact was proved in [1] for the groups \mathbf{R} and \mathbf{Z} .

It well known that automorphisms with discrete spectra are always AT. However the following result also hold.

Corollary 8 *Any ergodic automorphism with quasi-discrete, but not discrete, spectra is not AT.*

According to [14] these automorphisms always have a spectral component with a countable multiplicity, though they have zero entropy.

Note it is easy to construct an AT action for a group G because such action is a range of a product cocycle. Theorem 1 gives a possibility to construct new examples actions without an AT property.

Example 9. Let (X, μ) be a Lebesgue space, T an ergodic automorphism of (X, μ) , G a compact non-commutative group, $\alpha \in Z_1(X, T, G)$. Besides we suppose, that α has a dense range in G . Hence the automorphism

$$\tilde{T}(x, g) = (Tx, g\alpha(x, T)), x \in X, g \in G,$$

of the space $(X \times G, \mu \times \mu_G)$, where μ_G is the Haar measure of G , is ergodic. The automorphism \tilde{T} has not simple spectrum, because it is permutable with

$$L_h(x, g) = (x, hg), h, g \in G.$$

It follows from Theorem 1 that \tilde{T} has not AT property.

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