

**Parabolic Geometries, CR-Tractors,
and the Fefferman Construction**

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PARABOLIC GEOMETRIES, CR-TRACTORS, AND THE FEFFERMAN CONSTRUCTION

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ABSTRACT. This is a survey on recent joint work with A.R. Gover on the geometry of non-degenerate CR manifolds of hypersurface type. Specifically we discuss the relation between standard tractors on one side and the canonical Cartan connection, the construction of the Fefferman space and the ambient metric construction on the other side. To put these results into perspective, some parts of the general theory of parabolic geometries are discussed.

1. INTRODUCTION

This paper surveys recent joint work (partly in progress) with A.R. Gover on the standard tractor bundle and its canonical linear connection in the geometry of non-degenerate CR structures of hypersurface type. Apart from general geometric questions, this work specifically aims at questions concerning CR invariants and the related question of CR invariant differential operators, i.e. differential operators that are intrinsic to a CR structure.

Specifically, I want to show that standard tractors tie in very nicely with three classical constructions of CR geometry. First, the standard tractor bundle and its linear connection are equivalent to the canonical Cartan bundle and the canonical Cartan connection on a CR manifold. Secondly, a construction of Ch. Fefferman associates to any CR manifold M an indefinite conformal structure on the total space of a certain circle bundle over M . Using the standard tractor bundle and connection leads to new ways to exploit this construction. Finally, there is the so-called ambient metric construction for embedded CR manifolds, which is also due to Ch. Fefferman. Starting from any defining function for an embedded CR manifold, a simple explicit algorithm provides a defining function satisfying a certain normalization condition. This in turn gives rise to a Ricci-flat Kähler metric on an ambient space. The important point here is, that this metric as well as its Levi-Civita connection can be easily computed explicitly from the normalized defining function. We shall see how this leads (in the special case of embedded CR manifolds) to a completely explicit description of the standard tractor bundle and the standard tractor connection.

On the other hand, the standard tractor bundle and connection for CR structures immediately leads to a (CR invariant) calculus on any CR manifold. This actually

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is a special case of a much more general concept of tractor bundles and connections for so-called parabolic geometries. The general theory of these geometric structures, which has been substantially developed during the last few years, provides a number of tools and constructions, e.g. of invariant differential operators. To put things into perspective, I will also describe some aspects of this general theory here.

2. THE CARTAN CONNECTION, STANDARD TRACTORS, AND PARABOLIC GEOMETRIES

2.1. CR manifolds. Let us start by recalling the relevant definitions. An *almost CR-manifold* is a smooth manifold M of odd dimension, $\dim(M) = 2n + 1$, together with a rank n complex subbundle $HM \subset TM$. Note that passing to the complexification $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ of the tangent bundle, the subbundle $HM \otimes \mathbb{C}$ splits as $H^{1,0}M \oplus H^{0,1}M$ into a holomorphic and an anti-holomorphic part. We will denote by $J : HM \rightarrow HM$ the almost complex structure on the bundle HM , by $QM := TM/HM$ the quotient bundle (which by construction is a real line bundle), and by $q : TM \rightarrow QM$ the natural projection. The Lie bracket then induces a tensorial map $\mathcal{L} : HM \times HM \rightarrow QM$ via $\mathcal{L}(\xi, \eta) = q([\xi, \eta])$ for $\xi, \eta \in \Gamma(HM)$.

The structure (M, HM, J) is called a *CR structure* if \mathcal{L} is non-degenerate (and hence HM defines a contact structure on M) and the subbundle $H^{1,0}M \subset T_{\mathbb{C}}M$ is involutive. A weakening of this integrability condition (assuming the \mathcal{L} is non-degenerate) is the condition of *partial integrability* which just requires the Lie bracket of two sections of $H^{1,0}M$ to be a section of $H^{1,0}M \oplus H^{0,1}M$. Partial integrability turns out to be equivalent to compatibility of \mathcal{L} with the almost complex structure in the sense that $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ and is the weakest condition under which existence of a canonical normal Cartan connection is guaranteed.

If (M, HM, J) and (M', HM', J') are CR manifolds, then a smooth map $f : M \rightarrow M'$ is called a *CR map* if for all $x \in M$ we have $T_x f(H_x M) \subset H_{f(x)} M'$ and the restriction $T_x f : H_x M \rightarrow H_{f(x)} M'$ is complex linear (with respect to J and J'). A (local) *CR diffeomorphism* between two CR manifolds is a (local) diffeomorphism which also is a CR map.

If (M, HM, J) is a partially integrable almost CR manifold, the compatibility of \mathcal{L} and J implies that, choosing a local trivialization of QM , we may view \mathcal{L} as the imaginary part of a non-degenerate Hermitian form, the *Levi-form*. The signature (p, q) of this Hermitian form is unambiguously defined if we require $p \geq q$, and it is called the *signature of (M, HM, J)* .

The basic examples of CR manifolds are provided by the boundaries of strictly pseudoconvex domains. If $\Omega \subset \mathbb{C}^{n+1}$ is a smoothly bounded strictly pseudoconvex domain with boundary $M = \partial\Omega$, then for $z \in M$ we define $H_z M := T_z M \cap iT_z M$. Thus, $H_z M \subset T_z M$ is the maximal complex subspace in the tangent space, and a moment of thought shows that this has to be of complex dimension n , so the spaces $H_z M$ define a rank n complex subbundle of TM . Moreover, strict pseudoconvexity is actually equivalent to the Levi form being definite (and hence according to our conventions being of signature $(n, 0)$). Finally, integrability of $H^{1,0}M$ in this case easily follows from integrability of the complex structure on \mathbb{C}^{n+1} .

Of course, one may consider more general real hypersurfaces M in \mathbb{C}^{n+1} or in general complex manifolds, with the subbundle HM being given by the maximal complex subspaces in the tangent spaces. If this subbundle defines a contact structure on M , then the integrability condition is automatically satisfied, so (M, HM, J) is a CR manifold. CR structures obtained in this way on hypersurfaces in \mathbb{C}^{n+1} are called *embedded CR manifolds*.

2.2. The flat model of CR structures. The first step towards the construction of a canonical Cartan connection on CR manifolds is to describe the flat model of CR structures of signature (p, q) (with $p + q = n$ and $p \geq q$) and thus identify the groups involved in the construction.

Consider the space $\mathbb{V} = \mathbb{C}^{n+2}$ endowed with a Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(p+1, q+1)$. Let $\mathcal{C} \subset \mathbb{V}$ be the cone of nonzero null-vectors and let M be the image of \mathcal{C} in the projectivization $\mathcal{P}\mathbb{V} \cong \mathbb{C}P^{n+1}$ of \mathbb{V} . Then M canonically carries a CR structure of signature (p, q) . This may be deduced from the fact that M is a smooth real hypersurface in the complex manifold $\mathcal{P}\mathbb{V}$. Alternatively, it is a nice exercise to deduce the CR structure on M directly from its description as a quotient of \mathcal{C} and the obvious description of the tangent spaces of the hypersurface $\mathcal{C} \subset \mathbb{V}$.

Next, consider the group $G := SU(\mathbb{V}) \cong SU(p+1, q+1)$. The cone \mathcal{C} clearly is invariant under the action of G on \mathbb{V} , and it is elementary to verify that G acts transitively on \mathcal{C} . Consequently, G acts transitively on M and from either of the two descriptions of the CR structure on M it is easy to see that G acts by CR diffeomorphisms. It is much less elementary to see that actually any CR diffeomorphism of M comes from the action of an element of G . Hence, we conclude that the group of CR automorphisms of M is $\underline{G} := PSU(\mathbb{C})$, the quotient of G by its center (which is isomorphic to \mathbb{Z}_{n+2}).

Fixing an element $e \in \mathcal{C}$, i.e. a nonzero null vector, we denote by $P \subset G$ and $\underline{P} \subset \underline{G}$ the stabilizer subgroups of the complex line $\mathbb{C}e$, thus obtaining diffeomorphisms $G/P \cong \underline{G}/\underline{P} \cong M$. The subgroups P and \underline{P} turn out to be so-called *parabolic subgroups* of the semisimple groups G respectively \underline{G} , i.e. the corresponding Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$ (which is the same for both groups) contains a maximal solvable subalgebra of the simple Lie algebra $\mathfrak{g} = \mathfrak{su}(\mathbb{V}) \cong \mathfrak{su}(p+1, q+1)$.

2.3. The canonical Cartan connection on CR manifolds. One of the basic results in CR geometry is that CR manifolds can actually be viewed as “curved analogs” of the homogeneous flat model $\underline{G}/\underline{P}$ from 2.2 above. This is an instance of E. Cartan’s concept of “espaces généralisés” which associates to any homogeneous space a geometric structure, see [17]. In modern terminology, these structures are called *Cartan geometries*. Given a Lie group H and a closed subgroup $Q \subset H$, a Cartan geometry of type (H, Q) on a smooth manifold M is defined as a principal Q -bundle $p : \mathcal{H} \rightarrow M$ (which is an analog of the canonical bundle $H \rightarrow H/Q$) together with a Cartan connection $\omega \in \Omega^1(\mathcal{H}, \mathfrak{h})$, where \mathfrak{h} is the Lie algebra of H . This Cartan connection should be thought of as an analog of the left Maurer–Cartan form on H , and its defining properties

- (i) $\omega_u : T_u\mathcal{H} \rightarrow \mathfrak{h}$ is a linear isomorphism for all $u \in \mathcal{H}$
- (ii) $(r^h)^*\omega = \text{Ad}(h^{-1}) \circ \omega$ for $h \in Q$, with r^h denoting the principal right action of h

(iii) $\omega(\zeta_A) = A$ for all $A \in \mathfrak{q}$, with ζ_A denoting the fundamental vector field corresponding to A

are precisely the parts of the properties of the left Maurer Cartan form which remain to make sense in the more general setting.

A model case for this concept is given by taking H the group of Euclidean motions of \mathbb{R}^n and $Q = O(n)$, and hence H/Q the Euclidean space \mathbb{R}^n . In this case, a Cartan geometry of type (H, Q) on M is easily seen to be equivalent to a reduction of the frame bundle of M to the structure group $Q = O(n)$ together with a principal connection on this principal $O(n)$ -bundle. These data in turn are equivalent to a Riemannian metric on M together with a linear connection on TM which is compatible with the metric. Hence, imposing a normalization condition on the Cartan connection which amounts to requiring the linear connection to be torsion free, one sees that normal Cartan geometries of type (H, Q) are exactly n -dimensional Riemannian manifolds. In the case of Riemannian structures, the point of view of Cartan geometries (while conceptually very valuable) is not really necessary to efficiently deal with the geometry, since the Cartan connection is essentially equivalent to the Levi-Civita connection. In more general situations, such a simple translation is not possible, and more sophisticated methods for using the Cartan connection are required.

In the general case, interpreting a Cartan geometry of given type is rather difficult, there are however cases in which normal Cartan geometries can be canonically constructed from underlying geometric structures. With conformal and projective structures, CR structures are one of the main examples of this situation: We continue to use the notation for the groups G and \underline{G} , $P \subset G$ and $\underline{P} \subset \underline{G}$ from 2.2 above.

Theorem. *Let (M, HM, J) be a CR manifold of signature (p, q) . Then there exists a canonical principal \underline{P} -bundle $\underline{p} : \underline{\mathcal{G}} \rightarrow M$ endowed with a unique normal Cartan connection $\omega \in \Omega^1(\underline{\mathcal{G}}, \mathfrak{g})$.*

The normalization condition on the Cartan connection is a restriction on the curvature that can be either formulated in Lie theoretic terms (which then generalizes to parabolic geometries, see below) or directly as the vanishing of certain traces of the curvature. The principal bundle is then uniquely determined (up to isomorphism) by the fact that it admits a normal Cartan connection. This result was proved by E. Cartan for $n = 1$ (i.e. 3-dimensional CR structures), see [16]. For general n , it is due to N. Tanaka (see [35, 36]) and to S.S. Chern and J. Moser (see [18]). It should be remarked here that Tanaka's construction actually works in the more general setting of partially integrable almost CR structures.

For later use, it is very important to slightly extend this construction, in order to get a Cartan geometry of type (G, P) rather than $(\underline{G}, \underline{P})$. It turns out that in order to get a principal P -bundle $p : \mathcal{G} \rightarrow M$ endowed with a canonical Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ one in addition to the CR structure has to choose a complex line bundle $\mathcal{E}(1, 0) \rightarrow M$ such that $\mathcal{E}(1, 0)^{\otimes n+2} \cong \Lambda_{\mathbb{C}}^n HM \otimes QM$. While such a bundle need not exist globally, and if it does, it is not necessarily determined uniquely, existence and uniqueness are always clear locally. Moreover, we shall see later on that in the case

of a boundary of a domain that can be described by a defining function, there is always a canonical global choice.

2.4. Parabolic geometries. By definition, parabolic geometries are Cartan geometries of type (H, Q) , where H is a semisimple Lie group and $Q \subset H$ is a parabolic subgroup, i.e. the Lie algebra \mathfrak{q} of Q contains a maximal solvable subalgebra of the semisimple Lie algebra \mathfrak{h} and Q is the normalizer of \mathfrak{q} in H . It is well known that for complex semisimple Lie algebras parabolic subalgebras are in bijective correspondence with sets of simple roots, while in the real case there is an additional condition which can be easily described in terms of the Satake diagram, so one has a complete (and rather large) list of examples of such structures. It turns out that all these structures, which are very diverse from a geometrical point of view, can be studied in a surprisingly uniform way.

For any parabolic geometry $(p : \mathcal{H} \rightarrow M, \omega)$ one can construct an underlying structure, called an infinitesimal flag structure. This consists of a filtration of the tangent bundle of M together with a reduction of structure group of the associated graded $\text{gr}TM$ of the tangent bundle to the reductive part Q_0 of the parabolic subgroup Q . This underlying structure is rather easy to understand geometrically. If one requires the filtration to be compatible with the Lie bracket of vector fields, this Lie bracket induces an algebraic bracket on $\text{gr}TM$. On the other hand, from the reduction to the structure group to the group Q_0 , one gets another algebraic bracket on $\text{gr}TM$, and requiring these two algebraic brackets to coincide, one obtains the notion of a *regular infinitesimal flag structure*.

In the CR case, the filtration is simply given by the subbundle $HM \subset TM$ and the reduction to the group \underline{P}_0 amounts just to an almost complex structure on HM and a bracket $HM \times HM \rightarrow QM$ which is the imaginary part of a Hermitian form of signature (p, q) . From this description, it is easy to see that a regular infinitesimal flag structure in this case is exactly a partially integrable almost CR structure.

Next, it turns out that regularity of the underlying infinitesimal flag structure can be easily described in terms of the (curvature of the) Cartan connection, thus leading to the notion of regular parabolic geometries. Moreover, the Kostant–codifferential (see [29]) can be used as a normalization condition for Cartan connections of the type in question. Now there exist prolongation procedures which extend any regular infinitesimal flag structure canonically to a unique regular normal parabolic geometry. Using these, one obtains an equivalence between the category of regular normal parabolic geometries and the category of regular infinitesimal flag structures. (In both cases there is an obvious notion of morphisms.)

The first version of such a prolongation procedure (with some restrictions on the groups and a quite different description of the underlying structures) is due to N. Tanaka (see [37]). It can also be obtained (in full generality) as a special case of a construction of T. Morimoto of Cartan connections for geometric structures on filtered manifolds, see [32]. Finally, a procedure tailored to parabolic geometries can be found in [12].

In this way, a large number of geometric structures can be identified as parabolic geometries. First, there are several examples in which the underlying filtration is

trivial, and thus one has a classical first order structure. In particular, conformal structures (of arbitrary signature) and almost quaternionic structures fall into this group. Next, there is the group of parabolic contact structures, in which the filtration simply amounts to a contact structure. Apart from CR structures and Lie–sphere structures, this class also contains a quaternionic version of CR structures and a contact version of projective structures. Among more general parabolic geometries, there are some higher codimension partially integrable almost CR structures (see [33, 13]) as well as structures showing up in the geometry of differential equations, etc.

There are a large number of general tools available for parabolic geometries. First, it is possible to extract from the curvature of the Cartan connection (which is geometrically very complicated to understand) a geometrically much simpler part, which still is a complete obstruction to local flatness. Next, there is a general version of normal coordinates and distinguished curves (see [34] for a survey), as well as a general theory of a distinguished class of underlying linear connections (generalizing Weyl–structures in conformal geometry and Webster–Tanaka connections in CR geometry), see [14]. Finally, a general construction of so-called correspondence spaces allows one to construct on the total spaces of certain natural bundles over a manifold endowed with a normal parabolic geometry of some type, a normal parabolic geometry of different (more complicated) type. Conversely, one obtains a construction of twistor spaces and one can completely characterize geometries which are locally isomorphic to a correspondence space, see [6].

2.5. Irreducible bundles and tractor bundles. We now turn to the question of natural vector bundles on manifolds endowed with a parabolic geometry. If $(p : \mathcal{H} \rightarrow M, \omega)$ is a parabolic geometry of type (H, Q) , then the obvious natural vector bundles available in this situation are vector bundles associated to the principal bundle $p : \mathcal{H} \rightarrow M$. It is well known that these bundles are in bijective correspondence with (finite dimensional) representations of the parabolic subgroup Q .

The structure of parabolic subgroups is well understood in general. It turns out that Q always is a semidirect product of a reductive subgroup Q_0 and a nilpotent normal vector subgroup Q_+ . On the Lie algebra level, this corresponds to the reductive Levi decomposition of \mathfrak{q} into the reductive part \mathfrak{q}_0 and the nilradical \mathfrak{q}_+ . In the CR case, P_0 is isomorphic to the conformal unitary group $CU(p, q)$, while \mathfrak{p}_+ is two step nilpotent $\mathfrak{p}_+ \cong \mathbb{C}^{p+q} \oplus \mathbb{R}$, with \mathbb{R} the center and the bracket $\mathbb{C}^{p+q} \times \mathbb{C}^{p+q} \rightarrow \mathbb{R}$ being given by the imaginary part of a non-degenerate Hermitian form of signature (p, q) . (Notice that \mathfrak{p}_+ looks like the associated graded to any tangent space of a partially integrable almost CR manifold with the bracket induced by the Lie bracket.)

In any case, this shows that the representation theory of Q is very difficult, however there are always two simple classes of representations:

Irreducible representations. On any irreducible representation of Q , the nilpotent group Q_+ acts trivially. Thus representations of this type are obtained by taking irreducible representations of the reductive group Q_0 (which are well understood) and extending them trivially to Q . The corresponding natural vector bundles are

called *irreducible bundles*. They are usually easy to describe geometrically and they are the bundles one is mainly interested in. In the CR case, any irreducible bundle is a subbundle of a tensor product of copies of HM and of density bundles. However, it is very difficult to find invariant differential operators acting on sections of such bundles.

Restrictions of representations of the semisimple group. Since representations of the semisimple group H are well understood, the second simple way to obtain representations of Q is to use restrictions of representations of H . One should however be aware of the fact that as representations of Q these typically are indecomposable but not irreducible. So usually there are many invariant subspaces, but none of these has an invariant complement. The bundles corresponding to such representations are called *tractor bundles*. They are hard to describe geometrically, but they have the nice feature that they admit canonical linear connections. The point about this is that if \mathbb{W} is a representation of H , then $\mathcal{H} \times_Q \mathbb{W} \cong (\mathcal{H} \times_Q H) \times_H \mathbb{W}$. Now the Cartan connection ω induces a principal connection on the extended bundle $\mathcal{H} \times_Q H$ and thus a linear connection on the tractor bundle $\mathcal{H} \times_Q \mathbb{W}$. This linear connection is called the *normal tractor connection*. Alternatively, one may also describe the passage from the Cartan connection to the normal tractor connection directly (without using the extended bundle), see [7, section 2].

The fundamental example of a tractor bundle is the *adjoint tractor bundle* $\mathcal{H} \times_Q \mathfrak{h}$. This has the nice property that it contains the cotangent bundle as a subbundle and has the tangent bundle as a quotient. In the case of classical Lie algebras, and thus in particular in the CR case (after the choice of a bundle $\mathcal{E}(1,0)$) one also has the standard tractor bundle corresponding to the standard representation, which will play a main role in the rest of this paper.

In [7] it has been shown how tractor bundles and tractor connections can be used as an independent equivalent description of parabolic geometries. To do this, one first abstractly defines adjoint tractor bundles of type (H, Q) over a manifold M , essentially as bundles of filtered Lie algebras modeled on \mathfrak{h} with a canonical filtration induced by the parabolic subalgebra \mathfrak{q} . Then using an abstract notion of tractor connections, one gets a bijective correspondence between adjoint tractor bundles endowed with tractor connections and principal Q -bundles endowed with Cartan connections. Finally, one can characterize normality of the Cartan connection in terms of the tractor connection. For specific structures (such as conformal or CR) there is a simple variation using standard tractor bundles rather than adjoint tractor bundles.

Let us describe the case of the standard tractor bundle on CR manifolds explicitly. Let (M, HM, J) be a partially integrable almost CR manifold of signature (p, q) , let $\mathcal{E}(1,0) \rightarrow M$ be a complex line bundle such that $\mathcal{E}(1,0)^{\otimes n+2} \cong \Lambda_{\mathbb{C}}^n HM \otimes QM$ and put $\mathcal{E}(-1,0) := \mathcal{E}(1,0)^*$. Then a standard tractor bundle over M is a rank $n+2$ complex vector bundle $\mathcal{T} \rightarrow M$ endowed with a Hermitian bundle metric h of signature $(p+1, q+1)$, a complex line bundle $\mathcal{T}^1 \subset \mathcal{T}$ and a global non-vanishing smooth section τ of $\Lambda_{\mathbb{C}}^{n+2} \mathcal{T}^*$ which is compatible with h such that the following properties are satisfied:

- (i) $\mathcal{T}^1 \cong \mathcal{E}(-1, 0)$ and the fibers of \mathcal{T}^1 are null for h .
- (ii) $(\mathcal{T}^1)^\perp/\mathcal{T}^1 \cong HM \otimes \mathcal{E}(-1, 0)$, where the orthogonal complement is taken with respect to h .

A tractor connection on this tractor bundle is then a linear connection ∇ , which is Hermitian and compatible with τ and the complex structure \tilde{J} on \mathcal{T} , i.e. $\nabla h = 0$, $\nabla \tau = 0$, and $\nabla \tilde{J} = 0$ for the induced linear connections. Moreover, ∇ has to satisfy a non-degeneracy condition, namely that for any $x \in M$ and any tangent vector $\xi \in T_x M$, there is a smooth section $f \in \Gamma(\mathcal{T}^1)$, such that $\nabla_\xi f(x) \notin \mathcal{T}_x^1$.

Having these data and fixing a nonzero element $\alpha \in \Lambda_{\mathbb{C}}^{n+2} \mathbb{V}^*$ compatible with the Hermitian form $\langle \cdot, \cdot \rangle$, we define \mathcal{G}_x for $x \in M$ to be the set of all unitary isomorphisms $\mathbb{V} \rightarrow \mathcal{T}_x$, which map the distinguished line $\mathbb{C}e \subset \mathbb{V}$ to \mathcal{T}_x^1 and such that the induced map on the highest exterior power maps α to $\tau(x)$. Then the union $\mathcal{G} = \cup_{x \in M} \mathcal{G}_x$ is naturally a subspace in the linear frame bundle of \mathcal{T} , which by construction admits smooth local sections. Moreover, composition from the right defines a smooth right action of P on \mathcal{G} which is immediately seen to be free and transitive on each fiber, thus making $\mathcal{G} \rightarrow M$ into a P -principal bundle. Moreover, by construction $\mathcal{T} = \mathcal{G} \times_P \mathbb{V}$. The Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ corresponding to the tractor connection ∇ is then given as follows: For a section $s \in \Gamma(\mathcal{T})$ consider the corresponding function $f : \mathcal{G} \rightarrow \mathbb{V}$ given by $f(v) = v^{-1}(s(p(v)))$ for all $v \in \mathcal{G}$. Then for a point $u \in \mathcal{G}_x$ and a tangent vector $\xi \in T_u \mathcal{G}$, the value $\omega(\xi) \in \mathfrak{su}(\mathbb{V})$ is characterized by $u^{-1}(\nabla_{T_p \cdot \xi} s(x)) - (\xi \cdot f)(u) = \omega(\xi)(f(u))$, see [7, 2.5]. One can then characterize the normalization condition on ω in terms of ∇ , see [7, 2.9–2.11].

Finally, it should be remarked that for several structures and some classes of structures there are direct constructions of tractor bundles and tractor connections from underlying structures, see [1, 8, 26] and [7, section 4].

2.6. Tractor calculus. The drawback of the normal tractor connection is that while one may differentiate sections of a tractor bundle once, the cotangent bundle T^*M is not a tractor bundle, so there is no direct way to iterate the differentiation. For specific structures this problem was solved by introducing specific invariant differential operators that are now called *tractor D -operators* which act on sections of tractor bundles twisted by density bundles and which may be iterated. The first construction of such operators (for conformal structures) goes back to the work of T. Thomas in the 1930's, see [38]. They were rediscovered in [1] and a version for almost Grassmannian structures was introduced in [26]. Apart from other problems, these operators have in particular been applied to the description of projective and conformal invariants, see [21, 23].

Later on, it was realized that these tractor D -operators can be recovered from more basic invariant operators. Versions of these operators have been around in the literature earlier, but seemingly they have not been related to the adjoint tractor bundle and used systematically until the papers [23, 26, 7], where they are called *fundamental D -operators*. Roughly speaking, they may be viewed as an analog of the Levi-Civita connection in Riemannian geometry, but with the tangent bundle replaced by the adjoint tractor bundle.

For a parabolic geometry $(p : \mathcal{H} \rightarrow M, \omega)$ of type (H, Q) with adjoint tractor bundle $\mathcal{A} \rightarrow M$, and any vector bundle $E \rightarrow M$ that is associated to \mathcal{H} , one obtains an operator $\Gamma(\mathcal{A}) \otimes \Gamma(E) \rightarrow \Gamma(E)$ which we write as $(s, \varphi) \mapsto D_s \varphi$ to emphasize the similarity with a covariant derivative. To define this operator, one just has to note that via the Cartan connection, sections of \mathcal{A} are in bijective correspondence with Q -invariant vector fields on the total space of the Cartan bundle, which can then be used to differentiate the equivariant functions corresponding to sections of E .

From this construction, it is easy to see that the operators are algebraic in the \mathcal{A} -slot and first order differential operators in the E -slot, and that they are natural with respect to all vector bundle maps induced by homomorphisms of Q -modules. Now we can view D as an operator from $\Gamma(E)$ to $\Gamma(\mathcal{A}^* \otimes E)$, and since there is no restriction on the bundle involved, we can obviously iterate fundamental D 's. The iterated fundamental D 's then provide an invariant way to encode the infinite jet of a section into a fairly manageable bundle, which is a major step toward the general problem of invariants.

An important point to note here is that on tractor bundles the fundamental D -operators can be directly computed from the normal tractor connection, so no knowledge of the Cartan bundle and the Cartan connections is necessary in this case. Moreover, given the fundamental D 's on a tractor bundle, by the naturality one gets the fundamental D 's on all its subquotients. For example, knowing the normal tractor connection on \mathcal{A} , one not only gets the fundamental D on \mathcal{A} but also on the tangent bundle TM and thus on all tensor bundles.

2.7. Bernstein–Gelfand–Gelfand sequences. BGG-sequences offer a general construction of invariant differential operators acting on sections of irreducible bundles. On the flat model H/Q , naturality of a differential operator is equivalent to equivariance under the natural action of H on the spaces of sections of homogeneous vector bundles. It turns out that invariant differential operators (in that sense) acting between sections of irreducible bundles are via a dualization equivalent to homomorphisms of generalized Verma modules, so this problem reduces to representation theory. In the Borel case, such homomorphisms were completely classified by I.N. Bernstein, I.M. Gelfand, and S.I. Gelfand, see [2]. In particular most of these homomorphisms show up in the so-called BGG-resolutions of irreducible representations of H by homomorphisms of Verma modules. These resolutions were generalized to the case of arbitrary parabolics by J. Lepowsky, see [31].

To obtain curved analogs of these operators, one first notices that for a tractor bundle $\mathcal{T} \rightarrow M$, the normal tractor connection induces the so-called covariant exterior derivative on \mathcal{T} -valued differential forms. If \mathbb{W} is the H -representation inducing \mathcal{T} , then from Kostant's harmonic theory (see [29]) one concludes that the vector bundle corresponding to the cohomology module $(H^k(\mathfrak{q}_+, \mathbb{W}))^*$ is a natural subquotient of the bundle $\Lambda^k T^*M \otimes \mathcal{T}$. The Q -representation $H^k(\mathfrak{q}_+, \mathbb{W})$ (and thus also its dual) turns out to be completely reducible, and the irreducible components are explicitly computable using Kostant's version of the Bott–Borel–Weil theorem.

In the joint work [15] with J. Slovak and V. Souček, we introduced differential splittings from sections of these subquotients to \mathcal{T} -valued forms. Using these, one

can compress the twisted de–Rham sequence corresponding to \mathcal{T} to a sequence of higher order differential operators on the subquotients, whose properties are controlled by the twisted de–Rham sequence. This was significantly improved by D. Calderbank and T. Diemer in [4] in which differential projections onto the subbundles in question were constructed on the level of all \mathcal{T} –valued forms. This leads to an efficient calculus on \mathcal{T} –valued forms and in particular to an explicit procedure for constructing the BGG operators in terms of the covariant exterior derivatives and algebraic operations. In particular, all these operators are (at least in principle) explicitly computable from the fundamental D operator on \mathcal{T} . It should also be remarked here that with these differential projections at hand, one also obtains bi– and multilinear invariant differential operators, differential cup products, tools to translate invariant differential operators and so on.

3. THE AMBIENT METRIC AND THE FEFFERMAN SPACE

3.1. The ambient metric construction. The ambient metric construction works in the realm of embedded CR manifolds. Originally, it was introduced in the setting of smoothly bounded strictly pseudoconvex domains in \mathbb{C}^{n+1} using the Bergman kernel. Since the Bergman kernel is only computable asymptotically at the boundary, the interest moved to the local behavior of the ambient metric near the boundary. Finally, in [19] Ch. Fefferman introduced a version of the ambient metric in which the Bergman kernel was replaced by suitably normalized defining functions:

Suppose that $M \subset \mathbb{C}^{n+1}$ is an embedded CR manifold of signature (p, q) (i.e. M is a real hypersurface, such that the subbundle HM of maximal complex subspaces in tangent spaces defines a CR structure of signature (p, q)). Suppose further that r is a *defining function* for M , i.e. a smooth function defined locally around M such that $M = r^{-1}(0)$ and dr is nonzero on M . Then one defines $M_{\#} := \mathbb{C}^* \times M \subset \mathbb{C}^* \times \mathbb{C}^{n+1}$. Putting $r_{\#}(z^0, z) := |z^0|^2 r(z)$, one obtains a defining function for $M_{\#}$ and it turns out that $r_{\#}$ can be used as the potential of a Kähler metric g of signature $(p+1, q+1)$ defined locally around $M_{\#}$.

Now Fefferman introduced a normalization condition on the defining function r , which in particular implies that the resulting metric g is Ricci–flat. Namely, put

$$J(r) := (-1)^{q+1} \det \begin{pmatrix} r & \frac{\partial r}{\partial z^j} \\ \frac{\partial r}{\partial \bar{z}^i} & \frac{\partial^2 r}{\partial \bar{z}^i \partial z^j} \end{pmatrix}.$$

The equation $J(r) = 1$ is known as the *complex Monge–Ampère* equation. While this is a very difficult nonlinear PDE, Fefferman found a simple algorithm to modify any defining function ψ for M to a defining function r such that $J(r) = 1 + O(r^s)$ along M for $1 \leq s \leq n+2$. For such a defining function, it turns out that the Ricci curvature of the ambient metric g vanishes to order $s-3$ along M . For our purpose, solutions with $s=3$ will always be sufficient. Explicitly, the algorithm goes as follows: Starting with any defining function ψ , put $\psi_1 := \psi J(\psi)^{-1/(n+1)}$ and $\psi_2 := \psi_1 \left(1 + \frac{1-J(\psi_1)}{n+1}\right)$. Then $r := \psi_2 \left(1 + \frac{1-J(\psi_2)}{2n}\right)$ satisfies $J(r) = 1 + O(r^3)$ along M .

Fefferman showed that extending local biholomorphisms appropriately (which essentially amounts to viewing $M_{\#}$ as the frame bundle of a fixed choice of a complex line bundle $\mathcal{E}(1, 0)^*$), a certain jet along M of the ambient metric obtained from a normalized defining function is biholomorphism invariant. For our purposes, this invariance is not really important, since we shall later use the ambient metric to construct the standard tractor bundle and its canonical linear connection, which are known to be CR invariant. What is however of central importance for us is that everything related to the ambient metric is explicitly computable: Starting with any defining function ψ , we have the explicit algorithm to compute the normalized defining function r , and thus also $r_{\#}$. Since this is the potential for g , the components of the metric (in the coordinates (z^0, z^j)) are simply the mixed second partial derivatives of $r_{\#}$. The Christoffel symbols of the Levi-Civita connection of g are then partial derivatives of this components, i.e. higher partials of $r_{\#}$. From these, the curvature of g can be obtained as partial derivatives of $r_{\#}$, and so on.

3.2. The Fefferman space. Having the ambient metric g at hand, the construction of the Fefferman space in the embedded setting is now a very natural idea. Since the well-defined part of the ambient metric is some jet along $M_{\#}$, one tries to restrict g to $M_{\#}$. This restriction turns out to be degenerate, but in a very weak sense. Namely, the degenerate directions are exactly the real directions up the cone. Hence it is natural to consider the quotient $\tilde{M} := M_{\#}/\mathbb{R}^*$, which by construction is a bundle over M with fiber $\mathbb{C}^*/\mathbb{R}^* \cong U(1)$. Compressing the real part of g to this quotient, one gets a non-degenerate metric of signature $(2p+1, 2q+1)$, which however is only well defined up to a positive real factor, so we get a conformal structure of that signature on \tilde{M} . If one starts with a defining function r such that $J(r) = 1 + O(r^2)$ along M , then this conformal structure turns out to be CR invariant.

In particular, this implies that conformal invariants of the Fefferman space \tilde{M} are CR invariants of the original CR manifold M , (which is an important source of interest in conformal invariants). Since the ambient metric is completely computable, also the conformal structure on the Fefferman space can be computed explicitly starting from any defining function for M . A second main application of the Fefferman space is the description of chains on M . For any CR manifold, the chains form a family of distinguished curves. For any point $x \in M$ and any tangent vector $\xi \in T_x M$ which does not lie in the subspace $H_x M$, there is a unique such curve with initial point x and initial direction ξ up to parametrization. Now it turns out that the chains on M are exactly the projections of light-like geodesics on \tilde{M} , i.e. geodesics corresponding to null directions. In [19], Fefferman uses this to compute chains on embedded CR manifolds as trajectories of a Hamiltonian system.

The Fefferman space can also be constructed for abstract CR manifolds. This was first done by D. Burns, K. Diederich, and S. Shnider in [3], see also [30] for a version stressing the point of view of Cartan connections. These constructions are similar in spirit to the one in terms of tractors that will be described below, but working on the level of the Cartan bundle and the Cartan connection. Starting from an abstract CR manifold (M, HM, J) , one directly defines the Fefferman space \tilde{M} and its conformal structure via a certain natural complex line bundle on M . Then one proves that

the CR Cartan bundle of M naturally includes into the conformal Cartan bundle of \tilde{M} , and this inclusion is compatible with the normal Cartan connection on both bundles, which means that the normal conformal Cartan connection can be obtained from the normal CR Cartan connection by equivariant extension. In particular, the conformal Cartan curvature of the Fefferman space is essentially the same object as the CR Cartan curvature of the original manifold.

3.3. Fefferman space and standard tractors. We next describe a construction of the Fefferman space using standard tractors which adds a lot of power to this construction. Details and applications can be found in [9]. As in 2.2 consider $\mathbb{V} = \mathbb{C}^{n+2}$ endowed with a Hermitian inner product $\langle \cdot, \cdot \rangle$ of signature $(p+1, q+1)$, put $G = SU(\mathbb{V}) \cong SU(p+1, q+1)$, choose a fixed nonzero null vector e and define $P \subset G$ to be the stabilizer of the complex line $\mathbb{C}e$. For a CR manifold (M, HM, J) of signature (p, q) and a fixed choice of a complex line bundle $\mathcal{E}(1, 0)$ such that $\mathcal{E}(1, 0)^{\otimes n+2} \cong \Lambda_{\mathbb{C}}^n HM \otimes QM$, we get from 2.3 a canonical principal P -bundle $p : \mathcal{G} \rightarrow M$ endowed with a canonical normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Now we put $\mathcal{E}(-1, 0) := \mathcal{E}(1, 0)^*$, and we define \tilde{M} to be the quotient of the frame bundle of $\mathcal{E}(-1, 0)$ (which is a principal \mathbb{C}^* -bundle) by the action of the subgroup $\mathbb{R}^* \subset \mathbb{C}^*$. Thus, there is a natural projection $\tilde{M} \rightarrow M$, which is a principal bundle with structure group $\mathbb{C}^*/\mathbb{R}^* \cong U(1)$, i.e. \tilde{M} is a circle bundle over M .

Now we may as well consider \mathbb{V} as a real vector space and the real part of the Hermitian form as a real inner product. Define $\tilde{G} := O(\mathbb{V}) \cong O(2p+2, 2q+2)$, the orthogonal group of this inner product, and let $\tilde{P} \subset \tilde{G}$ be the stabilizer subgroup of the real null line $\mathbb{R}e$. Then normal parabolic geometries of type (\tilde{G}, \tilde{P}) are exactly conformal structures of signature $(2p+1, 2q+1)$. By construction, we have $G \subset \tilde{G}$, $G \cap \tilde{P} \subset P$ and one easily verifies that $P/(G \cap \tilde{P}) \cong \mathbb{C}^*/\mathbb{R}^*$.

Next, one easily shows that \tilde{M} is canonically isomorphic to the quotient $\mathcal{G}/(G \cap \tilde{P})$, which implies that there is a natural projection $\mathcal{G} \rightarrow \tilde{M}$ which defines a principal bundle with structure group $G \cap \tilde{P}$. Using this, we define a vector bundle $\tilde{\mathcal{T}} \rightarrow \tilde{M}$ by $\tilde{\mathcal{T}} := \mathcal{G} \times_{G \cap \tilde{P}} \mathbb{V}$. The real inner product on \mathbb{V} induces a bundle metric of signature $(2p+2, 2q+2)$ on $\tilde{\mathcal{T}}$. Moreover, the real line through the chosen vector e gives rise to a real line bundle sitting inside $\tilde{\mathcal{T}}$ which is null and can be shown to be isomorphic to a certain density bundle over \tilde{M} . These two data make $\tilde{\mathcal{T}} \rightarrow \tilde{M}$ into a conformal standard tractor bundle.

The mechanism introduced in [7] to construct tractor connections from Cartan connections can then be applied to construct a linear connection $\nabla^{\tilde{\mathcal{T}}}$ on the bundle $\tilde{\mathcal{T}}$, which can be easily shown to be a tractor connection. The most difficult part of the construction is then to analyze the relation between the normalization conditions for conformal and CR Cartan connections to prove that the tractor connection $\nabla^{\tilde{\mathcal{T}}}$ is normal. It is worth noticing that this is not a purely algebraic game, and in particular it does not work for partially integrable almost CR structures. Rather one has to show that for CR manifolds, the curvature of the Cartan connection actually satisfies a stronger version of the normalization condition.

Having this result at hand, we obtain a conformal structure on \tilde{M} (which can be easily described explicitly) but more importantly, we see that the Cartan curvatures of the conformal structure on \tilde{M} and of the CR structure on M are essentially the same object, thus recovering all facts known from earlier constructions of the Fefferman space. But this construction has important new features: By definition, the CR standard tractor bundle $\mathcal{T} \rightarrow M$ is the associated bundle $\mathcal{G} \times_P \mathbb{V}$. From this description it follows immediately that the $U(1)$ -action on \tilde{M} which has M as its orbit space lifts to an action by vector bundle homomorphisms on $\tilde{\mathcal{T}}$ and the orbit space of this action is exactly \mathcal{T} . This in turn means that one gets a $U(1)$ -action on the space $\Gamma(\tilde{\mathcal{T}})$ of smooth sections of $\tilde{\mathcal{T}} \rightarrow \tilde{M}$ such that the invariant elements are exactly the smooth sections of $\mathcal{T} \rightarrow M$. Moreover, in this process the normal tractor connection $\nabla^{\tilde{\mathcal{T}}}$ descends to the normal tractor connection on \mathcal{T} .

A similar relation can be built up for a large class of bundles (essentially all those which come from a representation of \tilde{P} whose restriction to $G \cap \tilde{P}$ admits an extension to P), and in all cases one gets a $U(1)$ -action on the sections of the conformal bundle whose invariant elements are exactly the sections of the corresponding CR bundle. In particular, this works for all density bundles and all conformal tractor bundles.

Next, one may compare the adjoint tractor bundles on \tilde{M} and on M , which can be described as $\tilde{\mathcal{A}} = \mathfrak{so}(\tilde{\mathcal{T}})$ and $\mathcal{A} = \mathfrak{su}(\mathcal{T})$, respectively. In particular, there is a canonical $U(1)$ -action on $\tilde{\mathcal{A}}$. On the other hand, the complex structure on \mathbb{V} induces an almost complex structure on $\tilde{\mathcal{T}}$, which is parallel for the normal tractor connection on $\tilde{\mathcal{A}}$ (which is induced by $\nabla^{\tilde{\mathcal{T}}}$). One shows that sections of \mathcal{A} are exactly those sections of $\tilde{\mathcal{A}}$ which are $U(1)$ -invariant and commute with this almost complex structure. Using this, one then shows that for arbitrary compatible bundles as above, the conformal fundamental D -operator is $U(1)$ -equivariant and descends to the CR fundamental D -operator.

In this way, one obtains a machinery to descend conformally invariant differential operators that can be described in terms of fundamental D 's to CR invariant differential operators, which then are automatically described in tractor terms. In particular, as we shall see below there is a way to describe these operators explicitly in the case of embedded CR manifolds.

3.4. Ambient metric and standard tractors. As a final topic, I want to describe how the ambient metric construction from 3.1 can be used to get a completely explicit description of the CR standard tractor bundle and its normal tractor connection in the case of embedded CR manifolds. Details and applications are presented in [10].

As in 3.1 let $M \subset \mathbb{C}^{n+1}$ be an embedded CR manifold of signature (p, q) and let r be a (not necessarily normalized) defining function for M . Define $M_{\#} := \mathbb{C}^* \times M \subset \mathbb{C}^* \times \mathbb{C}^{n+1}$, consider the defining function $r_{\#}$ for $M_{\#}$ given by $r_{\#}(z^0, z) = |z^0|^2 r(z)$ and let g be the corresponding ambient metric, i.e. the Kähler metric (defined locally around $M_{\#}$) with potential $r_{\#}$. For $\alpha \in \mathbb{C}^*$ let ρ^{α} denote the canonical action of α on $M_{\#}$ and $\mathbb{C}^* \times \mathbb{C}^{n+1}$, i.e. $\rho^{\alpha}(z^0, z) = (\alpha z^0, z)$.

First it is easy to show directly that the associated bundle $M_{\#} \times_{\mathbb{C}^*} \mathbb{C}$ with respect to the action given by multiplication by α^{-1} is an appropriate choice for $\mathcal{E}(1, 0)$ in this

situation. Otherwise put, sections of $\mathcal{E}(1, 0)$ correspond to smooth functions $M_{\#} \rightarrow \mathbb{C}$ which are homogeneous of degree $(1, 0)$, i.e. which satisfy $f(\alpha z^0, z) = \alpha f(z^0, z)$. In this way, $M_{\#}$ is exactly the frame bundle of the dual $\mathcal{E}(-1, 0)$ of $\mathcal{E}(1, 0)$. Similarly one can then describe sections of the density bundles $\mathcal{E}(k, \ell)$ for $k, \ell \in \mathbb{Z}$ (which are defined as appropriate tensor products of copies of $\mathcal{E}(1, 0)$, or $\mathcal{E}(-1, 0)$ and their conjugate bundles) as functions $f : M_{\#} \rightarrow \mathbb{C}$ which are homogeneous of degree (k, ℓ) , i.e. $f(\alpha z^0, z) = \alpha^k \bar{\alpha}^{\ell} f(z^0, z)$.

Next, consider the restriction of the ambient tangent bundle $T(\mathbb{C}^* \times \mathbb{C}^{n+1})$ to $M_{\#}$ and define a \mathbb{C}^* -action on this bundle by $\alpha \cdot \xi := \alpha^{-1} T\rho^{\alpha} \cdot \xi$. Then this is an action by vector bundle homomorphisms lifting the action on $M_{\#}$, and thus the quotient $\mathcal{T} := (T(\mathbb{C}^* \times \mathbb{C}^{n+1})|_{M_{\#}})/U(1)$ is a smooth rank $n + 2$ complex vector bundle over M . Moreover, sections of \mathcal{T} are in bijective correspondence with ambient vector fields along $M_{\#}$ which are homogeneous of degree -1 , i.e. which satisfy $\xi(\alpha z^0, z) = \alpha^{-1} T\rho^{\alpha} \cdot \xi(z^0, z)$.

From the construction it follows easily that the ambient Kähler metric g is homogeneous of degree $(2, 0)$, so inserting two fields homogeneous of degree $(-1, 0)$ one gets a function which is constant along the fibers, and hence g descends to a Hermitian metric h of signature $(p + 1, q + 1)$ on the bundle \mathcal{T} . On the other hand, the fundamental field X generating the \mathbb{C}^* -action is homogeneous of degree $(0, 0)$, so for a smooth function $f : M_{\#} \rightarrow \mathbb{C}$ homogeneous of degree $(-1, 0)$ the field fX is a section of \mathcal{T} . Thus one gets a subbundle $\mathcal{T}^1 \subset \mathcal{T}$ which by construction is isomorphic to $\mathcal{E}(-1, 0)$. It is easy to verify that for $x \in M$ the line \mathcal{T}_x^1 is null for h , and projecting vector fields induces an isomorphism $(\mathcal{T}^1)^{\perp}/\mathcal{T}^1 \cong HM \otimes \mathcal{E}(-1, 0)$.

Next, one verifies that the Levi-Civita connection of the ambient metric is compatible with homogeneities and that for fields homogeneous of degree $(-1, 0)$ the covariant derivative in vertical directions vanishes. Thus, taking a vector field on M and lifting it to a field on $M_{\#}$ homogeneous of degree $(0, 0)$, the covariant derivative with respect to that lift maps sections of \mathcal{T} to sections of \mathcal{T} and is independent of the choice of the lift. In that way, the Levi-Civita connection of g descends to a linear connection $\nabla^{\mathcal{T}}$ on \mathcal{T} , which by construction is Hermitian with respect to the bundle metric h .

This is not yet enough to make \mathcal{T} into a standard tractor bundle and $\nabla^{\mathcal{T}}$ into a tractor connection on that bundle, since that would also require a trivialization of $\Lambda_{\mathbb{C}}^{n+2} \mathcal{T}$ such that the corresponding (constant) global sections are parallel with respect to $\nabla^{\mathcal{T}}$, see 2.5, and such sections cannot exist in general. However, we may consider the bundle $\mathcal{A} = \mathfrak{su}(\mathcal{T})$ (which makes sense with the data defined up to now) and the induced connection $\nabla^{\mathcal{A}}$, and one obtains:

Theorem. (1) *The bundle $\mathcal{A} = \mathfrak{su}(\mathcal{T})$ is an adjoint tractor bundle on M , and $\nabla^{\mathcal{A}}$ is a tractor connection on \mathcal{A} .*

(2) *The tractor connection $\nabla^{\mathcal{A}}$ is normal if and only if the ambient metric g is Ricci-flat along M .*

The proof of part (1) is a rather straightforward verification, while for (2) one first relates the curvature of $\nabla^{\mathcal{A}}$ to the curvature of g (which is rather easy) and then has to analyze in detail the normalization condition for tractor connections.

3.5. The case of a normalized defining function. Let us now suppose that we start with a defining function r which satisfies $J(r) = 1 + O(r^3)$ along M . Then it is easy to see that the corresponding ambient metric g is Ricci-flat along M , so the construction of 3.4 leads to the normal adjoint tractor bundles and its normal tractor connection. But in this case, one may actually construct a non-vanishing smooth section of $\Lambda_{\mathbb{C}}^{n+2}\mathcal{T}$ which is parallel for $\nabla^{\mathcal{T}}$, thus making \mathcal{T} into a standard tractor bundle over M and $\nabla^{\mathcal{T}}$ into a tractor connection on this tractor bundle. The general theory of tractor connections then implies that the curvature of $\nabla^{\mathcal{T}}$ is essentially the same object as the curvature of $\nabla^{\mathcal{A}}$, and one obtains

Theorem. *If the defining function r satisfies $J(r) = 1 + O(r^3)$ along M , then the bundle \mathcal{T} and the connection $\nabla^{\mathcal{T}}$ constructed in 3.4 are the normal standard tractor bundle and the normal standard tractor connection for the CR structure on M .*

As we noted in 3.1 the ambient metric and its Levi-Civita connection are explicitly computable from the defining function r , so this result gives a completely explicit description of the standard tractor bundle \mathcal{T} , the tractor metric h and the normal tractor connection $\nabla^{\mathcal{T}}$, so all ingredients needed for tractor calculus are explicitly computable. The general theory then implies that the curvature of $\nabla^{\mathcal{T}}$ coincides with the curvature of the normal Cartan connection, so we get an explicit algorithm for computing the Cartan curvature of an embedded CR manifold starting from a normalized defining function (which in turn can be computed explicitly starting from any defining function). Already this simple consequence is rather remarkable in view of the recent paper [39]. In that paper, S.M. Webster computes the Cartan curvature for a simple class of embedded CR manifolds (using methods tailored to this class) and claims that this is the first case in which the Cartan curvature of a non-flat CR manifold of dimension bigger than 3 is computed completely.

It should also be pointed out that the relation to the Fefferman space can be easily exploited in that picture. In particular, going through the construction presented in 3.3, one sees that in the embedded case this construction really coincides with Fefferman's original construction. The ambient metric can in this situation also be used to describe the conformal standard tractor bundle and its normal tractor connection, see also 3.6 below.

Notice further that any irreducible representation of the group $SU(p+1, q+1)$ is a subrepresentation of a tensor product of copies of the standard representation and its dual. Thus any tractor bundle is a subbundle of a tensor product of copies of \mathcal{T} and \mathcal{T}^* , and since all normal tractor connections are induced from the normal Cartan connection, the normal tractor connection on such a subbundle is just the restriction of the connection on the tensor product induced by $\nabla^{\mathcal{T}}$. Thus, we get an explicit description of any tractor bundle and its normal tractor connection, which can then be used to compute BGG operators as outlined in 2.7.

There is yet another way to proceed further, which ties in standard tractors with further classical techniques used in CR geometry. Starting with the defining function r , the (real valued) one form $-i\partial r|_M$ actually defines a contact form for the contact structure defined by the subbundle $HM \subset TM$. On one hand, this may be viewed as a pseudo-Hermitian structure on the CR manifold (M, HM, J) , thus giving rise

to the so-called *Webster–Tanaka connection* on TM . On the other hand, in the terminology of [14], this contact form defines a section of a bundle of scales and thus an exact Weyl structure on (M, HM, J) . Hence we get the corresponding *Weyl connection* which gives a linear connection on any irreducible bundle as well as the associated rho-tensor. Now it turns out that from the explicit description of \mathcal{T} and $\nabla^{\mathcal{T}}$ one may explicitly compute this rho-tensor as well as the Weyl connections on density bundles and on the bundle HM , which then can be used to compute the Weyl connection on any irreducible bundle. This gives access to a general machinery (see [5]) for explicitly computing a large class of BGG-operators. On the other hand, one may also compute in general the relation between the Weyl connections on HM and QM and the Webster–Tanaka connection associated to the pseudo-Hermitian structure defined by $-i\partial r$, thus obtaining an explicit formula for this Webster–Tanaka connection. This also makes contact to the CR tractor calculus developed in [24], see also [22].

3.6. Remark on the conformal ambient metric. There is also an ambient metric construction for conformal structures, which was introduced in [20] and applied to the construction of invariant powers of the Laplacian in [27]. This ambient metric construction is closely related to Poincaré metrics and the theory of conformal infinities which has been recently related to string theory, see [28]. This construction starts with an embedding of the frame bundle of a certain density bundle into an ambient manifold and produces a conformally invariant jet of a Ricci-flat pseudo-Riemannian metric along the frame bundle. A construction analogous to the one discussed in 3.4 above can be used to obtain a standard tractor bundle and a tractor connection on that bundle from any ambient metric, and this tractor connection is normal if and only if the ambient metric is Ricci-flat along the frame bundle, see [11]. In fact, this result is significantly simpler to prove than Theorems 3.4 and 3.5, since on one hand in the conformal case the relation between adjoint tractor bundles and standard tractor bundles is simpler, and on the other hand the normalization condition for conformal tractor connections is much easier to analyze than the corresponding condition in the CR case. These results have already found applications to the study of invariant powers of the Laplacian and of the so-called Q -curvatures, see [25].

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