Boundaries and Harmonic Functions for Random Walks with **Random Transition Probabilities**

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BOUNDARIES AND HARMONIC FUNCTIONS FOR RANDOM WALKS WITH RANDOM TRANSITION PROBABILITIES

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ABSTRACT. The usual random walk on a group (homogeneous both in time and in space) is determined by a probability measure on the group. In a random walk with random transition probabilities this single measure is replaced with a stationary sequence of measures, so that the resulting (random) Markov chains are still space homogeneous, but no longer time homogeneous. We study various notions of measure theoretical boundaries associated with this model, establish an analogue of the Poisson formula for (random) bounded harmonic functions, and identify these boundaries for several classes of groups.

0. Introduction

Random walks on groups were intensively studied during the last 40 years (see, for instance, [Ka96] and the references therein). Their importance is due to numerous applications, in particular, to the description of boundaries and spaces of harmonic functions and to the study of ergodic properties of group actions. Such random walks are Markov chains which are homogeneous both in time and space and can be also represented as products of independent identically distributed (i.i.d.) group elements. In the particular important case of products of random matrices additional tools such as Lyapunov exponents can be employed.

A random walk on a group G is determined by a Markov operator $P = P(\mu)$ which consists in the (right) convolution with a fixed probability measure μ on G, so that the operator P is invariant with respect to the action of the group on itself by left translations. There are two models for further "randomization" of these "ordinary" random walks. The first model is usually referred to as random walks in random environment (RWRE) and consists in considering a probability measure λ on the space of all Markov operators on G. In this model the individual operators (environments) are not group invariant, although the group structure is taken into account by requiring the measure λ to be quasi-invariant with respect to the action of G on the space of environments (more specifically, λ is usually assumed to be either translation invariant or stationary

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with respect to the "moving environment" chain, see, for instance, [Kal81], [KMo84], [KSi00]). One chooses a random environment according to the distribution λ , and then runs a time (but not space!) homogeneous Markov chain in this environment.

The other model which we call random walks with random transition probabilities (RWRTP) is opposite to RWRE in the sense that here one keeps the space homogeneity but does not assume the time homogeneity. Namely, the additional randomness is introduced in this model by taking a random sequence μ_0, μ_1, \ldots of probability measures on G so that the (G-invariant!) transition probabilities of the arising chain on G at time n are given by the measure μ_n . The formal description of this model consists in fixing an invertible ergodic transformation T of a probability space (Ω, λ) and a measurable map $\omega \mapsto \mu^{\omega}$. One chooses $\omega \in \Omega$ according to the distribution λ and then runs the arising random walk with time dependent increments RWTDI(ω) determined by the sequence $\mu^{\omega}, \mu^{T\omega}, \mu^{T^2\omega}, \ldots$ The transformation T is usually assumed to be measure preserving, so that the above random sequence of measures is stationary. In the same way one can also talk about random sequences of Markov operators on a general state space (which does not have to be a group or to be endowed with any additional spatial structure).

This model was first introduced more than 20 years ago in connection with a model of random automata and various properties of such Markov chains were investigated since then in a number of papers (see, for instance, [Or91], [Ki96] and the references therein). Random walks with random transition probabilities first appeared in [MR88] (see also [MR94], [LRW94], [Ru95]), and products of independent random matrices with station-arily changing distributions were studied in [Ki01]. Ideologically and methodically this topic is rather close to random dynamical systems which were intensively studied in recent years, and Markov chains with random transition probabilities have the same relation to the classical Markov chains as random dynamical systems to deterministic ones. In both cases the guiding philosophy suggests that we have good chances to obtain an additional non-trivial information about the system if it acquires nice properties after conditioning by some ergodic stationary process (which we do not have much information about). Note that in the framework of random walks on groups one can also make one more step and to combine the RWRE and RWRTP models (so that individual random chains will be neither space nor time homogeneous).

RWRTP can be considered as a generalization of yet another model of "randomization" of the ordinary random walks called random walks with internal degrees of freedom (RWIDF) [KSz83] or covering Markov chains [Ka95]. These are G-invariant Markov chains on the product of the group G by another space X. The transition probabilities of RWIDF are $p((g, x), (gh, y)) = \overline{p}(x, y)\mu^{x,y}(h)$ (assuming that X is countable), where $\mu^{x,y}$ are probability measures on X, and $\overline{p}(x, y)$ are the transition probabilities of the quotient chain on X. If the quotient chain has a finite stationary measure, then the associated RWRTP is determined by the space $\Omega = X^{\mathbb{Z}}$ endowed with the corresponding shift-invariant Markov measure and the map $(\ldots, x_{-1}, x_0, x_1, \ldots) \mapsto \mu^{x_0, x_1}$.

The setup of RWRTP yields a natural notion of random harmonic functions f_{ω} on G which satisfy the relation $f_{\omega}(g) = \int f_{T\omega}(gh) d\mu^{\omega}(h)$. These functions can be considered

as harmonic functions of the global time homogeneous G-invariant Markov chain on the product $\Omega \times G$ with the transition probabilities $p((\omega, g), (T\omega, gh)) = \mu^{\omega}(h)$. This global chain is an immediate analogue of the usual "space-time" chain (the role of "time" is played here by the space Ω endowed with the transformation T). Therefore, description of all bounded random harmonic functions amounts to describing the *Poisson boundary* Γ of the global chain.

In this paper we consider only discrete groups G. We introduce the notion of the relative (or fiber) Avez type *entropy* of RWRTP which is close to the notion of the relative (fiber) entropy in the ergodic theory of random dynamical systems (this theory is also known under the name of relative ergodic theory, see [Ki86]). Similarly to the theory of ordinary random walks (see [KV83], [Ka00]) we give an entropy criterion for triviality of the *tail boundary* of almost all RWTDI(ω), $\omega \in \Omega$, which implies triviality of the Poisson boundary of the global chain, i.e., absence of non-trivial bounded random harmonic functions. As a corollary, we prove convergence of random convolutions to an invariant mean for nilpotent groups (earlier it was established for compact and abelian groups in the works of Mindlin and Rubshtein [MR94] and of Lin, Rubshtein and Wittman [LRW94] by completely different methods).

The relationship between the Poisson and the tail boundaries for RWRTP turns out to be more complicated than for ordinary random walks (where the tail and the Poisson boundaries coincide with respect to any single point initial distribution). Indeed, in the RWRTP setup the Poisson boundary does not make sense for individual RWTDI(ω) on G (because they are not time homogeneous). As for the global chain on $\Omega \times G$, its projection onto Ω is deterministic, so that the tail boundary E of the global chain admits a natural projection onto Ω whose fibers are the tail boundaries of RWTDI(ω) (in particular, triviality of the tail boundary of almost all $RWTDI(\omega)$ is equivalent to coincidence of E and Ω). On the other hand, the Poisson boundary of any time homogeneous Markov chain is a quotient of its tail boundary. Therefore, there are two natural projections of the tail boundary E of the global chain: onto Ω and onto the Poisson boundary Γ . We say that RWRTP is *stable* if these two projection separate point of E. If RWRTP is stable, then the tail boundaries of individual RWTDI(ω) can be identified with the Poisson boundary of the global chain, so that stability of RWRTP is a property analogous to coincidence of the tail and the Poisson boundaries for ordinary random walks.

We do not know whether RWRTP on groups are always stable. However, in the final section under the finite first moment condition we (by using the entropy technique) explicitly identify the Poisson and the tail boundaries of RWRTP on discrete groups of isometries of non-positively curved spaces with natural geometric boundaries (for example, for a free group this natural boundary is the space of ends). Therefore, these RWRTP are stable.

We do not study here random boundaries for continuous groups which, we hope, will be dealt with in another paper.

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1. Measure theoretical boundaries of Markov chains

In this Section we introduce the necessary notations and background from the general theory of Markov chains, see [Re84], [Ka92], [Ki01].

1.1. Markov operators.

Definition 1.1. Let (X, m) be a Lebesgue measure space with a σ -finite positive measure m. A linear operator $P: L^{\infty}(X, m) \leftrightarrow$ is called *Markov* if

- (i) P preserves positivity, i.e., $Pf \ge 0$ for any function $f \ge 0$;
- (ii) P preserves constants, i.e., $P\mathbf{1} = \mathbf{1}$ for the function $\mathbf{1}(x) \equiv 1$;
- (iii) P is continuous in the sense that $Pf_n \downarrow \mathbf{0}$ a.e. whenever $f_n \downarrow \mathbf{0}$ a.e.

The adjoint operator P^* of a Markov operator $P: L^{\infty}(X, m) \leftrightarrow \text{acts on the space of}$ integrable functions on the space (X, m), or, in other words, on the space of measures θ on X absolutely continuous with respect to m (notation: $\theta \prec m$). We shall use the notation θP for the measure on X with the density $P^*(d\theta/dm)$, so that $\langle \theta P, f \rangle_m = \langle \theta, Pf \rangle_m$ for any function $f \in L^{\infty}(X, m)$.

A (σ -finite) initial distribution $\theta \prec m$ gives rise in a standard way to a Markov measure \mathbf{P}_{θ} in the path space $X^{\mathbb{Z}_{+}} = \{\mathbf{x} = (x_0, x_1, \ldots)\}$ of the associated Markov chain on X. The one-dimensional distributions of \mathbf{P}_{θ} are θP^n , and the time shift $(S\mathbf{x})_n = \mathbf{x}_{n+1}$ acts on it as $S(\mathbf{P}_{\theta}) = \mathbf{P}_{\theta P}$. A measure $\theta \prec m$ is called a stationary measure of the Markov operator P if $\theta P = \theta$, or, equivalently, if the measure \mathbf{P}_{θ} is S-invariant.

By definition, the conditional expectations \mathbf{E}_{θ} of the measure \mathbf{P}_{θ} satisfy the relation $\mathbf{E}_{\theta}[f(x_{n+1})|x_n = x] = Pf(x)$ for any $n \geq 0$. Since the space (X, m), and therefore all spaces $(X_{+}^{\mathbb{Z}}, \mathbf{P}_{\theta}), \ \theta \prec m$ are *Lebesgue*, these conditional expectations can be replaced by the integrals with respect to the corresponding *conditional measures* π_x which are called *one-step transition probabilities*. Then the operator P and its adjoint operator take the form

(1.1)
$$Pf(x) = \int f(y) d\pi_x(y) , \qquad \theta P = \int \pi_x d\theta(x) .$$

Remark 1.2. The measures π_x are not necessarily absolutely continuous with respect to m. Still, for any function $f \in L^{\infty}(X, m)$ the integrals above make sense for m-a.e. $x \in X$ by Rokhlin's theorem on conditional decomposition of measures in Lebesgue spaces (e.g., see [CFS82]). We shall use this theorem on several occasions below without further notice.

1.2. The tail boundary and harmonic sequences.

Denote by α_k , $k \in \mathbb{Z}_+$ the k-th coordinate partition of the path space $(X^{\mathbb{Z}_+}, \mathbf{P}_m)$, and for $0 \leq k < l \leq \infty$ put $\alpha_{k,l} = \bigvee_{i=k}^{l} \alpha_i$, i.e., two paths \boldsymbol{x} and \boldsymbol{x}' are $\alpha_{k,l}$ -equivalent iff $x_i = x'_i$ for all $k \leq i \leq l$. The partitions $\alpha_{k,\infty}$, k > 0 coincide with the preimage partitions of the powers S^k of the time shift S.

Recall that the measurable partitions of the same space are ordered in such a way that "the bigger are the elements, the smaller is the partition"; this order is denoted by \preccurlyeq . Obviously, $\alpha_{k+1,\infty} \preccurlyeq \alpha_{k,\infty}$ for any $k \in \mathbb{Z}_+$. Let $\alpha_{\infty} = \bigwedge_k \alpha_{k,\infty}$ be the *measurable intersection* of the sequence $\alpha_{k,\infty}$, i.e., the biggest measurable partition of the space $(X^{\mathbb{Z}_+}, \mathbf{P}_m)$ which is smaller than any partition $\alpha_{k,\infty}$. The partition α_{∞} is called the *tail partition* of the path space.

Definition 1.3. The quotient E of the path space $(X^{\mathbb{Z}+}, \mathbf{P}_m)$ with respect to the tail partition α_{∞} is called the *tail boundary*. Denote by **tail** : $X^{\mathbb{Z}+} \to E$ the corresponding projection.

The space E is endowed with the *tail measure type* $[\varepsilon_m]$ which is the image of the type of the measure \mathbf{P}_m . For any probability measure $\theta \prec m$ the *tail measure* $\varepsilon_{\theta} = \mathbf{tail}(\mathbf{P}_{\theta})$ is absolutely continuous with respect to $[\varepsilon_m]$. We emphasize that the space E and the projection **tail** are defined in the measure theoretical category, so that they make sense " \mathbf{P}_m -mod 0" (i.e., up to the sets of \mathbf{P}_m -measure 0) only.

The quotient of the path space $X^{\mathbb{Z}}$ with respect to the partition $\alpha_{n,\infty}$ is the space $X^{[n,\infty)}$ of paths on X running from the time n only. Therefore, one can consider the space E as the inductive limit (in the measure theoretical category!) of the sequence of the spaces $X^{[n,\infty)}$ endowed with the images of the measure \mathbf{P}_m . Denote by $\mathbf{P}_{n,\theta}$ the measure on the space $X^{[n,\infty)}$ corresponding to starting the Markov chain at time n with the initial distribution θ . Projecting the measure $\mathbf{P}_{n,\theta}$ onto E gives the associated tail measure $\varepsilon_{n,\theta}$. Denote by $\varepsilon_{n,x}$ the tail measures on E corresponding to starting the Markov chain at time n markov chain at time n from a point $x \in X$ (cf. Remark 1.2). Then

(1.2)
$$\varepsilon_{n,x} = \int \varepsilon_{n+1,y} \, d\pi_x(y) \; .$$

Since $\operatorname{tail}(\boldsymbol{x}) = \operatorname{tail}(\boldsymbol{x}')$ if and only if $\operatorname{tail}(S\boldsymbol{x}) = \operatorname{tail}(S\boldsymbol{x}')$, the action of the time shift S descends from $X^{\mathbb{Z}_+}$ to an invertible transformation of E (also denoted S), and $\varepsilon_{n,\theta} = S^{-n} \varepsilon_{\theta}$.

Definition 1.4. A sequence of functions $f_n \in L^{\infty}(X, m)$, $n \in \mathbb{Z}_+$ is called a harmonic sequence if $f_n = Pf_{n+1}$ for any $n \in \mathbb{Z}_+$. Denote by $HS^{\infty}(X, m, P)$ the space of harmonic sequences endowed with the norm $\sup_n ||f_n||_{\infty}$.

Theorem 1.5 ([Re84], [Ka92]). The spaces $HS^{\infty}(X, m, P)$ and $L^{\infty}(E, [\varepsilon_m])$ are isometric. This isometry is established by the formulas

(1.3)
$$\lim_{n \to \infty} f_n(x_n) = \widehat{f}(\mathbf{tail}(x)) , \qquad f_n(x) = \langle \widehat{f}, \varepsilon_{n,x} \rangle ,$$

where $\{f_n\} \in HS^{\infty}(X, m, P)$ and $\widehat{f} \in L^{\infty}(E, [\varepsilon_m])$.

1.3. The Poisson boundary and harmonic functions.

Definition 1.6. The space Γ of ergodic components of the shift S in the path space $(X^{\mathbb{Z}_+}, \mathbf{P}_m)$ is called the *Poisson boundary* of the Markov operator P. Denote the corresponding projection by **bnd** : $X^{\mathbb{Z}_+} \to \Gamma$.

The Poisson boundary can be also defined as the space of ergodic components of the transformation S (induced by the shift in the path space) of the tail boundary E. Therefore, the map **bnd** is the result of the composition of the maps **tail** : $X^{\mathbb{Z}_+} \to E$ and $\mathfrak{p}_{\Gamma} : E \to \Gamma$. Denote by $[\nu_m] = \mathbf{bnd}([\mathbf{P}_m]) = \mathfrak{p}_{\Gamma}([\varepsilon_m])$ the harmonic measure type on Γ , and by $\nu_{\theta} = \mathbf{bnd}(\mathbf{P}_{\theta}) = \mathfrak{p}_{\Gamma}(\varepsilon_{\theta})$ the harmonic measure corresponding to an initial probability distribution $\theta \prec m$, so that $\nu_{\theta} \prec [\nu_m]$. By ν_x we shall denote the harmonic measures corresponding to individual points $x \in X$ (cf. Remark 1.2). Since Γ is the space of ergodic components of the action of S on E, for any $n \ge 0$ we have $\mathfrak{p}_{\Gamma}(\varepsilon_{n,\theta}) = \mathfrak{p}_{\Gamma}(\varepsilon_{\theta}) = \nu_{\theta}$. Therefore, (1.2) implies that the harmonic measures on Γ satisfy the stationarity relation

$$u_x = \int \nu_y \, d\pi_x(y) \, .$$

Definition 1.7. A function $f \in L^{\infty}(X, \mu)$ is called *harmonic* with respect to a Markov operator $P : L^{\infty}(X, m) \leftrightarrow$ if f = Pf. Denote by $H^{\infty}(X, m, P)$ the subspace of $L^{\infty}(X, m)$ consisting of harmonic functions.

Any function $f \in H^{\infty}(X, m, P)$ determines the harmonic sequence $f_n \equiv f$, so that $H^{\infty}(X, m, P)$ is isometrically embedded into the space $HS^{\infty}(X, m, P)$. Since the subspace of S-invariant functions in $L^{\infty}(E, [\varepsilon_m])$ is naturally isometric to the space $L^{\infty}(\Gamma, [\nu_m])$ of all bounded measurable functions on the Poisson boundary (which is the space of S-ergodic components in E), Theorem 1.5 implies

Theorem 1.8 ([Re84], [Ka92]). The spaces $H^{\infty}(X, m, P)$ and $L^{\infty}(\Gamma, [\nu_m])$ are isometric. The isometry is established by the formulas

(1.4)
$$\lim_{n \to \infty} f_n(x_n) = \widehat{f}(\mathbf{bnd}(x)) , \qquad f(x) = \langle \widehat{f}, \nu_x \rangle$$

where $f \in H^{\infty}(X, m, P)$ and $\widehat{f} \in L^{\infty}(\Gamma, [\nu_m])$.

Formula (1.4) and its time dependent counterpart (1.3) are called the *Poisson formu*las. See [Fu63], [Ka96] for a relationship with the classical Poisson formula for bounded harmonic functions on the unit disk (which is the origin of this term). Criteria of triviality and of coincidence of the tail and the Poisson boundaries for general Markov chains are given by the θ -2 laws [De76], [Ka92].

1.4. Non-homogeneous Markov chains.

The notions introduced above also apply to Markov chains which are *not* homogeneous in time. In this situation instead of a single Markov operator P we have a sequence

of Markov operators $P_n : L^{\infty}(X, m) \leftrightarrow$ on the same space (X, m). The operator P_n governs the transitions of the Markov chain at time n, so that the measure \mathbf{P}_{θ} in the path space $X^{\mathbb{Z}}$ - corresponding to an initial distribution θ on X satisfies the relations

$$\mathbf{E}_{\theta}\left(f(x_{n+1}) | x_n = x\right) = P_n f(x) \; .$$

The one-dimensional distribution of \mathbf{P}_{θ} at time n + 1 is $\theta P_{0,n}$, where

$$(1.5) P_{k,n} = P_k P_{k+1} \cdots P_n , 0 \le k \le n$$

are the "time k to time n + 1" transition operators. The standard way to "make" such chains homogeneous consists in extending the state space by passing to the "space-time" $\mathbb{Z}_+ \times X$ (or to $\mathbb{Z} \times X$ when dealing with negative times as well). Then one can talk about a single *space-time operator*

(1.6)
$$Pf(n,\cdot) = P_n f(n+1,\cdot)$$

on $\mathbb{Z}_+ \times X$ and about the corresponding time homogeneous Markov chain (which is called the *space-time chain*). The projection of the space-time chain onto \mathbb{Z}_+ is deterministic and consists in moving forward with unit speed.

The notions of the tail boundary and of harmonic sequences carry over to nonhomogeneous Markov chains without any changes, whereas the Poisson boundary and harmonic functions do not make much sense in this situation. For the space-time chain the tail boundary coincides with the Poisson boundary and is the product of the tail boundary of the original non-homogeneous chain by \mathbb{Z} , see [Ka92] for more details.

1.5. Random Markov operators.

Definition 1.9. A random Markov operator on a space (X, m) is determined by a measure type preserving transformation T of a probability space (Ω, λ) and a measurable map $\omega \mapsto P_{\omega}$ from Ω to the space of Markov operators on (X, m). We shall call (X, m) the state space and (Ω, λ) the base space of the random Markov operator $\{P_{\omega}\}$. Here by measurability of the map $\omega \mapsto P_{\omega}$ we mean that the integral $\langle P_{\omega} f, g \rangle_m$ is a measurable function of ω for any two functions $f \in L^{\infty}(X, m), g \in L^1(X, m)$.

For simplicity we shall always assume that the transformation T is ergodic and invertible. In most applications the measure λ on Ω is in addition assumed finite and T-invariant in order to guarantee the "stochastic homogeneity" of the sequence of operators $P_{T^n\omega}$.

For any $\omega \in \Omega$ we have a non-homogeneous Markov chain on X determined by the sequence of operators $P_{\omega}, P_{T\omega}, P_{T^2\omega}, \ldots$ Denote by $\mathbf{P}_{\omega,\theta}$ the measure in its path space $X^{\mathbb{Z}_+}$ corresponding to an initial distribution θ on X, and by E_{ω} its tail boundary. By $\varepsilon_{\omega,\theta} = \mathbf{tail}_{\omega} \mathbf{P}_{\omega,\theta}$ we denote the tail measure on E_{ω} corresponding to the initial

distribution θ on X (the subscript ω indicates that the map \mathbf{tail}_{ω} is defined on the path space of the chain determined by ω).

Simultaneously with the Markov operators $P_{\omega}: L^{\infty}(X, m) \leftrightarrow$ we shall also consider the "global" Markov operator

(1.7)
$$Pf(\omega, \cdot) = P_{\omega}f(T\omega, \cdot) .$$

acting on the space $L^{\infty}(\Omega \times X, \lambda \otimes m)$. The operator P is an immediate analogue of the space-time operator (1.6), the only difference being that the role of "time" here is played by the space Ω endowed with the transformation T. The transition probabilities of the operator P are

$$\pi_{\omega,x} = \delta_{T\omega} \otimes \pi_x(\omega)$$

where $\pi_x(\omega)$ are the transition probabilities of the operator P_{ω} . The sample paths of the operator P have the form

$$\overline{\boldsymbol{x}} = (\overline{x}_0, \overline{x}_1, \dots)$$
, $\overline{x}_n = (T^n \omega, x_n)$

where $\boldsymbol{x} = (x_n)$ is a sample path of the non-homogeneous Markov chain determined by the sequence of operators $(P_{\omega}, P_{T\omega}, \ldots)$. Therefore, the path space of the operator Pcan be identified with $\Omega \times X^{\mathbb{Z}+}$ by the map

(1.8)
$$\Pi: \overline{\boldsymbol{x}} \mapsto (\omega, \boldsymbol{x}) , \qquad \boldsymbol{x} = (x_0, x_1, \ldots) \in X^{\mathbb{Z}+}.$$

As usually, denote by $\mathbf{P}_{\overline{\theta}}$ the measure on the path space $(\Omega \times X)^{\mathbb{Z}}$ + of the operator P corresponding to an initial distribution $\overline{\theta}$ on $\Omega \times X$. Then

(1.9)
$$\Pi \mathbf{P}_{\overline{\theta}} = \int \delta_{\omega} \otimes \mathbf{P}_{\omega,\theta_{\omega}} d\theta(\omega) ,$$

where θ is the image of $\overline{\theta}$ under the projection from $\Omega \times X$ onto Ω , and θ_{ω} , $\omega \in \Omega$ are the conditional measures of this projection.

Denote by E the tail boundary of the operator P(1.7). Let

(1.10)
$$\Pi_{\Omega}: \overline{\boldsymbol{x}} \mapsto \omega , \qquad \overline{\boldsymbol{x}} = \left((\omega, x_0), (T\omega, x_1), \ldots \right)$$

be the composition of the map Π and the projection from $\Omega \times X^{\mathbb{Z}}$ + onto Ω . Since the transformation T is invertible, Π_{Ω} is measurable with respect to the tail partition of the path space $(\Omega \times X)^{\mathbb{Z}}$ +. Therefore, Π_{Ω} determines a natural projection $\mathfrak{p}_{\Omega} : E \to \Omega$. Formula (1.9) then implies

Proposition 1.10. The fibers of the projection \mathfrak{p}_{Ω} are the tail boundaries E_{ω} of the non-homogeneous Markov chains on X associated with the points $\omega \in \Omega$. More precisely, for an arbitrary initial distribution $\overline{\theta} \prec \lambda \otimes m$ on $\Omega \times X$ denote by $\theta_{\omega}, \omega \in \Omega$ its conditional measures on X. Then the conditional measures of the tail measure $\varepsilon_{\overline{\mu}}$ on E

with respect to the projection \mathfrak{p}_{Ω} coincide with the tail measures $\varepsilon_{\theta_{\omega}}$ on the tail boundaries $E_{\omega}, \omega \in \Omega$.

Denote by Γ the Poisson boundary of the operator P. By definition, there is a projection $\mathfrak{p}_{\Gamma} : E \to \Gamma$. Let σ_{Ω} and σ_{Γ} be the preimage partitions of the tail boundary E determined by the projections \mathfrak{p}_{Ω} and \mathfrak{p}_{Γ} , respectively.



Definition 1.11. A random Markov operator $\{P_{\omega}\}$ has a stable Poisson boundary if the common refinement $\sigma_{\Omega} \vee \sigma_{\Gamma}$ of the partitions σ_{Ω} and σ_{Γ} coincides with the point partition σ_E of the tail boundary E, i.e., if the projections $\mathfrak{p}_{\Omega} : E \to \Omega$ and $\mathfrak{p}_{\Gamma} : E \to \Gamma$ separate points of E.

If $\{P_{\omega}\}$ has a stable Poisson boundary Γ , then the tail boundaries E and E_{ω} , $\omega \in \Omega$ can be identified with the product $\Omega \times \Gamma$ and with Γ , respectively. Therefore, in this situation the same space Γ is responsible (via the Poisson formula) for an integral representation both of P-harmonic functions on $\Omega \times X$ and of $(P_{\omega}, P_{T\omega}, \ldots)$ -harmonic sequences on X for a.e. $\omega \in \Omega$. In other words, the boundary behaviour of the operators P_{ω} and the operator P (in the latter case modulo dependence on the initial state) is described by the same space Γ . If the Poisson boundary Γ of the operator P is trivial, then stability of $\{P_{\omega}\}$ means that the tail boundary E of P coincides with Ω . The general case can be reduced to this situation by conditioning the operator P by points of Γ .

One can easily give a simple (if somewhat degenerate) example of a random Markov operator whose Poisson boundary is unstable in the sense of Definition 1.11. Basically, it consists just in taking an ergodic skew product over an ergodic invertible transformation. Indeed, consider on $\Omega \times X$ the skew product transformation $\widetilde{T}(\omega, x) = (T\omega, \varphi(\omega, x))$ with the base T, where $\varphi : \Omega \times X \to X$ is a measurable map such that $\varphi(\omega, \cdot) : X \to X$ is invertible, and let $P_{\omega}f(x) = f(\varphi(\omega, x))$ be the associated family of deterministic Markov operators. Then the operator P is also deterministic and corresponds to the transformation \widetilde{T} . Therefore, the tail boundary of P is $E = \Omega \times X$, whereas the tail boundaries of the operators P_{ω} are $E_{\omega} = X$. If \widetilde{T} is ergodic (e.g., see [CFS82] for examples), then the Poisson boundary Γ of P is trivial, which gives an example we are looking for.

It would be interesting to find general sufficient conditions for stability of the Poisson boundary of random Markov operators. In Section 4 we shall prove it for group invariant random Markov operators on certain classes of groups.

2. RANDOM WALKS WITH TIME DEPENDENT INCREMENTS

In this Section we consider random walks on groups which are *space homogeneous* without being *time homogeneous*.

2.1. Definitions and notations.

Let G be a countable group, and $m = m_G$ be the counting measure on G. Denote by $\mathcal{P}G$ the space of probability measures on G. Any measure $\mu \in \mathcal{P}G$ determines the Markov operator

$$P_{\mu}f(g) = \sum_{h} \mu(h)f(gh)$$

commuting with the action of the group G on itself by left translations. The operator P_{μ} acts on measures on G by (right) convolution with μ :

$$\theta P_{\mu} = \theta \mu$$
 .

The associated Markov chain on G with the transition probabilities

$$\mathsf{Prob}(x_{n+1} = gh|x_n = g) = \mu(h)$$

which is homogeneous both in time and in space is called the (right) random walk (RW) on G determined by the measure μ and is denoted RW(μ) (e.g., see [KV83]).

Definition 2.1. For any sequence $\mu = {\mu_0, \mu_1, \ldots} \in \mathcal{P}G^{\mathbb{Z}_+}$ the associated sequence of Markov operators $P_n = P_{\mu_n}$ determines the Markov chain on G whose transition probabilities at time n are

$$\mathsf{Prob}(x_{n+1} = gh | x_n = g) = \mu_n(h)$$

This chain is called a random walk with time dependent increments (RWTDI) on G, and we denote it RWTDI(μ).

Random walks with time dependent increments are homogeneous in space, but not in time unless the sequence μ is constant, in which case we have a usual random walk on the group G homogeneous both in time and space. All objects connected with RWTDI(μ) obviously depend on the sequence μ . However, for the sake of keeping the notations concise, we shall usually omit the argument μ .

The "time k to time n + 1" transition operators (1.5) of RWTDI(μ) are

$$P_{k,n}f(g) = P_{\mu_k} \cdots P_{\mu_n}f(g) = \sum_h \mu_{k,n}(h)f(gh) = P_{\mu_{k,n}}f(g)$$

where

$$\mu_{k,n} = \mu_k \mu_{k+1} \dots \mu_n$$

is the convolution of the measures $\mu_k, \mu_{k+1}, \ldots, \mu_n$.

Denote by $\mathbf{P}_{n,\theta}$ the measure in the path space $G^{[n,\infty)}$ corresponding to starting RWTDI($\boldsymbol{\mu}$) at time $n \in \mathbb{Z}_+$ with an initial distribution θ . If $\theta = \delta_g$, $g \in G$, then we use the notation $\mathbf{P}_{n,g}$. We shall omit the subscript n if n = 0 and the subscript g if g = e. In particular, we denote by \mathbf{P} the probability measure on $G^{\mathbb{Z}_+}$ corresponding to starting RWTDI($\boldsymbol{\mu}$) from the group identity at time 0. Since G acts on the path space $G^{\mathbb{Z}_+}$ coordinate-wise as $(g\boldsymbol{x})_n = g\boldsymbol{x}_n$, and the transition probabilities of RWTDI($\boldsymbol{\mu}$) are G-invariant, $\mathbf{P}_{n,g} = g\mathbf{P}_n$, $g \in G$, and $\mathbf{P}_{n,\theta} = \theta\mathbf{P}_n$ for any initial distribution θ .

2.2. The tail boundary and conditional chains.

Denote by E the tail boundary of RWTDI(μ), and by $\varepsilon_{n,\theta} = \operatorname{tail}(\mathbf{P}_{n,\theta})$ (resp., $\varepsilon_{n,g}$, etc.) the tail measures on E. There are two types of the tail behaviour of RWTDI(μ): one (rather obvious) is connected with the dependence on the starting point and can be dealt with by passing to a smaller group (see Theorem 2.10 below); the other one is more interesting and reflects the "true" tail behaviour which can not be reduced to a dependence on the initial state. Because of the space homogeneity of RWTDI, for the study of the latter we may always assume that the starting point is the group identity. Denote by $\varepsilon = \operatorname{tail}(\mathbf{P})$ the associated tail measure. Below by the *tail boundary* of RWTDI(μ) we shall always mean the space (E, ε) (sometimes we shall also call it the *local tail boundary*), whereas the space E endowed with measure type [ε_m] will be referred to as the *total tail boundary*.

The action of G on the path space $G^{\mathbb{Z}_+}$ commutes with the time shift, so that this action descends to an action on E which preserves the tail measure type $[\varepsilon_m]$, and $\varepsilon_{n,g\theta} = g\varepsilon_{n,\theta}$ for any $n \in \mathbb{Z}_+, g \in G$ and any measure θ on G. In particular, $\varepsilon_{n,\theta} = \theta\varepsilon_n$. The stationarity relations (1.2) then imply that

(2.1)
$$\varepsilon_n = \sum \mu_n(g) g \varepsilon_{n+1} = \mu_n \varepsilon_{n+1} \, .$$

Proposition 2.2. The family of measures $\mathbf{P}^{\gamma}, \gamma \in E$ on $G^{\mathbb{Z}_+}$ defined on cylinder sets

$$C_{e,g_1,\ldots,g_n} = \{ \boldsymbol{x} \in G^{\mathbb{Z}_+} : x_0 = e, x_1 = g_1,\ldots,x_n = g_n \}$$

as

$$\mathbf{P}^{\gamma}(C_{e,g_1,\ldots,g_n}) = \mathbf{P}(C_{e,g_1,\ldots,g_n}) \frac{dg_n \varepsilon_n}{d\varepsilon}(\gamma)$$

is the canonical system of conditional measures of the measure \mathbf{P} with respect to the tail boundary.

Proof. If A is a measurable subset of the tail boundary E with $\varepsilon(A) = \mathbf{P}(\mathbf{tail}^{-1}A) > 0$, then by the Markov property

$$\mathbf{P}(C_{e,g_1,\ldots,g_n} \cap \mathbf{tail}^{-1}A) = \mathbf{P}(C_{e,g_1,\ldots,g_n})\mathbf{P}_{n,g_n}(\mathbf{tail}^{-1}A) = \mathbf{P}(C_{e,g_1,\ldots,g_n})g_n\varepsilon_n(A)$$

for any cylinder set C_{e,g_1,\ldots,g_n} , whence for the conditional measure $\mathbf{P}^A(\cdot) = \mathbf{P}(\cdot|\mathbf{tail}^{-1}A)$ we have

$$\mathbf{P}^{A}(C_{e,g_{1},\ldots,g_{n}}) = \frac{\mathbf{P}(C_{e,g_{1},\ldots,g_{n}})g_{n}\varepsilon_{n}(A)}{\mathbf{P}(\mathbf{tail}^{-1}A)} = \mathbf{P}(C_{e,g_{1},\ldots,g_{n}})\frac{g_{n}\varepsilon_{n}(A)}{\varepsilon(A)}$$

Therefore,

$$\mathbf{P}^{A}(C_{e,g_{1},\ldots,g_{n}}) = \frac{1}{\varepsilon(A)} \int_{A} \mathbf{P}^{\gamma}(C_{e,g_{1},\ldots,g_{n}}) d\varepsilon(\gamma)$$

for any measurable subset $A \subset E$, which implies the claim.

Remark 2.3. The definition of conditional measures \mathbf{P}^{γ} , $\gamma \in E$ can be rewritten as

$$\mathbf{P}^{\gamma}(C_{e,g_{1},\dots,g_{n}}) = \mathbf{P}(C_{e,g_{1},\dots,g_{n}})\frac{dg_{n}\varepsilon_{n}}{d\varepsilon}(\gamma)$$
$$= \mathbf{P}(C_{e,g_{1},\dots,g_{n}})\frac{dg_{1}\varepsilon_{1}}{d\varepsilon}(\gamma)\frac{dg_{2}\varepsilon_{2}}{dg_{1}\varepsilon_{1}}(\gamma)\cdots\frac{dg_{n}\varepsilon_{n}}{dg_{n-1}\varepsilon_{n-1}}(\gamma)$$

Thus, the measures \mathbf{P}^{γ} are the measures in the path space of *conditional Markov chains* on G with the transition probabilities

(2.2)
$$\mathsf{Prob}^{\gamma}(x_{n+1} = g_{n+1} | x_n = g_n) = \mathsf{Prob}(x_{n+1} = g_{n+1} | x_n = g_n) \frac{dg_{n+1}\varepsilon_{n+1}}{dg_n \varepsilon_n}(\gamma)$$
$$= \mu_n (g_n^{-1}g_{n+1}) \frac{dg_{n+1}\varepsilon_{n+1}}{dg_n \varepsilon_n}(\gamma) .$$

These conditional chains are, generally speaking, inhomogeneous both in space and in time. Note that (2.2) makes sense only when $g_{n+1}\varepsilon_{n+1} \prec g_n\varepsilon_n$. However, as it follows from (2.1), this relation is satisfied whenever $\mu_{0,n-1}(g_n)$ and $\mu_n(g_n^{-1}g_{n+1})$ are both non-zero, i.e., the conditional chains are well-defined on the whole *attainability space-time* cone in $\mathbb{Z}_+ \times G$ with the origin (0, e)

(2.3)
$$C = C(0, e) = \{ (n+1, x) : n \in \mathbb{Z}_+, \ \mu_{0,n}(x) > 0 \} .$$

Given a measurable partition ξ of the total tail boundary $(E, [\varepsilon_m])$ denote by E^{ξ} the associated quotient space, and by $\varepsilon_{n,\theta}^{\xi}$, etc. the images of the corresponding tail measures under the projection $\gamma \mapsto \xi(\gamma)$ from E to E^{ξ} . A partition ξ is called *Ginvariant* if the action of G on E maps elements of ξ onto elements of ξ (although individual elements of ξ do not have to be fixed by the action). If ξ is a *G*-invariant partition, then the action of G descends from E to the corresponding quotient space E^{ξ} . Reproducing the proof of Proposition 2.2 we get

Proposition 2.4. Let ξ be a *G*-invariant measurable partition of the tail boundary *E*. Then the family of measures $\mathbf{P}^{\xi(\gamma)}, \gamma \in E$ on $G^{\mathbb{Z}_+}$ defined on cylinder sets as

$$\mathbf{P}^{\xi(\gamma)}(C_{e,g_1,\ldots,g_n}) = \mathbf{P}(C_{e,g_1,\ldots,g_n}) \frac{dg_n \varepsilon_n^{\xi}}{d\varepsilon^{\xi}}(\xi(\gamma))$$

is the canonical system of conditional measures of the measure \mathbf{P} with respect to the quotient E^{ξ} of the tail boundary by the partition ξ .

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2.3. Triviality of the tail boundary.

Definition 2.5. If the tail boundary (E, ε) of RWTDI (μ) is a singleton, we shall say that it is *trivial*. In this Section we shall also use the term *local triviality* in order to distinguish it from the *total triviality* of the tail boundary when the total tail boundary $(E, [\varepsilon_m])$ is a singleton.

Theorem 2.6 ([De76],[Ka92]). The tail boundary of RWTDI(μ) is totally trivial iff

$$\|g\mu_{m,n} - \mu_{0,n}\| \underset{n \to \infty}{\longrightarrow} 0 \qquad \forall g \in G, \ m \in \mathbb{Z}_+$$

and it is locally trivial iff

$$\|g\mu_{m,n} - \mu_{0,n}\| \underset{n \to \infty}{\longrightarrow} 0 \qquad \forall g \in \mathrm{supp}\mu_{0,m-1}, \ m \in \mathbb{Z}_+ ,$$

where $\|\theta\|$ denotes the total variation of a measure θ .

Corollary. If the tail boundary of RWTDI(μ) is totally trivial, then for any $m \in \mathbb{Z}_+$ the sequence of measures $\mu_{m,n}$ strongly converges (as n tends to infinity) to a left-invariant mean on G, and therefore the group G must be amenable.

Remark 2.7. Theorem 2.6 implies that the total triviality of the tail boundary of RWTDI(μ) is equivalent to the total triviality of the tail boundary of RWTDI($S\mu$), where $S\mu = (\mu_1, \mu_2, ...)$ is the shift of $\mu = (\mu_0, \mu_1, ...)$. It also implies that local triviality of the tail boundary of RWTDI($S\mu$) follows from local triviality of the tail boundary of RWTDI(μ). However, the converse is not true in general. For the simplest example take for μ_0 any measure whose support consists of more than one point, and put $\mu_1 = \mu_2 = \cdots = \delta_e$.

Obviously, total triviality implies local triviality. We shall show that the converse is also true under natural irreducibility conditions. For any given $g \in G$ the sequence $\|g\mu_{0,n} - \mu_{0,n}\|$ is clearly non-decreasing. Denote by $\Delta(g) = \Delta(g, \mu)$ its limit.

Proposition 2.8. If the tail boundary of RWTDI(μ) is locally trivial, then $\Delta(g)$ equals either 0 or 2 for any $g \in G$.

Proof. Let

$$A = \{ \boldsymbol{x} \in G^{\mathbb{Z}_+} : (n, x_n) \in C \text{ for a certain } n \in \mathbb{Z}_+ \}$$

where C = C(0, e) is the attainability cone (2.3) in $\mathbb{Z}_+ \times G$. Clearly, for \mathbf{P}_m -a.e. path \boldsymbol{x} if $(k, x_k) \in C$ then also $(n, x_n) \in C$ for all n > k, so that the set A is measurable with respect to the tail partition. Since the tail boundary is locally trivial, $\mathbf{P}_g(A)$ equals 0 or 1 for any $g \in G$.

Suppose that $\Delta(g) < 2$, i.e., for a certain $n \ge 0$ the measures $g\mu_{0,n}$ and $\mu_{0,n}$ are non-singular. Then $\mathbf{P}_g(A) > 0$, and by the above $\mathbf{P}_g(A) = 1$, so that \mathbf{P}_g -a.e. path eventually hits the cone C. Denote by

$$\tau(\boldsymbol{x}) = \min\{n \in \mathbb{Z}_+ : (n, x_n) \in C\}$$

the first hitting time, and by θ the corresponding hitting distribution on C:

$$\theta(n, x) = \mathbf{P}_{q} \{ \boldsymbol{x} \in G^{\mathbb{Z}_{+}} : \tau(\boldsymbol{x}) = n, x_{n} = x \}$$

By the Markov property for any $n \in \mathbb{Z}_+$ the measure $g\mu_{0,n}$ decomposes as

$$g\mu_{0,n} = \sum_{(k,x)\in C:k\leq n} heta(k,x)x\mu_{k,n} + lpha_n$$

where $\|\alpha_n\| \to 0$. On the other hand, by the local triviality of the tail boundary

$$||x\mu_{k,n} - \mu_{0,n}|| \to 0$$

for any $(k, x) \in C$, and we are done.

Denote by $G(\boldsymbol{\mu})$ the subgroup of G generated by all $g \in G$ such that the measures $\mu_{0,n}$ and $g\mu_{0,n}$ are non-singular for a certain n. Proposition 2.8 implies

Proposition 2.9. If the tail boundary of $RWTDI(\mu)$ is locally trivial, then

$$G(\boldsymbol{\mu}) = \{g \in G : \Delta(g) = 0\} .$$

Corollary. If the tail boundary of $\text{RWTDI}(\mu)$ is locally trivial, then the group $G(\mu)$ is amenable.

Theorem 2.10. If the tail boundary of RWTDI(μ) is locally trivial, then its total tail boundary is isomorphic to the coset space $G/G(\mu)$, and the boundary map tail has the form

$$\mathbf{tail}(oldsymbol{x}) = oldsymbol{x}_0 G(oldsymbol{\mu})$$
 .

Proof. As it follows from Proposition 2.9, sample paths of RWTDI(μ) issued from different cosets of $G(\mu)$ never intersect, so that the map $\boldsymbol{x} \mapsto x_0 G(\mu)$ is indeed measurable with respect to the tail partition. On the other hand, again by Proposition 2.9, the tail boundary is trivial with respect to any initial distribution concentrated on a single coset.

We shall say that RWTDI(μ) is *irreducible* if $G(\mu) = G$. In particular, if all points of G are attainable with positive probability from the group identity (\equiv from an arbitrary starting point), i.e., if $\bigcup_{n \in \mathbb{Z}_+} \operatorname{supp} \mu_{0,n} = G$, then RWTDI(μ) is irreducible. Thus, Theorem 2.10 implies

Proposition 2.11. The tail boundary of an irreducible RWTDI with locally trivial tail boundary is totally trivial.

Remark 2.12. An immediate generalization of irreducible RWTDI is provided by *periodic* RWTDI. We shall say that RWTDI(μ) has period $d \geq 1$ if there exists a homomorphism $\varphi : G \to \mathbb{Z}_d$ such that all measures μ_n are concentrated on $\varphi^{-1}(1)$, and the RWTDI on $G_0 = \ker \varphi$ determined by the sequence of measures

$$\mu'_n = \mu_{nd}\mu_{nd+1}\cdots\mu_{(n+1)d-1}$$

is irreducible. In this case $G(\boldsymbol{\mu}) = G_0$, and the total tail boundary is isomorphic to \mathbb{Z}_d .

2.4. The entropy.

Below we shall need several facts from the entropy theory of measurable partitions of Lebesgue spaces [Ro67]. First recall that the entropy of a discrete probability distribution $p = (p_1, p_2, ...)$ is defined as $H(p) = -\sum_i p_i \log p_i$. The entropy $H(\xi) = H_m(\xi)$ of a countable partition $\xi = \{X_i\}$ of a Lebesgue probability space (X, m) is defined as the entropy of the probability distribution $p_i = m(X_i)$. In other words,

$$H(\xi) = -\int \log m(\xi(x)) dm(x) ,$$

where $\xi(x)$ is the element of the partition ξ containing x. Given another measurable (not necessarily countable!) partition ζ of X, the *conditional entropy* of ξ with respect to ζ is defined as

$$H(\xi|\zeta) = \int H_{\zeta(x)}(\xi) \, dm(x) = -\int \log m^{\zeta(x)}(\xi(x)) \, dm(x) \, .$$

where $x \mapsto \zeta(x) \subset X$ is the projection from the space (X, m) onto its quotient by the partition ζ (we identify the points of the quotient space with the corresponding elements of the partition ζ), the measures $m^{\zeta(x)}$ are the conditional measures of this projection, and $H_{\zeta(x)}(\xi)$ is the entropy of ξ with respect to the measure $m^{\zeta(x)}$.

Proposition 2.13 ([Ro67]). Let ξ , ζ be measurable partitions of a Lebesgue space (X, m). If ξ is countable with $H(\xi) < \infty$, then

- (i) $0 \le H(\xi|\zeta) \le H(\xi)$, and $H(\xi|\zeta) = 0$ (resp., $H(\xi|\zeta) = H(\xi)$) iff ζ is a refinement of ξ (resp., ζ and ξ are independent).
- (ii) If ζ' is a refinement of ζ , then $H(\xi|\zeta') \leq H(\xi|\zeta)$, and the equality holds iff $m^{\zeta'(x)}(\xi(x)) = m^{\zeta(x)}(\xi(x))$ for m-a.e. $x \in X$.
- (iii) If ζ is the limit of a monotonously decreasing sequence of measurable partitions ζ_n , then $H(\xi|\zeta_n) \nearrow H(\xi|\zeta)$.

Theorem 2.14. If a sequence $\mu = (\mu_0, \mu_1, ...) \in \mathcal{P}G^{\mathbb{Z}_+}$ is such that $H(\mu_n) < \infty$ for all measures μ_n , then for any $k \in \mathbb{Z}_+$ there exists a limit

$$h_k = h_k(\boldsymbol{\mu}) = \lim_{n \to \infty} \left[H(\mu_{0,n}) - H(\mu_{k,n}) \right] \ge 0$$
,

and the tail boundary of RWTDI(μ) is (locally) trivial iff $h_k(\mu) = 0$ for all k.

Proof. For the coordinate partitions of the path space $(\mathbf{P}, G^{\mathbb{Z}_+})$ the Markov property yields

(2.4)
$$H(\alpha_{0,k}) = \sum_{i=0}^{k-1} H(\mu_i) ,$$

and for k < n

$$H(\alpha_{0,k}|\alpha_{n,\infty}) = \sum_{i=0}^{k-1} H(\mu_i) + H(\mu_{k,n-1}) - H(\mu_{0,n-1})$$

= $H(\alpha_{0,k}) + H(\mu_{k,n-1}) - H(\mu_{0,n-1})$.

The partitions $\alpha_{n,\infty}$ are decreasing to the tail partition α_{∞} . Therefore by Proposition 2.13 (i), (iii) the limits h_k exist, and

(2.5)
$$H(\alpha_{0,k}|\alpha_{\infty}) = H(\alpha_{0,k}) - h_k \leq H(\alpha_{0,k})$$

Moreover, $h_k = 0$ iff

$$H(\alpha_{0,k}) = H(\alpha_{0,k}|\alpha_{\infty}) ,$$

i.e., iff the partitions $\alpha_{0,k}$ and α_{∞} are independent, see Proposition 2.13 (i). This is obviously the case if α_{∞} is trivial, i.e., if the tail boundary is locally trivial. Conversely, if all h_k equal 0, then α_{∞} is independent of all coordinate partitions $\alpha_{0,k}$, and therefore of the point partition of the path space, so that α_{∞} must be trivial.

3. RANDOM WALKS WITH RANDOM TRANSITIONS PROBABILITIES

Random walks with random transitions probabilities are a specialization of the notion of random Markov operators discussed in Section 1.5.

3.1. Definitions and preliminaries.

Definition 3.1. Let (Ω, λ) be a probability Lebesgue measure space endowed with an invertible ergodic measure preserving transformation T, and

$$\mu: \Omega \to \mathcal{P}G , \qquad \omega \mapsto \mu^{\omega}$$

be a measurable map. Put

$$\boldsymbol{\mu}^{\omega} = \left(\mu^{\omega}, \mu^{T\omega}, \mu^{T^{2}\omega}, \dots\right) \in \mathcal{P}G^{\mathbb{Z}+}.$$

The family of RWTDI(μ^{ω}) parameterized by the points $\omega \in \Omega$ is called a *random* walk on *G* with random transition probabilities (RWRTP), for which we shall use the notation RWRTP(Ω, λ, T, μ). Below we shall usually write just RWTDI(ω) instead of RWTDI(μ^{ω}).

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Simultaneously with the chains $\operatorname{RWTDI}(\omega)$ on G parameterized by the points $\omega \in \Omega$ we shall also consider the "global chain" on the space $\Omega \times G$ determined by the Markov operator

(3.1)
$$P: L^{\infty}(\Omega \times G, \lambda \otimes m) \longleftrightarrow, \qquad Pf(\omega, g) = \sum \mu^{\omega}(h)f(T\omega, gh) ,$$

with the transition probabilities $\pi_{\omega,g} = \delta_{T\omega} \otimes g\mu^{\omega}$ (cf. (1.7)). This chain is homogeneous both in time and space (with respect to the dissipative action of G on $\Omega \times G$ by left translations), and the σ -finite measure $\lambda \otimes m$ is easily seen to be P-stationary, so that the operator P is a covering Markov operator in the sense of [Ka95]. The corresponding quotient chain on the space Ω is deterministic with the transitions $\omega \to T\omega$. The projection of the "global chain" starting from a fixed point $\omega \in \Omega$ onto G is the RWTDI determined by the sequence μ^{ω} (cf. Section 1.5).

Let $\mathbf{P}_{\overline{\lambda}}$ be the measure in the path space $(\Omega \times G)^{\mathbb{Z}_+}$ of the operator P determined by the initial distribution $\overline{\lambda} = \lambda \otimes \delta_e$ on $\Omega \times G$. Then by formula (1.9) the map $\Pi : \overline{\boldsymbol{x}} \mapsto (\omega, \boldsymbol{x})$ (1.8) is an isomorphism between the spaces $((\Omega \times G)^{\mathbb{Z}_+}, \mathbf{P}_{\overline{\lambda}})$ and $(\Omega \times G^{\mathbb{Z}_+}, \overline{\mathbf{P}})$, where

(3.2)
$$\overline{\mathbf{P}} = \int \delta_{\omega} \otimes \mathbf{P}_{\omega} \, d\lambda(\omega)$$

and \mathbf{P}_{ω} denotes the probability measure on the path space $G^{\mathbb{Z}}$ + corresponding to starting RWTDI(ω) from the identity of the group.

One can also identify the space of paths $x \in G^{\mathbb{Z}_+}$ starting from $x_0 = e$ with the space $\mathcal{H} \cong G^{\mathbb{Z}_+}$ of increments $h = (h_0, h_1, \ldots)$ by the map

$$\boldsymbol{x} \mapsto \boldsymbol{h}$$
, $x_n = h_0 h_1 \cdots h_{n-1}$, $n > 0$

[Although the path space and the space of increments in G both coincide with $G^{\mathbb{Z}+}$, their meaning for us is quite different, which is why we use a separate notation for the space of increments.] Therefore, the space $(\Omega \times G^{\mathbb{Z}+}, \overline{\mathbf{P}}) \cong ((\Omega \times G)^{\mathbb{Z}+}, \mathbf{P}_{\overline{\lambda}})$ is isomorphic to the space $(\Omega \times \mathcal{H}, \mathbf{Q})$, where

$$\mathbf{Q} = \int \delta_{\omega} \otimes \bigotimes_{i=0}^{\infty} \mu^{T^{i}\omega} d\lambda(\omega)$$

Below we shall freely switch between the three descriptions of the same measure space

$$(\Omega \times G^{\mathbb{Z}_+}, \overline{\mathbf{P}}) \cong ((\Omega \times G)^{\mathbb{Z}_+}, \mathbf{P}_{\overline{\lambda}}) \cong (\Omega \times \mathcal{H}, \mathbf{Q})$$

using the correspondence

(3.3)

$$(\omega, \boldsymbol{x}) = (\omega, (e, x_1, x_2, \ldots))$$

$$\longleftrightarrow \quad \overline{\boldsymbol{x}} = (\overline{\boldsymbol{x}}_n) = ((\omega, e), (T\omega, x_1), (T^2\omega, x_2), \ldots)$$

$$\longleftrightarrow \quad (\omega, \boldsymbol{h}) = (\omega, (h_n)) = (\omega, (x_1, x_1^{-1}x_2, x_2^{-1}x_3, \ldots)).$$

For uniformity we shall usually refer to this measure space as $(\Omega \times G^{\mathbb{Z}_+}, \overline{\mathbf{P}})$ "changing variables" by formulas (3.3) if necessary.

Denote by E (resp., Γ) the tail (resp., the Poisson) boundary of the operator P (3.1). Below we shall only be interested in the boundary behaviour of the operator P and of RWTDI(ω) which is not reducible to a dependence on the starting point (cf. Section 2.3). Therefore, the tail boundary E will be always endowed with the tail measure $\varepsilon = \text{tail}\overline{\mathbf{P}}$. By $\nu = \text{bnd}\overline{\mathbf{P}} = \mathfrak{p}_{\Gamma}\varepsilon$ (where \mathfrak{p}_{Γ} is the projection $E \to \Gamma$) we denote the corresponding harmonic measure on the Poisson boundary.

The fibers of the projection $\mathfrak{p}_{\Omega} : (E, \varepsilon) \to (\Omega, \lambda)$ are the tail boundaries $(E_{\omega}, \varepsilon_{\omega})$ of RWTDI (ω) , where ε_{ω} is the tail measure on E_{ω} corresponding to starting RWTDI (ω) from the group identity at time 0 (Proposition 1.10). More generally, by $\varepsilon_{n,\omega}$ we denote the tail measure on E_{ω} corresponding to starting RWTDI (ω) from the group identity at an arbitrary time $n \geq 0$. In view of Proposition 1.10 we may also consider the measures $\varepsilon_{\omega}, \varepsilon_{n,\omega}$ as measures on E. By $\nu_{\omega} = \mathfrak{p}_{\Gamma}\varepsilon_{\omega} = \mathfrak{p}_{\Gamma}\varepsilon_{n,\omega}$ we denote the corresponding harmonic measures on the Poisson boundary Γ of the operator P. In other words, $\varepsilon_{\omega} = \mathbf{tail}(\mathbf{P}_{\omega})$ and $\nu_{\omega} = \mathbf{bnd}(\mathbf{P}_{\omega})$.

The action of G descends from the path space to the tail boundary and to the Poisson boundary, and clearly $g\varepsilon_o = \operatorname{tail}(g\mathbf{P}_{\omega})$ and $g\nu_{\omega} = \operatorname{bnd}(g\mathbf{P}_{\omega})$ are the measures on E(resp., on Γ) corresponding to starting RWTDI(ω) from the point $g \in G$ at time 0. Then formula (2.1) takes in this setup the form

$$\varepsilon_{n,\omega} = \mu^{\omega} \varepsilon_{n+1,T\omega}$$
.

Therefore,

$$u_\omega = \mu^\omega \nu_{T\omega}$$
 .

The Poisson formulas for bounded harmonic sequences and functions of the operator P (see Theorems 1.6 and 1.9) take the form, respectively,

$$f_n(\omega,g) = \langle \widehat{f}, g\varepsilon_{n,\omega} \rangle$$

and

$$f(\omega,g) = \langle \widehat{f}, g\nu_{\omega} \rangle$$

Proposition 3.2. The tail boundaries of the RWTDI(ω), $\omega \in \Omega$ are all trivial or non-trivial simultaneously.

Proof. As it follows from Theorem 2.6 (see Remark 2.7), triviality of the tail boundary of RWTDI(ω) implies triviality of the tail boundary of RWTDI($T\omega$). Therefore, the claim follows from ergodicity of the transformation T.

We shall say that $\text{RWRTP}(\Omega, \lambda, T, \mu)$ is *irreducible* if $\text{RWTDI}(\omega)$ is irreducible for λ -a.e. $\omega \in \Omega$ (see Definition 2.9). Proposition 1.10 then implies

Proposition 3.3. If RWRTP $(\Omega, \lambda, T, \mu)$ is irreducible and the tail boundary of λ -a.e. RWTDI (ω) , $\omega \in \Omega$ is locally trivial, then the Poisson boundary Γ of the operator P is trivial.

Remark 3.4. Proposition 3.3 remains true if RWRTP $(\Omega, \lambda, T, \mu)$ is *periodic*, i.e., if a.e. RWTDI (ω) is periodic with the same period d.

Problem 3.5. Give general conditions which would ensure stability (in the sense of Definition 1.11) of RWRTP $(\Omega, \lambda, T, \mu)$.

3.2. The asymptotic entropy.

Definition 3.6. We shall say that RWRTP $(\Omega, \lambda, T, \mu)$ has finite entropy if

$$\overline{H} = \overline{H}(\Omega, \lambda, T, \mu) = \int H(\mu^{\omega}) \, d\lambda(\omega) < \infty$$

Theorem 3.7. If RWRTP $(\Omega, \lambda, T, \mu)$ has finite entropy, then the limit

(3.4)
$$\overline{h} = \overline{h}(\Omega, \lambda, T, \mu) = \lim_{n \to \infty} \frac{H(\mu_{0, n-1}^{\omega})}{n}$$

exists a.e. and in the space $L^1(\Omega, \lambda)$ and is independent of ω .

Definition 3.8. The limit (3.4) is called the *asymptotic entropy* of RWRTP(Ω, λ, T, μ). *Proof of Theorem 3.7.* The measure $\mu_{0,k+n-1}^{\omega}$ is the convolution of the measures $\mu_{0,k-1}^{\omega}$ and $\mu_{k,k+n-1}^{\omega} = \mu_{0,n-1}^{T^{k}\omega}$. Therefore,

$$H(\mu_{0,k+n-1}^{\omega}) \le H(\mu_{0,k-1}^{\omega}) + H(\mu_{0,n-1}^{T^{\kappa}\omega})$$

so that the sequence of functions $\varphi_n(\omega) = H(\mu_{0,n-1}^{\omega})$ on Ω satisfies conditions of the Kingman subadditive ergodic theorem (e.g., see [De80]), which implies the claim. \Box

Theorem 3.9. If $\overline{H}(\Omega, \lambda, T, \mu) < \infty$, then $\overline{h}(\Omega, \lambda, T, \mu) = 0$ iff the tail boundary of λ -a.e. RWTDI(ω) is trivial.

Corollary. If $\overline{h}(\Omega, \lambda, T, \mu) = 0$, then the Poisson boundary of RWRTP $(\Omega, \lambda, T, \mu)$ is trivial.

Remark 3.10. Contrary to the situation with the "ordinary" random walks we do not know whether triviality of the Poisson boundary of RWRTP(Ω, λ, T, μ) implies that $\overline{h}(\Omega, \lambda, T, \mu) = 0$. The reason for this difference is that for ordinary random walks the tail and the Poisson boundary coincide with respect to any single point initial distribution (e.g., see [Ka92]), so that the entropy criterion of triviality of the tail boundary automatically becomes the entropy criterion of triviality of the Poisson boundary. However, for RWRTP the relation between the tail and the Poisson boundaries is more complicated, see Definition 1.11 (together with the discussion there) and Problem 3.5. Of course, if the Poisson boundary is stable in the sense of Definition 1.11, then it is trivial iff the tail boundary Γ coincides with Ω , i.e., iff $\overline{h}(\Omega, \lambda, T, \mu) = 0$.

Proof of Theorem 3.9. By Theorem 2.14 for λ -a.e. $\omega \in \Omega$ and any k > 0 there exists the limit

(3.5)
$$h(\omega) = \lim_{n \to \infty} \left[H(\mu_{0,n}^{\omega}) - H(\mu_{1,n}^{\omega}) \right] = \lim_{n \to \infty} \left[H(\mu_{0,n}^{\omega}) - H(\mu_{0,n-1}^{T\omega}) \right],$$

and the tail boundary of λ -a.e. RWTDI(ω) is locally trivial iff $h(\omega) = 0$ for λ -a.e. $\omega \in \Omega$. Since $h(\omega) \leq H(\mu^{\omega})$, the function h is integrable, and the convergence in (3.5) also holds in the space $L^1(\Omega, \omega)$, whence

$$\int h(\omega) \, d\lambda(\omega) = \lim_{n \to \infty} \left[\int H(\mu_{0,n}^{\omega}) \, d\lambda(\omega) - \int H(\mu_{0,n-1}^{\omega}) \, d\lambda(\omega) \right] \, .$$

Therefore, by (3.4)

(3.6)
$$\int h(\omega) \, d\lambda(\omega) = \overline{h}(\Omega, \lambda, T, \mu) \; .$$

In particular, $\overline{h}(\Omega, \lambda, T, \mu) = 0$ iff $h \equiv 0$.

Definition 3.11 [Ka98]. A probability measure Θ on $G^{\mathbb{Z}}$ + has asymptotic entropy $\mathbf{h}(\Theta)$ if the following Shannon-Breiman-McMillan type equidistribution condition is satisfied:

$$-\frac{1}{n}\log\theta_n(x_n)\to\mathbf{h}(\Theta)$$

for Θ -a.e. sequence $\boldsymbol{x} = \{x_n\} \in G^{\mathbb{Z}+}$ and in the space $L^1(G^{\mathbb{Z}+}, \Theta)$, where θ_n are the one-dimensional distributions of the measure Θ .

The following result shows that for RWRTP the asymptotic entropies in the sense of Definitions 3.8 and 3.11 coincide.

Theorem 3.12. If $\overline{H}(\Omega, \lambda, T, \mu) < \infty$, then

$$\mathbf{h}(\mathbf{P}_{\omega}) = \overline{h}(\Omega, \lambda, T, \mu)$$

for λ -a.e. $\omega \in \Omega$.

In combination with Theorem 3.9 it immediately implies

Corollary. If $\overline{H}(\Omega, \lambda, T, \mu) < \infty$ and for λ -a.e. $\omega \in \Omega$ there exists a sequence of finite subsets $A_n \subset G$ such that $\log |A_n| = o(n)$ and $\mu_{0,n}^{\omega}(A_n) > \varepsilon$ for a certain fixed number $\varepsilon > 0$, then the tail boundary of RWTDI(ω) is trivial for λ -a.e. $\omega \in \Omega$.

Lemma 3.13. Let $(Sh)_n = h_{n+1}$ be the shift in the space of increments \mathcal{H} . Then the transformation

$$\overline{T}(\omega, \mathbf{h}) = (T\omega, S\mathbf{h})$$

of the space $(\Omega \times G^{\mathbb{Z}_+}, \overline{\mathbf{P}}) \cong (\Omega \times \mathcal{H}, \mathbf{Q})$ (see (3.3)) is measure preserving and ergodic. Proof. The map

$$(\omega, (h_0, h_1, \ldots)) \mapsto ((\omega, h_0), (T\omega, h_1), \ldots)$$

identifies the space $(\Omega \times \mathcal{H}, \mathbf{Q})$ with the path space of the Markov chain on $\Omega \times G$ with the transition probabilities

(3.7)
$$\mathsf{Prob}[(T\omega, h')|(\omega, h)] = \mu^{T\omega}(h')$$

and the initial distribution $d\theta(\omega, h) = d\lambda(\omega)\mu^{\omega}(h)$. This identification conjugates \overline{T} with the shift in $(\Omega \times G)^{\mathbb{Z}+}$. By *T*-invariance of λ the measure θ is stationary with respect to the transition probabilities (3.7), so that the measure **Q** is \overline{T} -invariant (this fact can be also easily checked directly).

Further, since the measure θ is finite, ergodicity of \mathbf{Q} with respect to \overline{T} is equivalent to absence of non-constant bounded harmonic functions of the chain (3.7), e.g., see [Ka92]. The transition probabilities (3.7) from a point (ω, h) do not depend on h, so that any such function depends on ω only, i.e., is a non-constant *T*-invariant function on Ω , which is impossible by ergodicity of *T*.

Remark 3.14. In terms of the path space $(\Omega \times G)^{\mathbb{Z}_+}$ the transformation \overline{T} takes the form

$$((\omega, e), (T\omega, x_1), (T^2\omega, x_2), \ldots) \mapsto ((T\omega, e), (T^2\omega, x_1^{-1}x_2), (T^3\omega, x_1^{-1}x_3), \ldots),$$

i.e., it is the combination of the shift in the path space with the subsequent group translation consisting in moving the origin of the shifted path to the identity of G.

Proof of Theorem 3.12. Put

$$\varphi_n(\omega, \mathbf{h}) = -\log \mu_{0,n-1}^{\omega}(x_n) = -\log \mu_{0,n-1}^{\omega}(h_0 h_1 \cdots h_{n-1})$$

Since

$$\mu_{0,k+n-1}^{\omega}(h_0h_1\cdots h_{k+n-1}) \ge \mu_{0,k-1}^{\omega}(h_0h_1\cdots h_{k-1})\mu_{k,k+n-1}^{\omega}(h_kh_{k+1}\cdots h_{k+n-1})$$

we have

$$\varphi_{k+n}(\omega, \boldsymbol{h}) \leq \varphi_k(\omega, \boldsymbol{h}) + \varphi_n(T^k\omega, S^k\boldsymbol{h})$$

Finiteness of entropy of RWRTP $(\Omega, \lambda, T, \mu)$ means that

(3.8)
$$\int \varphi_1(\omega, \boldsymbol{h}) \, d\mathbf{Q}(\omega, \boldsymbol{h}) = -\int \log \mu^{\omega}(h_0) \, d\mathbf{Q}(\omega, \boldsymbol{h})$$
$$= -\int \sum_g \log \mu^{\omega}(g) \, \mu^{\omega}(g) \, d\lambda(\omega)$$
$$= \int H(\mu^{\omega}) \, d\lambda(\omega) = \overline{H}(\Omega, \lambda, T, \mu) < \infty$$

so that the conditions of the Kingman subadditive ergodic theorem are satisfied, and therefore there exists a limit

(3.9)
$$\lim_{n \to \infty} -\frac{1}{n} \log \mu_{0,n-1}^{\omega}(h_0 h_1 \cdots h_{n-1})$$

for **Q**-a.e. pair $(\omega, h) \in \Omega \times \mathcal{H}$ and in the space $L^1(\Omega \times \mathcal{H}, \mathbf{Q})$. Since

$$\int \varphi_n(\omega, \boldsymbol{h}) \, d\mathbf{Q}(\omega, \boldsymbol{h}) = \int H(\mu_{0, n-1}^{\omega}) \, d\lambda(\omega)$$

(cf. (3.8)), and

$$\frac{1}{n} \int H(\mu_{0,n-1}^{\omega}) \, d\lambda(\omega) \to \overline{h}(\Omega,\lambda,T,\mu)$$

by Theorem 3.7, the limit (3.9) coincides with $\overline{h}(\Omega, \lambda, T, \mu)$.

3.3. The asymptotic entropy of conditional chains.

We shall apply to the partitions of the path space of the operator P(3.1) the notations $\overline{\alpha}_k, \overline{\alpha}_{k,l}$, etc. introduced in Section 1.2 (sometimes we overline the objects connected with the operator P on the state space $\Omega \times G$ in order to distinguish them from the objects associated with the individual RWRTP(ω) on G). We continue to use the identifications (3.3). In the model ($\Omega \times \mathcal{H}, \mathbf{Q}$) of the path space $((\Omega \times G)^{\mathbb{Z}_+}, \mathbf{P}_{\overline{\lambda}}) \cong (\Omega \times G^{\mathbb{Z}_+}, \overline{\mathbf{P}})$ the partition $\overline{\alpha}_{0,n}$ coincides with the common refinement of the point partition of the space Ω and the partition of \mathcal{H} determined by the first n coordinates $h_0, h_1, \ldots, h_{n-1}$. In particular, $\overline{\alpha}_0$ coincides $\overline{\mathbf{P}}$ -mod 0 with the preimage partition of the projection $\Pi_{\Omega} : \overline{\boldsymbol{x}} \mapsto \omega$ (1.10). By $\overline{\alpha}_{\infty} = \lim \overline{\alpha}_{n,\infty}$ we denote the tail partition. Note that $\overline{\alpha}_{\infty}$ is a refinement of $\overline{\alpha}_0$, see Proposition 1.10.

Below all the entropies and the conditional entropies of partitions of the space $\Omega \times G^{\mathbb{Z}_+}$ are calculated with respect to the measure $\overline{\mathbf{P}}$. We begin with expressing the asymptotic entropy $\overline{h}(\Omega, \lambda, T, \mu)$ in terms of the conditional entropies of coordinate partitions. Integrating formula (2.5) with respect to the measure λ and using (2.4), (3.5) and (3.6), we obtain

Lemma 3.15. If $\overline{H}(\Omega, \lambda, T, \mu) < \infty$, then for any $k \ge 0$

$$H(\overline{\alpha}_{0,k}|\overline{\alpha}_{\infty}) = k\left[\overline{H}(\Omega,\lambda,T,\mu) - \overline{h}(\Omega,\lambda,T,\mu)\right]$$

By σ_E (resp., σ_Ω) we denote the point partition of the tail boundary (E, ε) (resp., the partition generated by the projection $\mathfrak{p}_\Omega: E \to \Omega$, see Proposition 1.10). Let ξ be a *G*-invariant partition of *E* which is a refinement of σ_Ω . Denote by $\overline{\alpha}_{\xi} = \mathbf{tail}^{-1}\xi$ the partition of the path space $((\Omega \times G)^{\mathbb{Z}_+}, \mathbf{P}_{\overline{\lambda}}) \cong (\Omega \times G^{\mathbb{Z}_+}, \overline{\mathbf{P}})$ which is the preimage of ξ under the map **tail**. Since $\mathbf{tail}^{-1}\sigma_E$ is the tail partition $\overline{\alpha}_{\infty}$, and $\mathbf{tail}^{-1}\sigma_\Omega = \overline{\alpha}_0$, we have

$$\overline{\alpha}_0 \preccurlyeq \overline{\alpha}_{\xi} \preccurlyeq \overline{\alpha}_{\infty}$$

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The quotient $(E^{\xi}, \varepsilon^{\xi})$ of the tail boundary (E, ε) by the partition ξ can be also considered as the quotient of the path space by the partition $\overline{\alpha}_{\xi}$. Denote by

$$\operatorname{tail}^{\xi} : (\Omega \times G^{\mathbb{Z}_+}, \overline{\mathbf{P}}) \to (E^{\xi}, \varepsilon^{\xi})$$

the associated quotient map. The spaces $(E^{\xi}, \varepsilon^{\xi})$ are "random analogues" of the μ boundaries in the case of usual time homogeneous random walks on groups, see [Fu71], [Ka00]. If $\xi = \sigma_E$, then **tail**^{ξ} = **tail**, and if $\xi = \sigma_{\Omega}$, then **tail**^{ξ} coincides with the projection Π_{Ω} (1.10). Denote by $\varepsilon_{\omega}^{\xi} = \varepsilon_{0,\omega}^{\xi}$ (resp., $\varepsilon_{n,\omega}^{\xi}$) the images of the tail measures ε_{ω} (resp., $\varepsilon_{n,\omega}$) under the projection $E \to E^{\xi}$.

Lemma 3.16. If $\overline{H}(\Omega, \lambda, T, \mu) < \infty$, then for any $k \ge 0$

$$\begin{split} H(\overline{\alpha}_{0,k}|\overline{\alpha}_{\xi}) &= kH(\overline{\alpha}_{0,1}|\overline{\alpha}_{\xi}) \\ &= k\left[\overline{H}(\Omega,\lambda,T,\mu) - \int \log \frac{dx_1 \varepsilon_{1,T\omega}^{\xi}}{d\varepsilon_{\omega}^{\xi}}(\mathbf{tail}^{\xi}(\omega,\boldsymbol{x})) \ d\overline{\mathbf{P}}(\omega,\boldsymbol{x})\right] \ . \end{split}$$

Proof. Since $\sigma_{\Omega} \preccurlyeq \xi$, conditioning by ξ uniquely determines the starting point ω of the sample path $\overline{\boldsymbol{x}} \longleftrightarrow (\omega, \boldsymbol{x})$. The traces of ξ on the elements of the partition σ_{Ω} (i.e., on the tail boundaries E_{ω} , see Proposition 1.10) are *G*-invariant partitions, so that we may apply Proposition 2.4, according to which the conditional probability of the element of the partition $\overline{\alpha}_{0,k}$ containing given $(\omega, \boldsymbol{x}) \longleftrightarrow (\omega, \boldsymbol{h})$ with respect to the partition $\overline{\alpha}_{\xi}$ is

$$\mu^{\omega}(h_1)\mu^{T\omega}(h_2)\dots\mu^{T^{k-1}\omega}(h_k)rac{dx_karepsilon_{k,T^{k}\omega}^{\xi}}{darepsilon_{\omega}^{\xi}}(\mathbf{tail}^{\xi}(\omega,oldsymbol{x}))$$

whence integrating we obtain

$$H(\overline{\alpha}_{0,k}|\overline{\alpha}_{\xi}) = k\overline{H}(\Omega,\lambda,T,\mu) - \int \log \frac{dx_k \varepsilon_{k,T^k\omega}^{\xi}}{d\varepsilon_{\omega}^{\xi}}(\mathbf{tail}^{\xi}(\omega,\boldsymbol{x})) d\mathbf{P}(\omega,\boldsymbol{x}) \ .$$

The integrand in the last term in the right hand side telescopes as

$$\log \frac{dx_k \varepsilon_{k,T^k \omega}^{\xi}}{d \varepsilon_{\omega}^{\xi}} (\mathbf{tail}^{\xi}(\omega, \boldsymbol{x})) = \sum_{i=0}^{k-1} \varphi(\overline{T}^i(\omega, \boldsymbol{x})) ,$$

where

$$arphi(\omega,x) = \log rac{dx_1 arepsilon_{1,T\omega}^{\xi}}{darepsilon_{\omega}^{\xi}}(\mathbf{tail}^{\xi}(\omega,x))$$

and \overline{T} is the transformation of the path space $\Omega \times G^{\mathbb{Z}_+}$ introduced in Lemma 3.13. Since \overline{T} preserves the measure $\overline{\mathbf{P}}$, we get the claim.

Corollary. If $\overline{H}(\Omega, \lambda, T, \mu) < \infty$, then

$$\overline{h}(\Omega, \lambda, T, \mu) = \int \log \frac{dx_1 \varepsilon_{1, T\omega}}{d\varepsilon_{\omega}} (\mathbf{tail}(\omega, \boldsymbol{x}))$$

Lemma 3.17. Let ξ and ξ' be two *G*-invariant measurable partitions of the tail boundary *E* such that $\sigma_{\Omega} \preccurlyeq \xi \preccurlyeq \xi'$. If $\overline{H}(\Omega, \lambda, T, \mu) < \infty$, then

$$H(\overline{\alpha}_{0,1}|\overline{\alpha}_{\xi}) \ge H(\overline{\alpha}_{0,1}|\overline{\alpha}_{\xi'})$$

and the equality holds iff $\xi = \xi'$.

Proof. Obviously, if ξ' is a refinement of ξ , then $\overline{\alpha}_{\xi'}$ is a refinement of $\overline{\alpha}_{\xi}$, so that the inequality follows from Proposition 2.13 (ii). If $H(\overline{\alpha}_{0,1}|\overline{\alpha}_{\xi}) = H(\overline{\alpha}_{0,1}|\overline{\alpha}_{\xi'})$, then by Lemma 3.16, $H(\overline{\alpha}_{0,k}|\overline{\alpha}_{\xi}) = H(\overline{\alpha}_{0,k}|\overline{\alpha}_{\xi'})$ for any $k \geq 1$, which by Proposition 2.13 (ii) means that for ε -a.e. point $\gamma \in E$ the k-dimensional distributions of the conditional measures $\overline{\mathbf{P}}^{\xi(\gamma)}$ and $\overline{\mathbf{P}}^{\xi'(\gamma)}$ are the same. Therefore, for ε -a.e. $\gamma \in E$ the conditional measures $\overline{\mathbf{P}}^{\xi(\gamma)}$ and $\overline{\mathbf{P}}^{\xi'(\gamma)}$ coincide, which is only possible if $\xi = \xi'$.

Corollary. Let ξ be a *G*-invariant measurable partitions of the tail boundary *E* such that $\sigma_{\Omega} \preccurlyeq \xi$. Then $\xi = \sigma_E$ iff

$$H(\alpha_{0,1}|\alpha_{\xi}) = H(\alpha_{0,1}|\alpha_{\infty}) ,$$

Theorem 3.18. Let ξ be a *G*-invariant measurable partition of the tail boundary *E* such that $\sigma_{\Omega} \preccurlyeq \xi$. If $\overline{H}(\Omega, \lambda, T, \mu) < \infty$, then for ε^{ξ} -a.e. point $\xi(\gamma) \in E^{\xi}$ the asymptotic entropy (in the sense of Definition 3.11) of the conditional measure $\overline{\mathbf{P}}^{\xi(\gamma)}$ exists and is equal to

(3.10)
$$\mathbf{h}(\overline{\mathbf{P}}^{\xi(\gamma)}) = \overline{h}(\Omega, \lambda, T, \mu) - \int \log \frac{dx_1 \varepsilon_{1,T\omega}^{\xi}}{d\varepsilon_{\omega}^{\xi}} (\mathbf{tail}^{\xi}(\omega, \boldsymbol{x})) d\overline{\mathbf{P}}(\omega, \boldsymbol{x}) \ .$$

Proof. Since $\sigma_{\Omega} \preccurlyeq \xi$, by Proposition 2.4 the one-dimensional distributions $\pi_n^{\xi(\gamma)}$ of the conditional measure $\overline{\mathbf{P}}^{\xi(\gamma)}$ are

$$\pi_n^{\xi(\gamma)}(g) = \mu_{0,n-1}^{\omega}(g) \frac{dg \varepsilon_{n,T^n\omega}^{\xi}}{d\varepsilon_{\omega}^{\xi}}(\xi(\gamma)) ,$$

where the point $\omega = \omega(\gamma) \in \Omega$ is determined by the projection $E^{\xi} \to \Omega$. Theorem 3.12 implies the convergence ($\overline{\mathbf{P}}$ -a.e. and in the space $L^1(\Omega \times G^{\mathbb{Z}_+}, \overline{\mathbf{P}})$)

(3.11)
$$-\frac{1}{n}\log\mu_{0,n-1}^{\omega}(x_n)\to\overline{h}(\Omega,\lambda,T,\mu) ,$$

and the telescoping at the end of the proof of Lemma 3.16 in combination with the Birkhoff ergodic theorem for the transformation \overline{T} yields the convergence (also a.e. and in the L^1 -space),

$$(3.12) \qquad \frac{1}{n}\log\frac{dx_n\varepsilon_{n,T^n\omega}^{\xi}}{d\varepsilon_{\omega}^{\xi}}(\mathbf{tail}^{\xi}(\omega,\boldsymbol{x})) \to \int\log\frac{dx_1\varepsilon_{1,T\omega}^{\xi}}{d\varepsilon_{\omega}^{\xi}}(\mathbf{tail}^{\xi}(\omega,\boldsymbol{x}))\,d\overline{\mathbf{P}}(\omega,\boldsymbol{x}) \ .$$

Combining (3.11) and (3.12), we obtain the convergence ($\overline{\mathbf{P}}$ -a.e. and in the L^1 -space) of 1 = tails(u, p) (u, p)

$$-\frac{1}{n}\log\pi_n^{\operatorname{\mathbf{tail}}^{\boldsymbol{\xi}}(\omega,\boldsymbol{x})}(x_n)$$

to the limit (3.10), which implies the claim, because the measures $\overline{\mathbf{P}}^{\xi(\gamma)}$ are the conditional measures of the measure $\overline{\mathbf{P}}$ with respect to the partition $\overline{\alpha}_{\xi}$.

Now, combining Theorem 3.18 with Lemmas 3.15, 3.16 and 3.17, we get the following generalization of Theorem 3.9

Theorem 3.19. Let ξ be a *G*-invariant measurable partition of the tail boundary *E* such that $\sigma_{\Omega} \preccurlyeq \xi$. If $\overline{H}(\Omega, \lambda, T, \mu) < \infty$, then $\xi = \sigma_E$ iff the asymptotic entropies of the conditional measures $\overline{\mathbf{P}}^{\xi(\gamma)}$ vanish.

Corollary. The partition ξ coincides with σ_E iff for ε^{ξ} -a.e. point $\xi(\gamma) \in E^{\xi}$ there exist $\varepsilon > 0$ and a sequence of sets $A_n = A_n(\xi(\gamma)) \subset G$ such that $\log |A_n| = o(n)$ and $\pi_n^{\xi(\gamma)}(A_n) > \varepsilon$ for all sufficiently large n.

4. TRIVIALITY AND DESCRIPTION OF THE TAIL AND THE POISSON BOUNDARIES

In this section we consider several concrete classes of groups and describe the boundaries of RWRTP on these groups.

4.1. Boundary triviality.

The entropy theory developed in Section 3 allows one to extend to RWRTP almost all results on triviality and identification of the boundaries earlier obtained for usual random walks, see [KV83], [Ka00].

Throughout this section we assume that the group G acts by isometries on a complete metric space (X, d). Fix once and forever a reference point $o \in X$ (its choice is irrelevant for what follows) and put

$$|g| = |g|_X = d(o, go) , \qquad g \in G .$$

Suppose that the group G has bounded exponential growth with respect to the space X, i.e.,

(4.1)
$$v(G,X) = \limsup_{t \to \infty} \frac{1}{t} \log \operatorname{card} \{g \in G : |g|_X \le t\} < \infty .$$

If G is a finitely generated group, and $X \cong G$ is its Cayley graph determined by a finite generating set and endowed with the associated word metric d, then condition (4.1) is obviously satisfied. Another example is provided by a discrete subgroup G of isometries of a Riemannian manifold of bounded geometry X. If v(G, X) = 0 we shall say that the group G has subexponential growth.

For a measure $\theta \in \mathcal{P}G$ denote by

$$|\theta| = \sum_{g} d(o, go)\theta(g)$$

its first moment. We shall say that $\operatorname{RWRTP}(\Omega, \lambda, T\mu)$ has a finite first moment (with respect to the space X) if

$$\int |\mu^{\omega}| \, d\lambda(\omega) < \infty \; .$$

Using the triangle inequality and the Kingman subadditive ergodic theorem, we derive

Theorem 4.1. If RWRTP $(\Omega, \lambda, T, \mu)$ on the group G has a finite first moment with respect to the space X then there exists a number $l = l(\Omega, \lambda, T, \mu, X)$ called the linear rate of escape such that for $\overline{\mathbf{P}}$ -a.e. $(\omega, \mathbf{x}) \in \Omega \times G^{\mathbb{Z}+}$,

$$\lim_{n \to \infty} \frac{|x_n|_X}{n} = l \; .$$

The convergence also holds in the space $L^1(\Omega \times G^{\mathbb{Z}_+}, \overline{\mathbf{P}})$, where $\overline{\mathbf{P}}$ is the measure (3.2).

Lemma 4.2 ([De86]). There exists a constant C = C(G, X) such that for any measure $\theta \in \mathcal{P}G$,

$$H(\theta) \le C(|\theta|_K + 1)$$

Now, using Theorem 3.12 we obtain in the same way as for ordinary random walks on groups (see [Gu80]) the following result

Theorem 4.3. If RWRTP $(\Omega, \lambda, T, \mu)$ on G has a finite first moment, then its entropy $\overline{h}(\Omega, \lambda, T, \mu)$, the rate of escape $l(\Omega, \lambda, T, \mu, X)$ and the rate of growth v(G, X) satisfy the inequality

$$\overline{h} \leq lv$$
 .

Corollary (cf. Proposition 2.9). If RWRTP $(\Omega, \lambda, T, \mu)$ is irreducible and the group G is non-amenable, then l > 0.

Theorem 4.3 in combination with Theorem 3.9 implies triviality of the tail boundaries of λ -a.e. RWTDI(ω) and of the Poisson boundary of RWRTP(Ω, λ, T, μ) when either $l(\Omega, \lambda, T, \mu, X)$ or v(G, X) vanish. Since any finitely generated nilpotent group has polynomial growth (with respect to the word metric determined by any finite generating set), we obtain **Theorem 4.4.** The Poisson boundary of any RWRTP with a finite first moment on a finitely generated nilpotent group is trivial.

Combining Theorem 4.4 with Proposition 2.9 we now obtain

Theorem 4.5. Let $(\Omega, \lambda, T, \mu)$ be an irreducible RWRTP with a finite first moment on a finitely generated nilpotent group G. Then for any $g \in G$ and λ -a.e. $\omega \in \Omega$

$$\|g\mu_{0,n}^{\omega}-\mu_{0,n}^{\omega}\|\underset{n\to\infty}{\longrightarrow}0$$
,

i.e., a.e. sequence $\mu_{0,n}^{\omega}$ strongly converges to a left-invariant mean on G.

Remark 4.6. By completely different methods Theorem 4.5 was proved in [MR94], [LRW94], and [Ru95] for compact and abelian groups without any additional moment assumptions.

Returning to Theorem 4.3, recall that another way of proving boundary triviality consists in showing that the rate of escape $l(\Omega, \lambda, T, \mu, X)$ vanishes. The methods used in [Ka91] allow one to do it for "centered" RWRTP on several classes of solvable groups in the same way as for usual time homogeneous random walks. For the sake of brevity we shall consider just the class of *polycyclic groups*. Without loss of generality we may assume that the polycyclic group $G = A \times N$ is the semi-direct product of an abelian group $A \cong \mathbb{Z}^d$ and a normal finitely generated nilpotent subgroup N (see [Ka91]). For a measure θ on G denote by θ_A its projection onto A, and by

$$\overline{\theta}_A = \sum_{a \in A} \theta_A(a) a \in \mathbb{R}^d$$

the barycenter of θ_A (this definition requires finiteness of the first moment of the measure θ_A). If $\overline{\theta}_A = 0$, then the measure θ is called *centered*. We shall say that RWRTP(Ω, λ, T, μ) on G with a finite first moment is centered if

$$\int \overline{\mu}^{\omega}_A \, d\lambda(\omega) < \infty \; .$$

Theorem 4.7. The Poisson boundary of any centered RWRTP with a finite first moment on a polycyclic group is trivial.

Theorem 4.8. For any centered irreducible RWRTP with a finite first moment on a polycyclic group G a.e. sequence $\mu_{0,n}^{\omega}$ strongly converges to a left-invariant mean on G.

4.2. Boundary identification.

We shall now look at the problem of identifying the tail and the Poisson boundaries of RWRTP on groups. Suppose, for the sake of argument, that our group G admits an invariant compactification \overline{G} with the boundary ∂G (i.e., the action of G on itself by left translations extends to a continuous action on \overline{G}), and that \mathbf{P}_{ω} -a.e. sample path $\boldsymbol{x} = (x_n)$ converges in this compactification to a limit point $x_{\infty} = x_{\infty}(\boldsymbol{x}) \in \partial G$ for λ -a.e. $\omega \in \Omega$. Obviously, the map $\boldsymbol{x} \mapsto \boldsymbol{x}_{\infty}$ is measurable with respect to the tail σ -algebra of the global operator P on $\Omega \times G$ (actually, the topological nature of ∂G is completely irrelevant for what follows). Therefore, the map $(\omega, \boldsymbol{x}) \mapsto (\omega, \boldsymbol{x}_{\infty})$ gives rise to a measurable partition ξ of the Poisson boundary E of the operator P. The partition ξ is G-invariant, and it is a refinement of the partition σ_{Ω} determined by the projection $E \to \Omega$. Coincidence of the partition ξ with the point partition σ_E of E means that the tail boundary E actually can be identified with the product $\Omega \times \partial G$. Therefore, in the latter case the Poisson boundary of the RWRTP is stable (in the sense of Definition 1.11), i.e., the Poisson boundary of the operator P and the tail boundaries E^{ω} can be both identified with ∂G .

The main method of proving boundary convergence for "groups with hyperbolic properties" goes back to Furstenberg [Fu71] and consists in using the martingale convergence theorem in combination with contracting (proximality) properties of the action of G on the boundary ∂G , see [CS89], [Wo93], [Ka00]. This method does not impose any moment conditions on the random walk and may be combined with the "strip approximation" criteria [Ka00] to give a full boundary identification. However, its application to RWRTP is rather tedious and we could not get rid of rather awkward conditions on the measures μ^{ω} (like existence of a single non-degenerate measure on G dominated by a.e. μ^{ω}) following this way. Instead of this we shall use the "ray approximation" approach (see [Ka00]) and its recent generalization obtained in [KMa99] which will save us from a good deal of technical details.

Recall that a metric space (X, d) is called convex if for any two points $x, y \in X$ there exists a *midpoint* $z \in X$ such that

$$d(x,z) = d(y,z) = \frac{1}{2}d(x,y)$$
.

In a complete convex space any two points can be joined by a geodesic (see the related definitions in [BH99]).

A metric space (X, d) is called *uniformly convex* if it is convex and in addition there exists a strictly decreasing continuous function φ on [0, 1] with $\varphi(0) = 1$ such that for any $x, y, w \in X$ and a midpoint z of x and y

$$\frac{d(z,w)}{R} \le \varphi\left(\frac{d(x,y)}{2R}\right)$$

where $R = \max\{d(x, w), d(y, w)\}$. The midpoints (and therefore geodesics with given endpoints) in a uniformly convex space are unique.

A convex metric space (X, d) is called *non-positively curved* (in the sense of Busemann) if for any $x, y, z \in X$ and any midpoints m_{xz} (resp., m_{yz}) of x and y (resp., of y and z)

$$d(m_{xz}.m_{yz}) \le rac{1}{2}d(x,y)$$
 .

From now on we shall assume that

(4.2) The metric space X on which the group G acts is uniformly convex and satisfies the Busemann non-positive curvature condition.

Denote by ∂X the space of asymptotic classes of geodesic rays in X. We shall identify ∂X with the space of geodesic rays issued from the point o. Examples of spaces (X, d) satisfying condition (4.2) include all Cartan-Hadamard manifolds (in particular, noncompact Riemannian symmetric spaces without compact factors) and all metric trees. In the first case ∂X is the visibility sphere of X, and in the second case it is the space of ends of X.

An application of [KMa99] to the transformation \overline{T} of the space $(\Omega \times G^{\mathbb{Z}_+}, \overline{\mathbf{P}})$ gives

Theorem 4.9. Suppose that RWRTP $(\Omega, \lambda, T, \mu)$ on the group G has a finite first moment, its rate of escape $l = l(\Omega, \lambda, T, \mu, X)$ is positive, and the space X satisfies conditions (4.1) and (4.2). Then for $\overline{\mathbf{P}}$ -a.e. $(\omega, \mathbf{x}) \in \Omega \times G^{\mathbb{Z}}$ + there exists a unique geodesic ray $\gamma = \gamma(\omega, \mathbf{x}) \in \partial X$ such that

$$d(x_n,\gamma(nl)) = o(n) \; .$$

Note that in view of the Corollary of Theorem 4.3 the condition l > 0 in Theorem 4.9 is not really restrictive as discrete groups of isometries of non-positively curved spaces are usually non-amenable. Theorem 4.9 in combination with Theorem 3.19 immediately implies

Theorem 4.10. Under conditions of Theorem 4.9 the Poisson boundary of RWRTP is stable and is isomorphic to the space ∂X with the resulting hitting measure.

Therefore, the Poisson boundary identifies with the natural geometric boundaries for RWRTP with a finite first moment on free groups and on discrete groups of isometries of Cartan-Hadamard manifolds (in particular, in discrete subgroups of semi-simple Lie groups), see [Ka00] for a more detailed description of these boundaries in the case of usual time homogeneous random walks. Note that Theorem 3.19 allows one to extend identification of the Poisson boundary with the "natural" boundaries from usual time homogeneous random walks to RWRTP for several other classes of groups, including the polycyclic groups and the groups with infinitely many ends, cf. [Ka91] and [Ka00].

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