

Dirac Submanifolds and Poisson Involutions

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Dedicated to Rencontres Mathématiques de Glanon on the occasion of her fifth birthday

Abstract

Dirac submanifolds are a natural generalization in the Poisson category for symplectic submanifolds of a symplectic manifold. In a certain sense they correspond to symplectic subgroupoids of the symplectic groupoid of the given Poisson manifold. In particular, Dirac submanifolds arise as the stable locus of a Poisson involution. In this paper, we provide a general study for these submanifolds including both local and global aspects.

In the second part of the paper, we study Poisson involutions and the induced Poisson structures on their stable locuses. We discuss the Poisson involutions on a special class of Poisson groups, and more generally Poisson groupoids, called symmetric Poisson groups (and symmetric Poisson groupoids). Many well-known examples, including the standard Poisson group structures on semi-simple Lie groups, Bruhat Poisson structures on compact semi-simple Lie groups, and Poisson groupoids connecting with dynamical r -matrices of semi-simple Lie algebras are symmetric, so they admit a Poisson involution. For symmetric Poisson groups, the relation between the stable locus Poisson structure and Poisson symmetric spaces is discussed. As a consequence, we show that the Dubrovin-Ugaglia-Boalch-Bondal Poisson structure on the space of Stokes matrices U_+ appearing in Dubrovin's theory of Frobenius manifolds is indeed a Poisson symmetric space for the Poisson group $B_+ * B_-$.

1 Introduction

The underlying structure of any Hamiltonian system is a Poisson manifold. To deal with mechanics with constraints, it is always desirable to understand how to put a Poisson structure on a submanifold of a Poisson manifold. A naive way is to consider Poisson submanifolds. However, these are not too much different from the original Poisson manifold from the viewpoint of the Hamiltonian systems. On the other hand, for symplectic manifolds, there do not exist any nontrivial Poisson submanifolds. However Dirac was able to write down a Poisson bracket for a submanifold of a symplectic manifold which is given by a set of constraints:

$$Q = \{x \in P \mid \varphi_i(x) = 0, i = 1, \dots, k\} \tag{1}$$

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such that the matrix $(\{\varphi_i, \varphi_j\})$ is invertible on Q . This is the famous Dirac bracket [7]. In this case, Q is a symplectic submanifold, i.e., the pull back on Q of the symplectic form is non-degenerate.

There has appeared a lot of work attempting to generalize Dirac brackets, for example, the notion of cosymplectic manifolds of Weinstein [30], Poisson reduction of Marsden-Ratiu [25], just to name a few. In particular, Courant presented a unified approach to this question by introducing the notion of Dirac structures [5], by which one could obtain a Poisson bracket on admissible functions on a submanifold Q . In some situation, one indeed can get a Poisson structure on all functions on Q . Then Q becomes a Poisson manifold itself.

In his study of Frobenius manifolds, which is connected with 2-dimensional topological quantum field theories, Dubrovin recently found a Poisson structure on U_+ , the space of upper triangular matrices with ones on the diagonal, by viewing it as a space of Stokes matrices. Indeed Dubrovin identifies U_+ with the local moduli spaces of semisimple Frobenius manifolds. In particular, an explicit formula was found for the Poisson bracket in the three dimensional case [8]:

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \quad (2)$$

$$\{x, y\} = xy - 2z, \{y, z\} = yz - 2x, \{z, x\} = zx - 2y. \quad (3)$$

Such a Poisson structure admits various nice properties. For instance, it naturally admits a braid group action. The casimir function is the Markoff polynomial $x^2 + y^2 + z^2 - xyz$. Its linear and quadratic parts give rise to a biHamiltonian structure, etc. Then Ugaglia extended Dubrovin's formula to the $n \times n$ case [29]. Recently, in connection with his study of the monodromy map, Boalch [2] proved that U_+ arises as the stable locus of a Poisson involution on the Poisson group $B_+ * B_-$ and that the above Poisson structure on U_+ is induced from the standard Poisson structure on $B_+ * B_-$.

From a completely different angle, independently Bondal discovered exactly the same Poisson structure on U_+ in his study of the theory of triangulated categories [3]. He also studied extensively this Poisson structure including the braid group action and symplectic leaves etc. In his approach, instead of writing down the Poisson structure on U_+ , first of all Bondal discovered a symplectic groupoid \mathcal{M} whose space of objects is U_+ . Then the general theory of symplectic groupoids [31] implies that U_+ is a Poisson manifold. What is more interesting is, in a subsequent paper [4], he discovered an extremely simple connection between his symplectic groupoid \mathcal{M} and the standard symplectic groupoid Γ over the Poisson group $B_+ * B_-$ of Lu-Weinstein [22]. Namely, \mathcal{M} is simply a symplectic subgroupoid of Γ which can be realized as the stable locus of an involutive symplectic groupoid automorphism of Γ .

Bondal's work suggests a simple fact, which was somehow overlooked in the literature, namely a submanifold inherits a natural Poisson structure if it can be realized as the base space of a symplectic subgroupoid. A natural question arises as to what are these submanifolds and how they can be characterized. One of the main purpose of the paper is to answer this question. These submanifolds will be called Dirac submanifolds. Symplectic subgroupoids are very simple to describe: they are subgroupoids and in the mean time symplectic submanifolds. In contrast, Dirac submanifolds are not so simple as we shall see. There are some interesting and rich geometry there (both global and local), which we believe deserve further studies.

Dirac submanifolds are a special case of those submanifolds, on which the admissible functions for the pulled back Dirac structure happen to be all functions in terms of [5, 25]. In other words, the intersection of a Dirac submanifold with symplectic leaves of P are symplectic submanifolds of the leaves. This feature explains where the induced Poisson structure comes from for a Dirac submanifold. However, not all such submanifolds are Dirac submanifolds. For instance, symplectic leaves (except for the zero point) of $\mathfrak{su}(2)$ are not Dirac submanifolds. It is still not clear at the moment how to describe the global obstruction in general. On the other hand, when the Poisson manifold is symplectic, Dirac submanifolds are precisely symplectic submanifolds. Other examples include cosymplectic submanifolds and stable locus of a Poisson involution.

The second aim of the paper is to study systematically Poisson involutions and the induced Poisson structures on stable locuses. When the underlying Poisson manifolds are Poisson groups or more generally Poisson groupoids, there is an effective way of producing a Poisson involution, namely through their infinitesimal invariants: Lie bialgebras or Lie bialgebroids. These are called symmetric Poisson groupoids and symmetric Lie bialgebroids. As we see, such a Poisson involution exists in almost every well-known example of Poisson groupoids and Poisson groups, including the standard Poisson group structures on semi-simple Lie groups, Bruhat Poisson structures on compact semi-simple Lie groups, and Poisson groupoids connecting with dynamical r -matrices of semi-simple Lie algebras. For Poisson groups, they were studied by Fernandes in connection with Poisson symmetric spaces [11, 12], i.e., symmetric spaces which are Poisson homogeneous spaces. It turns out that the induced Poisson structure on the stable locus Q of the Poisson involution of a symmetric Poisson group is closely connected with Poisson symmetric spaces. In particular, we prove that the identity connected component of Q is always a Poisson symmetric space. As a consequence, we show that the DUBB-Poisson structure on Stokes matrices U_+ is a Poisson symmetric space for the Poisson group $B_+ * B_-$.

The paper is organized as follows. In Section 2, we introduce the definition of Dirac submanifolds and study their basic properties. Local Dirac submanifolds are also introduced and their connection with transverse Poisson structures is discussed. Section 3 is devoted to the study of some further properties. In particular, we study how the modular class of a Dirac submanifold is related to that of the Poisson manifold P . We also study Poisson group actions on Dirac submanifolds. Finally we prove that Dirac submanifolds are indeed infinitesimal version of symplectic subgroupoids. In Section 4, we investigate stable locuses of Poisson involutions, and study Poisson involutions on Poisson groupoids by introducing the notion of symmetric Poisson groupoids. In Section 5, we consider particularly symmetric Poisson groups and the induced Poisson structures on stable locuses. The connection with Poisson symmetric spaces is discussed.

We remark that one should not confuse Dirac submanifolds here with the notion of Dirac manifolds of Courant [5]. Courant's Dirac manifolds are manifolds equipped with a Dirac structure, which generalize the notion of both Poisson and presymplectic manifolds. In an earlier version, some other names such as Q -submanifolds and IR-submanifolds were suggested, but we feel that neither of these names reflects the complete nature of the objects we study here. At the end, we decided to call them Dirac submanifolds, which at least contains a famous name that people have heard of.

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2 Dirac submanifolds

This section is devoted to the study of general aspects of Dirac submanifolds.

2.1 Definition and properties

Let us introduce the definition first.

Definition 2.1 *A submanifold Q of a Poisson manifold P is called a Dirac submanifold if the tangent bundle of P along Q admits a vector bundle decomposition:*

$$T_Q P = TQ \oplus V_Q \quad (4)$$

*so that V_Q^\perp is a Lie subalgebroid of T^*P , where T^*P is equipped with the standard cotangent bundle Lie algebroid structure.*

Note that the last condition above is equivalent to that $V_Q \subset TP$ is a coisotropic submanifold of the tangent Poisson manifold TP . So alternatively, we have

Proposition 2.2 *A submanifold $Q \subset P$ is a Dirac submanifold iff there is a decomposition as in Equation (4) so that $V_Q \subset TP$ is a coisotropic submanifold of the tangent Poisson manifold TP .*

In what follows, we will see that Q itself must be a Poisson manifold. However, Q in general is not a Poisson submanifold. We need to introduce some notations. By pr , we denote the bundle map $T_Q P \rightarrow TQ$ obtained simply by taking the projection along the decomposition (4). And let $\text{pr}^* : T^*Q \rightarrow T^*P$ denote the dual of pr by considering $T_Q^* P$ as a subbundle of T^*P . By pr_* , we denote the map from $\mathfrak{X}^d(P)$ to $\mathfrak{X}^d(Q)$ naturally induced from pr , which is defined by $\text{pr}_*(D) = \text{pr}(D|_Q)$, $\forall D \in \mathfrak{X}^d(P)$.

We summarize some important properties of Dirac submanifolds in the following

Theorem 2.3 *Let Q be a Dirac submanifold of a Poisson manifold (P, π) . Then*

- (i). $\pi|_Q = \pi_Q + \pi'$, where $\pi_Q \in \Gamma(\wedge^2 TQ)$ and $\pi' \in \Gamma(\wedge^2 V_Q)$;
- (ii). π_Q is a Poisson tensor on Q ;
- (iii). $\text{pr}^* : T^*Q \rightarrow T^*P$ is a Lie algebroid morphism, where both T^*Q and T^*P are equipped with the cotangent bundle Lie algebroid structures;

(iv). for any $X \in \mathfrak{X}(P)$,

$$pr_*[X, \pi] = [pr_*X, \pi_Q]; \quad (5)$$

(v). for any $x \in Q$, $\pi_Q^\#(T_x^*Q) = \pi^\#(T_x^*P) \cap T_xQ$;

(vi). for any $x \in Q$, $\pi_Q^\#(T_x^*Q)$ is a symplectic subspace of $\pi^\#(T_x^*P)$. Hence, each symplectic leaf of Q is the intersection of Q with a symplectic leaf of P , which is a symplectic submanifold of that leaf.

Before proving this theorem, we need a couple of lemmas. The following lemma, which can also be easily verified directly, follows from the fact that the natural inclusion $TQ \rightarrow TP$ is a Lie algebroid morphism.

Lemma 2.4 *Let $Q \subseteq P$ be a submanifold. Assume that $D \in \mathfrak{X}^d(P)$ and $D' \in \mathfrak{X}^{d'}(P)$ are multi-vector fields tangent to Q , i.e., $D|_Q \in \mathfrak{X}^d(Q)$ and $D'|_Q \in \mathfrak{X}^{d'}(Q)$. Then*

$$[D, D']|_Q = [D|_Q, D'|_Q].$$

Next is the following

Lemma 2.5 *Assume that Q is a submanifold of a Poisson manifold (P, π) such that there is a vector bundle decomposition $T_QP = TQ \oplus V_Q$. Moreover assume that $\pi|_Q = \pi_Q + \pi'$, where $\pi_Q \in \Gamma(\wedge^2 TQ)$ and $\pi' \in \Gamma(\wedge^2 V_Q)$. Then π_Q is a Poisson tensor on Q .*

Proof. Write $\pi' = \sum_i X_i \wedge Y_i$ where $X_i, Y_i \in \Gamma(V_Q)$. Now let $\tilde{X}_i, \tilde{Y}_i \in \mathfrak{X}(P)$ be (local) extensions of X_i, Y_i , and $\tilde{\pi}' = \sum_i \tilde{X}_i \wedge \tilde{Y}_i$. Let $\pi'' = \pi - \tilde{\pi}' \in \mathfrak{X}^2(P)$. Then clearly $\tilde{\pi}'|_Q = \pi'$ and $\pi''|_Q = \pi_Q$. Now it follows from $[\pi, \pi] = 0$ that $[\pi'', \pi''] = -2[\pi'', \tilde{\pi}'] - [\tilde{\pi}', \tilde{\pi}']$. On the other hand, it is clear by definition that $pr_*[\pi'', \tilde{\pi}'] = pr_*[\tilde{\pi}', \tilde{\pi}'] = 0$. Thus $pr_*[\pi'', \pi''] = 0$. According to Lemma 2.4, the latter implies that $[\pi_Q, \pi_Q] = pr_*[\pi'', \pi''] = 0$. This concludes the proof. □

Proof of Theorem 2.3 By definition, V_Q^\perp is a Lie subalgebroid of the cotangent Lie algebroid T^*P . By identifying T^*Q with V_Q^\perp , one obtains a Lie algebroid structure on T^*Q , and a Lie algebroid morphism $\varphi : T^*Q \rightarrow T^*P$. Clearly, $\varphi = pr^*$. By ρ_Q , we denote the anchor map of the Lie algebroid T^*Q . Thus we have $i \circ \rho_Q = \pi^\# \circ \varphi$, where $i : TQ \rightarrow TP$ is the natural inclusion. It follows that $\rho_Q = pr \circ i \circ \rho_Q = pr \circ \pi^\# \circ \varphi = pr \circ \pi^\# \circ pr^*$. Hence $\rho_Q : T^*Q \rightarrow TQ$ is skew-symmetric. Thus it defines a bivector field $\pi_Q \in \Gamma(\wedge^2 TQ)$ so that $\rho_Q = \pi_Q^\#$. Under the decomposition (4), we have $\wedge^2 T_QP = \wedge^2 TQ \oplus (TQ \wedge V_Q) \oplus \wedge^2 V_Q$. It is clear that π_Q is the $\Gamma(\wedge^2 TQ)$ -part of $\pi|_Q$ under the above decomposition. Since $\pi^\#(V_Q^\perp) \subset TQ$, $\pi|_Q$ does not involve any mixed term, i.e., the $\Gamma(TQ \wedge V_Q)$ -part. Hence we have $\pi|_Q = \pi_Q + \pi'$ with $\pi' \in \Gamma(\wedge^2 V_Q)$. This proves (i).

By Lemma 2.5, π_Q is indeed a Poisson tensor on Q . Hence (ii) follows. Next we need to show that the Lie algebroid structure on T^*Q is indeed the cotangent Lie algebroid corresponding to the Poisson structure π_Q . Since φ is a Lie algebroid morphism, $\varphi^* = pr$ induces a morphism of the

(graded) differential algebras $pr_* : (\Gamma(\wedge^* TP), d_{*P}) \longrightarrow (\Gamma(\wedge^* TQ), d_{*Q})$. Since T^*P is the cotangent Lie algebroid, we know that $d_{*P} = [\pi, \cdot]$. To prove the claim, it suffices to show that $d_{*Q} = [\pi_Q, \cdot]$. To this end, given any $X \in \mathfrak{X}(Q)$, choose an extension $\tilde{X} \in \mathfrak{X}(P)$. Write $\pi = \pi'' + \tilde{\pi}'$ as in Lemma 2.5 so that $\pi''|_Q = \pi_Q$ and $\tilde{\pi}'|_Q \in \Gamma(\wedge^2 V_Q)$. Then $d_{*Q}X = (d_{*Q}pr_*)\tilde{X} = (pr_*d_{*P})\tilde{X} = pr_*[\pi, \tilde{X}] = pr_*[\pi'' + \tilde{\pi}', \tilde{X}] = pr_*[\pi'', \tilde{X}] = [\pi_Q, X]$, where the last step follows from Lemma 2.4. This proves (iii), and therefore (iv) as a consequence.

Next we prove the relation $\pi_Q^\#(T_x^*Q) = \pi^\#(T_x^*P) \cap T_xQ$. Since $i \circ \pi_Q^\# = \pi^\# \circ \varphi$, it is obvious that $\pi_Q^\#(T_x^*Q) \subset \pi^\#(T_x^*P) \cap T_xQ$. Conversely, let $v \in \pi^\#(T_x^*P) \cap T_xQ$ be any vector. Then $v = \pi^\#\xi$ for some $\xi \in T_x^*P$. Now since $T_x^*P = T_xQ^\perp \oplus V_x^\perp$, we may write $\xi = \xi_1 + \xi_2$, where $\xi_1 \in T_xQ^\perp$ and $\xi_2 \in V_x^\perp$. Since $V_x^\perp = \varphi(T_x^*Q)$, $\pi^\#\xi_2 \in (\pi^\# \circ \varphi)(T_x^*Q) = \pi_Q^\#(T_x^*Q) \subset T_xQ$. Hence $\pi^\#\xi_1 = v - \pi^\#\xi_2 \in T_xQ$. On the other hand, it is clear that $\pi^\#\xi_1 \in V_x$. Hence, $\pi^\#\xi_1 = 0$ and therefore $v = \pi^\#\xi_2 \in \pi_Q^\#(T_x^*Q)$. Thus we have proved the relation $\pi_Q^\#(T_x^*Q) = \pi^\#(T_x^*P) \cap T_xQ$, which implies that the symplectic leaves of Q are the intersection of the symplectic leaves of P with Q .

Finally, let $D_x = \pi_Q^\#(T_x^*Q)$ and $D'_x = \pi'^\#(T_x^\perp Q)$. It is simple to see that $\pi^\#(T_x^*P) = D_x \oplus D'_x$, and $\pi_Q(x) \in \wedge^2 D_x$ and $\pi'(x) \in \wedge^2 D'_x$ are both nondegenerate. Thus $\pi_Q(x)|_{D_x}^{-1} \oplus \pi'(x)|_{D'_x}^{-1}$ is the inverse of $\pi(x)$ when being restricted to $\pi^\#(T_x^*P)$. It follows that $\pi_Q^\#(T_x^*Q)$ is indeed a symplectic subspace of $\pi^\#(T_x^*P)$. This implies that any symplectic leaf of Q is indeed a symplectic submanifold of a symplectic leaf of P . This concludes our proof of the theorem. □

As an immediate consequence, we have

Corollary 2.6 *Assume that Q is a Dirac submanifold of a Poisson manifold P . Then we have*

(i). *there is a morphism on the level of Poisson cohomology*

$$pr_* : H_\pi^*(P) \longrightarrow H_{\pi_Q}^*(Q);$$

(ii). *if $X \in \mathfrak{X}(P)$ is a vector field such that $X|_Q \in \Gamma(V_Q)$, then $pr_*[X, \pi] = 0$.*

Remark 2.7 From Theorem 2.3 (vi), we see that the choice of the complementary V_Q is immaterial for the purpose of getting the Poisson structure on Q . Indeed, any submanifold whose intersections with symplectic leaves of P are symplectic submanifolds of the leaves admits a potential Poisson tensor, which, however, might be discontinuous. This is simply the bivector field obtained by taking the inverse of the restriction of the leafwise symplectic form to Q . In terms of the language of Dirac structures, such submanifolds precisely correspond to those for which the pulled back Dirac structure [5] of the one corresponding to the graph of the Poisson tensor on P is a bivector on each tangent space. In general, it might be discontinuous though. However, note that even when it is smooth so that one obtains a Poisson structure on Q , it may still not be a Dirac submanifold. See Example 2.17 below.

We also note that Dirac submanifolds are a special case of the situation in [25], where general Poisson reduction was studied. This provides another route to obtain the Poisson structures on these submanifolds.

Next proposition gives an alternative definition of Dirac submanifolds, which is presumably easier to check in practice.

Proposition 2.8 *A submanifold Q of a Poisson manifold (P, π) is a Dirac submanifold if the following conditions are all satisfied:*

- (i). *there is a vector bundle decomposition $T_Q P = TQ \oplus V_Q$;*
- (ii). *$\pi|_Q = \pi_Q + \pi'$, where $\pi_Q \in \Gamma(\wedge^2 TQ)$ and $\pi' \in \Gamma(\wedge^2 V_Q)$.*
- (iii). *for any $X' \in \Gamma(V_Q)$, there is an extension $X \in \mathfrak{X}(P)$ of X' such that $\text{pr}_*[X, \pi] = 0$.*

Proof. From (i)-(ii), we know that π_Q is a Poisson tensor on Q , and therefore T^*Q is a Lie algebroid. The decomposition (i) induces a natural identification between V_Q^\perp and T^*Q , which equips V_Q^\perp with a Lie algebroid structure by pulling back the cotangent Lie algebroid on T^*Q . It remains to show that this Lie algebroid structure on V_Q^\perp is indeed a Lie subalgebroid of T^*P . To this end, it suffices to prove Equation (5) for any vector field $X \in \mathfrak{X}(P)$.

If $X \in \mathfrak{X}(P)$ such that $X|_Q$ is tangent to Q , Equation (5) follows from Lemma 2.4. On the other hand, assume that $X|_Q \in \Gamma(V_Q)$. Then $\text{pr}_*[X, \pi] = \text{pr}_*[X, \pi'' + \tilde{\pi}'] = \text{pr}_*[X, \pi'']$, where π'' and $\tilde{\pi}'$ are the bivector fields as in the proof of Lemma 2.5. Since $\pi''|_Q = \pi_Q$ is tangent to Q , $[X, \pi'']|_Q$ depends only on $X|_Q$. From assumption (iii), we thus have $\text{pr}_*[X, \pi] = 0$. This concludes the proof.

□

Remark 2.9 Conditions (ii) and (iii) in Proposition 2.8 can be replaced, respectively, by the following equivalent conditions:

- (ii'). $\pi^\#(V_Q^\perp) \subseteq TQ$;
- (iii'). for any point x in Q , there are a set of local vector fields $X_1, \dots, X_k \in \mathfrak{X}(P)$ around x such that $X_i|_Q \in \Gamma(V_Q)$, $i = 1, \dots, k$ consists of a fiberwise basis for V_Q and satisfies the property that $\text{pr}_*[X_i, \pi] = 0$, $i = 1, \dots, k$.

Recall that cosymplectic submanifolds of a Poisson manifold P are those, which are characterized by the two properties [30]:

- (i). Q intersects each symplectic leaf of P transversely;
- (ii). at each point of Q , the intersection of TQ with the tangent space of the symplectic leaf is a symplectic subspace.

Lemma 2.10 *A submanifold Q of a Poisson manifold (P, π) is cosymplectic iff it satisfies the conditions (i)-(ii) as in Proposition 2.8 with the property that $\pi' \in \Gamma(\wedge^2 V_Q)$ is non-degenerate.*

Proof. If Q is cosymplectic, then $T_x P = T_x Q \oplus \pi^\#(T_x Q^\perp)$, $\forall x \in Q$ [30]. It is simple to see that $V_Q = \pi^\#(TQ^\perp)$ is a complementary of TQ in $T_Q P$ which possesses all the required properties for Q being a Dirac submanifold.

Conversely, let Q be a submanifold which satisfies the conditions (i)-(ii) as in Proposition 2.8 with the property that $\pi' \in \Gamma(\wedge^2 V_Q)$ is non-degenerate. For any $x \in Q$, it is clear that $\pi^\#(T_x Q^\perp) = \pi'^\#(T_x Q^\perp) \subset V_x$. Since π' is non-degenerate, $\pi'^\# : T_x Q^\perp \rightarrow V_x$ is an isomorphism. Thus we have $V_x = \pi^\#(T_x Q^\perp)$. Thus it follows that $T_x P = T_x Q \oplus \pi^\#(T_x Q^\perp)$. Hence Q is cosymplectic.

□

Corollary 2.11 *Cosymplectic submanifolds are Dirac submanifolds.*

Proof. According to Lemma 2.10, it suffices to verify the last condition (iii) in Proposition 2.8. Since $V_Q = \pi^\#(TQ^\perp)$, $\Gamma(V_Q)$ is spanned by the vector fields $gX_f|_Q$ where $f, g \in C^\infty(P)$ and f is constant along Q . Now clearly $\text{pr}_*[gX_f, \pi] = \text{pr}_*(X_f \wedge X_g) = 0$, and therefore the condition (iii) in Proposition 2.8 is satisfied. This concludes the proof.

□

The following proposition gives a nice characterization for a Dirac submanifold.

Proposition 2.12 *Assume that there is a set of functions $f_1, \dots, f_k \in C^\infty(P)$ which defines a coordinate system on Q . Then Q is a Dirac submanifold if*

- (i). *the Hamiltonian vector field X_{f_i} , $\forall i$, is tangent to Q ;*
- (ii). *$d\{f_i, f_j\} \cong 0 \pmod{df_i}$ along Q .*

Proof. Let $V_Q = \{v \in T_Q P \mid v f_i = 0, \forall i = 1, \dots, k\}$. Clearly, V_Q is a vector bundle such that $T_Q P = TQ \oplus V_Q$. Moreover $V_Q^\perp = \text{span}\{df_i|_Q, i = 1, \dots, k\}$. Thus from (i) it follows that $\pi^\#(V_Q^\perp) \subset TQ$. Combining with (ii), we see that V_Q^\perp is indeed a Lie subalgebroid of T^*P . Thus Q is a Dirac submanifold.

□

Remark 2.13 It is natural to ask what is the rule of the subbundle V_Q in the definition of a Dirac submanifold. For a given Dirac submanifold, is V_Q unique? If not, what is the relation between different choices of V_Q ?

Let Q be a Dirac submanifold and $\pi|_Q = \pi_Q + \pi'$, where $\pi_Q \in \Gamma(\wedge^2 TQ)$ and $\pi' \in \Gamma(\wedge^2 V_Q)$. Assume that there is another decomposition $T_Q P = TQ \oplus V'_Q$ satisfying the condition of Definition

2.1. Then V'_Q must correspond to a bundle map $\varphi : V_Q \longrightarrow TQ$, i.e., $V'_Q = \{\varphi(v) + v \mid v \in V_Q\}$. It is simple to see that the condition (ii') in Remark 2.9 implies that $\varphi_*(\pi')^\# = 0$. In particular, if Q is cosymplectic, φ must be zero so V_Q is unique. However, in general, it is not clear how to elaborate the other condition (iii) in order to give a clean description of φ .

2.2 Examples

Now we will discuss some examples of Dirac submanifolds. By Corollary 2.11, we already know that cosymplectic manifolds are Dirac submanifolds. The following gives a list of other examples.

Example 2.14 Assume that P is a symplectic manifold. If Q is a Dirac submanifold, then Q must be a symplectic submanifold according to Theorem 2.3 (vi). On the other hand, symplectic submanifolds are automatically Dirac submanifolds since they are cosymplectic. In other words, Dirac submanifolds of a symplectic manifold are precisely symplectic submanifolds.

Another extreme case is the following

Example 2.15 If x is a point where the Poisson tensor vanishes, then $\{x\}$ is a Dirac submanifold.

Example 2.16 Let $P = \mathbb{R}^n$ be equipped with a constant Poisson structure. Then P is a regular Poisson manifold, where symplectic leaves are affine subspaces $x + S$. Here S is the symplectic leaf through 0 which is also a linear subspace of \mathbb{R}^n . Assume that an affine subspace $Q = u + V$ is a Dirac submanifold, where V is a linear subspace of \mathbb{R}^n . By Theorem 2.3 (i), we see that V must admit a complementary subspace U such that the $P = V \times U$ as a product of Poisson manifolds, where V and U are equipped with the constant Poisson structures $\pi_Q(u)$ and $\pi'(u)$ respectively. This condition is equivalent to that the intersection of V with S is a symplectic subspace of S . Conversely, given any such a linear subspace V , then one can decompose $P = V \times U$ as a product of constant Poisson structures. For $Q = V \times \{u\}$, by taking $V_Q \cong Q \times U$ to be constant, one easily sees that the conditions in Proposition 2.8 are indeed satisfied. Hence Q is a Dirac submanifold.

In conclusion, an affine space $u + V$ is a Dirac submanifold iff $V \cap S$ is a symplectic linear subspace of S .

The following example, which indicates that being a Dirac submanifold is indeed a global property, was pointed out to the author by Weinstein.

Example 2.17 Let $P = M \times C$, where each M -slice is a Poisson submanifold. Namely the Poisson tensor at each point $(x, t) \in M \times C$ is of the form $\pi(x, t) = \pi_t(x)$, where $\pi_t(x)$ is a family of C -dependent Poisson structures on M . Consider a particular M -slice $Q = M \times \{t_0\}$ which is a Poisson submanifold. We will investigate when Q becomes a Dirac submanifold.

Since we are only concerned with a small neighborhood of t_0 in C , we may identify C with \mathbb{R}^n by choosing a local coordinate system (t_1, \dots, t_n) . If Q is a Dirac submanifold, then $T_Q P = TQ \oplus V_Q$ for some vector bundle V_Q along Q . Hence V_Q must be of the form:

$$V_Q = \left\{ \frac{\partial}{\partial t_i} + X_i \mid i = 1, \dots, n \right\},$$

where X_i , $i = 1, \dots, n$, are some vector fields on M . Clearly Condition (ii) in Proposition 2.8 is satisfied automatically. Thus according to Remark 2.9, for Q to be a Dirac submanifold, it suffices that $\text{pr}_*[\frac{\partial}{\partial t_i} + X_i, \pi_t(x)] = 0$, for $i = 1, \dots, n$, which is equivalent to

$$\frac{\partial \pi_t(x)}{\partial t_i} \Big|_{t=t_0} = -[X_i, \pi_{t_0}(x)], \quad i = 1, \dots, n. \quad (6)$$

This equation precisely means that $\frac{\partial \pi_t(x)}{\partial t_i} \Big|_{t=t_0}$ is a coboundary with respect to the Poisson cohomology operator defined by π_{t_0} . Thus we conclude that

Q is a Dirac submanifold iff the map $f : T_{t_0}C \longrightarrow H^2_{\pi_{t_0}}(M) : v \longrightarrow [v(\pi_t)]$ vanishes.

Note that $v(\pi_t)$ is always a 2-cocycle with respect to the Poisson cohomology operator $[\pi_{t_0}, \cdot]$ because of the identity $[\pi_t, \pi_t] = 0$.

As a special case, let us consider the situation where all M -slices are symplectic leaves. Then one obtains a map $\varphi : C \longrightarrow H^2(M)$ by taking the symplectic class of the fiber. On the other hand, it is known that $H^2_{\pi_{t_0}}(M)$ is canonically isomorphic to $H^2(M)$. By identifying these two cohomology groups, we have

$$f = -\varphi_*. \quad (7)$$

To see this relation, let ω_t denote the leafwise symplectic forms, and let $\omega_t^b : TM \longrightarrow T^*M$ and $\pi_t^\# : T^*M \longrightarrow TM$ be the induced bundle maps by ω_t and π_t , respectively. It follows from the equation $\omega_t^b \circ \pi_t^\# = 1$ that $(v(\pi_t))^\# = -\pi_t^\# \circ (v(\omega_t))^b \circ \pi_t^\#$, for any $v \in T_{t_0}C$. Equation (7) thus follows immediately.

Hence we conclude that a symplectic leaf $M \times \{t_0\}$ is a Dirac submanifold iff t_0 is a critical point of the map φ . For instance, the symplectic leaves in the Lie-Poisson $\mathfrak{su}(2)$ can never be Dirac submanifolds except for the zero point.

Example 2.18 Let $P = \mathfrak{g}^*$ be a Lie-Poisson structure corresponding to a Lie algebra \mathfrak{g} . Consider an affine space $Q = \mu + V$. Assume that Q is a Dirac submanifold where V_Q can be taken constant. This amounts to saying that we have a decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ such that $V = \mathfrak{m}^\perp$ and $V_Q \cong Q \times \mathfrak{m}$ as a vector bundle. Let $\{e_1, \dots, e_k\}$ be a basis of \mathfrak{l} and $\{m_1, \dots, m_t\}$ a basis of \mathfrak{m} . Then $\{e_1, \dots, e_k, m_1, \dots, m_t\}$ consists of a basis of \mathfrak{g} . Now let $\{\lambda_1, \dots, \lambda_k, r_1, \dots, r_t\}$ be its corresponding linear coordinates on \mathfrak{g}^* . Thus their Poisson brackets are given by

$$\{\lambda_i, \lambda_j\} = a_{ij}^k \lambda_k + b_{ij}^k r_k, \quad \{\lambda_i, r_j\} = c_{ij}^k \lambda_k + d_{ij}^k r_k,$$

where $a_{ij}^k, b_{ij}^k, c_{ij}^k, d_{ij}^k$ are constants. It is clear that $\{\lambda_1, \dots, \lambda_k\}$ is a set of coordinate functions on Q such that V_Q^\perp is spanned by $d\lambda_i, i = 1, \dots, k$. Since $d\{\lambda_i, \lambda_j\} = a_{ij}^k d\lambda_k + b_{ij}^k dr_k$, Condition (ii) of Proposition 2.12 implies that $b_{ij}^k = 0$. On the other hand, we have

$$X_{\lambda_i}|_Q = (\{\lambda_i, \lambda_j\} \frac{\partial}{\partial \lambda_j} + \{\lambda_i, r_j\} \frac{\partial}{\partial r_j})|_Q = (\{\lambda_i, \lambda_j\} \frac{\partial}{\partial \lambda_j} + (c_{ij}^l \lambda_l + d_{ij}^l \mu_l) \frac{\partial}{\partial r_j})|_Q,$$

where $\mu_l = r_l(\mu)$, $l = 1, \dots, t$. It thus follows that X_{λ_i} is tangent to Q iff $c_{ij}^k = 0$ and $d_{ij}^k \mu_k = 0$. The latter is equivalent to $\langle ad_{e_j}^* \mu, m_j \rangle = 0$. Therefore we conclude that $\mu + \mathfrak{m}^\perp$ is a Dirac submanifold with constant V_Q iff $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$ is a reductive decomposition (i.e., \mathfrak{l} is a Lie subalgebra and $[\mathfrak{l}, \mathfrak{m}] \subseteq \mathfrak{m}$), and $ad_{\mathfrak{l}}^* \mu \in \mathfrak{m}^\perp$. In this case, the induced Poisson structure can be identified with the Lie-Poisson structure on \mathfrak{l}^* .

2.3 Local Dirac submanifolds

Definition 2.19 *A submanifold Q of a Poisson manifold P is called a local Dirac submanifold if at each point of Q there is an open neighborhood which is a Dirac submanifold.*

Immediately we have

Proposition 2.20 *A local Dirac submanifold naturally carries an induced Poisson structure.*

Example 2.21 If Q is a symplectic leaf of P , by Weinstein splitting theorem [30], locally $P \cong Q \times N$ as a product Poisson manifold. It thus follows that Q is a local Dirac submanifold.

The following proposition gives a characterization of local Dirac submanifolds.

Proposition 2.22 *A submanifold Q of a Poisson manifold P is a Dirac submanifold if there exist local coordinates $(x_1, \dots, x_k, y_1, \dots, y_t)$ of P at any point $q \in Q$ such that Q is defined by $y_1 = \dots = y_t = 0$ and the Poisson bracket between coordinate functions satisfy:*

$$\lambda_{ij}(x, 0) = 0, \quad \forall 1 \leq i \leq k, \quad 1 \leq j \leq t; \quad \frac{\partial \varphi_{ij}}{\partial y_l}(x, 0) = 0, \quad \forall 1 \leq i, j \leq k, \quad 1 \leq l \leq t, \quad (8)$$

where $\varphi_{ij} = \{x_i, x_j\}$, $\forall 1 \leq i, j \leq k$, and $\lambda_{ij} = \{x_i, y_j\}$, $\forall 1 \leq i \leq k, \quad 1 \leq j \leq t$.

Proof. Assume that Q is a local Dirac submanifold. Given any point $q \in Q$, there exists an open neighborhood U of q in P such that $U \cap Q$ is a Dirac submanifold. Let $V_{U \cap Q}$ denote the subbundle as in the decomposition (4). By shrinking U to a smaller open neighborhood of q if necessary, one may always choose local coordinates $(x_1, \dots, x_k, y_1, \dots, y_t)$ of U such that $U \cap Q$ is defined by $y_1 = \dots = y_t = 0$ and $V_{U \cap Q}$ is spanned by $\{\frac{\partial}{\partial y_i} | i = 1, \dots, t\}$. In other words, $\{x_1, \dots, x_k\}$ is a set of coordinates on Q such that $V_{U \cap Q}^\perp$ is spanned by $\{dx_i | i = 1, \dots, k\}$. Then we have

$$d\{x_i, x_j\}|_Q = \frac{\partial \varphi_{ij}}{\partial y_l}(x, 0) dy_l, \quad (\text{mod } dx_i); \quad X_{x_i}|_Q = \lambda_{ij}(x, 0) \frac{\partial}{\partial y_j}, \quad (\text{mod } \frac{\partial}{\partial x_i}).$$

It thus follows that $\lambda_{ij}(x, 0) = 0, \quad 1 \leq i \leq k, \quad 1 \leq j \leq t$ and $\frac{\partial \varphi_{ij}}{\partial y_l}(x, 0) = 0, \quad 1 \leq i, j \leq k, \quad 1 \leq l \leq t$.

Conversely, if such local coordinates exist in an open neighborhood U of q in P , one can show that $U \cap Q$ is a Dirac submanifold.

□

The following result reveals a connection between local Dirac submanifolds and transverse Poisson structures [30].

Proposition 2.23 *If Q is a local Dirac submanifold which is a cross section of a symplectic leaf S at a point q (i.e., Q has complementary dimension to S and intersects with S at a single point q*

transversely), then the induced Poisson structure on Q in a neighborhood of q is isomorphic to the transverse Poisson structure.

Conversely, if Q is a cross section of a symplectic leaf S at a point q , then Q is a Dirac submanifold in a neighborhood of q and the induced Poisson structure is isomorphic to the transverse Poisson structure.

Proof. From Weinstein splitting theorem [30], it follows that a cross section of a symplectic leaf S must be a Dirac submanifold in a small neighborhood of the intersection point. It remains to show that the induced Poisson structure on Q as a Dirac submanifold is indeed isomorphic to the transverse Poisson structure.

We choose local coordinates as in the proof of Proposition 2.22. Thus X_{x_i} are all tangent to Q for $i = 1, \dots, k$. By definition, the transverse Poisson structure is $\{x_i, x_j\}|_Q = \varphi_{ij}(x, 0)$, which is precisely the induced Poisson structure on Q as a Dirac submanifold.

□

An immediate consequence, by combing with Example 2.18, is the following theorem of Molino [26] and Weinstein [30].

Corollary 2.24 *Let $\mu \in \mathfrak{g}^*$ and \mathfrak{g}_μ be the isotropic Lie algebra at μ . If \mathfrak{g} admits a reductive decomposition: $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m}_\mu$, then the transverse Poisson structure at μ to the symplectic leaf $G \cdot \mu$ (i.e., the coadjoint orbit through μ) is isomorphic to the Lie-Poisson structure on \mathfrak{g}_μ^* .*

3 Properties of Dirac submanifolds

This section is devoted to the further study on properties of Dirac submanifolds.

3.1 Relative modular vector fields

First we want to see how modular class of a Dirac submanifold is related to that of P . We start with the following:

Lemma 3.1 *Let Q be a Dirac submanifold of a Poisson manifold P . Assume that $f \in C^\infty(P)$ satisfies the property $df|_Q \in V_Q^\perp$. Denote by φ_t the flow generated by the Hamiltonian vector field X_f . Then both TQ and V_Q (hence TQ^\perp and V_Q^\perp) are stable under φ_t .*

Proof. It is clear that X_f is tangent to Q , and therefore $[X_f, Y]|_Q$ is well-defined for any $Y \in \Gamma(T_Q P)$. If $Y \in \Gamma(TQ)$, clearly $[X_f, Y]|_Q \in \Gamma(TQ)$. Hence TQ is stable under φ_t .

Now assume that $Y \in \Gamma(V_Q)$. Let $\tilde{Y} \in \mathfrak{X}(P)$ be any of its extension. By the graded Jacobi identity, we have

$$[X_f, \tilde{Y}] = [[\pi, f], \tilde{Y}] = [[f, \tilde{Y}], \pi] - [[\tilde{Y}, \pi], f].$$

Now $[[f, \tilde{Y}], \pi] = -[\tilde{Y}(f), \pi] = -[\tilde{Y}(f), \pi'' + \tilde{\pi}'] = -[\tilde{Y}(f), \pi''] - [\tilde{Y}(f), \tilde{\pi}']$, where π'' and $\tilde{\pi}'$ are bivector fields on P as in the proof of Lemma 2.5, i.e., $\pi''|_Q = \pi_Q$ and $\tilde{\pi}' \in \Gamma(\wedge^2 V_Q)$. Since $\tilde{Y}(f)|_Q = Y(f)|_Q = 0$, it is obvious that $[\tilde{Y}(f), \pi'']|_Q = [\tilde{Y}(f), \pi_Q] = 0$. Thus $[[f, \tilde{Y}], \pi]|_Q \in \Gamma(V_Q)$. On the other hand, according to Equation (5), we have $\text{pr}_*[\tilde{Y}, \pi] = [\text{pr}_* \tilde{Y}, \pi_Q] = 0$. Therefore, one can write $[\tilde{Y}, \pi]|_Q = \sum Z_i \wedge Z'_i$ with $Z'_i \in \Gamma(V_Q)$. Then $[[\tilde{Y}, \pi], f]|_Q = \sum (Z_i(f)Z'_i - Z'_i(f)Z_i) = \sum Z_i(f)Z'_i \in \Gamma(V_Q)$, since $Z'_i(f) = 0$ by assumption. This shows that $[X_f, Y]|_Q = [X_f, \tilde{Y}]|_Q \in \Gamma(V_Q)$, which implies that V_Q is stable under the flow φ_t . This concludes the proof. \square

We are now ready to introduce the relative modular class. Let $\Omega' \in \Gamma(\wedge^{\text{top}} TQ^\perp) \cong \Gamma(\wedge^{\text{top}} V_Q^*)$ be a nonzero section, which we always assume exist. Otherwise, one needs to consider densities as in [32]. For any $f \in C^\infty(Q)$, let $\tilde{f} \in C^\infty(P)$ be an extension of f satisfying the property $d\tilde{f}|_Q \in V_Q^\perp$. Then according to Lemma 3.1 the Hamiltonian flow of $X_{\tilde{f}}$ preserves both vector bundles TQ and V_Q , hence it preserves TQ^\perp . It thus follows that $L_{X_{\tilde{f}}}\Omega'$ is a section of $\wedge^{\text{top}} TQ^\perp$, and therefore $(L_{X_{\tilde{f}}}\Omega')/\Omega'$ is a well-defined function on Q . Also it is clear that this function only depends on the 1-jet of \tilde{f} along Q , and therefore is independent of the extension. Thus one obtains a linear map $\nu_r : C^\infty(Q) \rightarrow C^\infty(Q)$, $f \mapsto (L_{X_{\tilde{f}}}\Omega')/\Omega'|_Q$. From the fact that $X_{\tilde{f}}|_Q$ is tangent to Q and $\Omega' \in \Gamma(\wedge^{\text{top}} TQ^\perp)$, one can easily show that

$$\nu_r(fg) = f\nu_r(g) + g\nu_r(f), \quad \forall f, g \in C^\infty(Q).$$

Hence ν_r is a vector field on Q , which will be called *the relative modular vector field* corresponding to Ω' .

Proposition 3.2 ν_r is a Poisson vector field with respect to π_Q . For different choices of Ω' , the corresponding relative modular vector fields ν_r differ by a Hamiltonian vector field.

As a consequence, $[\nu_r]$ is a well defined class in the Poisson cohomology $H^1_{\pi_Q}(Q)$, which will be called *the relative modular class* of the Dirac submanifold Q . The proof of Proposition 3.2 follows from the lemma below.

Choose a nonzero section $\Omega_Q \in \Gamma(\wedge^{\text{top}} V_Q^\perp) \cong \Gamma(\wedge^{\text{top}} T^*Q)$, which we again assume exist. Then $\Omega = \Omega_Q \wedge \Omega' \in \Gamma(\wedge^{\text{top}} T^*P|_Q)$ is a nonzero section. Extend Ω to a volume form on P (at least locally along the submanifold Q), which will be denoted by the same symbol Ω . By ν_P and ν_Q , we denote the modular vector fields of the Poisson manifolds P and Q corresponding to Ω and Ω_Q , respectively.

Lemma 3.3 *The modular vector fields are related by*

$$\nu_r = \text{pr}_* \nu_P - \nu_Q. \quad (9)$$

Proof. $\forall f \in C^\infty(Q)$, let $\tilde{f} \in C^\infty(P)$ be an extension of f satisfying the property $d\tilde{f}|_Q \in V_Q^\perp$. Then $L_{X_{\tilde{f}}}\Omega|_Q = \nu_P(\tilde{f})\Omega|_Q = (\text{pr}_* \nu_P)(f)\Omega|_Q$, and $L_{X_{\tilde{f}}}\Omega_Q|_Q = \nu_Q(f)\Omega_Q$. From the derivation law: $L_{X_{\tilde{f}}}\Omega = (L_{X_{\tilde{f}}}\Omega_Q) \wedge \Omega' + \Omega_Q \wedge L_{X_{\tilde{f}}}\Omega'$, it follows that $\text{pr}_* \nu_P(f) = \nu_Q(f) + \nu_r(f)$. Equation (9) thus follows.

□

Another consequence, besides Proposition 3.2, is the following:

Proposition 3.4 *The modular classes of the Poisson structures on P and Q are related by*

$$pr_*[\nu_P] - [\nu_Q] = [\nu_r], \quad (10)$$

where $pr_* : H_\pi^1(P) \rightarrow H_{\pi_Q}^1(Q)$ is the morphism as in Corollary 2.6.

Remark 3.5 It would be interesting to see how other characteristic classes [6, 13] on P and Q are related, and in particular, how to describe $pr_*[C_k(P)] - [C_k(Q)] \in H_{\pi_Q}^*(Q)$ for other characteristic class C_k .

3.2 Poisson actions

Next we consider Poisson actions on Dirac submanifolds. As we shall see below, Dirac submanifolds indeed behave nicely under Poisson group actions, which include the usual Hamiltonian actions as a special case.

Theorem 3.6 *Assume that (P, π) is a Poisson manifold which admits a Poisson action of a Poisson group G . Assume that Q is a Dirac submanifold stable under the G -action. Then the action of G on Q is also a Poisson action. Moreover, if $J : P \rightarrow G^*$ is a momentum map, then $J|_Q : Q \rightarrow G^*$ is a momentum map of the G -action on Q .*

Proof. Let $\mu_P : T^*P \rightarrow \mathfrak{g}^*$ and $\mu_Q : T^*Q \rightarrow \mathfrak{g}^*$ be the linear morphisms dual to the infinitesimal \mathfrak{g} -actions on P and Q , respectively. Since the infinitesimal \mathfrak{g} -action on Q : $\mathfrak{g} \rightarrow \mathfrak{X}(Q)$ is the composition of the infinitesimal \mathfrak{g} -action on P : $\mathfrak{g} \rightarrow \mathfrak{X}(P)$ with the projection $pr_* : \mathfrak{X}(P) \rightarrow \mathfrak{X}(Q)$, it follows that $\mu_Q = \mu_P \circ pr^*$, where $pr^* : T^*Q \rightarrow T^*P$ is the dual of the projection $pr : T_Q P \rightarrow TQ$. Since pr^* is a Lie algebroid morphism according to Theorem 2.3 (iii), it follows immediately from Proposition 6.1 in [34] that the G -action on Q is also a Poisson action.

Assume that $J : P \rightarrow G^*$ is a momentum map for the Poisson G -action [20]. I.e., for any $\xi \in \mathfrak{g}$, $\pi^\#(J^*\xi^l) = \hat{\xi}$, where $\xi^l \in \Omega^1(G^*)$ is the left invariant one-form corresponding to ξ , and $\hat{\xi} \in \mathfrak{X}(P)$ is the vector field on P generated by ξ . Then we have $pr_*\pi^\#(J^*\xi^l) = \hat{\xi}$ since $\hat{\xi}$ is tangent to Q . On the other hand, it is clear that $pr_*\pi^\#(J^*\xi^l) = \pi_Q^\#(J^*\xi^l)$. This shows that $J|_Q : Q \rightarrow G^*$ is indeed a momentum map for the Poisson G -action on Q .

□

3.3 Symplectic subgroups

Finally we consider symplectic groupoids of Dirac submanifolds. As we see below, Dirac submanifolds are indeed infinitesimal version of symplectic subgroups.

Theorem 3.7 *If $\Gamma' \rightrightarrows Q$ is a symplectic subgroupoid of a symplectic groupoid $(\Gamma \rightrightarrows P, \alpha, \beta)$, then Q is a Dirac submanifold of P . Conversely, if P is an integrable Poisson manifold with symplectic groupoid Γ and Q is a Dirac submanifold whose corresponding cotangent Lie algebroid T^*Q integrates to a Lie subgroupoid Γ' of Γ , then Γ' is a symplectic subgroupoid.*

Proof. Assume that $\Gamma' \rightrightarrows Q$ is a symplectic subgroupoid of a symplectic groupoid $(\Gamma \rightrightarrows P, \alpha, \beta)$. By ω and ω' we denote the symplectic forms on Γ and Γ' respectively, and by A and A' , we denote their corresponding Lie algebroids. Then A' is a Lie subalgebroid of A . As vector bundles, $A \cong T_P^\alpha \Gamma$ and $A' \cong T_Q^\alpha \Gamma'$, and the Lie algebroid morphism $A' \rightarrow A$ is simply the inclusion: $T_Q^\alpha \Gamma' \rightarrow T_P^\alpha \Gamma$. It is well known that $\omega^b : T_P^\alpha \Gamma \rightarrow T^*P$ and $(\omega')^b : T_Q^\alpha \Gamma' \rightarrow T^*Q$ are isomorphisms of Lie algebroids, where T^*P and T^*Q are equipped with the cotangent Lie algebroids corresponding to the induced Poisson structures. Thus, one obtains a Lie algebroid morphism $\varphi : T^*Q \rightarrow T^*P$ so that the following diagram

$$\begin{array}{ccc}
T_Q^\alpha \Gamma' & \longrightarrow & T_P^\alpha \Gamma \\
(\omega')^b \downarrow & & \downarrow \omega^b \\
T^*Q & \xrightarrow{\varphi} & T^*P
\end{array} \tag{11}$$

commutes. In particular, $\varphi(T^*Q)$ is a Lie subalgebroid of T^*P . In what follows, we will show that $\varphi^* \circ i$ is the identity map, where $i : TQ \rightarrow TP$ is the inclusion.

Let $\xi \in T_x^*Q$ be any covector. Assume that $\xi = (\omega')^b u$ for some $u \in T_q^\alpha \Gamma'$. Then using the commuting diagram (11), we have, for any $v \in T_x Q$,

$$\langle (i^* \circ \varphi) \xi, v \rangle = \langle \varphi \xi, v \rangle = \langle \varphi (\omega')^b u, v \rangle = \langle \omega^b u, v \rangle = \omega(u, v) = \omega'(u, v) = \langle (\omega')^b u, v \rangle = \langle \xi, v \rangle.$$

Therefore $i^* \circ \varphi = id$, or equivalently $\varphi^* \circ i = id$. Let $V_Q = \ker \varphi^*$, which is a subbundle of $T_Q P$. Then $T_Q P = TQ \oplus V_Q$. In fact $V_Q^\perp = \varphi(T^*Q)$, so V_Q^\perp is a Lie subalgebroid of T^*P . Hence Q is a Dirac submanifold.

Conversely, assume that Q is a Dirac submanifold of P , and $\varphi = \text{pr}^* : T^*Q \rightarrow T^*P$ is the Lie algebroid morphism as in Theorem 2.3 (iii). Let $\Gamma' \subset \Gamma$ be a Lie subgroupoid integrating the Lie subalgebroid $\varphi(T^*Q)$. For any $x \in Q$, we have $T_x \Gamma = T_x P \oplus T_x^\alpha \Gamma$ and $T_x \Gamma' = T_x Q \oplus T_x^\alpha \Gamma'$. By identifying $T_x^\alpha \Gamma$ with T_x^*P via ω^b as above, one obtains a decomposition $T_x \Gamma \cong T_x P \oplus T_x^*P$, under which the symplectic form $\omega_x \in \wedge^2 T_x^* \Gamma$ takes the form:

$$\begin{pmatrix} 0 & I \\ -I & \pi(x) \end{pmatrix} \tag{12}$$

Now $T_x P = T_x Q \oplus V_x$ and $T_x^*P = T_x Q^\perp \oplus V_x^\perp \cong V_x^* \oplus T_x^*Q$. Thus $T_x \Gamma \cong T_x Q \oplus V_x \oplus V_x^* \oplus T_x^*Q$. It is clear that under this decomposition $T_x \Gamma'$ corresponds to the subspace $T_x Q \oplus T_x^*Q$. Thus the restriction of ω_x to the subspace $T_x \Gamma'$ has the form:

$$\begin{pmatrix} 0 & I \\ -I & \pi_Q(x) \end{pmatrix}, \tag{13}$$

which is clearly non-degenerate. It follows immediately that the pull back of the symplectic form ω is non-degenerate along the identity section Q . To show its non-degeneracy at every point of Γ' , it suffices to show that through each point of Γ' , there exists a Lagrangian (local) bisection S of Γ such that $S|_Q$ is a bisection of Γ' . This is true since any closed one-form on Q extends to a closed one-form on P .

□

4 Poisson involutions

This section is devoted to the study on a special class of Dirac submanifolds arising as the stable locus of a Poisson involution. In particular, we discuss Poisson involutions on Poisson groupoids as well as on Poisson groups. As we will see, such involutions do often exist. Examples include the standard Poisson group structures on semi-simple Lie groups, Bruhat Poisson structures on compact semi-simple Lie groups, and Poisson groupoids connecting with dynamical r -matrices of semi-simple Lie algebras.

4.1 Stable locus of a Poisson involution

Recall that a Poisson involution on a Poisson manifold P is a Poisson diffeomorphism $\Phi : P \rightarrow P$ such that $\Phi^2 = id$. Another important class of Dirac manifolds arises as follows.

Proposition 4.1 *Let $\Phi : P \rightarrow P$ be a Poisson involution. Then its stable locus Q is a Dirac submanifold.*

Proof. It is well known that Q is a smooth manifold. For any $x \in Q$, since the linear morphism $\Phi_* : T_x P \rightarrow T_x P$ is an involution, its eigenvalues are either $+1$ or -1 . Let V_x denote the -1 -eigenspace of Φ_* , and $V_Q = \cup_{x \in Q} V_x$. Clearly, $T_x Q$ coincides with the $+1$ -eigenspace of Φ_* , and $T_x P = T_x Q \oplus V_x$. Since $\Phi_* \pi = \pi$, it is clear that $\pi|_Q = \pi_Q + \pi'$, where $\pi_Q \in \Gamma(\wedge^2 TQ)$ and $\pi' \in \Gamma(\wedge^2 V_Q)$. It remains to verify Condition (iii) in Proposition 2.8. For this, notice that any vector field X on P can be decomposed as $X = X^+ + X^-$ where $\Phi_* X^+ = X^+$ and $\Phi_* X^- = -X^-$. Indeed,

$$X^+ = \frac{1}{2}(X + \Phi_* X) \quad \text{and} \quad X^- = \frac{1}{2}(X - \Phi_* X). \quad (14)$$

It thus suffices to prove that $\text{pr}_*[X^-, \pi] = 0$. This is obvious since $\Phi_*[X^-, \pi] = [\Phi_* X^-, \Phi_* \pi] = -[X^-, \pi]$.

□

Remark 4.2 The fact that the stable locus of a Poisson involution inherits a Poisson structure was already hidden in the work of Bondal [4] and Boalch [2] in their study of the Poisson structures on Stokes matrices. On the other hand, an algebraic version of this fact appeared in the work of Fernandes-Vanhaecke [14].

The Poisson structure on Q indeed can be described more explicitly in this case.

Proposition 4.3 *Let Q be the stable locus of a Poisson involution $\Phi : P \rightarrow P$. Assume that the Poisson tensor π on P is $\pi = \sum_i X_i \wedge Y_i$, where X_i and Y_i are vector fields on P . Then the Poisson tensor π_Q on Q is given by $\pi_Q = \sum_i X_i^+ \wedge Y_i^+|_Q$, where X_i^+ and Y_i^+ are defined by Equation (14).*

As a consequence of Theorem 3.7, we have the following

Corollary 4.4 *If Q is the stable locus of a Poisson involution on an integrable Poisson manifold P , then Q is always an integrable Poisson manifold itself.*

Proof. Assume that Q is the stable locus of a Poisson involution $\Phi : P \rightarrow P$. Let Γ be an α -connected and simply connected symplectic groupoid of P . To the Poisson involution $\Phi : P \rightarrow P$, there corresponds to an involutive symplectic groupoid automorphism $\tilde{\Phi} : \Gamma \rightarrow \Gamma$. Then the stable locus of $\tilde{\Phi}$, which is a smooth manifold, is a symplectic subgroupoid of Γ integrating Q .

□

4.2 Poisson involutions on Poisson groupoids

For Poisson groupoids, there is an effective way of producing Poisson involutions. This is via the so called symmetric Poisson groupoids. Symmetric Poisson groups and their infinitesimal version: symmetric Lie bialgebras, were studied by Fernandes [11, 12]¹

Definition 4.5 (i). *A symmetric Poisson groupoid consists of a pair (Γ, Φ) , where Γ is a Poisson groupoid and $\Phi : \Gamma \rightarrow \Gamma$ is a groupoid anti-morphism which is also a Poisson involution.*

(ii). *A symmetric Lie bialgebroid consists of a triple (A, A^*, φ) , where (A, A^*) is a Lie bialgebroid and $\varphi : A \rightarrow A$ is an involutive Lie algebroid anti-morphism such that $\varphi^* : A^* \rightarrow A^*$ is a Lie algebroid morphism.*

Theorem 4.6 *Under the assumption that the relevant Lie algebroid is integrable, there is one-to-one correspondence between α -simply connected symmetric Poisson groupoids and symmetric Lie bialgebroids.*

Proof. Assume that (A, A^*, φ) is a symmetric Lie bialgebroid. Let Γ be an α -simply connected Poisson groupoid corresponding to the Lie bialgebroid (A, A^*) . It is known that any Lie algebroid isomorphism integrates to a Lie groupoid isomorphism for α -simply connected Lie groupoids. Hence the Lie algebroid involution $\varphi' = -\varphi : A \rightarrow A$ integrates to a Lie groupoid involution $\Phi' : \Gamma \rightarrow \Gamma$. By assumption, $(\varphi')^* = (-\varphi)^* = -\varphi^*$ is a Lie algebroid anti-morphism. By the Poisson groupoid duality [23, 24], Φ' is an anti-Poisson map. Let $\tau : \Gamma \rightarrow \Gamma$ be the map: $\tau(g) = g^{-1}$, $\forall g \in \Gamma$,

¹Note that our definition here, however, is precisely the opposite to that in [11, 12]. We require that Φ be group(oid) anti-morphism and Poisson, while in [11, 12] Φ is required to be group morphism and anti-Poisson.

which is clearly a groupoid anti-morphism and an anti-Poisson map. Set $\Phi = \Phi' \circ \tau$. Then Φ is an integration of φ , which possesses all the required properties.

Conversely, if (Γ, Φ) is a symmetric Poisson groupoid, then it is clear that (A, A^*, φ) is a symmetric Lie bialgebroid, where $\varphi : A \rightarrow A$ is the derivative of Φ .

□

Remark 4.7 Note that the roles of A and A^* can be switched for a symmetric Lie bialgebroid. Namely, if (A, A^*, φ) is a symmetric Lie bialgebroid, then $(A^*, A, -\varphi^*)$ is also a symmetric Lie bialgebroid. This means that from a symmetric Lie bialgebroid one can in fact construct a pair of Poisson involutions: one on Γ and the other on its dual Poisson groupoid Γ^* (provided that both A and A^* are integrable).

Theorem 4.6 indicates that a useful source of producing Poisson involutions on Poisson groupoids is to construct symmetric Lie bialgebroids. Next we will consider the case of coboundary Lie bialgebroids [18], namely those Lie bialgebroids (A, A^*) where the Lie algebroid on the dual A^* is generated by an r -matrix $\Lambda \in \Gamma(\wedge^2 A)$ with the property $[X, [\Lambda, \Lambda]] = 0, \forall X \in \Gamma(A)$.

Proposition 4.8 *A coboundary Lie bialgebroid (A, A^*) with an r -matrix $\Lambda \in \Gamma(\wedge^2 A)$ is a symmetric Lie bialgebroid if there is an involutive Lie algebroid anti-morphism $\varphi : A \rightarrow A$ such that $\varphi\Lambda = -\Lambda$.*

Proof. Let $d_* : \Gamma(\wedge^* A) \rightarrow \Gamma(\wedge^{*+1} A)$ be the exterior differential induced from the Lie algebroid structure on A^* . Then for any $X \in \Gamma(\wedge^* A)$, $d_* X = [\Lambda, X]$. Hence $(\varphi \circ d_*)X = \varphi[\Lambda, X] = -[\varphi\Lambda, \varphi X] = [\Lambda, \varphi X] = (d_* \circ \varphi)X$, which implies that $\varphi \circ d_* = d_* \circ \varphi$. Hence $\varphi^* : A^* \rightarrow A^*$ is a Lie algebroid morphism.

□

4.3 Symmetric Courant algebroids

A nice way of understanding a Lie bialgebroid (A, A^*) is via its double $E = A \oplus A^*$, which is a Courant algebroid [16]. Roughly, a Courant algebroid is a vector bundle $E \rightarrow M$ equipped with a non-degenerate symmetric bilinear form (\cdot, \cdot) of signature (n, n) on the fibers, a bundle map $\rho : E \rightarrow TM$, and a bracket $[\cdot, \cdot]$ on $\Gamma(E)$, which satisfy some complicated compatibility conditions resembling that of a Lie algebroid up to a homotopy. Lie bialgebroids precisely correspond to splittable Courant algebroids, namely those which admit two transversal Dirac structures. We refer the reader to [16] for details.

Definition 4.9 (i). *A symmetric Courant algebroid is a Courant algebroid $(E, (\cdot, \cdot), \rho, [\cdot, \cdot])$ together with an involutive anti-morphism $\chi : E \rightarrow E$, i.e.,*

$$\rho \circ \chi = -f_* \circ \chi; \quad (\chi e_1, \chi e_2) = -(e_1, e_2); \quad \text{and } \chi[e_1, e_2] = -[\chi e_1, \chi e_2]$$

for any $e_1, e_2 \in \Gamma(E)$, where $f : M \rightarrow M$ is the base map corresponding to χ ;

(ii). *A symmetric splittable Courant algebroid is a symmetric Courant algebroid (E, χ) , such that E admits a pair of χ -stable transversal Dirac structures.*

Theorem 4.10 *There is a one-one correspondence between symmetric Lie bialgebroids and symmetric splittable Courant algebroids.*

Proof. Assume that (A, A^*, φ) is a symmetric Lie bialgebroid. Let M denote the base of the Lie bialgebroid (A, A^*) , and a, a_* their anchors respectively. Denote, by $f : M \rightarrow M$, the involution on the base manifold corresponding to φ . Then φ^* is a bundle map over the same base map $f : M \rightarrow M$ since f is an involution. Let $E = A \oplus A^*$ be the double of the Lie bialgebroid, which is a Courant algebroid [16] over the base manifold M , with anchor $\rho = a + a_*$. Define $\chi : E \rightarrow E$ by

$$\chi(X + \xi) = \varphi X - \varphi^* \xi, \quad \forall X \in A|_m \text{ and } \xi \in A^*|_m. \quad (15)$$

Then χ is clearly an involutive bundle map over the base map $f : M \rightarrow M$. It is also simple to check that χ anti-commutes with the anchor on E , and $\langle \chi e_1, \chi e_2 \rangle = -\langle e_1, e_2 \rangle$ for any $e_1, e_2 \in \Gamma(E)$. It remains to check that $\langle \chi e_1, \chi e_2 \rangle = -\langle e_1, e_2 \rangle$ for any $e_1, e_2 \in \Gamma(E)$. To this end, it suffices to show that $\langle \chi X, \chi \xi \rangle = -\langle X, \xi \rangle$ for any $X \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$. First, we will need the following identities:

$$L_{\varphi^* \xi} \varphi X = \varphi(L_\xi X); \quad (16)$$

$$L_{\varphi X} \varphi^* \xi = -\varphi^*(L_X \xi). \quad (17)$$

Note that for any $\eta \in \Gamma(A^*)$,

$$\begin{aligned} & \langle L_{\varphi^* \xi} \varphi X, \eta \rangle \\ &= \langle a_* \varphi^* \xi \rangle \langle \varphi X, \eta \rangle - \langle \varphi X, [\varphi^* \xi, \eta] \rangle \\ &= f_*(a_* \xi) \langle \varphi X, \eta \rangle - \langle \varphi X, [\varphi^* \xi, \eta] \rangle \\ &= \langle a_* \xi \rangle \langle X, \varphi^* \eta \rangle - \langle X, \varphi^* [\varphi^* \xi, \eta] \rangle \\ &= \langle a_* \xi \rangle \langle X, \varphi^* \eta \rangle - \langle X, [\xi, \varphi^* \eta] \rangle \\ &= \langle L_\xi X, \varphi^* \eta \rangle \\ &= \langle \varphi(L_\xi X), \eta \rangle. \end{aligned}$$

Equation (16) thus follows. Equation (17) can be proved similarly. Now

$$\begin{aligned} & \langle \chi X, \chi \xi \rangle \\ &= -\langle \varphi X, \varphi^* \xi \rangle \\ &= L_{\varphi^* \xi} \varphi X - \frac{1}{2} d_* \langle \varphi^* \xi, \varphi X \rangle - L_{\varphi X} \varphi^* \xi + \frac{1}{2} d \langle \varphi^* \xi, \varphi X \rangle \quad (\text{by Equations (16)-(17)}) \\ &= \varphi(L_\xi X) - \frac{1}{2} \varphi d_* \langle \xi, X \rangle + \varphi^*(L_X \xi) - \frac{1}{2} \varphi^* d \langle \xi, X \rangle. \end{aligned}$$

On the other hand,

$$\chi[X, \xi]$$

$$\begin{aligned}
&= \chi[(-L_\xi X + \frac{1}{2}d_*\langle \xi, X \rangle) + (L_X \xi - \frac{1}{2}d\langle \xi, X \rangle)] \\
&= -\varphi(L_\xi X) + \frac{1}{2}\varphi d_*\langle \xi, X \rangle - \varphi^*(L_X \xi) + \frac{1}{2}\varphi^* d\langle \xi, X \rangle.
\end{aligned}$$

Thus $[\chi X, \chi \xi] = -\chi[X, \xi]$.

Conversely, assume that E is a splittable Courant algebroid such that $E = A \oplus A^*$ for a Lie bialgebroid (A, A^*) , and $\chi : E \rightarrow E$ is an involutive anti-morphism preserving both components A and A^* . Let $\varphi = \chi|_A : A \rightarrow A$ and $\psi = \chi|_{A^*} : A^* \rightarrow A^*$. Then both φ and ψ are involutive Lie algebroid anti-morphisms. For any $X \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$, since $(\chi \xi, \chi X) = -(\xi, X)$, and $\chi X = \varphi X$, $\chi \xi = \psi \xi$, it follows immediately that $\varphi^* \psi = -id$, which implies that $\psi = -\varphi^*$. This concludes the proof. □

4.4 Poisson involutions on dynamical Poisson groupoids

As a special case, we will consider dynamical Poisson groupoids introduced by Etingof-Varchenko [9]. Recall that a dynamical r -matrix is a function $r : \mathfrak{h}^* \rightarrow \wedge^2 \mathfrak{g}$ satisfying:

- (i). $r : \mathfrak{h}^* \rightarrow \wedge^2 \mathfrak{g}$ is H -equivariant;
- (ii). $\sum_i h_i \wedge \frac{\partial r}{\partial \lambda^i} - \frac{1}{2}[r, r]$ is a constant $(\wedge^2 \mathfrak{g})^{\mathfrak{g}}$ -valued function over \mathfrak{h}^* ,

where $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, H is the Lie subgroup of G with Lie algebra \mathfrak{h} , $\{h_1, \dots, h_k\}$ is a basis of \mathfrak{h} , and $\{\lambda_1, \dots, \lambda_k\}$ is its induced coordinates on \mathfrak{h}^* .

It is known [1, 19] that a dynamical r -matrix naturally defines a coboundary Lie bialgebroid (A, A^*, Λ) , where $A = T\mathfrak{h}^* \times \mathfrak{g}$, and $\Lambda = \pi_{\mathfrak{h}^*} + \sum_{i=1}^k (\frac{\partial}{\partial \lambda^i} \wedge h_i) + r(\lambda) \in \Gamma(\wedge^2 A)$. Here $\pi_{\mathfrak{h}^*}$ is the Lie-Poisson tensor on \mathfrak{h}^* .

The following theorem can be verified directly.

Theorem 4.11 *Let $r : \mathfrak{h}^* \rightarrow \wedge^2 \mathfrak{g}$ be a dynamical r -matrix. Assume that $s : \mathfrak{g} \rightarrow \mathfrak{g}$ is an involutive Lie algebra anti-morphism, which preserves \mathfrak{h} and satisfies the property $s(r(\lambda)) = -r(s^* \lambda)$, $\forall \lambda \in \mathfrak{h}^*$. Here $s_{\mathfrak{h}} : \mathfrak{h} \rightarrow \mathfrak{h}$ is the restriction of s to \mathfrak{h} . Then $(T\mathfrak{h}^* \times \mathfrak{g}, T^*\mathfrak{h}^* \times \mathfrak{g}^*, \varphi)$, where $\varphi = (-Ts_{\mathfrak{h}}^*, s) : T\mathfrak{h}^* \times \mathfrak{g} \rightarrow T\mathfrak{h}^* \times \mathfrak{g}$, is a symmetric Lie bialgebroid.*

Corollary 4.12 *Under the same hypothesis as in Theorem 4.11, let $S : G \rightarrow G$ be the group anti-morphism corresponding to s . Then,*

- (i). $\Phi : \mathfrak{h}^* \times \mathfrak{h}^* \times G \rightarrow \mathfrak{h}^* \times \mathfrak{h}^* \times G$, $\Phi(u, v, g) = (s_{\mathfrak{h}}^* v, s_{\mathfrak{h}}^* u, S(g))$, $\forall u, v \in \mathfrak{h}^*$ and $g \in G$, is a Poisson involution of the dynamical Poisson groupoid $\mathfrak{h}^* \times \mathfrak{h}^* \times G$.
- (ii). $\Gamma^0 = \{(u, s_{\mathfrak{h}}^* u, g) | \forall u \in \mathfrak{h}^*, g \in G^0\}$ is a Dirac submanifold, where $G^0 \subset G$ is the stable locus of S .

Example 4.13 Let \mathfrak{g} be a semi-simple Lie algebra over \mathbb{C} of rank k with a Cartan subalgebra \mathfrak{h} . Let $\{e_\alpha, f_\alpha, h_i | \alpha \in \Delta_+, 1 \leq i \leq k\}$ be a Chevalley basis. Then

$$r(\lambda) = \sum_{\alpha \in \Delta_+} d_\alpha \coth\left(\frac{1}{2} \langle \alpha, \lambda \rangle\right) e_\alpha \wedge f_\alpha$$

is a dynamical r -matrix over \mathfrak{h}^* , where $(e_\alpha, f_\alpha) = d_\alpha$, and $\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ is the hyperbolic cotangent function [9].

Let $s : \mathfrak{g} \rightarrow \mathfrak{g}$ be a \mathbb{C} -linear morphism, which, on generators, is defined as follows²:

$$s e_\alpha = f_\alpha, \quad s f_\alpha = e_\alpha, \quad s h_i = h_i. \quad (18)$$

It is clear that s is an involutive Lie algebra anti-morphism and $s|_{\mathfrak{h}} = id$. Moreover, it is also simple to see that $s(r(\lambda)) = -r(\lambda)$ for any $\lambda \in \mathfrak{h}^*$. Therefore, according to Theorem 4.11, $(T\mathfrak{h}^* \times \mathfrak{g}, T^*\mathfrak{h}^* \times \mathfrak{g}^*, \varphi)$ is a symmetric Lie bialgebroid, where $\varphi : T\mathfrak{h}^* \times \mathfrak{g} \rightarrow T\mathfrak{h}^* \times \mathfrak{g}$ is given by $\varphi(v, X) = (-v, sX)$, $\forall (v, X) \in T\mathfrak{h}^* \times \mathfrak{g}$. Thus one obtains a pair of Poisson involutions on their corresponding Poisson groupoids $\Phi : \Gamma \rightarrow \Gamma$ and $\Psi : \Gamma^* \rightarrow \Gamma^*$. Now $\Gamma = \mathfrak{h}^* \times \mathfrak{h}^* \times G$, and $\Phi(u, v, g) = (v, u, Sg)$, $\forall u, v \in \mathfrak{h}^*$ and $g \in G$. Hence, the stable locus of S is diffeomorphic to $\mathfrak{h}^* \times G^0$, where G^0 is the stable locus of S . It would be interesting to compute explicitly the induced Poisson structure on $\mathfrak{h}^* \times G^0$. On the other hand, it is quite mysterious what the stable locus of Ψ should look like, since it is even not clear how to describe the groupoid Γ^* .

Let \mathfrak{l} be a reductive Lie subalgebra of \mathfrak{g} containing \mathfrak{h} , i.e.,

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta'_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \quad (19)$$

where Δ'_+ is some subset of Δ_+ .

The claim in Example 4.13 in fact holds in a more general situation when \mathfrak{h} is replaced by \mathfrak{l} .

Proposition 4.14 *Let \mathfrak{l} be a reductive Lie subalgebra of a semi-simple Lie algebra \mathfrak{g} as in Equation (19), and $r : \mathfrak{l}^* \rightarrow \wedge^2 \mathfrak{g}$ a dynamical r -matrix. Then the map $s : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by Equation (18) satisfies the conditions as in Theorem 4.11, and therefore $(T\mathfrak{l}^* \times \mathfrak{g}, T^*\mathfrak{l}^* \times \mathfrak{g}^*, \varphi)$ is a symmetric Lie bialgebroid. Here $\varphi = (-Ts_{\mathfrak{l}}^*, s) : T\mathfrak{l}^* \times \mathfrak{g} \rightarrow T\mathfrak{l}^* \times \mathfrak{g}$.*

Proof. We prove this proposition by using the classification result in [9]. Let $r_0 : \mathfrak{h}^* \rightarrow \wedge^2 \mathfrak{g}$ be the function:

$$r_0(\lambda) = \sum_{\alpha \in \Delta'_+} \frac{1}{(\alpha, \lambda)} e_\alpha \wedge f_\alpha.$$

According to [9], $\tilde{r} = r|_{\mathfrak{h}^*} + r_0 : \mathfrak{h}^* \rightarrow \wedge^2 \mathfrak{g}$ is a classical dynamical r -matrix on \mathfrak{h}^* . Hence, from Example 4.13 (the rational case can also be similarly checked), we know that $s(\tilde{r}(\lambda)) = -\tilde{r}(\lambda)$, $\forall \lambda \in \mathfrak{h}^*$, which in turn implies that $s(r(\lambda)) = -r(\lambda)$, $\forall \lambda \in \mathfrak{h}^*$.

²Note that $-s$ is precisely the Cartan involution on the split real form.

Now assume that $\mu = Ad_{x-1}^* \lambda \in \mathfrak{l}^*$, where $\lambda \in \mathfrak{h}^*$ and $x \in L$. Then

$$\begin{aligned}
s(r(\mu)) &= s(r(Ad_{x-1}^* \lambda)) \quad (\text{since } r \text{ is } L\text{-equivariant}) \\
&= s[Ad_x r(\lambda)] \\
&= Ad_{Sx-1} s(r(\lambda)) \\
&= Ad_{Sx-1} (-r(\lambda)) \\
&= -r(Ad_{Sx}^* \lambda) \\
&= -r(s^* Ad_{x-1}^* s^* \lambda) \\
&= -r(s^* Ad_{x-1}^* \lambda) \\
&= -r(s^* \mu).
\end{aligned}$$

Here we used the identities: $s \circ Ad_x = Ad_{Sx-1} \circ s$ and $Ad_{Sx}^* = s^* Ad_{x-1}^* s^*$. Since those points $\mu = Ad_{x-1} \lambda, \forall \lambda \in \mathfrak{h}^*, x \in L$, consist of a dense subset of \mathfrak{l}^* , the conclusion thus follows immediately. □

5 Poisson involutions on Poisson groups

In this section, we turn our attention to Poisson involutions on Poisson groups.

5.1 Symmetric Poisson groups

As a special case of Definition 4.5, we have

Definition 5.1 (i). *A symmetric Poisson group consists of a pair (G, Φ) , where G is a Poisson group and $\Phi : G \rightarrow G$ is a group anti-morphism which is also a Poisson involution.*

(ii). *A symmetric Lie bialgebra consists of a triple $(\mathfrak{g}, \mathfrak{g}^*, \varphi)$, where $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra and $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ is an involutive Lie algebra anti-morphism such that $\varphi^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a Lie algebra morphism.*

In this case, a combination of Theorems 4.6 and 4.10 leads to the following:

Theorem 5.2 (i). *There is a one-one correspondence between simply connected symmetric Poisson groups and symmetric Lie bialgebras.*

(ii). *There is one-one correspondence between symmetric Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*, \varphi)$ and involutive anti-morphisms $\chi : \sigma \rightarrow \sigma$ (i.e., $(\chi e_1, \chi e_2) = -(e_1, e_2)$; and $\chi[e_1, e_2] = -[\chi e_1, \chi e_2]$) of the double $\sigma = \mathfrak{g} \oplus \mathfrak{g}^*$ preserving both components \mathfrak{g} and \mathfrak{g}^* .*

(iii). *If $(\mathfrak{g}, \mathfrak{g}^*)$ is a coboundary Lie bialgebra with an r -matrix $r \in \wedge^2 \mathfrak{g}$, then $(\mathfrak{g}, \mathfrak{g}^*, \varphi)$ is a symmetric Lie bialgebra if $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ is an involutive Lie algebra anti-morphism such that $\varphi r = -r$.*

Now assume that $(\mathfrak{g}, \mathfrak{g}^*, \varphi)$ is a symmetric Lie bialgebra. Then according to the proof of Theorem 4.10, $\chi : \sigma \rightarrow \sigma$, $\chi(X + \xi) = \varphi X - \varphi^* \xi$, $\forall X + \xi \in \mathfrak{g} \oplus \mathfrak{g}^*$, is an involutive Lie algebra anti-morphism, where $\sigma = \mathfrak{g} \oplus \mathfrak{g}^*$ is the double of the Lie bialgebra. On the other hand, it is well known that (σ, σ^*) itself is a Lie bialgebra with the r -matrix: $r = \sum_i X_i \wedge \xi^i \in \wedge^2 \sigma$, where $\{X_1, \dots, X_n\}$ is a basis of \mathfrak{g} and $\{\xi^1, \dots, \xi^n\}$ is its dual basis of \mathfrak{g}^* . Then $\chi(r) = -\sum_i \varphi X_i \wedge \varphi^* \xi^i = -r$, since $\{\varphi^* \xi^1, \dots, \varphi^* \xi^n\}$ is a dual basis to $\{\varphi X_1, \dots, \varphi X_n\}$. Thus we have proved the following:

Proposition 5.3 *The double of a symmetric Lie bialgebra is still a symmetric Lie bialgebra.*

Remark 5.4 Let D denote the Lie group of σ . Then the same space D possesses three different structures (under certain assumptions on completeness): a Poisson group, a symplectic groupoid Γ_G over G and a symplectic groupoid Γ_{G^*} over G^* . If $(\mathfrak{g}, \mathfrak{g}^*, \varphi)$ is a symmetric Lie bialgebra, then φ induces a Poisson involution on D , an involutive automorphism on symplectic groupoid Γ_G , and an involutive automorphism on the symplectic groupoid Γ_{G^*} . These three involutions are all **different** (see [4]). Their stable locuses correspond to a Dirac submanifold of D , a symplectic groupoid over the stable locus of Φ , and a symplectic groupoid over the stable locus of Ψ . Here $\Phi : G \rightarrow G$ and $\Psi : G^* \rightarrow G^*$ are the corresponding involutions induced by φ .

5.2 Poisson structures on stable locuses

Below we outline a scheme to explicitly compute the Poisson tensor on the stable locus Q of the Poisson involution Φ for a symmetric Poisson group (G, Φ) . Since Φ is an involutive group anti-morphism, we have

$$Ad_{\Phi(x)^{-1}} \circ \varphi = \varphi \circ Ad_x : \mathfrak{g} \rightarrow \mathfrak{g}, \quad \forall x \in G. \quad (20)$$

Definition 5.5 (i). *A smooth map $\xi : G \rightarrow \wedge^* \mathfrak{g}$ is said to be Φ -equivariant if*

$$\xi(\Phi(x)) = Ad_{\Phi(x)} \varphi(\xi(x)), \quad \forall x \in G; \quad (21)$$

(ii). *It is said to be anti- Φ -equivariant if*

$$\xi(\Phi(x)) = -Ad_{\Phi(x)} \varphi(\xi(x)), \quad \forall x \in G. \quad (22)$$

Indeed, any smooth map $\xi : G \rightarrow \wedge^* \mathfrak{g}$ can be decomposed as $\xi = \xi^+ + \xi^-$ such that ξ^+ is Φ -equivariant and ξ^- is anti- Φ -equivariant, where

$$\xi^+(x) = \frac{1}{2}[\xi(x) + \varphi(Ad_{\Phi(x)^{-1}} \xi(\Phi(x)))]; \quad (23)$$

$$\xi^-(x) = \frac{1}{2}[\xi(x) - \varphi(Ad_{\Phi(x)^{-1}} \xi(\Phi(x)))] \quad (24)$$

It is simple to see that $\xi : G \rightarrow \wedge^* \mathfrak{g}$ is Φ -equivariant (or anti- Φ -equivariant) iff its right translation $r_{x*} \xi(x)$ is a Φ -invariant (or anti- Φ -invariant) multi-vector field on G .

Let $\delta : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ denote the cobracket of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, which is also a Lie algebra 1-cocycle, and let $\lambda : G \rightarrow \wedge^2 \mathfrak{g}$ be its corresponding Lie group 1-cocycle. It is well-known that $\pi(x) = r_{x*} \lambda(x)$, $\forall x \in G$, is the Poisson tensor on the Poisson group G . Since π is Φ -invariant, it thus follows that $\lambda : G \rightarrow \wedge^2 \mathfrak{g}$ is Φ -equivariant.

Proposition 5.6 *Assume that the group 1-cocycle $\lambda : G \rightarrow \wedge^2 \mathfrak{g}$ is $\lambda = \sum_i \xi_i \wedge \eta_i$, where $\xi_i, \eta_i : G \rightarrow \mathfrak{g}$. Then $\pi_Q(x) = \sum_i r_{x*} \xi_i^+(x) \wedge r_{x*} \eta_i^+(x)|_Q$ is the Poisson tensor on Q , where ξ_i^+ and η_i^+ are defined as in Equations (23-24). Moreover, the symplectic leaves of Q are the intersection of Q with dressing orbits of G^* .*

When G is a coboundary Poisson group, one can write π_Q more explicitly.

Corollary 5.7 *Under the same hypothesis as in Theorem 5.2, moreover assume that G is a coboundary Poisson group with r -matrix $r = \sum_i e_i \wedge f_i \in \wedge^2 \mathfrak{g}$. Then the Poisson tensor on Q is given by*

$$\pi_Q = \frac{1}{4} \sum_i (\overleftarrow{e}_i + \overrightarrow{\varphi e}_i) \wedge (\overleftarrow{f}_i + \overrightarrow{\varphi f}_i)|_Q - \frac{1}{4} \sum_i (\overrightarrow{e}_i + \overleftarrow{\varphi e}_i) \wedge (\overrightarrow{f}_i + \overleftarrow{\varphi f}_i)|_Q, \quad (25)$$

where \overleftarrow{e}_i and \overrightarrow{e}_i are the left- and right-invariant vector fields on G , respectively, corresponding to $e_i \in \mathfrak{g}$; similarly for \overleftarrow{f}_i and \overrightarrow{f}_i , etc.

In particular, if e_i, f_i are chosen such that $\varphi e_i = e_i$ and $\varphi f_i = -f_i$, then

$$\pi_Q = \frac{1}{2} \sum_i (\overleftarrow{e}_i + \overrightarrow{e}_i) \wedge (\overleftarrow{f}_i - \overrightarrow{f}_i)|_Q. \quad (26)$$

Proof. It is simple to see, by using Equation (23), that for any $\xi \in \mathfrak{g}$, $\xi^+(x) = \frac{1}{2}(\xi + Ad_x(\varphi\xi))$ and $(Ad_x\xi)^+(x) = \frac{1}{2}(Ad_x\xi + \varphi\xi)$. Hence it follows that $r_{x*}(Ad_x\xi)^+(x) = \frac{1}{2}(\overleftarrow{\xi} + \overrightarrow{\varphi\xi})$ and $r_{x*}\xi^+(x) = \frac{1}{2}(\overrightarrow{\xi} + \overleftarrow{\varphi\xi})$. It is well known that for a coboundary Poisson group $\lambda(x) = \sum_i (Ad_x e_i \wedge Ad_x f_i - e_i \wedge f_i)$. Equation (25) thus follows immediately. □

5.3 Poisson symmetric spaces

In what follows, we discuss the relation between the stable locus of the Poisson involution of a symmetric Poisson group and Poisson symmetric spaces. A Poisson symmetric space is a symmetric space, which is in the mean time also a Poisson homogeneous space. Poisson symmetric spaces were studied systematically by Fernandes in his Ph. D. thesis [11, 12], to which we refer the reader for details.

Assume that (G, Φ) is a symmetric Poisson group, and $Q = \{g | \Phi(g) = g\}$ is the stable locus of Φ . The following result is standard (c.f. [27, 28]). For completeness, we outline a proof below.

Proposition 5.8 *Any connected component of Q is a symmetric space.*

Proof. Let $g_0 \in Q$ be any fixed point of Φ , and Q_{g_0} the connected component of Q through g_0 .

Consider the twisted G -action on (the space) G given by [27]:

$$g \cdot x = gx\Phi(g), \quad \forall g, x \in G. \quad (27)$$

a Since Φ is a group anti-morphism, this is clearly an action. Now $\Phi(g \cdot x) = \Phi(gx\Phi(g)) = g\Phi(x)\Phi(g) = g \cdot \Phi(x)$, so Q is stable under this action. Therefore in particular Q_{g_0} is stable as well. Let Q'_{g_0} denote the G -orbit through g_0 . Then Q'_{g_0} is a homogeneous space $Q'_{g_0} \cong G/H_{g_0}$, where $H_{g_0} = \{g|g \in G, gg_0\Phi(g) = g_0\}$ is the isotropic group at g_0 . Set

$$\Phi_{g_0} : G \longrightarrow G, \quad \Phi_{g_0}(g) = Ad_{g_0}\Phi(g^{-1}), \forall g \in G. \quad (28)$$

Then Φ_{g_0} is an involutive group homomorphism, since

$$\Phi_{g_0}^2(g) = \Phi_{g_0}(Ad_{g_0}\Phi(g^{-1})) = Ad_{\Phi_{g_0}(g_0)}\Phi_{g_0}(\Phi(g^{-1})) = Ad_{g_0^{-1}}Ad_{g_0}\Phi(\Phi(g^{-1}))^{-1} = g, \quad \forall g \in G.$$

It is clear that H_{g_0} is the stable locus of Φ_{g_0} . Hence Q'_{g_0} is indeed a symmetric space, and its dimension equals to the dimension of -1 -eigenspace of φ_{g_0} , where $\varphi_{g_0} = -Ad_{g_0} \circ \varphi : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the Lie algebra involution corresponding to Φ_{g_0} . On the other hand, the tangent space $T_{g_0}Q_{g_0}$ is spanned by those vectors $v \in T_{g_0}G$ such that $\Phi_*v = v$. By identifying $T_{g_0}G$ with \mathfrak{g} by right translations, $T_{g_0}Q_{g_0}$ can be identified with the subspace of \mathfrak{g} consisting of those elements X satisfying $Ad_{g_0} \circ \varphi X = X$, i.e., the -1 -eigenspace of φ_{g_0} . Therefore Q'_{g_0} is a submanifold of Q_{g_0} of the same dimension, so it must be an open submanifold. Since it is also closed, they must be identical. This concludes the proof. □

We are now ready to prove the following

Theorem 5.9 *Let (G, Φ) be a symmetric Poisson group, and $Q = \{g|\Phi(g) = g\}$ the stable locus of Φ . If the Poisson tensor π on G vanishes at a point $g_0 \in Q$, then the connected component Q_{g_0} is a Poisson symmetric space up to a multiplier of 2. In particular, the identity component of Q is a Poisson symmetric space.*

Proof. Consider the map

$$f : G \longrightarrow Q_{g_0}, \quad g \longrightarrow g \cdot g_0 = gg_0\Phi(g), \quad \forall g \in G.$$

It suffices to prove that f is a Poisson map, where Q_{g_0} is equipped with the Poisson tensor $2\pi_Q$.

First, it is simple to see that

$$f_*\delta_g = R_{g_0\Phi(g)}\delta_g + L_{gg_0}\Phi_*\delta_g, \quad \forall \delta_g \in T_gG. \quad (29)$$

On the other hand, we have

$$L_{gg_0}\Phi_*\delta_g = \Phi_*(R_{g_0\Phi(g)}\delta_g). \quad (30)$$

To see this, take a curve $g(t)$ starting at g with $\frac{\partial}{\partial t}|_{t=0}g(t) = \delta_g$. Since Φ is an involutive anti-morphism, we have $gg_0\Phi(g(t)) = \Phi(g(t)g_0\Phi(g))$. Equation (30) thus follows by taking the derivative at $t = 0$. Combining Equation (29) with Equation (30), we are thus lead to

$$f_*\delta_g = 2(R_{g_0\Phi(g)}\delta_g)^+. \quad (31)$$

Now write $\pi(g) = \sum_{ij} \delta_g^i \wedge \delta_g^j$, where $\delta_g^i, \delta_g^j \in T_g G$. Then we have

$$f_*\pi(g) = 4 \sum_{ij} (R_{g_0\Phi(g)}\delta_g^i)^+ \wedge (R_{g_0\Phi(g)}\delta_g^j)^+.$$

On the other hand, from the multiplicity condition of the Poisson tensor $\pi(g)$, it follows that

$$\begin{aligned} \pi(gg_0\Phi(g)) &= R_{g_0\Phi(g)}\pi(g) + L_g\pi(g_0\Phi(g)) \\ &= R_{g_0\Phi(g)}\pi(g) + L_{gg_0}\pi(\Phi(g)) \\ &= R_{g_0\Phi(g)}\pi(g) + L_{gg_0}\Phi_*\pi(g) \\ &= R_{g_0\Phi(g)}\pi(g) + \Phi_*(R_{g_0\Phi(g)}\pi(g)) \\ &= \sum R_{g_0\Phi(g)}\delta_g^i \wedge R_{g_0\Phi(g)}\delta_g^j + \sum \Phi_*R_{g_0\Phi(g)}\delta_g^i \wedge \Phi_*R_{g_0\Phi(g)}\delta_g^j. \end{aligned}$$

Here we used the assumption $\pi(g_0) = 0$ in the second equality. Therefore we have

$$\pi_Q(gg_0\Phi(g)) = 2 \sum (R_{g_0\Phi(g)}\delta_g^i)^+ \wedge (R_{g_0\Phi(g)}\delta_g^j)^+.$$

This concludes the proof. □

Remark 5.10 (i). Theorem 5.9 would follow from Theorem 3.6, if the action defined by Equation (27) were a Poisson action where the Poisson group is equipped with the Poisson tensor $\pi(g)$ while the space it acts, which is G again, is equipped with $2\pi(g)$. However, this is false in general. So we can see that a Poisson group action on a Poisson manifold P may not be Poisson action, but it can still be Poisson when restricted to the stable locus Q .

- (ii). One drawback of Theorem 5.9 is that the stable locuses do not seem to produce any new examples of Poisson manifolds for symmetric Poisson groups in contraction to what one may have initially expected. A good point, on the other hand, is that one might be able to quantize these Poisson structures on stable locuses including the one on Stokes matrices U_+ (see Example 5.11) using quantum homogeneous spaces.
- (iii). One can construct a symplectic groupoid of a Poisson symmetric space by means of reduction [33]. On the other hand, according to Corollary 4.4, for a stable locus Poisson structure, one can construct a symplectic groupoid directly via the lifted involution on the corresponding symplectic groupoid. It is interesting to compare these two approaches in our case here.

5.4 Examples

We end the paper with a list of examples. We refer the reader to [11] for a complete list of orthogonal symmetric Lie bialgebras, which also contains examples below.

Example 5.11 Let \mathfrak{g} be a semi-simple Lie algebra of rank k over \mathbb{C} with a Cartan subalgebra \mathfrak{h} . Let $\{e_\alpha, f_\alpha, h_i | \alpha \in \Delta_+, 1 \leq i \leq k\}$ be a Chevalley basis. It is well known that $(\mathfrak{g}, \mathfrak{g}^*)$ is a coboundary Lie bialgebra with r -matrix:

$$r = \sum_{\alpha \in \Delta_+} d_\alpha (e_\alpha \wedge f_\alpha),$$

where $d_\alpha = (e_\alpha, f_\alpha)$.

As in Example 4.13, let $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ be the \mathbb{C} -linear morphism, which, on generators, is defined as follows:

$$\varphi e_\alpha = f_\alpha, \varphi f_\alpha = e_\alpha, \varphi h_i = h_i.$$

It is clear that φ is an involutive Lie algebra anti-morphism and $\varphi r = -r$. Therefore, $(\mathfrak{g}, \mathfrak{g}, \varphi)$ is a symmetric Lie bialgebra, which in turn induces a pair of symmetric Poisson groups (G, Φ) and (G^*, Ψ) . Thus one obtains a pair of Poisson involutions: $\Phi : G \rightarrow G$ and $\Psi : G^* \rightarrow G^*$, which are the group anti-morphisms corresponding to the Lie algebra anti-morphisms: $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ and $-\varphi^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$, respectively.

For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, it is well-known that $G = SL(n, \mathbb{C})$ and $G^* = B_+ * B_-$. It is simple to see that Φ and Ψ are given by the following:

$$\Phi : SL(n, \mathbb{C}) \rightarrow SL(n, \mathbb{C}), \quad \Phi(A) = A^T, \quad \forall A \in SL(n, \mathbb{C});$$

and

$$\Psi : B_+ * B_- \rightarrow B_+ * B_-, \quad \Psi(B, C) = (C^T, B^T), \quad \forall (B, C) \in B_+ * B_-.$$

The stable locus of Φ thus consists of all symmetric matrices in $SL(n, \mathbb{C})$. On the other hand, the set U_+ of Stokes matrices (i.e. upper triangular matrices with all main diagonal entries being 1) can be identified with the identity component of the stable locus of Ψ . As a consequence, both the space S of symmetric matrices in $SL(n, \mathbb{C})$ and the space U_+ of Stokes matrices admit natural Poisson structures. These Poisson manifolds, together with their symplectic groupoids, were studied in details by Bondal [4] in connection with his study of triangulated categories. Independently, the Poisson structure on U_+ was also obtained independently by Dubrovin [8] in the 3×3 -case and Ugaglia [29] in general in connection with the study of Frobenius manifolds. From a very different aspect, the relation between the Poisson structure on the space of Stokes matrices U_+ and the Poisson group $B_+ * B_-$ was independently found by Boalch in his study of the so called ‘‘monodromy map’’ [2]. We refer the reader to [4, 2] for details. As a consequence of Theorem 5.9, we conclude that both S and U_+ are indeed Poisson symmetric spaces.

Theorem 5.12 *Up to a multiplier 2,*

- (i). *the map $SL(n, \mathbb{C}) \rightarrow S, A \rightarrow AA^T, \forall A \in SL(n, \mathbb{C})$, is a Poisson map. Indeed S is a Poisson symmetric space with the Poisson $SL(n, \mathbb{C})$ -action:*

$$SL(n, \mathbb{C}) \times S \rightarrow S, \quad A \cdot X = AXA^T, \quad \forall A \in SL(n, \mathbb{C}), X \in S;$$

(ii). the map $B_+ * B_- \longrightarrow U_+$, $(B, C) \rightarrow BC^T, \forall (B, C) \in B_+ * B_-$ is a Poisson map. Indeed U_+ is a Poisson symmetric space with the Poisson $B_+ * B_-$ -action:

$$B_+ * B_- \times U_+ \longrightarrow U_+, \quad (B, C) \cdot X = BXC^T, (B, C) \in B_+ * B_-, X \in U_+.$$

Example 5.13 Let K be a compact semi-simple Lie group with Lie algebra \mathfrak{k} , and \mathfrak{t} its Cartan subalgebra. It is well known that K admits a standard Poisson group structure called Bruhat Poisson structure [21]. Let $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$ be its complexification, which is a complex semi-simple Lie algebra. Choose a Chevalley basis $\{e_\alpha, f_\alpha, h_i | \alpha \in \Delta_+, 1 \leq i \leq k\}$ of \mathfrak{g} as in Example 5.11, then $\{X_\alpha, Y_\alpha, t_i | \alpha \in \Delta_+, 1 \leq i \leq k\}$, where

$$X_\alpha = e_\alpha - f_\alpha, \quad Y_\alpha = \sqrt{-1}(e_\alpha + f_\alpha), \quad \text{and } t_i = \sqrt{-1}h_i, \quad (32)$$

is a basis (over \mathbb{R}) of \mathfrak{k} , and

$$\hat{r} = \sqrt{-1}r = \sqrt{-1} \sum_{\alpha \in \Delta_+} d_\alpha (e_\alpha \wedge f_\alpha) = \sum_{\alpha \in \Delta_+} \frac{1}{2} d_\alpha X_\alpha \wedge Y_\alpha \in \wedge^2 \mathfrak{k} \quad (33)$$

is indeed the r -matrix generating the corresponding Lie bialgebra $(\mathfrak{k}, \mathfrak{k}^*)$. Let $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}$ be the anti-morphism as in Example 4.13. It is then clear that $\varphi(X_\alpha) = -X_\alpha$, $\varphi(Y_\alpha) = Y_\alpha$, and $\varphi(t_i) = t_i$, so \mathfrak{k} is stable under φ . It is also clear that $\varphi \hat{r} = -\hat{r}$. Hence $(\mathfrak{k}, \mathfrak{k}^*, \hat{\varphi})$, where $\hat{\varphi} = \varphi|_{\mathfrak{k}} : \mathfrak{k} \longrightarrow \mathfrak{k}$, is a symmetric Lie bialgebra. Thus it induces a pair of Poisson involutions $\hat{\Phi} : K \longrightarrow K$ and $\hat{\Psi} : K^* \longrightarrow K^*$.

To describe the stable locuses of these involutions, we need to consider the double of the Lie bialgebra $(\mathfrak{k}, \mathfrak{k}^*)$, which is isomorphic to \mathfrak{g} as a real Lie algebra. According to Theorem 5.2, $\hat{\varphi}$ induces an involutive Lie algebra antimorphism (over \mathbb{R}) $\chi : \mathfrak{g} \longrightarrow \mathfrak{g}$, under which both \mathfrak{k} and \mathfrak{k}^* are stable and whose restrictions to these Lie subalgebras are $\hat{\varphi}$ and $-\hat{\varphi}^*$, respectively. In our case, a straightforward computation yields that on generators χ is given by:

$$\begin{aligned} \chi(\sqrt{-1}e_\alpha) &= \sqrt{-1}e_\alpha, \quad \chi(\sqrt{-1}f_\alpha) = \sqrt{-1}f_\alpha, \quad \chi(\sqrt{-1}h_i) = \sqrt{-1}h_i \\ \chi(e_\alpha) &= -e_\alpha, \quad \chi(f_\alpha) = -f_\alpha, \quad \chi(h_i) = -h_i. \end{aligned}$$

In other words, $\chi(X) = -\bar{X}, \forall X \in \mathfrak{g}$. On the group level, χ induces an involutive Lie group antimorphism $\Upsilon : G \longrightarrow G$ such that $\Upsilon(g) = \bar{g}^{-1}, \forall g \in G$, where G is a simply connected Lie group (considered as a real Lie group) integrating the Lie algebra \mathfrak{g} . By Q , we denote the stable locus of Υ , i.e., $Q = \{g \in G | \bar{g} = g^{-1}\}$. Then the stable locus of $\hat{\Phi}$ and $\hat{\Psi}$ are $K \cap Q$ and $K^* \cap Q$, respectively. In particular, according to Corollary 5.7,

$$\pi_Q = \sum_{\alpha \in \Delta_+} \frac{1}{4} d_\alpha (\bar{X}_\alpha^\rightarrow - \bar{X}_\alpha^\leftarrow) \wedge (\bar{Y}_\alpha^\rightarrow + \bar{Y}_\alpha^\leftarrow) \quad (34)$$

is the Poisson tensor on $K \cap Q$. Theorem 5.9 implies that the map $g \rightarrow g\bar{g}^{-1}$ is indeed Poisson maps (up to a factor of 2) when being restricted to K and K^* .

For $K = SU(n)$, its dual group K^* is isomorphic to $SB(n, \mathbb{C})$, and the double $G \cong SL(n, \mathbb{C})$, considered as a real Lie group. Thus $Q = \{A \in SL(n, \mathbb{C}) | \bar{A}A = I\}$. Hence we have $K \cap Q \cong$

$\{A|A^*A = \bar{A}A = I, \det A = 1\}$, which is the submanifold of $SU(n)$ consisting of all symmetric matrices. On the other hand, $K^* \cap Q \cong \{A \in SB(n, \mathbb{C})|\bar{A}A = I\}$.

We note that $SB(n, \mathbb{C})$ is Poisson diffeomorphic to the linear Poisson structure on $\mathfrak{sb}(n, \mathbb{C})$ according to Ginzburg-Weinstein theorem [15]. The recent result of Boalch [2] suggests that there may exist a Poisson diffeomorphism $SB(n, \mathbb{C}) \rightarrow \mathfrak{sb}(n, \mathbb{C})$ commuting with the Poisson involutions, where the Poisson involution on $SB(n, \mathbb{C})$ is given by $A \rightarrow \bar{A}^{-1}$ while the Poisson involution on $\mathfrak{sb}(n, \mathbb{C})$ is: $A \rightarrow -\bar{A}$. If so, the induced Poisson structures on their stable locus should be isomorphic. The latter is a lot easier to compute and in fact is a linear Poisson structure.

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