

**Dynamical Entropy for \mathbb{Z}^n -actions and
Bogoliubov Automorphisms of the CAR-algebra**

**Sergey I. Bezuglyi
Valentin Ya. Golodets**

Vienna, Preprint ESI 109 (1994)

June 21, 1994

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Sergey I. Bezuglyi and Valentin Ya. Golodets*
Erwin Schrödinger International
Institute for Mathematical Physics

January 12, 1995

Abstract

The notion of dynamical entropy for Z^n -actions by automorphisms of C^* -algebras is studied. These results are applied to Bogoliubov actions of Z^n on the CAR-algebra. It is shown that the dynamical entropy of Z^n -Bogoliubov actions is computed by a formula analogous to that found by Störmer and Voiculescu in the case $n = 1$, and also it is proved that singular parts of these actions give zero contribution to the entropy.

1 Introduction

The notion of entropy introduced by Kolmogorov [K] and Sinai [S] for transformations of a measure space is an important invariant in the ergodic theory. Connes, Narnhofer, Störmer and Thirring [CSt, C, CNT] defined and investigated the notion of dynamical entropy which is a natural generalization of Kolmogorov-Sinai entropy to automorphisms of operator algebras. In last years this entropy has been studied actively by many authors from different points of view (see the bibliography, for example, in [B, OP, St]). Some interesting applications of dynamical entropy in physics can be found in [B]. Also there are other promising approaches to non-abelian entropy [V, SaTh].

In just the same way as in the case of Kolmogorov-Sinai entropy a very interesting problem is to find models in which the dynamical entropy can be calculated exactly. As far as we know there are few such models up to now. One of them is the class of Bogoliubov automorphisms of the algebra of canonical anticommutation relations (CAR-algebra). In [StV] Störmer and Voiculescu obtained the following remarkable

*Permanent address of authors: Institute for Low Temperature Physics and Engineering, 47, Lenin ave., 310164 Kharkov, Ukraine

results. They showed that the dynamical entropy of a Bogoliubov automorphism is computed by a nice formula (predicted by Connes for the tracial state), and only continuous part of unitary operator defining Bogoliubov automorphism should be taken into account because the contribution of its singular part is zero. Later this model was also studied in papers [NT, PSh] for more general states on the CAR-algebra.

The following question arises naturally. Can we expect that for \mathbf{Z}^n -actions by Bogoliubov automorphisms on the CAR-algebra the Störmer-Voiculescu's results are still valid? We just study this problem in the present work. We will prove that the dynamical entropy of \mathbf{Z}^n -actions by Bogoliubov automorphisms on the CAR-algebra is found by a formula which is analogous to Störmer-Voiculescu one. Also it will be proved that the singular part of such a \mathbf{Z}^n -action has zero dynamical entropy and the value of entropy depends only on the absolutely continuous part.

To do it, we need to extend the notion of dynamical entropy to arbitrary \mathbf{Z}^n -actions on C^* -algebras. The similar problem for Kolmogorov-Sinai entropy was solved by Conze in [Co]. For the first time the interesting approach generalizing the methods of [Co] to \mathbf{Z}^n -actions on C^* -algebras was proposed by Hudetz in [H], who also applied these results to a physical model. The entropy for \mathbf{Z}^n -actions also was discussed in [B]. In the first part of this work we present (following the scheme of [Co]) our version.

Under the presentation of this paper we refer very often to the articles [CNT, StV]. We have to do it because otherwise the size of our work would be considerable more.

2 Dynamical entropy for \mathbf{Z}^n -actions

In this section we first remind briefly the definition of dynamical entropy for automorphisms of C^* -algebras following [CNT, St]. We do it mainly to introduce the necessary notations. The entropy of \mathbf{Z}^n -actions on C^* -algebras is studied more detailed. We note that similar considerations can be done for Connes-Störmer entropy of \mathbf{Z}^n -actions on type II₁ von Neumann algebras. As it was indicated in the Introduction our approach repeats on the whole the methods of [Co].

Let \mathcal{A} be a unital C^* -algebra, C_1, \dots, C_k finite dimensional (f.d.) C^* -algebras, and $\gamma_j : C_j \rightarrow \mathcal{A}$ a unital completely positive (u.c.p.) map, $j = 1, \dots, k$. Let ϕ be a state on \mathcal{A} and P a u.c.p. map from \mathcal{A} into a f.d. abelian C^* -algebra B such that there is a state μ on B for which $\mu \circ P = \phi$. If p_1, \dots, p_r are the minimal projections in B , then there are states $\phi_i, i = 1, \dots, r$ on \mathcal{A} such that

$$P(x) = \sum_{i=1}^r \phi_i(x) p_i, \quad x \in \mathcal{A},$$

and

$$\phi = \sum_{i=1}^r \mu(p_i) \phi_i.$$

Put

$$\epsilon_\mu(P) = \sum_{i=1}^r S(\phi | \phi_i)$$

where $S(\phi | \phi_i)$ is the relative entropy of ϕ and ϕ_i . For $S(\mu) = -\sum_{i=1}^r \mu(p_i) \log p_i$, the entropy defect $s_\mu(P)$ is given by

$$s_\mu(P) = S(\mu) - \epsilon_\mu(P).$$

Let $B_j, j = 1, \dots, k$, be a C^* -subalgebra of B and $E_j : B \rightarrow B_j$ a μ -invariant conditional expectation. Then (B, E_j, P, μ) is called an abelian model for $(\mathcal{A}, \gamma_1, \dots, \gamma_k)$. The entropy of such an abelian model is defined to be

$$S(\mu | \bigvee_{j=1}^k B_j) - \sum_{j=1}^k s_\mu(\rho_j)$$

where $\rho_j = E_j \circ P \circ \gamma_j : C_j \rightarrow B_j$. The supremum of entropies of all such abelian models is denoted by $H_\phi(\gamma_1, \dots, \gamma_k)$. If α is a ϕ -invariant automorphism of \mathcal{A} and $\gamma : C \rightarrow \mathcal{A}$ is a u.c.p. map of a f.d. C^* -algebra C , then we denote by

$$h_{\phi, \alpha}(\gamma) = \lim_k \frac{1}{k} H_\phi(\gamma, \alpha \circ \gamma, \dots, \alpha^{k-1} \circ \gamma).$$

The entropy of α with respect to ϕ is defined by the formula:

$$h_\phi(\alpha) = \sup_\gamma h_{\phi, \alpha}(\gamma).$$

Let \mathcal{A} be as above, $G = \mathbf{Z}^n$ the free abelian group with n generators and $\sigma : G \rightarrow \text{Aut}(\mathcal{A}, \phi)$ an ϕ -invariant action of G on \mathcal{A} by $*$ -automorphisms, i.e. σ is an injective homomorphism from G into $\text{Aut}(\mathcal{A})$ such that $\sigma(g)$ is a ϕ -invariant $*$ -automorphism of \mathcal{A} for any $g \in G$. To simplify the exposition we will consider throughout this paper the case $n = 2$ only. It is clear that the general case may be considered analogously.

Let γ be a u.c.p. map from a f.d. C^* -algebra C into \mathcal{A} . Take a sequence $\{\rho_n\}_{n=1}^\infty$ of a finite parallelograms in $\mathbf{Z}^2 = G$ such that $\rho_n \subset \rho_{n+1}, n \in \mathbf{N}$ and $\bigcup_n \rho_n = \mathbf{Z}^2$. We denote by

$$\sigma_{\rho_n}(\gamma) = \{\sigma_{(i,j)} \circ \gamma \mid (i,j) \in \rho_n\}.$$

According to [CNT, Proposition III.6,(b)],

$$H_\phi(\theta \circ \gamma_1, \dots, \theta \circ \gamma_k) = H_\phi(\gamma_1, \dots, \gamma_k) \quad (2.1)$$

for any $\theta \in \text{Aut}(\mathcal{A}), \phi \circ \theta = \phi$.

Lemma 2.1. *Let $\gamma : C \rightarrow \mathcal{A}$ be a u.c.p. map and $\{\rho_n\}$ arbitrary increasing sequence of parallelograms with $\bigcup_n \rho_n = G$ as above. Then for any finite parallelogram $\rho \subset G$ and $\epsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that for $n \geq n_0$*

$$\frac{1}{|\rho_n|} H_\phi(\sigma_{\rho_n}(\gamma)) \leq \frac{1}{|\rho|} H_\phi(\sigma_\rho(\gamma)) + \epsilon \quad (2.2)$$

where $|\rho|$ is the cardinality of ρ .

Proof. We will prove the lemma in case when ρ_n and ρ are rectangles. The proof for parallelograms differs only in some technical details. Let a_n and b_n be the lengths of sides of ρ_n , so $|\rho_n| = a_n b_n$ (a and b are defined for ρ analogously). We may assume that $\rho_n \supset \rho$ for all $n \geq n_1, n_1 \in \mathbf{N}$. Let $a_n = k_n a + r_n, b_n = m_n b + q_n$ where $r_n < a, q_n < b$. It means that one can cover by shifts of ρ a subrectangle $\bar{\rho}_n \subset \rho_n$ with $|\bar{\rho}_n| = k_n m_n |\rho|$. In view of (2.1) and [CNT, Proposition III.6,(d)] we have

$$H_\phi(\sigma_{\rho_n}(\gamma)) \leq k_n m_n H_\phi(\sigma_\rho(\gamma)) + H_\phi(\sigma_{\rho_n - \bar{\rho}_n}(\gamma)).$$

Since $|\rho_n - \bar{\rho}_n| \leq ab_n + a_n b$, we obtain

$$H_\phi(\sigma_{\rho_n - \bar{\rho}_n}(\gamma)) \leq (ab_n + a_n b) H_\phi(\gamma).$$

Thus,

$$\frac{1}{|\rho_n|} H_\phi(\sigma_{\rho_n}(\gamma)) \leq \frac{k_n m_n}{|\rho_n|} H_\phi(\sigma_\rho(\gamma)) + \frac{ab_n + a_n b}{|\rho_n|} H_\phi(\gamma). \quad (2.3)$$

For given $\epsilon > 0$, there is $n_0 \in \mathbf{N}$ (one can take $n_0 > n_1$) such that for all $n > n_0$

$$\frac{ab_n + a_n b}{a_n b_n} H_\phi(\gamma) < \epsilon. \quad (2.4)$$

It is clear that

$$\frac{k_n m_n}{|\rho_n|} \leq \frac{1}{|\rho|}. \quad (2.5)$$

Taking into account (2.4) and (2.5), we see that (2.3) implies (2.2). \diamond

Lemma 2.2. *Let γ and $\{\rho_n\}$ be as in Lemma 1.1. Then*

$$\lim_n \frac{1}{|\rho_n|} H_\phi(\sigma_{\rho_n}(\gamma))$$

exists and does not depend on the choice of $\{\rho_n\}$.

Proof. Lemma 2.1 tells us that

$$\limsup_n \frac{1}{|\rho_n|} H_\phi(\sigma_{\rho_n}(\gamma)) \leq \frac{1}{|\rho|} H_\phi(\sigma_\rho(\gamma)) \quad (2.6)$$

where ρ is arbitrary finite parallelogram in G . Let $\{\rho'_k\}$ be another sequence of parallelograms satisfying the conditions of Lemma 2.1. Replacing ρ in (2.6) by ρ'_k , we have

$$\limsup_n \frac{1}{|\rho_n|} H_\phi(\sigma_{\rho_n}(\gamma)) \leq \liminf_k \frac{1}{|\rho'_k|} H_\phi(\sigma_{\rho'_k}(\gamma)). \quad (2.7)$$

Then the statement of lemma follows from (2.7) because one can exchange the roles of $\{\rho_n\}$ and $\{\rho'_k\}$. \diamond

Definition 2.3. *For $\sigma : G \rightarrow \text{Aut}(\mathcal{A}, \phi)$, $\phi \circ \sigma(g) = \phi$, $g \in G$, we define*

$$h_{\phi, \sigma}(\gamma) = \lim_n \frac{1}{|\rho_n|} H_\phi(\sigma_{\rho_n}(\gamma))$$

where $\{\rho_n\}$ is as above, and then

$$h_\phi(\sigma(G)) = \sup_\gamma h_{\phi, \sigma}(\gamma)$$

is called the dynamical entropy of the action σ .

Lemma 2.4. *Let σ be an action of G on (\mathcal{A}, ϕ) , $\phi \circ \sigma(g) = \phi$ and $\alpha = \sigma(1, 0)$, $\beta = \sigma(0, 1)$ the generators of $\sigma(G)$. Then*

$$h_\phi(\sigma(G)) \leq h_\phi(\alpha), \quad h_\phi(\sigma(G)) \leq h_\phi(\beta).$$

Proof. This statement is a consequence of the inequality (2.2) since one can choose ρ in the form $\{(0, 0), (1, 0), \dots, (k, 0)\}$ or $\{(0, 0), (0, 1), \dots, (0, k)\}$ and first take the limit of the left side in (2.2) and then go to the limit as $k \rightarrow \infty$. \diamond

The next statement is an analogue of Theorem V.2 from [CNT].

Theorem 2.5. *Let $\{\tau_n\}$ be a sequence of u.c.p. maps $\tau_n : \mathcal{A}_n \rightarrow \mathcal{A}$ such that there are u.c.p. maps $\sigma_n : \mathcal{A} \rightarrow \mathcal{A}_n$ with $\tau_n \circ \sigma_n \rightarrow 1_{\mathcal{A}}$ in the pointwise norm topology. Then for an action $\sigma : G \rightarrow \text{Aut}(\mathcal{A}, \phi)$, $\phi \circ \sigma(g) = \phi$, $g \in G$ one has*

$$\lim_n h_{\phi, \sigma}(\tau_n) = h_{\phi}(\sigma(G)).$$

The **proof** is the same as in [CNT] because one can use Proposition IV.3 from [CNT] in the following form.

Let C be a f.d. C^* -algebra, γ, γ' u.c.p. maps from C into \mathcal{A} and $d = \dim C$. Suppose $\|\gamma - \gamma'\| < \epsilon$ and ρ is a finite rectangle in G . Then

$$|H_{\phi}(\sigma_{\rho}(\gamma)) - H_{\phi}(\sigma_{\rho}(\gamma'))| \leq 6|\rho|\epsilon(\log(1 + d\epsilon^{-1}) + 1/2). \quad (2.7)$$

We formulate now without proof the statement generalizing [CNT, Lemma VII.3 and Theorem VII.4]. We will identify below a subalgebra $N \subset M$ and the u.c.p. inclusion $N \rightarrow M$.

Proposition 2.6. *Let M be a hyperfinite von Neumann algebra, ϕ a normal state on M , $\sigma : G \rightarrow \text{Aut}(M)$ an action of G on M such that $\phi \circ \sigma_g = \phi$, $g \in G$. Then*

(i)

$$\sup_{\gamma} h_{\phi, \sigma}(\gamma) = \sup_N h_{\phi, \sigma}(N)$$

where N runs through all f.d. subalgebras of M ;

(ii) for an ascending sequence $\{N_k\}$ of f.d. subalgebras with $\bigcup_k N_k$ weakly dense in M , one has

$$h_{\phi}(\sigma) = \lim_k h_{\phi, \sigma}(N_k).$$

Let G_p be a subgroup of $G = \mathbf{Z}^2$ with the finite index p . It is known [Ku] that one can take the generators $\{e_1, e_2\}$ and $\{f_1, f_2\}$ in G and G_p respectively such that $f_1 = n_1 e_1$, $f_2 = n_2 e_2$ with $n_1, n_2 \in \mathbf{N}$.

Theorem 2.7. *Let M be a hyperfinite von Neumann algebra, ϕ a normal state on M and $\sigma : G \rightarrow \text{Aut}(M)$ an action of G on M with $\phi \circ \sigma_g = \phi$. If G_p is a subgroup of G with the finite index p , then $h_{\phi}(\sigma(G)) = h_{\phi}(\sigma(G_p))$.*

Proof. Firstly, one can define the entropy of action of G_p as it was done for G considering instead of parallelograms ρ_n their intersections with G_p , i.e. $\rho_n^p = \rho_n \cap G_p$.

In the other words, we may take the parallelograms ρ_n constructed by vectors $k_n f_1$ and $l_n f_2$ where k_n and l_n go to infinity as $n \rightarrow \infty$. Thus,

$$h_{\phi, \sigma_p}(\gamma) = \lim_n \frac{1}{|\rho_n^p|} H_\phi(\sigma_{\rho_n^p}(\gamma)), \quad \sigma_p = \sigma(G_p)$$

and

$$h_\phi(\sigma(G_p)) = \sup_\gamma h_{\phi, \sigma_p}(\gamma).$$

It follows from [CNT, Proposition III.6,(d)] that

$$H_\phi(\sigma_{\rho_n}(\gamma)) \geq H_\phi(\sigma_{\rho_n^p}(\gamma)). \quad (2.8)$$

Take γ such that $h_\phi(\sigma(G_p)) < h_{\phi, \sigma_p}(\gamma) + \epsilon$ for given $\epsilon > 0$. From (2.8) it follows

$$\begin{aligned} h_\phi(\sigma) &\geq \lim_n \frac{1}{|\rho_n|} H_\phi(\sigma_{\rho_n}(\gamma)) \\ &\geq \lim_n \frac{1}{p|\rho_n^p|} H_\phi(\sigma_{\rho_n^p}(\gamma)) \\ &> p^{-1}(h_{\phi, \sigma_p}(\gamma) - \epsilon), \end{aligned}$$

i.e.

$$ph_\phi(\sigma(G)) \geq h_{\phi, \sigma_p}(\gamma). \quad (2.9)$$

To prove the inverse inequality, we need to use the following result from [CNT]. For any $\xi > 0$ there exists a f.d. subalgebra $B \subset M$ such that the unit ball of $\sigma_{s_i}(N)$ lie in B up to ξ , i.e. $\sigma_{s_i}(N)_1 \subset^\xi B$ where s_i is taken from the parallelogram built by vectors f_1 and f_2 . It follows from the proof of Theorem VII.4, [CNT] that for every s_i there exists a u.c.p. map $\gamma_i : N \rightarrow B$ such that

$$\|\gamma_i - \sigma_{s_i}\|_\phi < \delta(\xi)$$

where $\delta(\xi) \rightarrow 0$ as $\xi \rightarrow 0$.

From the invariance of ϕ under σ , we get

$$\|\sigma_{(l,m)} \sigma_{s_i} - \sigma_{(l,m)} \gamma_i\|_\phi < \delta$$

where (l, m) runs through the set ρ_n^p , so the elements of ρ_n can be written as $(l, m) + s_i$.

Thus, by [CNT, Theorem VI.3 and Proposition III.6,(d)]

$$\begin{aligned} H_\phi(\sigma_{\rho_n}(N)) &\leq H_\phi(\sigma_{\rho_n^p}(\gamma_i)) + |\rho_n| \epsilon \\ &\leq H_\phi(\sigma_{\rho_n^p}(B)) + |\rho_n| \epsilon. \end{aligned}$$

Take N in such a way that

$$\begin{aligned}
h_\phi(\sigma(G)) - \epsilon &\leq \lim_n \frac{1}{|\rho_n|} H_\phi(\sigma_{\rho_n}(N)) \\
&\leq \lim_n \frac{1}{p|\rho_n^p|} H_\phi(\sigma_{\rho_n^p}(B)) + \epsilon \\
&\leq \frac{1}{p} h_\phi(\sigma(G_p)) + \epsilon.
\end{aligned} \tag{2.10}$$

Then (2.9) and (2.10) prove the theorem. \diamond

Corollary 2.8. *Let α and β be two generators of the group $\sigma(G)$ acting on (\mathcal{A}, ϕ) with \mathcal{A} being a nuclear C^* -algebra and $\phi \circ \sigma_g = \phi$. Then if only $h_\phi(\alpha)$ or $h_\phi(\beta)$ are finite, $h_\phi(\sigma(G)) = 0$.*

Proof. Consider the subgroup $\sigma(G_p)$ generated by α and $\beta^p, p \in \mathbf{N}$. According to Lemma 2.4 and Theorem 2.7, we have

$$h_\phi(\sigma(G)) = \frac{1}{p} h_\phi(\sigma(G_p)) \leq \frac{1}{p} h_\phi(\alpha).$$

It is clear that if $h_\phi(\alpha) < \infty$, then $h_\phi(\sigma(G)) = 0$, since p is arbitrary. \diamond

Proposition 2.9. *The following properties are valid:*

- (i) $h_\phi(\theta^{-1} \circ \sigma \circ \theta) = h_{\phi \circ \theta}(\sigma)$ where $\theta \in \text{Aut}(\mathcal{A})$;
- (ii) $h_{\lambda\phi_1 + (1-\lambda)\phi_2}(\sigma) = \lambda h_{\phi_1}(\sigma) + (1-\lambda)h_{\phi_2}(\sigma)$, $0 \leq \lambda \leq 1$ with σ being an action of G on (\mathcal{A}, ϕ) as above.

Proof. The statement (i) is proved as in [CNT, VII.5]. To prove (ii) we note that according to [CNT, Proposition III.6,(e)]

$$\begin{aligned}
&|h_{\lambda\phi_1 + (1-\lambda)\phi_2, \sigma}(\gamma) - \lambda h_{\phi_1, \sigma}(\gamma) - (1-\lambda)h_{\phi_2, \sigma}(\gamma)| \\
&\leq -\lambda \log \lambda - (1-\lambda) \log(1-\lambda).
\end{aligned} \tag{2.11}$$

Then from (2.11) it follows that

$$\begin{aligned}
&|h_{\lambda\phi_1 + (1-\lambda)\phi_2}(\sigma(G)) - \lambda h_{\phi_1}(\sigma(G)) - (1-\lambda)h_{\phi_2}(\sigma(G))| \\
&\leq -\lambda \log \lambda - (1-\lambda) \log(1-\lambda).
\end{aligned} \tag{2.12}$$

Let $G_p \subset G$ be a subgroup of G of index p . Then (2.12) one can be applied to G_p :

$$\frac{1}{p} |h_{\lambda\phi_1 + (1-\lambda)\phi_2}(\sigma(G_p)) - \lambda h_{\phi_1}(\sigma(G_p)) - (1-\lambda)h_{\phi_2}(\sigma(G_p))|$$

$$\leq -\frac{1}{p}(\lambda \log \lambda + (1 - \lambda) \log(1 - \lambda)). \quad (2.13)$$

But the left sides in (2.12) and (2.13) coincide. The statement (ii) follows now from (2.13) when p goes to infinity. \diamond

We will use in the next section the following statements.

Lemma 2.10. *Let ϕ be a pure state on an unital C^* -algebra \mathcal{A} , and σ a ϕ -invariant action of G on \mathcal{A} . Then $h_\phi(\sigma) = 0$.*

The **proof** follows immediately from [StV, Lemma 3.1] and Corollary 2.8.

Lemma 2.11. *Let \mathcal{A} be a C^* -algebra, ϕ a state, and σ a ϕ -invariant action of G on \mathcal{A} . Suppose \mathcal{B} is a C^* -subalgebra of \mathcal{A} such that there is an expectation $E : \mathcal{A} \rightarrow \mathcal{B}$ satisfying the conditions: $\sigma_g E = E \sigma_g, g \in G$, and $\phi \circ E = \phi$. Then $\sigma|_{\mathcal{B}}$ is an action of G on \mathcal{B} and*

$$h_\phi(\sigma|_{\mathcal{B}}) \leq h_\phi(\sigma).$$

The **proof** is the same as in [StV, Lemma 3.2].

Lemma 2.12. *Let \mathcal{A} be a C^* -algebra, ϕ a state, and σ a ϕ -invariant action of G on \mathcal{A} . Let $\{\mathcal{A}_j\}_{j=1}^\infty$ be an increasing sequence of C^* -subalgebras such that the expectations $E_j : \mathcal{A} \rightarrow \mathcal{A}_j$ satisfy the following conditions:*

- (i) $\sigma_g E_j = E_j \sigma_g, g \in G, j = 1, 2, \dots;$
- (ii) $E_{j+1} E_j = E_j E_{j+1} = E_j, j = 1, 2, \dots;$
- (iii) $E_j \rightarrow 1_{\mathcal{A}}$ in pointwise-norm topology.

Suppose that the norm closure of $\bigcup_j \mathcal{A}_j$ is \mathcal{A} ; then $\sigma|_{\mathcal{A}_j}$ is an action of σ on \mathcal{A}_j and

$$h_\phi(\sigma) \leq \liminf_j h_\phi(\sigma|_{\mathcal{A}_j}).$$

If moreover $\phi \circ E_j = \phi, j \in \mathbf{N}$, then

$$h_\phi(\sigma) = \lim_j h_\phi(\sigma|_{\mathcal{A}_j}).$$

Proof. Lemma can be proved by the same method as Lemma 3.3 from [StV] taking into account the inequality (2.7). \diamond

Lemma 2.13. *Let σ_i be an action of G on (\mathcal{A}_i, ϕ_i) such that $\phi_i \circ \sigma_i = \phi_i, i = 1, 2$. Then*

$$h_{\phi_1 \otimes \phi_2}(\sigma_1 \otimes \sigma_2) \geq h_{\phi_1}(\sigma_1) + h_{\phi_2}(\sigma_2).$$

Proof. See the proof of Lemma 3.4 from [StV]. \diamond

3 Entropy of \mathbf{Z}^n -actions by Bogoliubov automorphisms

Let H be a complex separable Hilbert space. Suppose a unitary representation σ of $G = \mathbf{Z}^n$ on H is given (as before we consider the case $n = 2$ only). Denote by $U_1 = \sigma(1, 0)$ and $U_2 = \sigma(0, 1)$ two commuting generators of $\sigma(G)$. According to the well known results of Mackey (see, for example, [Ki, p.128]) we have for $n_1, n_2 \in \mathbf{Z}$ that

$$U_1^{n_1} U_2^{n_2} = \int \int_{\mathbf{T}^2} \oplus \exp(2\pi i(n_1 x + n_2 y)) 1_{(x,y)} d\mu(x, y)$$

where $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$ is the two-dimensional torus (which is the dual space for \mathbf{Z}^2), $d\mu(x, y)$ is a Borel measure on \mathbf{T}^2 , and $1_{(x,y)}$ is the identity operator on a Hilbert space $H(x, y)$. In other words, H is decomposed in a direct integral of $H(x, y)$ in which the representation σ is diagonal.

Each unitary representation σ is a direct sum $\sigma = \sigma_a \oplus \sigma_s$ with σ_a acting on a Hilbert space H_a and σ_s on H_s . The representation σ_a has spectral measure μ_a absolutely continuous with respect to Lebesgue measure $d\theta_1 d\theta_2$ on \mathbf{T}^2 , and σ_s has spectral measure μ_s singular with respect to $d\theta_1 d\theta_2$. We will call σ_a the absolutely continuous part of σ and σ_s the singular part. If μ_a is equivalent to $d\theta_1 d\theta_2$, then σ is called a representation with Lebesgue spectrum. The measurable function $\dim H_a(x, y)$ is called the multiplicity function and denoted by $m(\sigma)$.

We remind following [StV] some definitions concerning the CAR-algebra notion. The CAR-algebra $\mathcal{A}(H)$ is a C^* -algebra with the property that there is a linear map $f \mapsto a(f)$ of H into $\mathcal{A}(H)$ whose range generates $\mathcal{A}(H)$ as a C^* -algebra and satisfying the canonical anticommutation relations:

$$a(f)a(g)^* + a(g)^*a(f) = (f, g)\mathbf{1},$$

$$a(f)a(g) + a(g)a(f) = 0, \quad f, g \in H$$

where (\cdot, \cdot) is the inner product on H , and $\mathbf{1}$ is the unit of $\mathcal{A}(H)$. Let $0 \leq A \leq 1$ be an operator on H . The quasifree state ω_A on $\mathcal{A}(H)$ is defined by its values on products of the form $a(f_n)^* \dots a(f_1)^* a(g_1) \dots a(g_m)$, $n, m \in \mathbf{N}$ given by

$$\omega_A(a(f_n)^* \dots a(f_1)^* a(g_1) \dots a(g_m)) = \delta_{nm} \det((Ag_i, f_j)).$$

If U is a unitary operator on H , then U defines an automorphism $\alpha(U)$ of $\mathcal{A}(H)$ called a Bogoliubov automorphism by formula: $\alpha(U)(a(f)) = a(Uf)$, $f \in H$. If $UA = AU$, then $\omega_A \circ \alpha(U) = \omega_A$. It is well known that $\mathcal{A}(H) \simeq \bigotimes_{n=1}^{\infty} M_2(\mathbf{C})_n$, and if A has a pure point spectrum, then $\omega_A = \bigotimes_{n=1}^{\infty} \omega_{\lambda_n}$, where $\{\lambda_n\}$ is the set of eigenvalues for A and

$$\omega_{\lambda} \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = (1 - \lambda)a + \lambda d$$

Let σ be a unitary representation of $G = \mathbf{Z}^2$ on H . Then σ defines an action α of G on $\mathcal{A}(H)$ by the formula

$$\alpha(\sigma_g)(a(f)) = a(\sigma_g(f)), \quad g \in G.$$

We will call $\alpha(\sigma_g)$ a Bogoliubov action.

The case of singular spectrum.

We first prove that the dynamical entropy of Bogoliubov actions with a pure singular spectrum is zero.

Lemma 3.1. *Let σ be a representation of G on H with spectral measure singular with respect to Lebesgue measure on \mathbf{T}^2 . Let $U_1 = \sigma(1, 0), U_2 = \sigma(0, 1)$ be two generators of $\sigma(G)$. Assume that P is a finite rank orthogonal projection and $\epsilon > 0$. Then there is $k_0 \in \mathbf{N}$ such that for each $k \geq k_0$ there is a finite rank projection Q_k with the properties:*

- (i) $\|(1 - Q_k)U_1^{s_1}U_2^{s_2}P\| < \epsilon, \quad 0 \leq s_1 \leq k, 0 \leq s_2 \leq k;$
- (ii) $\dim Q_k \leq \epsilon k^2.$

Proof. It follows from the conditions of the lemma that there is a measurable set $D \subset \mathbf{T}^2$ such that for given $\delta > 0$ the following hold:

- (a) $D = D_1 \cup \dots \cup D_N, D_j = \{(e^{i\theta_1}, e^{i\theta_2}) \mid \alpha_j^1 \leq \theta_1 \leq \beta_j^1, \alpha_j^2 \leq \theta_2 \leq \beta_j^2, \beta_j^1 - \alpha_j^1 = \beta_j^2 - \alpha_j^2\}$ where all D_j are disjunct and $\alpha_j^l, \beta_j^l \in [0, 2\pi], l = 1, 2, j = 1, \dots, N;$
- (b) $N \max\{(\beta_j^1 - \alpha_j^1)^2 \mid j = 1, \dots, N\} \leq \delta;$
- (c) if $E(D)$ is the spectral projection of σ for the set D , then $\|(1 - E(D))P\| < \delta.$

Take a number $M \in \mathbf{N}$ and divide every D_j in M^2 squares $D_{j,p}$ of equal measure, where $1 \leq j \leq N$ and $1 \leq p \leq M^2$. Let

$$X_M = \bigoplus_{j,p} E(D_{j,p})P(H).$$

Then $\dim X_M \leq NM^2 \dim P(H)$. If $f \in P(H), \|f\| = 1$, then we have that for $(e^{i\theta'_{j,p}}, e^{i\theta''_{j,p}}) \in D_{j,p}$ with $j \in [1, N], p \in [1, M^2]$

$$\begin{aligned} d(U_1^{s_1}U_2^{s_2}f, X_M)^2 &\leq \left\| \sum_{j,p} e^{i\theta'_{j,p}s_1} e^{i\theta''_{j,p}s_2} E(D_{j,p})f - U_1^{s_1}U_2^{s_2}f \right\|^2 \\ &\leq 2 \sum_{j,p} \left\| e^{i\theta'_{j,p}s_1} e^{i\theta''_{j,p}s_2} E(D_{j,p})f - U_1^{s_1}U_2^{s_2} E(D_{j,p})f \right\|^2 + 2 \left\| U_1^{s_1}U_2^{s_2} E(D)f - U_1^{s_1}U_2^{s_2} f \right\|^2 \\ &\leq 2 \sum_{j,p} \left\| (e^{i\theta'_{j,p}s_1} e^{i\theta''_{j,p}s_2} 1 - U_1^{s_1}U_2^{s_2}) E(D_{j,p}) \right\|^2 \|E(D_{j,p})f\|^2 + 2\delta^2 \\ &\leq 2 \max_{j,p} \left\| (e^{i\theta'_{j,p}s_1} e^{i\theta''_{j,p}s_2} 1 - U_1^{s_1}U_2^{s_2}) E(D_{j,p}) \right\|^2 + 2\delta^2. \end{aligned}$$

Since

$$\left\| (e^{i\theta'_{j,p}s_1} e^{i\theta''_{j,p}s_2} 1 - U_1^{s_1}U_2^{s_2}) E(D_{j,p}) \right\|$$

$$\begin{aligned}
&\leq \sup_{z_1, z_2 \in D_{j,p}} |e^{i\theta'_{j,p}s_1} e^{i\theta''_{j,p}s_2} - z_1^{s_1} z_2^{s_2}| \\
&\leq \sup_{z_1, z_2 \in D_{j,p}} \{|e^{i\theta'_{j,p}s_1} (e^{i\theta''_{j,p}s_2} - z_2^{s_2})| + |z_2^{s_2} (e^{i\theta'_{j,p}s_1} - z_1^{s_1})|\} \\
&\leq \frac{1}{M} (s_1 + s_2) \max_j |\beta_j^1 - \alpha_j^1| \\
&\leq \frac{(s_1 + s_2)}{M} \left(\frac{\delta}{N}\right)^{1/2},
\end{aligned}$$

we get

$$\begin{aligned}
d(U_1^{s_1} U_2^{s_2} f, X_M) &\leq \sqrt{2 \left(\frac{s_1 + s_2}{M}\right)^2 \frac{\delta}{N} + 2\delta^2} \\
&\leq \sqrt{2} \frac{s_1 + s_2}{M} \left(\frac{\delta}{N}\right)^{1/2} + \sqrt{2}\delta.
\end{aligned}$$

This estimate is uniform in $f \in P(H)$. Thus, we proved for the orthogonal projection Q onto X_M that

$$\|(1 - Q)U_1^{s_1} U_2^{s_2} P\| \leq \sqrt{2} \left(\frac{s_1 + s_2}{M} \left(\frac{\delta}{N}\right)^{1/2} + \delta\right)$$

and $\dim Q \leq M^2 N \dim P$.

Given k let us take Q_k to be the orthogonal projection onto X_M , and if ϵ is as in the condition of the lemma, let

$$M^2 = \left\lceil \frac{\epsilon k^2}{N \dim P} \right\rceil.$$

Then $\dim Q \leq M^2 N \dim P \leq \epsilon k^2$. Take

$$k_0 = \left\lceil \sqrt{\frac{N \dim P}{\epsilon}} + 1 \right\rceil,$$

then for $k \geq k_0$ we have

$$\frac{\epsilon k^2}{N \dim P} \geq 1.$$

Taking into account that

$$\begin{aligned}
&\max_{1 \leq s_1, s_2 \leq k} \sqrt{2} \left(\frac{s_1 + s_2}{M} \left(\frac{\delta}{N}\right)^{1/2} + \delta\right) \\
&= \sqrt{2} \left(\frac{2k}{\sqrt{\left\lceil \frac{\epsilon k^2}{N \dim P} \right\rceil}} \left(\frac{\delta}{N}\right)^{1/2} + \delta \right)
\end{aligned}$$

$$\leq \frac{4\sqrt{\dim P} + \sqrt{2\epsilon}}{\sqrt{\epsilon}} \delta^{1/2},$$

we obtain that if

$$\delta^{1/2} \leq \frac{\epsilon\sqrt{\epsilon}}{4\sqrt{\dim P} + \sqrt{2\epsilon}},$$

then $\|(1 - Q_k)U_1^{s_1}U_2^{s_2}P\| < \epsilon \diamond$

Theorem 3.2. *Let σ be a Bogoliubov action on $\mathcal{A}(H)$, and ϕ a state on $\mathcal{A}(H)$ such that $\phi \circ \alpha(\sigma_g) = \phi$ for any $g \in G$. If the spectral measure of σ is singular with respect to Lebesgue measure, then $h_\phi(\alpha(\sigma)) = 0$.*

Proof. Let P be an orthogonal projection in $B(H)$ of finite rank, and $j : P(H) \rightarrow H$ the inclusion map. Then there are u.c.p. maps $\gamma_j : \mathcal{A}(P(H)) \rightarrow \mathcal{A}(H)$, $\gamma_p : \mathcal{A}(H) \rightarrow \mathcal{A}(P(H))$ [E] such that

$$\gamma_j(a(f)) = a(jf), \quad \gamma_p(a(f)) = a(Pf).$$

If $P_n \nearrow 1$ is a sequence of such projections then with $j_n : P_n(H) \rightarrow H$ the inclusion, $\gamma_{j_n} \circ \gamma_{P_n} \rightarrow 1_{\mathcal{A}(H)}$ in pointwise-norm topology. By Theorem 2.5,

$$h_\phi(\alpha(\sigma)) = \lim_n h_{\phi, \alpha(\sigma)}(\gamma_{j_n}).$$

Whence it suffices to show that $h_{\phi, \alpha(\sigma)}(\gamma_j) = 0$. Since $\dim P < \infty$, for given $\delta > 0$ there is $\eta > 0$ such that if $W_1, W_2 : P(H) \rightarrow H$ are isometries with $\|W_1 - W_2\| < \eta$, then $\|\alpha(W_1) - \alpha(W_2)\| < \delta$, where $\alpha(W)(a(f)) = a(Wf)$ [E]. Take Q_k as in Lemma 3.1. Denote by $\text{pol}(Q_k U_1^{s_1} U_2^{s_2} |_{P(H)})$ the partial isometry W_2 appearing in the polar decomposition

$$Q_k U_1^{s_1} U_2^{s_2} |_{P(H)} = W_2 |Q_k U_1^{s_1} U_2^{s_2} |_{P(H)}|$$

with U_1, U_2 being the generators of $\sigma(G)$ as in Lemma 3.1. Let $W_1 = U_1^{s_1} U_2^{s_2} |_{P(H)}$. According to Lemma 3.1, $\|U_1^{s_1} U_2^{s_2} - Q_k U_1^{s_1} U_2^{s_2} P\| < \epsilon$ for $0 \leq s_1, s_2 \leq k$, hence

$$\|U_1^{s_1} U_2^{s_2} |_{P(H)} - \text{pol}(Q_k U_1^{s_1} U_2^{s_2} |_{P(H)})\| < 3\epsilon.$$

Choosing $\epsilon < \eta/3$ and $k \geq k_0$, we obtain that

$$\|\alpha(U_1^{s_1} U_2^{s_2} |_{P(H)}) - \alpha(\text{pol}(Q_k U_1^{s_1} U_2^{s_2} |_{P(H)}))\| < \delta$$

for $0 \leq s_1, s_2 \leq k$. By [CNT, Proposition IV.3] for given $\xi > 0$ and $\epsilon > 0$, there is $k_0 \in \mathbf{N}$ such that if $k \geq k_0$ and Q_k as in Lemma 3.1, then

$$H_\phi(\alpha(\sigma_{\rho_k})(\gamma_j)) \leq k^2 \xi + H_\phi(\alpha(\text{pol}(Q_k \sigma_{\rho_k} j))) \quad (3.1)$$

where $\rho_k = [0, 1, \dots, k-1] \times [0, 1, \dots, k-1]$. If we define $v : Q_k(H) \rightarrow H$ to be the inclusion map, then $\alpha(\text{pol}(Q_k \sigma_{(p,q)} j)) = \alpha(v) \circ \alpha(\text{pol}(Q_k \sigma_{(p,q)} j))$, $(p, q) \in \rho_k$, whence by [CNT, Proposition III.6 (a) and III.4]

$$H_\phi(\alpha(\text{pol}(Q_k \sigma_{\rho_k} j))) \leq H_\phi(\alpha(v)) = S(\phi \circ \alpha(v)) \quad (3.2)$$

where $\phi \circ \alpha(v)$ is a state on $\mathcal{A}(Q_k(H))$, a C^* -algebra of dimension less than $2^{k^2\epsilon}$. Thus,

$$H_\phi(\alpha(v)) \leq k^2\epsilon \log 2. \quad (3.3)$$

It follows from (3.1), (3.2) and (3.3) that

$$\frac{1}{|\rho_k|} H_\phi(\alpha(\sigma_{\rho_k})(\gamma_j)) \leq \xi + \epsilon \log 2.$$

Since ξ and ϵ are arbitrary, $h_{\phi, \alpha(\sigma)}(\gamma_j) = 0$. \diamond

The case of absolutely continuous spectrum.

Now we formulate and prove several lemmas concerning the actions with absolutely continuous spectrum.

Lemma 3.3. *Let $H = H_1 \oplus H_2$ and $0 \leq A_i \leq 1$ be an operator on H_i . Suppose σ_i be a unitary action of G on H_i such that $\sigma(g)A_i = A_i\sigma_i(g)$, $g \in G$, $i = 1, 2$. Then*

$$h_{\omega_{A_1 \oplus A_2}}(\alpha(\sigma_1 \oplus \sigma_2)) \geq h_{\omega_{A_1}}(\alpha(\sigma_1)).$$

Proof. According to [StV] the following formula is valid:

$$\alpha((\sigma_1 \oplus \sigma_2)(g))|_{\mathcal{A}(H_1) \otimes \mathcal{A}(H_2)_e} = \alpha(\sigma_1(g))|_{\mathcal{A}(H_1)} \otimes \alpha(\sigma_2(g))|_{\mathcal{A}(H_2)_e}$$

where $g \in G$ and $\mathcal{A}(H)_e$ is the even subalgebra of $\mathcal{A}(H)$. Then the statement of lemma follows from Lemmas 2.11 and 2.13. \diamond

Lemma 3.4. *Let $\lambda \in [0, 1]$ and σ_1, σ_2 be two unitary representations of G on a Hilbert space H .*

(i) *If there is a unitary operator W such that $\sigma_2(g) = W\sigma_1(g)W^{-1}$, $g \in G$, then $h_{\omega_\lambda}(\sigma_1) = h_{\omega_\lambda}(\sigma_2)$.*

(ii) *If σ_1 and σ_2 have the same singular parts and $m(\sigma_1) \geq m(\sigma_2)$, then $h_{\omega_\lambda}(\sigma_1) \geq h_{\omega_\lambda}(\sigma_2)$.*

Here and below ω_λ is the quasi-free state constructed by $A = \lambda 1$.

Proof. (i) is obvious due to Proposition 2.9. To prove (ii), we can assume that up to unitary equivalence σ_2 is the restriction of σ_1 to a reducing subspace, and now (ii) follows from Lemma 3.3. \diamond

Lemma 3.5. *Let $\{\sigma_n\}$ be a sequence of unitary representations of G all with Lebesgue spectrum. Suppose $\{m(\sigma_n)\}$ is an increasing sequence with pointwise limit $m(\sigma)$, where σ is a unitary representation of G also with Lebesgue spectrum. Then $\{h_{\omega_\lambda}(\alpha(\sigma_n))\}$ is an increasing sequence and*

$$h_{\omega_\lambda}(\alpha(\sigma)) = \lim_n h_{\omega_\lambda}(\alpha(\sigma_n)).$$

Proof. The statement of the lemma follows from Lemmas 2.11 and 2.13 in the same way as in [StV, Lemma 4.4]. \diamond

Let H be an infinite dimensional Hilbert space with an orthonormal basis $\{f_{m,n}\}_{m,n \in \mathbf{Z}}$. Let also V_1 and V_2 be two unitary operators acting on H by formulas:

$$V_1 f_{m,n} = f_{m+1,n}, \quad V_2 f_{m,n} = f_{m,n+1}, \quad m, n \in \mathbf{Z}$$

Then V_1 and V_2 generate an representation of G on H which we will call the two dimensional bilateral shift.

Lemma 3.6. *For $i \in \{1, \dots, r\}$, Let H be an infinite dimensional separable complex Hilbert space with identity 1_i , and let $\sigma_i = \sigma_i(\{U_1(i), U_2(i)\})$ be a unitary action of G on H_i with $U_1(i), U_2(i)$ being the generators such that for each i there are $p_1(i), p_2(i) \in \mathbf{N}$ and common $q_1, q_2 \in \mathbf{N}$ for all i such that the pair $(U_1^{q_1}, U_2^{q_2})$ is unitarily equivalent to $(V_1^{p_1(i)}, V_2^{p_2(i)})$ where V_1, V_2 are generators of the two-dimensional bilateral shift. Let $A = \bigoplus_{i=1}^r c_i 1_i, c_i \in [0, 1]$ and $U_1 = \bigoplus_{i=1}^r U_1(i), U_2 = \bigoplus_{i=1}^r U_2(i)$. Then if σ is the representation of G defined by U_1 and U_2 we have the formula for the entropy of Bogoliubov action $\alpha(\sigma)$*

$$h_{\omega_A}(\alpha(\sigma)) = \frac{1}{q_1 q_2} \sum_{i=1}^r p_1(i) p_2(i) S(\omega_{c_i}) = \frac{1}{q_1 q_2} \sum_{i=1}^r p_1(i) p_2(i) (\eta(c_i) + \eta(1 - c_i))$$

where $\eta(x) = -x \log x, x > 0$. The same formula holds for the restriction of $\alpha(\sigma)$ and ω_A to the even subalgebra $\mathcal{A}(H)_e$.

Proof. Let $\{f_{m,n}^i\}_{m,n \in \mathbf{Z}}$ be an orthonormal basic in $H_i, i = 1, \dots, r$, such that $V_1 f_{m,n}^i = f_{m+1,n}^i, V_2 f_{m,n}^i = f_{m,n+1}^i, m, n \in \mathbf{Z}$. Let

$$\begin{aligned} \mathcal{A} \left[\{f_{m,n}^1 | m = 1, \dots, p_1(1), n = 1, \dots, p_2(1)\}, \dots, \{f_{m,n}^r | m = 1, \dots, p_1(r), n = 1, \dots, p_2(r)\} \right] \\ = N \end{aligned} \quad (3.4)$$

where $[\{f\}]$ denotes the subspace spanned by vectors $\{f\}$. Since $A f_{m,n}^i = c_i f_{m,n}^i$, the subalgebra N can be written as a tensor product

$$N = \bigotimes_{i=1}^p M_2(\mathbf{C})_i \quad (3.5)$$

where $p = \sum_{i=1}^r p_1(i) p_2(i)$ and

$$\omega_A | N = \left(\bigotimes_{i=1}^{p_1(1)p_2(1)} \omega_{c_1} \right) \otimes \dots \otimes \left(\bigotimes_{i=1}^{p_1(r)p_2(r)} \omega_{c_r} \right). \quad (3.6)$$

Consider for $s, t \in \mathbf{Z}$ the subspaces

$$X_{s,t} = U_1^{q_1 s} U_2^{q_2 t} [\{f_{m,n}^1\}, \dots, \{f_{m,n}^r\}]$$

where the vectors $\{f_{m,n}^i\}$ are taken as in (3.4). According to conditions of lemma, we have that

$$X_{s,t} = \bigoplus_{i=1}^r V_1^{s p_1(i)} V_2^{t p_2(i)} [\{f_{m,n}^i\}]$$

and $X_{s,t} = X_{s',t'}$ if $(s,t) \neq (s',t')$. It is easy to check that the subspaces $\{X_{s,t}\}_{s,t \in \mathbf{Z}}$ span all $H = \bigoplus_{i=1}^r H_i$ and therefore, $\mathcal{A}(X_{s,t})$ generate all $\mathcal{A}(H)$. Let G_{q_1, q_2} be the subgroup of G with generators $(q_1, 0)$ and $(q_2, 0)$. The unitary operators $U_1^{q_1}$ and $U_2^{q_2}$ define the representation $\sigma(G_{q_1, q_2})$ on H . Let $\rho_{s,t}(q_1, q_2) = [0, q_1, \dots, (s-1)q_1] \times [0, q_2, \dots, (t-1)q_2]$, $s, t \in \mathbf{N}$. Repeating the arguments from the proof of [Lemma 4.5, StV], we get that if $N_{s,t}$ is the algebra generated by $\alpha(\sigma_{\rho_{s,t}(q_1, q_2)})(N)$, then it follows from [CNT, Corollary VIII.8] that

$$\begin{aligned} \frac{1}{st} H_{\omega_A}(\alpha(\sigma_{\rho_{s,t}(q_1, q_2)})(N)) &= \frac{1}{st} S(\omega_A | N_{s,t}) \\ &= \frac{1}{st} \sum_{u=0}^{s-1} \sum_{v=0}^{t-1} S(\omega_A | \alpha(U_1^{u q_1} U_2^{v q_2})(N)) \\ &= S(\omega_A | N) = \sum_{i=1}^r p_1(i) p_2(i) S(\omega_{c_i}). \end{aligned} \quad (3.7)$$

We used here relations (3.5) and (3.6). Thus, we have proved that

$$h_{\omega_A, \alpha(\sigma(G_{q_1, q_2}))}(N) = \sum_{i=1}^r p_1(i) p_2(i) S(\omega_{c_i}). \quad (3.8)$$

For $n \in \mathbf{N}$ denote by

$$M_n = \bigvee_{s=-n}^n \bigvee_{t=-n}^n \alpha(U_1^{s q_1} U_2^{t q_2})(N).$$

$\{M_n\}$ is an increasing sequence of f.d. subalgebras of $\mathcal{A}(H)$ with dense union. Then by Proposition 2.6

$$h_{\omega_A}(\alpha(\sigma(G_{q_1, q_2}))) = \lim_n h_{\omega_A, \alpha(\sigma(G_{q_1, q_2}))}(M_n). \quad (3.9)$$

The C^* -algebra generated by $\{\alpha(\sigma_{\rho_{s,t}(q_1, q_2)})(M_n)\}$ equals the one generated by $\{\alpha(\sigma_{\rho'_{s,t}(q_1, q_2)})(N)\}$ where $\rho'_{s,t}(q_1, q_2) = [-n, \dots, n+s-1] \times [-n, \dots, n+t-1]$. By previous arguments with N (see (3.7)) we obtain that

$$\frac{1}{(2n+s)(2n+t)} H_{\omega_A}(\alpha(\sigma_{\rho_{s,t}(q_1, q_2)})(M_n)) = S(\omega_A | N). \quad (3.10)$$

Taking the limit in (3.10) as $s, t \rightarrow \infty$ we get

$$h_{\omega_A, \alpha(\sigma_{(G_{q_1, q_2}))}}(M_n) = S(\omega_A|N).$$

It follows from Theorem 2.7 and (3.8) that

$$h_{\omega_A}(\alpha(\sigma)) = \frac{1}{q_1 q_2} S(\omega_A|N) = \frac{1}{q_1 q_2} \sum_{i=1}^r p_1(i) p_2(i) S(\omega_{c_i}). \diamond$$

Lemma 3.7. *Let σ be a representation of G on H with Lebesgue spectrum consisting of disjoint sets $D_j = \exp(2\pi i[a_1(j), b_1(j)]) \times \exp(2\pi i[a_2(j), b_2(j)])$, $j \in J$, such that $b_i(j) - a_i(j) = p_i(j)/q_i$ where $p_i(j), q_i \in \mathbf{N}$, $i = 1, 2$, and $J \subset \mathbf{N}$. Let $H_j = L^2(D_j, d\theta_1 d\theta_2)$ be considered as a subspace of $L^2(\mathbf{T}^2, d\theta_1 d\theta_2)$. If U_1 and U_2 are the generators of σ , then one can define $U_i(j) = U_i|_{H_j}$, $i = 1, 2$, $j \in J$, i.e. $U_1 = \bigoplus_{j \in J} U_1(j)$, $U_2 = \bigoplus_{j \in J} U_2(j)$ and $U_1(j), U_2(j)$ are the generators of representation σ_j on H_j . Suppose that σ_j has constant finite multiplicity n_j , and let $0 \leq A_j \leq 1$ acts on H_j and commutes with σ_j . Writing $\sigma_j = \tau_j \oplus \dots \oplus \tau_j$ (n_j times) (τ_j is a representation of G with the multiplicity 1), we assume that $A_j = \bigoplus_{k=1}^{n_j} c_{jk} 1_j$ where 1_j is the identity on the space on which τ_j acts. Let B_j denote the diagonal $n_j \times n_j$ matrix with the numbers c_{j1}, \dots, c_{jn_j} on the diagonal. Then $A_j = B_j \otimes 1_j$ and the following formula holds;*

$$h_{\omega_A}(\alpha(\sigma)) = \sum_{j \in J} (b_1(j) - a_1(j))(b_2(j) - a_2(j)) \text{Tr}_{n_j}(\eta(B_j) + \eta(1 - B_j)) \quad (3.11)$$

with Tr_{n_j} being usual trace on $M_{n_j}(\mathbf{C})$ and $\eta(x) = -x \log x$, $x > 0$. Also the same formula holds for the restrictions $\alpha(\sigma)$ and ω_A to $\mathcal{A}(H)_e$.

Proof. We first assume that J is finite, $J = \{1, 2, \dots, r\}$. According to conditions of the lemma, we may write

$$U_i = \underbrace{V_i(1) \oplus \dots \oplus V_i(1)}_{n_1} \oplus \dots \oplus \underbrace{V_i(r) \oplus \dots \oplus V_i(r)}_{n_r}, \quad i = 1, 2$$

where $V_1(j), V_2(j)$ generate the action τ_j . Also we have that

$$A = (c_{1,1} 1_1 \oplus \dots \oplus c_{1n_1} 1_1) \oplus \dots \oplus (c_{r1} 1_r \oplus \dots \oplus c_{rn_r} 1_r).$$

For every $j \in \{1, \dots, r\}$, the operators $V_1^{q_1}(j)$ and $V_2^{q_2}(j)$ generate the two-dimensional bilateral shift of multiplicity $p_1(j)p_2(j)$. Thus, by Lemma 3.6

$$\begin{aligned} h_{\omega_A}(\alpha(\sigma)) &= \sum_{j=1}^r \frac{p_1(j)p_2(j)}{q_1 q_2} \sum_{l=1}^{n_j} S(\omega_{c_{jl}}) \\ &= \sum_{j=1}^r (b_1(j) - a_1(j))(b_2(j) - a_2(j)) \text{Tr}_{n_j}(\eta(B_j) + \eta(1 - B_j)). \end{aligned}$$

The case of infinite set J is considered in the same way as in [StV, Lemma 4.6] taking into account Lemma 2.12. \diamond

Theorem 3.8. *Let σ be a unitary representation of G on a complex Hilbert space H . Assume that σ has Lebesgue spectrum. Then for $\lambda \in [0, 1]$ we have*

$$h_{\omega_\lambda}(\alpha(\sigma)) = \frac{1}{(2\pi)^2}(\eta(\lambda) + \eta(1 - \lambda)) \int_0^{2\pi} \int_0^{2\pi} m(\sigma)(\theta_1, \theta_2) d\theta_1 d\theta_2 \quad (3.12)$$

with $\eta(x) = -x \log x$ and $m(\sigma)$ being the multiplicity function.

Proof. If $\lambda = 0$ or 1 , then ω_λ is a pure state [PSt], so $h_{\omega_\lambda}(\alpha(\sigma)) = 0$ by Lemma 2.10. It means that formula (3.11) holds in this case. Assume $0 < \lambda < 1$. We will use the following result which is an obvious generalization of Theorem 2.1 from [StV].

Let \mathcal{G} be the additive semigroup of measurable functions $f : \mathbf{T}^2 \rightarrow \mathbf{N} \cup \{0\}$. Let for $n_1, n_2 \in \mathbf{N}$

$$T_{n_1, n_2} f(\rho_1, \rho_2) = \sum_{z_1^{n_1} = \rho_1} \sum_{z_2^{n_2} = \rho_2} f(z_1, z_2).$$

Suppose that a map $\mu : \mathcal{G} \rightarrow \mathbf{R}_+ = [0, \infty)$ satisfies the conditions:

- (i) $\mu(n1) = n$;
- (ii) $f \leq g \Rightarrow \mu(f) \leq \mu(g)$;
- (iii) $f_i \nearrow f \Rightarrow \mu(f_i) \nearrow \mu(f)$;
- (iv) $\mu(T_{n_1, n_2} f) = n_1 n_2 \mu(f)$;
- (v) $f = g$ a.e. $\Rightarrow \mu(f) = \mu(g)$.

Then

$$\mu(f) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f d\theta_1 d\theta_2. \quad (3.13)$$

The proof of formula (3.13) can be obtained using the methods of [StV].

Considering the representations of G with Lebesgue spectrum one can assume that h_{ω_λ} defines a map from the set of multiplicity functions into \mathbf{R}_+ . Theorem 2.7 and Lemmas 3.4, 3.5, and 3.6 tell us that h_{ω_λ} satisfies all conditions (i) - (v), therefore (3.12) is valid. \diamond

The case of mixed spectrum.

Lemma 3.9. *Let σ be a unitary representation of G on H with absolutely continuous part σ_a acting on H_a and singular part σ_s acting on H_s . Let $A = A_a \oplus A_s$ commute with $\sigma = \sigma_a \oplus \sigma_s$, $0 \leq A \leq 1$. Assume that A_a and σ_a are as in Lemma 3.7. Then $h_{\omega_A}(\alpha(\sigma)) = h_{\omega_{A_a}}(\alpha(\sigma_a))$ is given by formula (3.11).*

The **proof** of this lemma (which is analogous to [Lemma 5.3, StV]) repeats on the whole that of Theorem 3.2 and is based on Lemmas 3.3, 3.6, and Proposition 2.6.

Theorem 3.10. *Let σ be arbitrary unitary representation of G on H . Then for $\lambda \in [0, 1]$ the entropy $h_{\omega_\lambda}(\alpha(\sigma))$ is given by formula (3.12).*

Proof. Let $\sigma = \sigma_a \oplus \sigma_s$ be the decomposition of σ into absolutely continuous and singular parts acting on H_a and H_s respectively. Let $\epsilon > 0$ be given. By measure theory there is a unitary representation ψ with Lebesgue spectrum on H_a such that

$$m(\psi) = \sum_{j=1}^r d_j \chi_{E_j} \geq m(\sigma)$$

where E_j is of the form $\exp 2\pi i [b_1(j), a_1(j)] \times \exp 2\pi i [b_2(j), a_2(j)]$ with $b_k(j) - a_k(j)$ being rational, $k = 1, 2$, and

$$\int_0^{2\pi} \int_0^{2\pi} m(\sigma) d\theta_1 d\theta_2 + \epsilon > \int_0^{2\pi} \int_0^{2\pi} m(\psi) d\theta_1 d\theta_2.$$

According to Theorem 3.8 and Lemmas 3.3, 3.4 and 3.9, we get

$$\begin{aligned} & \frac{1}{(2\pi)^2} (\eta(\lambda)) + \eta(1 - \lambda) \int_0^{2\pi} \int_0^{2\pi} m(\sigma) d\theta_1 d\theta_2 = h_{\omega_\lambda}(\alpha(\sigma_a)) \\ & \leq h_{\omega_\lambda}(\alpha(\sigma_a \oplus \sigma_s)) \leq h_{\omega_\lambda}(\alpha(\psi \oplus \sigma_s)) = h_{\omega_\lambda}(\alpha(\psi)) \\ & = \frac{1}{(2\pi)^2} (\eta(\lambda) + \eta(1 - \lambda)) \int_0^{2\pi} \int_0^{2\pi} m(\psi) d\theta_1 d\theta_2 \\ & < \frac{1}{(2\pi)^2} (\eta(\lambda) + \eta(1 - \lambda)) \left[\int_0^{2\pi} \int_0^{2\pi} m(\sigma) d\theta_1 d\theta_2 + \epsilon \right]. \end{aligned}$$

Since ϵ is arbitrary, (3.12) follows. \diamond

If an operator A commutes with a representation $\sigma = \sigma_a \oplus \sigma_s$ of G , then by direct integral theory $A = A_a \oplus A_s$ and

$$A_a = \int \int_{\mathbf{T}^2} \oplus A(\theta_1, \theta_2) d\theta_1 d\theta_2$$

with $A(\theta_1, \theta_2) \in B(H(\theta_1, \theta_2))$, where $H(\theta_1, \theta_2) = 0$ if $m(\sigma)(\theta_1, \theta_2) = 0$. We note that if A has a pure point spectrum, the following formula can be proved using Lemmas 2.11, 2.12, 2.13, 3.3, 3.9 and Theorem 3.10 by the method of [StV, Theorem 6.3]

$$h_{\omega_A}(\alpha(\sigma)) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \text{Tr}(\eta(A(\theta_1, \theta_2)) + \eta(1 - A(\theta_1, \theta_2))) d\theta_1 d\theta_2$$

where Tr is the usual trace on $B(H(\theta_1, \theta_2))$.

Acknowledgement. This work was done during our visit to the Erwin Schrödinger International Institute for Mathematical Physics. We are most grateful to Professors H. Narnhofer and W. Thirring and the Erwin Schrödinger Institute for the

hospitality and support. We would like to thank H. Narnhofer and E. Störmer for useful discussions on the entropy. They also pointed out some inaccuracies in the manuscript. We express our deep gratitude to Dr. T.Hudetz for his comments of the first version of this paper. Also he explained to us his approach to the calculation of entropy of a finite index subgroup of \mathbf{Z}^2 .

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