

**QCD₁₊₁ with Massless Quarks and Gauge
Covariant Sugawara Construction****E. Langmann
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QCD₁₊₁ with massless quarks and gauge covariant Sugawara construction

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Abstract

We use the Hamiltonian framework to study massless QCD₁₊₁, i.e. Yang-Mills gauge theories with massless Dirac fermions on a cylinder (= (1+1) dimensional spacetime $S^1 \times \mathbb{R}$) and make explicit the full, non-perturbative structure of these quantum field theory models. We consider N_F fermion flavors and gauge group either $U(N_C)$, $SU(N_C)$ or another Lie subgroup of $U(N_C)$. In this approach, anomalies are traced back to kinematical requirements such as positivity of the Hamiltonian, gauge invariance, and the condition that all observables are represented by well-defined operators on a Hilbert space. We also give equal time commutators of the energy momentum tensor and find a gauge-covariant form of the (affine-) Sugawara construction. This allows us to represent massless QCD₁₊₁ as a gauge theory of Kac-Moody currents and prove its equivalence to a gauged Wess-Zumino-Witten model with a dynamical Yang-Mills field.

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0. Introduction.

The path integral formalism provides a very convenient starting point for perturbative calculations in quantum field theory. Alternatively, an algebraic framework in the spirit of the Hamiltonian formalism can be very useful for understanding the non-perturbative structure of interacting quantum field theories, especially quantum gauge theories. In this letter we wish to illustrate this idea in the simple context of (1+1) dimensions. We study massless QCD₁₊₁, i.e. massless Dirac fermions on spacetime $S^1 \times \mathbb{R}$ coupled to a Yang-Mills (YM) field [1, 2, 3], and we demonstrate that in this case all the mathematical tools and results required to complete such an algebraic approach do exist (they have been mostly developed in the context conformal quantum field theory and the representation theory of the affine Kac-Moody algebras (= current algebras on S^1) and the Virasoro algebra (for references close to the spirit of the present paper see [4, 5, 6], for a recent discussion of the history of the subject we refer to [7]). Using the latter, we outline of a simple, non-perturbative and rigorous construction of these quantum gauge theory models in terms of operators on a Hilbert space.

We restrict ourselves to the massless case for simplicity, mainly because representations of the affine Kac-Moody algebras in Fock spaces of fermions with mass $m > 0$ are more complicated and less understood than for $m = 0$ [5].

In general, we allow for N_F fermion flavors and a gauge group $H = U(N_C)$, $SU(N_C)$, or another Lie subgroup of $U(N_C)$. Put differently, we use fermions transforming in the fundamental representation of the group $G = U(N_C) \times U(N_F)$ and gauge the Lie subgroup H of G . To simplify our notation, we first concentrate on the special case $N_F = 1$ and $H = G = U(N_C)$, and then discuss the modifications required for the general case in Paragraph 7.

We note that a rigorous construction of 2 dimensional QCD has also been given by Klimek and Kondracki [8]. In contrast to our approach, they study the model in 2 Euclidean dimensions and use methods from constructive field theory (see e.g. [9]) which are close in spirit to the path integral formalism. We believe that our results here provide an example that Hamiltonian methods can provide a powerful alternative to these methods (of course, at the moment the latter are much further developed than the former [9]).

1. Preliminaries. To fix our notation and summarize the algebraic structure of the model, we first recall the canonical formalism for massless QCD₁₊₁ on the semiclassical level (= on the unphysical Hilbert space, no filled Dirac sea).

Let $G = U(N_C)$ be the structure group of the YM-field and T^a the generators of the Lie algebra g of G in the fundamental representation of $U(N_C)$ obeying¹

$$[T^a, T^b] = i\lambda^{ab}_c T^c, \quad (T^a)^* = T^a, \quad \text{tr}(T^a T^b) = \tau^{ab} \quad (1)$$

with (τ^{ab}) an invertible matrix ('metric tensor in color space'). Denoting as (τ_{ab}) the inverse matrix of (τ^{ab}) , we also introduce $T_a \equiv \tau_{ab} T^b$, and for $X \in g$ we write $X = X_a T^a = X^a T_a$ where

¹[\cdot, \cdot] is the commutator, $*$ and $\text{tr}(\cdot)$ the adjoint and the trace of $N \times N$ -matrices, respectively; repeated indices are summed over throughout unless stated otherwise

$X_a = \text{tr}(XT_a) = \tau_{ab}X^b$ etc. Then $\text{tr}(XY) = X_aY^a$ for $X, Y \in g$.

With $A_\nu \equiv A_\nu^a T_a$ the YM-field and $\psi, \bar{\psi} \equiv \psi^* \gamma^0$ the fermion fields, the Lagrangian density for massless QCD₁₊₁ is

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4e^2} \text{tr}(F_{\mu\nu}F^{\mu\nu}) - i\bar{\psi}\gamma^\nu D_\nu\psi \quad (2)$$

with $D_\nu\psi = (\partial_\nu + iA_\nu)\psi$, e the coupling constant, $\partial_\nu \equiv \partial/\partial x^\nu$, $x^0 = t \in \mathbb{R}$ time, $x^1 = x \in \Lambda \equiv [-L/2, L/2]$ the spatial coordinate, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ the YM field strength². More explicitly, $\gamma^\nu \equiv \gamma_{\sigma\sigma'}$, $T^a \equiv T_{AB}^a$, and $\psi^{(*)} \equiv \psi_{\sigma,A}^{(*)}$, $\sigma, \sigma' \in \{1, 2\}$ and $A, B \in \{1, 2, \dots, N\}$ are spin and color indices, respectively. To be specific, we choose the Dirac matrices as $\gamma^0 = \sigma_1$, $\gamma^1 = i\sigma_2$, $\gamma_5 \equiv -\gamma^0\gamma^1 = \sigma_3$ (σ_i the Pauli matrices as usual).

With the usual canonical procedure [10] we get the momenta Π_a^ν conjugate to A_ν^a , *viz.* $\Pi_a^0(x) = 0$ and $\Pi_a^1(x) = \frac{1}{e^2}(F_{01})_a(x)$, and the following canonical (anti-) commutator relations (C(A)CR)³

$$\begin{aligned} [\Pi_a^\mu(x), A_\nu^b(y)] &= i\delta_\nu^\mu \delta_a^b \delta(x-y) \\ \{\psi_{\sigma,A}(x), \psi_{\sigma',B}^*(y)\} &= \delta_{\sigma\sigma'} \delta_{AB} \delta(x-y) \end{aligned} \quad (3)$$

etc. as usual. Moreover, the resulting Hamiltonian is⁴

$$H_{\text{QCD}} = H_{\text{F}}^{(0)} + \int_\Lambda dx \text{tr} \left(\frac{e^2}{2} \Pi_1(x)^2 + A_1(x)j(x) - A_0(x)G(x) \right) \quad (4)$$

where we introduced the free fermion Hamiltonian

$$H_{\text{F}}^{(0)} \equiv \int_\Lambda dx \psi^*(x) \gamma_5 (-i\partial_1) \psi(x), \quad (5)$$

the fermion currents

$$\begin{aligned} \rho^a(x) &\equiv \psi^*(x) T^a \psi(x) \\ j^a(x) &\equiv \psi^*(x) \gamma_5 T^a \psi(x) \end{aligned} \quad (6)$$

and $j = j_a T^a$ etc., and the Gauss law operators

$$G(x) \equiv -D_1 \Pi_1(x) + \rho(x) \quad (7)$$

where $(D_\nu X)_a \equiv \partial_\nu X_a + i[A_\nu, X]_a = \partial_\nu X_a - \lambda_{abc} A_\nu^b X^c$.

The primary constraint $\Pi_a^0 = 0$ implies the secondary constraint $[\Pi_a^0(x), H_{\text{QCD}}] = -iG_a(x) \simeq 0$ (Gauss' law). One also gets $[G_a(x), H_{\text{QCD}}] = -i\lambda_{abc} A_0^b(x) G^c(x) \simeq 0$, hence there are no tertiary constraints.

From (2) we deduce that the vector- and the axial fermion currents, $J_a^\nu = \bar{\psi}\gamma^\nu T_a \psi$ and $(J_5^\nu)_a = \bar{\psi}\gamma^\nu \gamma_5 T_a \psi$, obey on the semi-classical level the equations of motion $D_\nu J^\nu = D_\nu J_5^\nu = 0$.

² $\mu, \nu \in \{0, 1\}$ are space-time indices; our metric is $g_{\mu\nu} = \text{diag}(1, -1)$, and the anti-symmetric tensor $\varepsilon^{\mu\nu}$ with $\varepsilon^{01} = 1$

³ $\{\cdot, \cdot\}$ is the anti-commutator

⁴we assume periodic boundary conditions for the YM fields

We also obtain the symmetric and gauge invariant energy-momentum tensor

$$\Theta^{\mu\nu} = \frac{1}{e^2} \text{tr} \left(\frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - F^{\mu\alpha} F^\nu{}_\alpha \right) - \frac{i}{2} \bar{\psi} [\gamma^\mu D^\nu + \gamma^\nu D^\mu] \psi \quad (8)$$

which is derived on-shell, i.e. by taking into account the eqs. of motion $D_\mu F^{\mu\nu} + e^2 J^\nu = 0$ and $\gamma^\mu D_\mu \psi = 0$. We also use light-cone coordinates, i.e.

$$J^\pm(x) \equiv \frac{1}{2} (\rho(x) \mp j(x)) \quad (9)$$

and similarly for Θ . We write

$$L^\pm \equiv \mp \Theta^{\pm\pm} = -i \psi^* \frac{1}{2} (1 \mp \gamma_5) D_1 \psi \equiv L_0^\pm + \text{tr} (A_1 J^\pm) \quad (10a)$$

$$M \equiv \frac{1}{e^2} \Theta^{+-} = \frac{1}{4} \text{tr} (\Pi_1^2) = \frac{1}{e^2} \Theta^{-+}, \quad (10b)$$

where again we used the equation of motion for ψ .

2. Fourier Transformation. In the following we find it convenient to work in Fourier space. Having in mind the thermodynamic limit $L \rightarrow \infty$ at the end of our construction, we use the following suggestive notation: Fourier space is $\Lambda^* \equiv \left\{ k = \frac{2\pi}{L} n \mid n \in \mathbb{Z} \right\}$, and for functions \hat{f} on Λ^* we write $\hat{\int}_{\Lambda^*} \hat{d}k \hat{f}(k) \equiv \sum_{k \in \Lambda^*} \frac{2\pi}{L} \hat{f}(k)$ so that $\delta(x-y) = \hat{\int}_{\Lambda^*} \frac{\hat{d}k}{2\pi} e^{ik(x-y)}$. Then the appropriate δ -function on Λ^* is $\hat{\delta}(k-q) \equiv \delta_{k,q} L/2\pi = \int_\Lambda \frac{dx}{2\pi} e^{-ix(k-q)}$ for $k, q \in \Lambda^*$.

For the Fourier transformed operators we use the following conventions ($k \in \Lambda^*$),

$$\hat{\psi}^{(*)}(k) = \int_\Lambda \frac{dx}{\sqrt{2\pi}} \psi^{(*)}(x) e^{(+i)kx} \quad (11a)$$

(to simplify our notation, we use periodic boundary conditions for the fermions; it is trivial to modify our eqs. so as to allow for anti-periodic boundary conditions),

$$\hat{A}_\nu(k) = \int_\Lambda \frac{dx}{2\pi} A_\nu(x) e^{-ikx}, \quad (11b)$$

and in all other cases

$$\hat{X}(k) = \int_\Lambda dx X(x) e^{-ikx} \quad \text{for } X = \Pi_\nu, G, \rho, j, J^\nu, J_5^\nu, \text{ and } \Theta. \quad (11c)$$

For convenience of the reader, we write down the non-trivial C(A)CR in Fourier space,

$$\begin{aligned} [\hat{\Pi}_\mu^a(k), \hat{A}_\nu^b(q)] &= ig_{\mu\nu} \tau^{ab} \hat{\delta}(k+q) \\ \{\hat{\psi}_{\sigma,A}(k), \hat{\psi}_{\sigma',B}^*(q)\} &= \delta_{\sigma\sigma'} \delta_{AB} \hat{\delta}(k-q) \quad \forall k, q \in \Lambda^*. \end{aligned} \quad (12)$$

3. Filling the Dirac Sea. Our approach is in the spirit of the algebraic approach to quantum field theory [11] where the non-trivial aspects of quantum field theory (as compared to quantum mechanics) arise due to the existence of unitarily inequivalent representations of quantum field algebras [12]. The essential *physical requirement* selecting the appropriate representation is *positivity of the Hamiltonian on the physical states*. The crucial simplification in (1+1) (and not possible in

higher) dimensions is that one can use a quasi-free representation [5] for the fermion field operators corresponding to “filling up the Dirac sea” associated with the *free* fermion Hamiltonian $H_F^{(0)}$, and for the YM operators one can use the naive Schrödinger representation. At this point, we have to take this as an assumption to be checked at the end of the construction. However, this assumption is plausible due to the facts that, (i) quasi-free representations for fermion fields in different *external* YM field are unitarily equivalent in (1+1) dimensions [13], (ii) the YM field on a cylinder has only a finite number of *physical* degrees of freedom [3], and as all representations of a finite number of quantum degrees of freedom are unitarily equivalent (von Neumann’s theorem), the simplest representation for the YM field algebra should do.

We therefore construct a representation of the C(A)CR algebra given above on a Hilbert space \mathcal{H} which is a tensor product of a YM and a fermion Hilbert space, $\mathcal{H} = \mathcal{H}_{\text{YM}} \otimes \mathcal{H}_F$, with \mathcal{H}_{YM} the usual Hilbert space of functionals of $\hat{A}_\nu^a(k)$ with $\hat{\Pi}_a^\nu(k) = \frac{L}{2\pi} i\partial/\partial \hat{A}_\nu^a(-k)$, and \mathcal{H}_F the Fermion Fock space with vacuum Ω_F such that

$$\begin{aligned} \frac{1}{2}(1 + \gamma_5)\hat{\psi}(k)\Omega_F &= \frac{1}{2}(1 - \gamma_5)\hat{\psi}^*(k)\Omega_F = 0 \quad \forall k > 0 \\ \frac{1}{2}(1 + \gamma_5)\hat{\psi}^*(k)\Omega_F &= \frac{1}{2}(1 - \gamma_5)\hat{\psi}(k)\Omega_F = 0 \quad \forall k \leq 0 \end{aligned} \quad (13)$$

(by abuse of notation, we do not distinguish the quantities introduced on the semi-classical level in the last Paragraph from the well-defined operators representing them on \mathcal{H}). It is well-known that the presence of the Dirac sea requires normal-ordering $:\cdots:$ of the fermion bilinears, hence $\tilde{H}_F^{(0)} = \hat{\int}_{\Lambda^*} \hat{d}q : q\hat{\psi}^*(q)\gamma_5\hat{\psi}(q) :$ (which is positive by construction [6]), and similarly for $\tilde{J}_a^\pm(k)$ and $\tilde{L}_0^\pm(k)$, where the tilde indicates *normal ordering with respect to the free fermion vacuum* Ω_F . This modifies their naive commutator relations following from the CAR (12) as Schwinger terms show up [4, 6] (for a mathematical rigorous discussion of this construction of fermion bilinears in the presence of a Dirac sea and how normal ordering leads to Schwinger terms, see [5]). In our case [4, 6],

$$[\tilde{J}_a^\pm(k), \tilde{J}_b^\pm(q)] = i\lambda_{ab}{}^c \tilde{J}_c^\pm(k+q) \mp k\hat{\delta}(k+q)\tau_{ab} \quad (14a)$$

and

$$[\tilde{L}_0^\pm(k), \tilde{L}_0^\pm(q)] = (k-q)\tilde{L}_0^\pm(k+q) \mp \frac{N_C}{6}k \left(k^2 - \left(\frac{2\pi}{L} \right)^2 \right) \hat{\delta}(k+q) \quad (14b)$$

with the second terms on the r.h.s. of (14a) and (14b) the Kac-Moody and Virasoro cocycles, respectively [4]. Moreover,

$$[\tilde{L}_0^\pm(k), \tilde{J}_a^\pm(q)] = -q\tilde{J}_a^\pm(k+q) \quad (14c)$$

with no Schwinger term arising here. Note that these relations are exactly the ones of the semi-direct product of an affine Kac-Moody algebra and the Virasoro algebra playing a prominent role in conformal field theory.

We can now write the Gauss’ law as $\hat{G}_a(k) = -(D_1\hat{\Pi}_1)_a(k) + \tilde{\rho}_a(k) \simeq 0$ where $(D_1\hat{\Pi}_1)_a(k) = ik(\hat{\Pi}_1)_a(k) - \lambda_{abc}\hat{\int}_{\Lambda^*} \hat{d}q \hat{A}_1^b(k+q)\hat{\Pi}_1^c(-q)$, so eqs. (14a) imply that

$$[\hat{G}_a(k), \tilde{J}_b^\pm(q)] = i\lambda_{ab}{}^c \tilde{J}_c^\pm(k+q) \mp k\hat{\delta}(k+q)\tau_{ab}.$$

Thus the presence of the Schwinger terms implies that these fermion currents no longer have the classical commutator relations with the Gauss' law generators and therefore do not transform covariantly under gauge transformations.

To restore gauge covariance and obtain fermion currents having canonical transformation properties (without Schwinger terms), we note that $[\hat{G}_a(k), (\hat{A}_1)_b(q)] = -k\hat{\delta}(k+q)\tau_{ab} + i\lambda_{abc}\hat{A}_1^c(k+q)$, hence the operators

$$\hat{J}^\pm(k) \equiv \tilde{J}^\pm(k) \mp \hat{A}_1(k) \quad (15)$$

obey the desired relations

$$[\hat{G}_a(k), \hat{J}_b^\pm(q)] = i\lambda_{ab}{}^c \hat{J}_c^\pm(k+q). \quad (16)$$

Similarly, the naive energy-momentum components $\tilde{L}^\pm(k) = \tilde{L}_0^\pm(k) + \int_{\Lambda^*} \hat{d}q \operatorname{tr} \left(\hat{A}_1(k+q) \tilde{J}^\pm(-q) \right)$ are not gauge invariant, $[\hat{G}^a(k), \tilde{L}_0^\pm(q)] = \mp k \hat{A}_1^a(k+q)$, but obviously there are unique polynomials in \hat{A}_1 which can be added to make them gauge invariant,

$$\hat{L}^\pm(k) = \tilde{L}_0^\pm(k) + \int_{\Lambda^*} \hat{d}q \operatorname{tr} \left(\hat{A}_1(k+q) \tilde{J}^\pm(-q) \mp \frac{1}{2} \hat{A}_1(k+q) \hat{A}_1(-q) \right). \quad (17)$$

Recalling that normal ordering is only unique up to finite terms, it is natural to regard the $\hat{J}_a^\pm(k)$ and $\hat{L}^\pm(k)$ as the currents and energy-momentum components obtained by a *gauge covariant normal ordering* preserving the transformation properties under gauge transformations.

To construct the full energy momentum tensor — especially the Hamiltonian — we also need the operators

$$\hat{M}(k) = \int_{\Lambda^*} \hat{d}q \frac{1}{8\pi} \operatorname{tr} \left(\hat{\Pi}_1(k+q) \hat{\Pi}_1(-q) \right). \quad (15)$$

At this point a technical difficulty arises: the operators $\hat{L}^\pm(k)$ and $\hat{M}(k)$ do not have a common, dense invariant domain of definition in \mathcal{H} (we recall that the sum of two unbounded Hilbert space operators can be defined directly only if these operators have such a domain). It is, however, possible to define the operators $:\hat{L}^\pm(k) \pm e^2 \hat{M}(k):$ — and therefore the components $\hat{\Theta}^{00}(k)$ and $\hat{\Theta}^{01}(k)$ of the energy momentum tensor — by normal ordering $:\dots:$ the YM field operators with respect to the YM vacuum Ω_{YM} obeying

$$\left(\frac{e}{\sqrt{4\pi}} i \hat{\Pi}_1(k) + \hat{A}_1(k) \right) \Omega_{\text{YM}} = 0 \quad \forall k \in \Lambda^*. \quad (16)$$

Note that Ω_{YM} is just the ground state of the free YM Hamiltonian

$$H_{\text{YM}}^{(0)} = :\int_{\Lambda^*} \hat{d}k \operatorname{tr} \left(\frac{e^2}{4\pi} \hat{\Pi}_1(k) \hat{\Pi}_1(-k) + \hat{A}_1(k) \hat{A}_1(-k) \right):. \quad (17)$$

Especially, we get the gauge invariant Hamiltonian $H_{\text{QCD}} = \hat{\Theta}^{00}(0) - G(A_0)$, or equivalently

$$H_{\text{QCD}} = \tilde{H}_{\text{F}}^{(0)} + H_{\text{YM}}^{(0)} + \int_{\Lambda^*} \hat{d}k \operatorname{tr} \left(\hat{A}_1(k) \tilde{j}(-k) - \hat{A}_0(k) \hat{G}(-k) \right). \quad (18)$$

We see that, similarly as in the Schwinger model [14], there is an additional ‘gluon mass term’ resulting from gauge invariant normal ordering.

4. Observable Algebra Relations. For the covariant currents we get the following commutators,

$$[\hat{J}_a^\pm(k), \hat{J}_b^\pm(q)] = i\lambda_{ab}{}^c \hat{J}_c^\pm(k+q) \mp \hat{S}_{ab}(k, q) \quad (19)$$

with the Schwinger term

$$\hat{S}_{ab}(k, q) = k\hat{\delta}(k+q)\tau_{ab} - i\lambda_{abc}\hat{A}_1^c(k+q). \quad (20)$$

It is natural to regard the latter as the *gauge covariant form of the Kac-Moody cocycle*, and we can represent it in explicitly covariant form as $\hat{S}_{ab}(k, q) = -[\hat{\Pi}_a^0(k), (\widehat{D_1 A_0})_b(q)]$.

Moreover, we get

$$[\hat{L}^\pm(k), \hat{L}^\pm(q)] = (k-q)\hat{L}^\pm(k+q) \mp \frac{N_C}{6}k \left(k^2 - \left(\frac{2\pi}{L}\right)^2\right) \hat{\delta}(k+q) \quad (21)$$

and

$$[\hat{L}^\pm(k), \hat{J}_a^\pm(q)] = -q\hat{J}_a^\pm(k+q) - [A_1, \widehat{J^\pm}]_a(k+q). \quad (22)$$

5. Equations of Motion. Using (18) and the algebraic relations above it is straightforward to work out the equations of motion for all observables of the model. For example, we obtain $[H_{\text{QCD}}, \hat{J}^\pm(k)] = \mp i(\widehat{D_1 J^\pm})(k) + [A_0, \widehat{J^\pm}](k) \pm i\frac{e^2}{2\pi}\hat{\Pi}(k)$. Using $\partial_0 J^\pm + i[H_{\text{QCD}}, J^\pm] = 0$, and transforming to position space, we can write this as $D_0 J^\pm \pm D_1 J^\pm = \pm \frac{e^2}{2\pi}\Pi_1$. Noting that $\Pi_1 = -\Pi^1 = -\frac{1}{2e^2}\varepsilon^{\mu\nu}F_{\mu\nu}$ and $J^0 = J^+ + J^- = -J_5^1$, $J^1 = -J^+ + J^- = -J_5^0$, this can be written as

$$D_\nu J^\nu = 0 \quad (23a)$$

$$D_\nu J_5^\nu = \frac{1}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu}. \quad (23b)$$

The second of these eqs. shows a covariant axial anomaly.

Evaluating $\partial_0 \Pi_1(k) + i[H_{\text{QCD}}, \Pi_1] = 0$ we obtain $D_0 \Pi_1 - j = 0$. Together with the Gauss’ law — which we can write as $D_1 \Pi_1 - \rho \simeq 0$ — this comprises the usual equations of motion of the YM field

$$D_\mu F^{\mu\nu} + e^2 J^\nu \simeq 0 \quad (24)$$

identical to those obtained on the semi-classical level. Using this and eq. (23a) rewritten as $D_0 j - D_1 \rho = -\frac{e^2}{\pi}\Pi_1$, we obtain $D_0^2 \Pi_1 - D_1^2 \Pi_1 = -\frac{e^2}{\pi}\Pi_1$, or equivalently

$$D_\nu D^\nu \Pi_1 + \frac{e^2}{\pi}\Pi_1 \simeq 0 \quad (25)$$

generalizing the Klein-Gordon equation one has in the Abelian case [14]. Equations (23a) and (25) have also been obtained by Sorensen and Thomas [15] in a path integral approach.

6. Bosonization. The celebrated (affine-) Sugawara construction allows to write the free Virasoro generators \tilde{L}_0^\pm in terms of the Kac-Moody currents \tilde{J}^\pm ,

$$\tilde{L}_0^\pm(k) = \mp \frac{1}{2} \int_{\Lambda^*} \hat{d}q \times \text{tr} \left(\tilde{J}^\pm(k+q) \tilde{J}^\pm(-q) \right) \times \quad (26)$$

(see e.g. [4]; for $k = 0, \pm 2\pi/L$ this was already given in [16]) with normal ordering $\times \tilde{J}_a^\pm(k) \tilde{J}_b^\pm(q) \times \equiv \tilde{J}_b^\pm(q) \tilde{J}_a^\pm(k)$ for $k \gtrsim q$ and $\tilde{J}_a^\pm(k) \tilde{J}_b^\pm(q)$ otherwise [4] (note that $\tilde{J}^+(-k)\Omega_F = \tilde{J}^-(k)\Omega_F = 0 \forall k > 0$). Combining this with eqs. (17) and (15), we observe that the terms involving \hat{A}_1 can be arranged such that

$$\hat{L}^\pm(k) = \mp \frac{1}{2} \int_{\Lambda^*} \hat{d}q \times \text{tr} \left(\hat{J}^\pm(k+q) \hat{J}^\pm(-q) \right) \times \quad (27)$$

Thus the Virasoro generators $\hat{L}^\pm(k)$ are obtained simply by replacing the non-covariant currents on the r.h.s. of eq. (26) by the covariant ones! It is natural to regard this as the *gauge covariant version of the Sugawara construction*.

Especially, we get the Hamiltonian of the model in the following form

$$H_{\text{QCD}} = \frac{1}{2} \int_{\Lambda^*} \hat{d}k \text{tr} \left(\times \left(\hat{J}^+(k) \hat{J}^+(-k) + \hat{J}^-(k) \hat{J}^-(-k) \right) \times + \frac{e^2}{2\pi} \hat{\Pi}_1(k) \hat{\Pi}_1(-k) - \hat{A}_0(k) \hat{G}(-k) \right) \dot{\cdot} . \quad (28)$$

It is now manifestly positive definite on the physical Hilbert space where $\hat{G}(k) \simeq 0$ (note that $\hat{J}^\pm(-k) = \hat{J}^\pm(k)^*$) thus justifying our choice of representation of the field algebra.

It is easy to see that this Hamiltonian is identical with the one obtained from a gauged Wess-Zumino-Witten model with dynamical gauge field⁵ (see e.g. [17]) and a gauge group $U(N_C)$ equal the flavor group and coupling constant $g = e/\sqrt{\pi}$ (to make this explicit, one has to rescale $\sqrt{\pi}A_\nu(x) \rightarrow A_\nu(x)$). This equivalence has been known from the path integral approach [18, 19] (see also [20]).

7. General Case. Our present approach can be immediately generalized to the case with a gauge group H being a Lie subgroup of $U(N_C)$ and N_F fermion flavors. In this case, we have fermions transforming under the fundamental representation of $G = U(N_C) \times U(N_F)$ with the gauge group H a subgroup of G . Labeling the generators of the Lie algebra of G such that T_a for $1 \leq a \leq \dim(H)$ span the Lie algebra of H and $\text{tr}(T^a T^b) = 0$ for $a \leq \dim(H), b > \dim(H)$, we have $A_1 = \sum_{a=1}^{\dim(H)} A_1^a T_a$ and similarly for $F_{\mu\nu}, G$. With that all equations and the discussion from Paragraphs 1–4 essentially remain the same.

For the bosonization, it is most convenient to use the G/H coset construction [4],

$$\tilde{L}_0^\pm = (\tilde{L}_0^\pm)_G = (\tilde{L}_0^\pm)_H + (\tilde{L}_0^\pm)_{G/H} \quad (29)$$

which implies a similar equation for $\hat{L}^\pm = \hat{L}_G^\pm$. Then obviously $\hat{L}_{G/H}^\pm = (\tilde{L}_0^\pm)_{G/H}$, i.e. the coset Virasoro generators are completely decoupled from the gauge field, and the discussion of gauge covariant normal ordering etc. above applies to \hat{L}_H^\pm only.

⁵i.e. the Lagrangian has a term $-\frac{1}{4g^2} \text{tr}(F_{\mu\nu} F^{\mu\nu})$ in addition to what is usually referred to as gauged WZW model [17]

Especially the Hamiltonian of the model is $H_{\text{QCD}} = H_H + H_{G/H}$ where $H_{G/H} = (\tilde{H}_F^{(0)})_{G/H}$ is completely decoupled from all YM fields, commutes with H_H , and is identical to the one one gets in the free fermion case [4]. The non-trivial interactions of the fermions with the YM field are completely contained in H_H . Thus we can write $\mathcal{H} = \mathcal{H}_H \otimes \mathcal{H}_{G/H}$ with all non-trivial dynamics occurring on \mathcal{H}_H , and $\mathcal{H}_{G/H}$ provides the superselection sectors of the model (cf. also [18]).

8. Technicalities. To complete the construction of massless QCD_{1+1} and make it mathematically rigorous, one has to establish several technical properties. Firstly (as most of the operators of the model are unbounded), one has to prove that there is a common, dense, invariant domain $\mathcal{D} \subset \mathcal{H}$ for all operators of interest so that the commutator relations given above are well-defined on \mathcal{D} . In fact, this can be proven for the Hamiltonian H_{QCD} and all other operators considered above except the energy momentum components $\hat{\Theta}^{11}(k)$ (see also the discussion in Paragraph 3; taking also the latter into account makes things slightly more complicated [24]).

Secondly, one has to prove that all observables of the model, especially the Hamiltonian and the (smeared) Gauss law generators, are represented not only by symmetric but in fact self-adjoint operators on \mathcal{H} [21, 22]. Finally, one would like to establish that the thermodynamic limit $L \rightarrow \infty$ is well-defined and leads to a relativistically invariant theory. The proof of these results can be done by using techniques developed in [5, 23] in combination with results summarized in [25] and will appear elsewhere [24].

9. Final Comments. Recently an interesting reformulation of QCD on spacetime $\mathbb{R} \times \mathbb{R}$ in terms of a gauge invariant, bilocal master field was given and used as a starting point for a systematic semi-classical approximation [26] (see also [27]). It would be interesting understand this reformulation in our framework (technically this is more complicated due to the presence of the physical YM degrees of freedom on spacetime $S^1 \times \mathbb{R}$). Alternatively one can eliminate the gauge degrees of freedom by ‘solving the Gauss’ law’ [28]. For massless QCD_{1+1} this results in a theory of interacting Kac-Moody currents J^\pm coupled to a finite number of quantum mechanical degrees of freedom, the latter representing the physical YM degrees of freedom [3]. From a mathematical point of view this gauge fixing procedure is quite delicate, and it would be important to get a deeper understanding, e.g. in the general framework of [29]. Work in this direction is in progress [24].

There are several deep reasons preventing a straightforward extension of our construction to higher dimensions. Most importantly, a YM field there has also an infinite number of physical degrees of freedom, and choosing the appropriate representation of the YM field algebra is a highly non-trivial problem, even for pure YM theory. Moreover, in higher dimensions the physical representations for fermions interacting with different external, static YM-fields are not unitarily equivalent and gauge transformations cannot be implemented by unitary operators in the fermion sector but only by sesquilinear forms [30, 31]. This suggests that the observable algebra of QCD in higher dimensions at fixed, sharp time does not allow for a reasonable Hilbert space representation, hence a standard Hamiltonian formalism might be too narrow a framework for higher dimensional quantum gauge theories. There is, however, a natural generalization of the theory of the affine Kac-Moody algebras to (3+1) dimensions [30, 31] which can be expected to provide a first step to

a non-perturbative understanding of the fermion sector of QCD_{3+1} .

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