ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Boltzmanngasse 9
A-1090 Wien, Austria

An Axiomatization of the Lattice of Higher Relative Commutants of a Subfactor

Sorin Popa

Vienna, Preprint ESI 115 (1994)

August 1, 1994

Supported by Federal Ministry of Science and Research, Austria

AN AXIOMATIZATION OF THE LATTICE OF HIGHER RELATIVE COMMUTANTS OF A SUBFACTOR

Sorin Popa*

International Erwin Schrödinger Institute for Mathematical Physics

October 2, 2001

Abstract. We consider certain conditions for abstract lattices of commuting squares, that we prove are necessary and sufficient for them to arise as lattices of higher relative commutants of a subfactor. We call such lattices standard and use this axiomatization to prove that their sublattices are standard too. We consider a method for producing sublatties and deduce from this and [Po5] some criteria for bipartite graphs to be graphs of subfactors.

0. Introduction

Let $N \subset M$ be an inclusion of von Neumann factors of type II_1 with finite Jones index, $[M : N] < \infty$. The standard invariant of $N \subset M$, $\mathcal{G}_{N,M}$, is given by the lattice of higher relative commutants $(M_i' \cap M_j)_{0 \le i \le j}$ in the Jones' tower associated to $N \subset M$, $M_0 = M \subset M_1 \subset M_2 \subset \dots$. The inclusions between the finite dimensional algebras $\mathbb{C} = M_i' \cap M_i \subset M_i' \cap M_{i+1} \subset$ in each row i of this lattice of inclusions are described by a pointed bipartite graph Γ^i. Due to periodicity the first two of these graphs, $\Gamma = \Gamma^{2i}$, $\Gamma' = \Gamma^{2i+1}$, $i \ge 0$, give all the inclusions. $\mathcal{G}_{N,M}$ has in fact more structure than just (Γ, Γ'). Describing $\mathcal{G}_{N,M}$ and in particular characterising the pairs of graphs (Γ, Γ') that can occur as graphs of subfactors (i.e. are *standard*) is a central problem of this theory. We attempt here a new approach to this problem.

Thus, we obtain in this paper a characterisation of $(M_i' \cap M_j)_{0 \le i \le j}$ as abstract lattices of inclusions $(A_{ij})_{0 \le i \le j}$ by considering a set of axioms that we prove are necessary and sufficient for a system of inclusions of finite dimensional algebras to occur as higher relative commutants of a subfactor. More precisely, let $(A_{ij})_{i \le j, i=0,1}$ be finite dimensional algebras with $A_{1j} \subset A_{0j}$ and $A_{ij} \subset A_{i,j+1}$, $i = 0, 1, j \ge i$, $A_{00} = A_{11} = \mathbb{C}$ and with a trace τ on $\cup A_{0j}$. Then the axioms that we consider are: 1). Commuting square conditions: $E_{A_{1j}} E_{A_{0k}} = E_{A_{1k}}$ for $1 \le k \le j$; 2). Existence of Jones λ-projections $e_k \in A_k, k \ge 2$, implementing the τ-preserving conditional expectations of $A_{i,k-1}$ onto $A_{i,k-2}$, with $\tau(e_2 x) = \lambda \tau(x)$ for $x \in A_{1j}$; 3). Dimension

---

Supported by the Federal Ministry of Science and Research, Austria
*Supported in part by NSF Grant DMS-9206984

conditions: $\dim A_{0j} = \dim A_{0,j+1}e_{j+1} = \dim A_{1,j+1}$ for all $j \geq 1$; 4). Commutation conditions: $[A_{0j}, A_{jk}] = 0$, where $A_{jk} = \{e_2, \dots, e_j\}' \cap A_{1k}$, for $k \geq j \geq 2$. Thus we prove that for the system of finite dimensional algebras $(A_{ij})_{i \leq j, i=0,1}$ to coincide with the higher relative commutants $(M_i' \cap M_j)_{i \leq j, i=0,1}$ of some subfactor $N \subset M$ of index $[M : N] = \lambda^{-1}$, it is necessary and suficient that (A_{ij}) satisfy the axioms 1)-4). And if so then $A_{ij} = M_i' \cap M_j$ for all $i \leq j$.

We call a system of finite dimensional inclusions (A_{ij}) satisfying the axioms 1)-4) a standard lattice of commuting squares. We mention that the subfactors that we construct to realize (A_{ij}) as higher relative commutants are hyperfinite only when the graph Γ of the lattice is strongly amenable. In general, the subfactors $N \subset M$ are constructed by universality considerations similar to [Po4] and are thus not hyperfinite.

A rather surprising application of this axiomatization is that a sublattice (in the obvious sense) of a lattice of higher relative commutants of a subfactor is itself the lattice of higher relative commutants of some subfactor. Sublattices can be constructed from an initial one similarly to the way one obtains new groups from a group that is given by generators and relations, by keeping the same generators but only part of the relations. This will enable us to obtain some rather strong obstruction criteria for (pairs of) graphs to be standard, i.e., to be graphs of subfactors, especially when the index is small (see 4.5-4.9). Thus, we will prove that is a standard pair of graphs (Γ, Γ') satisfies a certain stability condition at some distance n from the initial vertex then they must be finite graphs that continue with A_{fin} tails from that distance on.

Note that it is not clear whether one can find the subfactor realising a sublattice to be hyperfinite in case the initial subfactor is hyperfinite (the one that we construct are in any case not)! In fact, the problem of caracterizing all the standard lattices coming from hyperfinite subfactors remains open.

Recall that in the case Γ is finite, i.e., when $N \subset M$ has finite depth, $\mathcal{G}_{N,M}$ was shown in [Oc] to be equivalent to the (finite) graded tensor category of all irreducible bimodules (or correspondences) generated under Connes' fusion rule by $N \subset M$, and was described as an abstract object, called paragroup, by providing it with a full set of axioms ([Oc]). These can, of course, be viewed as axioms of the corresponding higher relative commutants. Note however that, even when regarded this way Ocneanu's axioms for higher relative commutants of finite depth subfactors do not coincide with our set of axioms for such lattices. Thus, even for arbitrary (not necessarily finite depth) lattices we do not assume the existence of the antisymmetry (=contragradient) maps in the lattice and we ask for commutation relations, rather than relative commutant conditions.

The interpretation ([Oc]) of $\mathcal{G}_{N,M}$ as a group like object in the finite depth case led, together with ([L]), to the consideration of using the "fusion rule" method for finding obstructions for bipartite graphs to be standard i.e., to be graphs of subfactors (cf. [Oc], [Iz], [Bi]). This method usually requires a case by case analysis, but it was useful in the index < 4 case, to prove the nonoccurence of the D_{odd} graphs as graphs of subfactors, and also for some index > 4 exclusions.

Some general obstruction criteria, called "triple point obstructions", were obtained in [H1,2] from local matrix computations. They were used there together with the fusion rule method and a number of ad-hoc arguments to exclude most of the graphs of square norm between 4 an 4.7 from being standard. Our results do cover the triple point obstruction in [H2], except for the case $\Gamma = T_{fin,fin}$, and recapture results from [H1,2] in a direct way, without extra-work. Thus, our global approach also offers some conceptual explanation to Haagerup's surprising result that most subfactors of index between 4 and 4.7 have graph A_∞.

In an independent recent work V. Jones considers a different "global" approach to the obstruction problem ([J2]), which in particular gives a powerful obstruction criterion that covers the triple point obstruction in [H2] and other results from [H1,2]. We included Corollary 4.9 to test if his criterion can be obtained from ours: again, we can recover it, except for the case $\Gamma = T_{fin,fin}$. However, the ideas of approaching the obstruction problem in [J2] and in this paper are from rather distinct points of view.

I am very grateful to Uffe Haagerup for patiently explaining to me his results on repeated occasions and for pointing out to me 1.4.3 in [Sc] , and to Vaughn Jones for keeping me informed on his recent exciting work.

1. Lattices of commuting squares

Let $(A_{ij})_{0\leq i\leq j<\infty}$ be a system of finite dimensional algebras with $A_{ii} = \mathbb{C}$, $A_{ij} \subset A_{kl}$, $\forall k \leq i$, $j \leq l$, and with a given faithful trace τ on $\cup_{n=0}^{\infty} A_{0n} = \cup_{i,j} A_{ij}$. We consider the following properties for A_{ij}, λ:

1.1.1. The commuting square condition.

$$E_{A_{ij}} E_{A_{kl}} = E_{A_{kl}} E_{A_{ij}} = E_{A_{rs}}$$

where $r = \max\{i, k\}$, $s = \min\{j, l\}$ and E_B is the τ-preserving expectation onto B.

1.1.2. Existence of Jones λ-projections. There exists a representation of the λ-sequence of Jones projections $\{e_i\}_{i\geq 2}$ in $\cup_n A_{0,n}$ such that

a) $e_j \in A_{i-2,k}$ in $2 \leq i \leq j \leq k$

b) $e_{j+1} x e_{j+1} = E_{A_{i,j-1}}(x) e_{j+1}, \quad \forall x \in A_{ij}, i \leq j-1$

c) $e_{i+1} x e_{i+1} = E_{A_{i+1,j}}(x) e_{i+1}, \quad x \in A_{ij}, i+1 \leq j$

1.1.3. Index condition. (The definition of $\mathrm{Ind}(A \subset B)$ is that of [PiPo1])

a) $\mathrm{Ind}(A_{i,j} \subset A_{i,j+1}) \leq \lambda^{-1}, E_{A_{i,j}}(e_{j+1}) = \lambda 1$

b) $\mathrm{Ind}\ (A_{i,j} \subset A_{i-1,j}) \leq \lambda^{-1}, E_{A_{i-1,j}}(e_i) = \lambda 1$

1.1. Definition. A system of finite dimensional algebras $(A_{ij})_{0\leq i\leq j}$ as above, satisfying (1.1.1) - (1.1.3) is called a *λ-lattice of commuting squares.* Note that by Jones' theorem, the existence of the λ-projections implies $\lambda^{-1} \in \{4\cos^2 \pi/n | n \geq 3\} \cup [4, \infty)$.

Recall from [Po1] that an inclusion of type II_1 on Neumann algebras $Q \subset P$ is λ-Markov if $\sum_j m_j m_j^* = \lambda^{-1} 1$, $\forall \{m_j\}_j$ orthonormal basis of P over Q and that it is called homogeneous λ-Markov if in addition e_Q has scalar central trace in $< P, e_Q >$.

1.2. Proposition. *Let $(A_{i,j})_{i,j}$ be a λ-lattice of commuting squares, with $\lambda \neq 1$, and denote by $A_{i,\infty} = \overline{\cup_{j\geq i} A_{i,j}}$ the completion of $A_{i,i} \subset A_{i,i+1} \subset \dots$ in the *-strong topology given by $\tau, i \geq 0$. Then $A_{1,\infty} \subset A_{0,\infty}$ is a homogeneous λ-Markov inclusion and $A_{0,\infty} \supset^{e_2} A_{1,\infty} \supset^{e_3} \dots$ is a tunnel for this inclusion.*

Proof. By [PiPo1] and (1.1.3) b) we have $\text{Ind}\ (A_{i,\infty} \subset A_{i-1,\infty}) = \lambda^{-1}$, $\forall i \geq 1$, and the rest is trivial by [Po1] and [PiPo1]. □

Many of the conditions (1.1.1) - (1.1.3) are, in fact, redundant. To see this let us consider one more:

1.3. Definition. Let

$$\begin{array}{ccccccc} \mathbb{C} = A_{00} & \subset & A_{01} & \subset & A_{02} & \subset & \dots \\ & & \cup & & \cup & & \\ & & \mathbb{C} = A_{11} & \subset & A_{12} & \subset & \dots \end{array}$$

be a sequence of inclusions of finite dimensional algebras, with a trace τ on $\cup_n A_{0n}$, satisfying the conditions:

1.3.1. Commuting square conditions. If $E_{A_{0i}}E_{A_{1j}} = E_{A_{1j}}E_{A_{0i}} = E_{A_{1i}}$, $\forall 1 \leq i \leq j$.

1.3.2. Existence of Jones projections. There exists a representation of the Jones' λ-projections $\{e_i\}_{i\geq 2}$ in $\cup_n A_{0n}$ such that:
a) $e_j \in A_{0j}$, $j \geq 2$, $e_j \in A_{1j}$, $j \geq 3$
b) $e_{j+1}xe_{j+1} = E_{A_{0,j-1}}(x)e_{j+1}$, $\forall x \in A_{0j}$,

1.3.3. Index conditions.
a) $\text{Ind}\ (A_{0,j} \subset A_{0,j+1}) \leq \lambda^{-1}$.
b) $\text{Ind}\ (A_{1,j} \subset A_{0,j}) \leq \lambda^{-1}$, $\tau(e_2x) = \lambda\tau(x)$, $\forall x \in A_{i,j}$.
Then $(A_{ij})_{i\leq j, i=0,1}$ is called a *λ-sequence of commuting squares.*

1.4. Proposition. *Let*

$$\begin{array}{ccccccc} A_{00} & \subset & A_{01} & \subset^{e_2} & A_{02} & \subset^{e_3} & A_{03} & \subset \dots \\ & & \cup & & \cup & & \cup & \\ & & A_{11} & \subset & A_{12} & \subset^{e_3} & A_{13} & \subset \end{array}$$

be a λ-sequence of commuting squares and define $A_{ij} = \{e_2, \dots, e_i\}' \cap A_{1j}$, $2 \leq i \leq j$. Then $(A_{ij})_{i,j}$ is a λ-lattice of commuting squares.

Proof. Let $A_{0,\infty} = \overline{\cup_n A_{0n}}$, $A_{1,\infty}=\overline{\cup_n A_{1,n}}$. By (1.3.3) b) and [PiPo1] we have $\text{Ind}\ (A_{1,\infty} \subset A_{0,\infty}) = \lambda^{-1}$. Moreover, if $P_0 = vN\{e_2, \dots\}$, $P_1 = vN\{e_3, \dots\}$ then by the fact that $\{e_j\}_{j\geq 2}$ is a λ-sequence of Jones projections and by (1.3.3) b) we have that

$$\begin{array}{ccc} A_{1,\infty} & \subset & A_{0,\infty} \\ \cup & & \cup \\ P_1 & \subset & P_0 \end{array}$$

is a commuting square, with both rows of index λ^{-1} and the bottom row an inclusion of factors ([J1]). Thus both row inclusions are homogeneous λ-Markov. Since $E_{A_{1,\infty}}(e_2) = \lambda 1$, it follows that e_2 is a Jones projection for $A_{1,\infty} \subset A_{0,\infty}$, i.e. if $A_{2,\infty} \overset{\text{def}}{=} \{e_2\}' \cap A_{1.\infty}$ then $A_{2,\infty} \subset A_{1,\infty} \subset^{e_2} A_{0,\infty}$ is a Jones' basic construction ([Po1]. Ch. 1). Moreover

$$E^{A_{1,\infty}}_{A_{2,\infty}}(x) = \lambda^{-1}E_{A_{1,\infty}}(e_2xe_2), \forall x \in A_{1,\infty}.$$

Since $A_{2,j} = \{e_2\}' \cap A_{1,j}$, by the above formula for $E^{A_{1,\infty}}_{A_{2,\infty}}$ it follows that $E^{A_{1,\infty}}_{A_{2,\infty}}(A_{1,n}) \subset A_{1,n}$. But we also have $E^{A_{1,\infty}}_{A_{2,\infty}}(A_{1,n}) \subset \{e_2\}' \cap A_{1,\infty} = A_{2,\infty}$. Thus, we have the commuting squares, $\forall j \geq 2$:

$$\begin{array}{ccccc}
A_{2,\infty} & \subset & A_{1,\infty} & \subset & A_{0,\infty} \\
\cup & & \cup & & \cup \\
A_{2,j} & \subset & A_{1,j} & \subset & A_{0,j}
\end{array}$$

Thus $(A_{ij})_{0 \leq i \leq j}$ satisfies (1.1.1) - (1.1.3) for $i = 0, 1, 2$, with $E_{A_{2,\infty}}(e_3) = \lambda 1$, and we continue this way inductively. ☐

The index conditions (1.1.3) (resp. (1.3.3)) may seem difficult to check in certain situations. We have the following alternative description:

1.5. Proposition. *Assume*

$$\begin{array}{ccccccc}
\mathbb{C} = A_{00} \subset & A_{01} & \subset^{e_2} & A_{02} & \subset^{e_3} & A_{03} & \subset \\
& \cup & & \cup & & \cup & \\
& \mathbb{C} = A_{11} & \subset & A_{12} & \subset^{e_3} & A_{13} & \subset
\end{array}$$

are commuting squares of finite dimensional algebras with $\{e_i\}_{i \geq 2}$ *a* λ*-sequence of Jones projections satisfying (1.1.3), (1.3.2). Then* $(A_{ij})_{\leq j, i=0,1}$ *satisfies (1.1.3) (i.e. it is a* λ*-sequence of commuting squares) if and only if it satisfies for* $i = 0$ *the dimension equalities.*

$$(1.3.3\,') \qquad \begin{aligned}
&a)'.\dim A_{i,j+2}e_{j+2} = \dim A_{i,j+1}, i = 0, \forall j \geq i \\
&b)'.\dim A_{i,j}e_{i+2} = \dim A_{i+1,j}, \forall j \geq 2, E_{A_{i+1,j}}(e_{i+2}) = \lambda 1, j \geq i
\end{aligned}$$

Also, (A_{ij}) *satisfies 1.3.1 if and only if it satisfies for* $i = 0$ *the (1.1 in* [PiPo1]*)-type identities:*

$$(1.3.3\,'') \qquad \begin{aligned}
&a)''\lambda^{-1}E_{A_{i,j+1}}(xe_{j+2})e_{j+2} = xe_{j+2}, \forall x \in A_{i,j+2}, \forall j \geq i \\
&b)''\lambda^{-1}E_{A_{i+1,j}}(xe_{i+2})e_{i+2} = xe_{i+2}, \forall x \in A_{i,j}, \forall j \geq i+2
\end{aligned}$$

Moreover, if $(A_{ij})_{i \leq j, i=0,1}$ *is a sequence of commuting squares and* $(A_{ij})_{0 \leq i \leq j}$ *is the corresponding* λ*-lattice (with* $A_{ij} = \{e_2, \ldots, e_i\}' \cap A_{i,j}, i \geq 2$*) then* $(A_{ij})_{i \leq j}$ *satisfy* $(1.3.3)', (1.3.3)''$ *for all* i*.*

Proof. If $(A_{ij})_{i \leq j, i=0,1}$ satisfies (1.3.1) - (1.3.3) then it gives rise to the homogeneous Markov tunnel $A_{0,\infty} \supset^{e_2} A_{1,\infty} \supset^{e_3} A_{2,\infty} \supset \ldots$ (see 1.4), so in particular we have $(1.3.3)''$ by [PiPo1], [Po1] and by commuting squares.

Clearly $(1.3.3)'' \Rightarrow (1.3.3)'$.

Finally, assume $(1.3.3)'$ holds true. Since $\dim A_{0,j+1} = \dim A_{0,j+1}e_{j+2}$ and $\dim A_{1,j} = \dim A_{1,j}e_2$, it follows that $(1.2.2)'$ implies

$$\begin{gathered}
A_{0,j+2}e_{j+2} = A_{0,j+1}e_{j+2}, \forall j \geq 0 \\
A_{0,j}e_2 = A_{1,j}e_2, \forall j \geq 2.
\end{gathered}$$

Since we have the trivial identities

$$x = \lambda^{-1} E_{A_{0,j+1}}(x) e_{j+2}, \forall x \in A_{0,j+1} e_{j+2}$$
$$x = \lambda^{-1} E_{A_{ij}}(x) e_2, \forall x \in A_{1,j} e_2$$

we are done. □

The fact that the index axiom (1.3.3) can be alternatively formulated in "probabilistic" and "dimension" terms is quite useful. Note also that the dimension condition is sufficient (and necessary as well) to ensure the ([PiPo1])-identity (1.3.3)$''$.

1.6. Corollary. *If $(A_{ij})_{0 \leq i \leq j}$ is a λ-lattice of commuting squares then* $\dim A_{ij} = \dim A_{i+n,j+n}$, $\forall i+1 \leq j$, $\forall n \geq 1$.

Proof. By (1.3.3)$''$ we have $\dim A_{ij} = \dim\, A_{i,j+1}\ e_{j+1} = \dim\, A_{i,j+1}\ e_{i+2} = \dim\, A_{i+1,j+1}$, where the equality $\dim A_{i,j+1} e_{j+1} = \dim A_{i,j+1} e_{i+2}$ is due to the equivlence to e_{i+2}, e_{j+1} in $A_{i,j+1} \supset$ Alg $\{1, e_{i+2}, \dots, e_{j+1}\}$ □

1.7. Corollary. *If $(A_{ij})_{0 \leq i \leq j}$ is a λ-lattice of commuting squares then the Jones projections $\{e_i\}_{i \geq 2}$ implement the following cononical embeddings between the centers of A_{ij}.*

a) $Z(A_{i,j}) \ni z \mapsto z' \in Z(A_{i,j+2})$, z' is the unique element in $Z(A_{i,j+2})$ such that $z e_{j+2} = z' e_{j+2}$

b) $Z(A_{i,j}) \ni z \mapsto z' \in Z(A_{i-2,j})$, z' is the unique element in $Z(A_{i-2,j})$ such that $z e_i = z' e_i$.

Moreover, if K_n (resp. L_n) and K'_n (resp. L'_n) are the sets of simple summands of $A_{0,2n}$ (resp. $A_{0,2n+1}$) and $A_{1,2n+1}$ (resp. $A_{1,2n+2}$), respectively, and if we identify K_n (resp. L_n) and K'_n (resp. L'_n) as subsets of K_{n+1} (resp. L_{n+1}) and K'_{n+1} (resp. L'_{n+1}) respectively, then there exist unique pointed bipartite graphs $\Gamma = (a_{kl})_{k \in k, l \in L}$, $\Gamma' = (b_{k'l'})_{k' \in k', l' \in L'}$, where $K = \cup_n K_n$, $L = \cup_n L_n$, $K' = \cup_n K'_n$, $L' = \cup_n L'_n$, such that the inclusion graphs of $A_{0,2n} \subset A_{0,2n+1}$ (resp. $A_{0,2n+1} \subset A_{0,2n+2}$) and $A_{1,2n+1} \subset A_{1,2n+2}$ (resp. $A_{1,2n+2} \subset A_{1,2n+3}$) are given by ${}_{K_n}\Gamma$ (resp. ${}_{L_n}\Gamma^t$) and ${}_{K'_n}\Gamma$ (resp. ${}_{L'_n}\Gamma'^t$) respectively. Furthermore, if Γ_i, Γ'_i are the similar graphs for the rows $(A_{2i,j})_j$ resp. $(A_{2i+1,j})_j$ and we identify the centers of $A_{2,2} \subset A_{2,3} \subset \dots$ with the centers of $A_{0,0} \subset A_{0,1} \subset A_{0,2} \dots$, and so on, by $z \mapsto z'$, with $z e_2 = z' e_2$, then there is a natural identification $\Gamma = \Gamma_i$, $\Gamma' = \Gamma'_i, \forall i \geq 0$.

Also, there exist unique vectors $(s_k)_{k \in k}$, $(t_l)_{l \in L}$, $(s'_{k'})_{k' \in k'}$, $(t'_{l'})_{l' \in L'}$ such that $s_{k'_0} = s'_{k'_0} = 1$ (where $\{k_0\} = K_0$, $\{k'_0\} = K'_0$), $\Gamma \Gamma^t \vec{s} = \lambda^{-1} \vec{s}$, $\lambda \Gamma^t \vec{s} = \vec{t}$, $\Gamma' \Gamma'^t \vec{s'} = \lambda^{-1} \vec{s'}$, $\lambda \Gamma'^t \vec{s'} = \vec{t'}$ and $(\lambda^n s_k)_{k \in K_n}$, $(\lambda^n t_l)_{l \in L_n}$, $(\lambda^n s'_{k'})_{k' \in K'_n}$, $(\lambda^n t'_{l'})_{l' \in L'_n}$ give the traces of the minimal projections in $A_{0,2n}, A_{0,2n+1}$ $A_{0,2n+1}, A_{1,2n+2}$, respectively.

Proof. In the proof of the existence of such a unique graph for $A_{00} \subset A_{01} \subset A_{02} \subset \dots$ in [GHJ] or [Po2] the only facts used were the axioms (1.3.1) – (1.3.3)$''$. □

2. Standard Lattices

The typical example of a λ-lattice of commuting squares is the lattice of higher relative commutants of an extremal subfactor $N \subset M$ of finite Jones index, $\lambda^{-1} = [M : N] < \infty$. Indeed, if $N \subset M \subset M_1 \subset^{e_2} M_2 \subset \dots$ is the associated Jones' tower, then $A_{ij} \overset{\text{def}}{=} M_i' \cap M_j$, $0 \leq i \leq j$, are well known to satisfy the axioms (1.1.1) - (1.1.3), with the observation that the extremality condition is needed only for the condition (1.1.2) c) and the second part of (1.1.3) b). In addition, however, $M_i' \cap M_j$ satisfy the condition $[M_i' \cap M_j, M_j' \cap M_l] = 0$, $\forall i \leq j \leq l$.

2.1. Definition. A λ-lattice of commuting squares is standard if it satisfies the following:

2.1.1. Commutation relations.

$$[A_{ij}, A_{kl}] = 0, \forall i \leq j \leq k \leq l$$

Since we proved that λ-lattices can be recovered from their first two rows, we want to write (2.1.1) as a condition involving A_{0i}, A_{1i} only.

2.2. Proposition. *Let* $(A_{ij})_{i \leq j, i=0,1}$ *be a* λ*-sequence of commuting squares. If* $A_{ij} \overset{\text{def}}{=} \{e_2, \dots, e_i\}' \cap A_{i,j}$, $\forall 2 \leq i \leq j$, *then* $(A_{ij})_{i,j}$ *is a* λ*-lattice of commuting squares if and only if* $(A_{ij})_{i \leq j, i=0,1}$ *satisfies;*

$$[A_{01}, A_{ij}] = 0, 1 \leq j \tag{2.1.1$'$}$$

$$[A_{01}, \{e_2, \dots, e_i\}' \cap A_{1,j}] = 0, \forall 2 \leq i \leq j$$

Proof. Trivial by the definitions ☐

Note that if we take $(A_{ij})_{i \leq j, i=0,1}$ to be a λ-sequence of commuting squares and we denote $A_{1,\infty} \subset^{e_2} A_{0,\infty} \subset^{e_1} A_{-1,\infty} \subset^{e_0} A_{-2,\infty} \subset^{e_{-1}} \dots$ its Jones tower then we can obtain $A_{i,\infty}, i \geq 0$, as $f_{i,0} A_{0,\infty} f_{i,0}$ with $f_{i,0}$ the word of maximal length in $e_i, e_{i-1}, \dots, e_{-i+2}$, which by [PiPo2] implements the expectation of $A_{0,\infty}$ onto $A_{i,\infty}$. More precisely, we have $f_{i,0} A_{0,n} f_{i,0} = A_{i,n} f_{i,0}$. We can then get rid of $f_{i,0}$ by taking $f_{0,-i} f_{i,0} A_{0n} f_{i,0} f_{0,-i} = A_{i,n} f_{0,-i}$, where $f_{0,-i}$ is the word in $e_0, e_{-1}, \dots, e_{-2i+2}$ implementing the expectation of $A_{-i,\infty}$ onto $A_{0,\infty}$. Thus, we can reformulate $(2.1.1)'$ as follows

$$[A_{0i}, f_{0,-1} f_{i,0} A_{0n} f_{i,0} f_{0,-1}] = 0, 0 \leq i \leq n \tag{2.1.1$''$}$$

Let us record the observation we started with, in the form of a statement.

2.3. Proposition. *Let* $N \subset M$ *be an extremal inclusion of type* II_1 *factors with finite Jones index,* $\lambda^{-1} = [M : N] < \infty$. *Then* $A_{ij} = M_i' \cap M_j$, $0 \leq i \leq j$, *is a standard* λ*-lattice.*

2.4. Definition. Let $(A_{ij})_{0 \leq i \leq j}$ be a λ-lattice. If $A_{ij}^0 \subset A_{ij}$ are subalgebras such that $A_{ij}^0 \subset A_{kl}^0$, $k \leq i \leq j \leq l$, $e_i \in A_{kl}^0$, $k+2 \leq i \leq l$, and $(A_{ij}^0)_{i,j}$ verify the commuting square axiom (1.1.1), then (A_{ij}^0) is called a λ-sublattice.

2.5. Corollary. *If* (A_{ij}) *is a standard* λ*-lattice and* (A_{ij}^0) *is a sublattice of* (A_{ij}) *then* (A_{ij}^0) *is a standard* λ*-lattice as well.*

Proof. Trivial by the definitions ☐

Although we will deduce it again in the next section from different considerations, we can already give a first proof to the fact that sublattices of the lattices of higher relative commutants are themselves lattices of higher relative commutants. The first proof of this result is based on the main theorem in [Po4]. It is this observation that led us to the considerations in this paper.

2.6. Theorem. *Any sublattice of a lattice of higher relative commutants is itself a lattice of higher relative commutants.*

Proof. Indeed, let $N \subset M$ be an extremal subfactor and assume $A^0_{ij} \subset M'_i \cap M_j$ is a sublattice. By 2.1 in [Po4] there exists a unitary $u \in M_1^\omega$ such that $M^u_\infty \overset{\text{def}}{=}$ vN $(uM_1u^*, M' \cap M_\infty) = uM_1u^* \vee M'_1 \cap M_\infty *_{M'_1 \cap M_\infty} M' \cap M_\infty$, where $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \dots$ is the Jones' tower for $N \subset M$ and $M_\infty = \overline{\cup_n M_n}$ its enveloping algebra. Let $P_0 = vN\{e_2, e_3, \dots\}$ and more generally $P_i = vN\{e_{i+2} \dots\}, i \leq 0$. Let $\{m^i_j\}_j$ be an orthonormal basis of P_i over P_{i+1}.

Let $\Phi_i(x) = \sum_j m^i_j e_{i+2} x e_{i+2} m^{i*}_j$, $x \in M^\omega_\infty$ and note that if $x \in M^\omega_{i+1}$ then $\Phi_i(x) = E^{M^\omega_{i+1}}_{M^\omega_i}(x)$. Also, if $Q^i \overset{\text{def}}{=}$ vN $(uM_1u^*, M' \cap M_{i+1})$ and if $x \in Q^i$, then $\Phi_i(x) \in M^u_\infty$, because $m^i_j, e_{i+2}, x \in M^u_\infty$. Thus vN $(Q^i, \Phi_i(Q^i)) \subset M^\omega_{i+1} \cap M^u_\infty$ and if we denote $B^i_0 = Q^i, B^i_{j+1} =$ vN $(B^i_j, \Phi_i(B^i_j))$ then by induction it follows that $B^i_{j+1} \subset M^\omega_{i+1} \cap M^u_\infty$. Thus if we put $M^i_{i+1} \overset{\text{def}}{=} \overline{\cup_j B^i_j}$ then $M^i_{i+1} \subset M^\omega_{i+1} \cap M^u_\infty$ and $\Phi_i(M^i_{i+1}) \subset M^i_{i+1}$, i.e., $E_{M^\omega_i} \subset M^i_{i+1}$, meaning that $M^i_i \overset{\text{def}}{=} \Phi_i(M^i_{i+1})$ is an algebra and

$$\begin{array}{ccc} M^\omega_i & \subset & M^\omega_{i+1} \\ \cup & & \cup \\ M^i_i & \subset & M^i_{i+1} \end{array}$$

is a commuting square, $\forall i \geq 0$.

Moreover, if $i = 0$ then we get that $M^0_0 \subset M^0_1$ is an inclusion of factors with $[M^0_1 : M^0_0] = [M : N]$ and if $M^0_i = < M^0_0, e_1, \dots, e_i >$ then $M^0_0 \subset M^0_1 \subset \dots$ is the Jones tower for $M^0_0 \subset M^0_1$, with $M^{0'}_0 \cap M^0_i =$ Alg $\{1, e_2, \dots, e_i\}$ and

$$\begin{array}{ccc} M^i_i & \subset & M^i_{i+1} \\ \cup & & \cup \\ M^0_i & \subset & M^0_{i+1} \end{array}$$

is a commuting square. Thus, $M^i_i \subset M^i_{i+1}$ is a homogeneous $[M : N]^{-1}$-Markov inclusion $\forall i$ (they will turn out to be factors shortly!). Since e_{i+2} is a Jones projection implementing $E^{M^\omega_{i+1}}_{M^\omega_i}$, from the previous commuting squares it follows that e_{i+2} also implements $E^{M^i_{i+1}}_{M^i_i}$ and thus, if we put $M^i_j \overset{\text{def}}{=} < M^i_{i+1}, e_{i+2}, \dots, e_j >$, $j \geq i+2$ and $M^i_{j-2} \overset{\text{def}}{=} M^i_i \cap \{e_{i+1}, e_i, \dots, e_j\}'$, $2 \leq j \leq i+1$, then $M^i_0 \subset M^i_1, \subset \dots \subset M^i_i \subset M^i_{i+1} \subset \dots$ is a Jones tower-tunnel for $M^i_i \subset M^i_{i+1}$. Also since $M^i_j \subset M^j_j$, for $i \leq j$, we have $M^i_0 \subset M^{i+1}_0$, $\forall i \leq 0$.

Finally, we define $M^u = \overline{\cup_i M^i_0}$, $M^u_1 = \overline{\cup_i M^i_0}$. We then have $M^u \subset M^\omega \cap M^u_\infty$, $M^u_1 \subset M^\omega_1 \cap M^u_\infty$ and the commuting square

$$\begin{array}{ccc} M^\omega & \subset & M^\omega_1 \\ \cup & & \cup \\ M^u & \subset & M^u_1 \end{array}$$

Also, since $M^i_i \subset < M^u_1, e_2, \dots, e_i >$, $\forall i \geq 2$, and uM_1u^*, $M' \cap M_i \subset M^i_i$, it follows that the enveloping algebra of $M^u \subset M^u_1$ coincides with M^u_∞. Let us then calculate

$M_1^{u'} \cap M_i^u$. We have by [Po3], $M_1^{u'} \cap M_\infty^u \subset uM_1u^{*'} \cap M_\infty^u = M_1' \cap M_\infty$. Since $M_1^{u'} \cap M_\infty \supset M_1^{'} \cap M_\infty$, we get $M_1^{u'} \cap M_\infty^u = M_1^{'} \cap M_\infty$. But, since $M' \cap M_\infty =< M_1^{'} \cap M_\infty, e_2 >$ we also get from this $M^{u'} \cap M_\infty^u = M' \cap M_\infty$. By expecting on M_i^ω we further get $M_j^{u'} \cap M_i^u = E_{M_i^\omega}(M_j^{u'} \cap M_\infty^u) \subset E_{M_i^\omega}(M_j^{'} \cap M_\infty) = M_j^{'} \cap M_i$, $j = 0, 1$. Since $M_j^{'} \cap M_i \subset M_j^{u'} \cap M_i^u$ as well, we have equalities.

But this implies that we have $M_j^{u'} \cap M_i^u = M_j^{'} \cap M_i$, $\forall i, j$.

Now we make the following change in the above construction:

Let $M_\infty^{u,0} \overset{\text{def}}{=}$ vN $(u\ M_1u^*,\ A_{0,\infty}^0)$, $= uM_1u^* \vee A_{1,\infty} *_{A_{1,\infty}} A_{0,\infty}$, $Q^{i,0} \overset{\text{def}}{=}$ vN$(uM_1u^*, A_{0,i+1})$, $B_0^{i,0} = Q^{i,0}$, $B_{j+1}^{i,0} = $ vN$(B_j^{i,0}, \Phi_i(B_j^i))$, $M_{i+1}^{i,0} \overset{\text{def}}{=} \overline{\cup_j B_j^{i,0}}$ and proceed the same way. Then finally put $M^{u,0} \overset{\text{def}}{=} \overline{\cup_i M_0^{i,0}}$, $M_1^{u,0} \overset{\text{def}}{=} \overline{\cup_i M_1^{i,0}}$ where $M_0^{i,0} = M_i^{i,0} \cap \{e_{i+1}, \dots, e_2\}'$, $M_1^{i,0} = M_i^{i,0} \cap \{e_{i+1}, \dots, e_3\}'$.

By the same arguments as above, if $M_i^{u,0} =< M_1^{u,0}, e_2, \dots, e_i >$, $i \geq 2$, then $M^{u,0'} \cap M_i^{u,0} = A_{0,i}$, $M^{u,0'} \cap M_i^{u,0} = A_{1,i}^0$. $\square$

2.7 Remark. Related to the above proof of 2.6, it is interesting to note that even if $N \subset M$ is an inclusion of hyperfinite type II_1 factors, the inclusion of factors $N^0 \subset M^0$ constructed in 2.6 so that its higher relative commutants coincide with a given sublattice of $(M_i^{'} \cap M_j)_{i,j}$, is not hyperfinite. Thus, in order to realise sublattices of a "hyperfinite" lattice, we may have to get out of the class of hyperfinite algebras.

3. CONSTRUCTION OF SUBFACTORS WITH GIVEN STANDARD LATTICE

It was already proved in [Po3] that the λ-lattice $A_{ij}^0 = Alg\{1, e_{i+2}, \dots, e_j\}$, $0 \leq i \leq j$, which is obviously standard, is indeed the lattice of higher relative commutants of a subfactor, by using a "universal construction" involving the Jones projections and amalgamated free products. In fact what is needed in order to extend that argument from (A_{ij}^0) to more general lattices is the property of being standard.

We will prove in this section that every standard lattice (A_{ij}) is a lattice of higher relative commutants of a subfactor, thus showing that the axioms (1.1.1), (1.1.2), (1.1.3), (2.1.1) are a complete set of axioms for the lattices of higher relative commutants.

Although we can prove this result by going along the lines of [Po3], we will present here a simpler argument which, in the case $(A_{ij}) = (A_{ij}^0)$, differs from the proof in [Po3] and from its subsequent simplifications in [Bo].

So let $(A_{ij})_{0 \leq i \leq j}$ be a λ-lattice, with $\lambda^{-1} > 4$, and with $e_i \in A_{kl}$, $k + 2 \leq i \leq l$, its Jones projections. We do not assume (A_{ij}) to be standard for now.

Let $P_i =$ vN $\{e_{i+2}, \dots\} \subset A_{i,\infty}$ $i \geq 0$. Let $\{m_j^i\}_j$ be an orthonormal basis of P_i over P_{i+1}. Let Q be an arbitrary separable type II_1 factor with the trace still denoted by σ. (All that follows works for Q a finite nonatomic von Neumann algebra, as in [Po3], but we take it to be a factor for the few simplifications that this hypothesis facilitates).

Let now $M_\infty \overset{\text{def}}{=} Q \otimes A_{1,\infty} *_{A_{1,\infty}} A_{0,\infty}$ and denote for $x \in M_\infty$, $\Phi_i(x) = \sum_j m_j^i x m_j^{i*}$, $i \geq 0$.

Let $\tilde{P}_i \overset{\text{def}}{=} P_i' \cap M_\infty$, $i \geq 0$. Since $P_{i+1} \subset P_i$ are locally trivial (because $\lambda^{-1} > 4$, [PiPo1]), if we let $f_{i+2} \in P_i \cap P_{i+1}'$, $\tau(f_{i+2}) = t < 1/2$, where $t(1-t) = \lambda$, then

$P_{i+1}f_{i+2} = f_{i+2}P_i f_{i+2}$, $P_{i+1}(1-f_{i+2}) = (1-f_{i+2})P_i(1-f_{i+2})$. Thus, we also have $\tilde{P}_i f_{i+2} = f_{i+2}\tilde{P}_{i+1}f_{i+2}$, $\tilde{P}_i(1-f_{i+2}) = (1-f_{i+2})\tilde{P}_{i+1}(1-f_{i+2})$. Also, Φ_i implements on $\tilde{P}_{i+1}$ the unique conditinal expectation onto $\tilde{P}_i$ that takes f_{i+2} into $(1-t)1$ (and which is not trace preserving!). Furthermore $e_{i+2}xe_{i+2} = \Phi_i(x)e_{i+2}$, $\forall x \in \tilde{P}_{i+1}$, and $\tilde{P}_0 \subset \tilde{P}_1 \subset^{e_2} \tilde{P}_2 \subset \dots$ is the Jones tower for $\tilde{P}_0 \subset^{\mathcal{E}} \tilde{P}_1$, where $\mathcal{E} = \Phi_0|_{\tilde{P}_1}$. Denote by $\mathcal{E}_i^j$ the expectaion of $\tilde{P}_j$ onto $\tilde{P}_i$ in this tower, i.e. $\mathcal{E}_i^j = \Phi_i \circ \dots \circ \Phi_{j|\tilde{P}_j}$.

Note that Φ_i (or $\mathcal{E}_i^{i+1}$) implements a conditional expectation of $A'_{i+1,\infty} \cap M_\infty$ onto $A'_{i,\infty} \cap M_\infty$ as well.

Since $\Phi_{j+1}(x) = x, \forall x \in Q \vee A_{0,j}$ we have:

$$\Phi_i \circ \dots \circ \Phi_j(Q \vee A_{0,j}) \subset \Phi_i \dots \Phi_{j+1}(Q \vee A_{0,j+1}), i \leq j \tag{3.1.1}$$

By [PiPo2], $\mathcal{E}_i^j$ is implemented by the projection e_i^j obtained as a scalar multiple of the word of maximal length in $e_{i+2}, \dots, e_j, e_{j+1}, \dots, e_{2j-i}$. Thus, if $x, y \in \tilde{P}_j$ then

$$\begin{aligned}\mathcal{E}_i^j(x)\mathcal{E}_i^j(y)e_j^{2j-1} &= \lambda^{2i-2j}(e_j^{2j-i}e_i^j x e_i^j e_j^{2j-i})(e_j^{2j-i}e_i^j y e_i^j e_j^{2j-i}) \\ &= \lambda^{i-j}e_j^{2j-i}(e_i^j x e_i^j y e_i^j)e_j^{2j-i} \\ &= \lambda^{i-j}\mathcal{E}_j^{2j-i}(e_i^j x e_i^j y e_i^j)e_j^{2j-i}\end{aligned}$$

In particular, if $x, y \in Q \vee A_{0,j}$ then $\mathcal{E}_i^j(x)\mathcal{E}_i^j(y) \in \mathcal{E}_j^{2j-i}(e_i^j(Q \vee A_{0,j})e_i^j(Q \vee A_{0,j})e_i^j)$ $\subset \mathcal{E}_j^{2j-i}(Q \vee A_{0,2j-i}) = \Phi_j \circ \dots \circ \Phi_{2j-i}(Q \vee A_{0,2j-i})$. But since $z = \mathcal{E}_i^j(x)\mathcal{E}_i^j(y) \in \tilde{P}_i$ we also have $\Phi_i \circ \dots \circ \Phi_j(z) = z$. This shows that:

$$(\Phi_i \circ \dots \circ \Phi_j(Q \vee A_{0,j}))^2 \subset \Phi_i \circ \dots \circ \Phi_{2j-i}(Q \vee A_{0,2j-i}) \tag{3.1.2}$$

Consider then the following notation:

$$M_i \overset{\text{def}}{=} (\cup_j \Phi_i \circ \dots \circ \Phi_j(Q \vee A_{0,j}))^-, i \geq 0 \tag{3.1.3}$$

By (3.1.1), (3.1.2) each M_i follows an algebra. By the definition we clearly have $M_i \subset \tilde{P}_i$ and $\Phi_i(M_{i+1}) = M_i$, so that we have the commuting squares

$$\begin{array}{ccccccc} \tilde{P}_0 & \subset^{\mathcal{E}_0^1} & \tilde{P}_1 & \subset^{\mathcal{E}_0^2} & \tilde{P}_2 & \subset \\ \cup & & \cup & & \cup & \\ M_0 & \subset & M_1 & \subset & M_2 & \subset \end{array} \tag{3.1.4}$$

with $e_j \in M_j$, $j \geq 2$. We will prove that, although $\mathcal{E}_i^{i+1} = \Phi_i|_{\tilde{P}_{i+1}}$ is not trace preserving, it is trace preserving when restricted to M_{i+1}. To do this we first need to prove that τ is a Markov trace on the inclusions $M_i \subset M_{i+1}$, i.e.,

$$\tau(e_{i+2}x) = \lambda\tau(x), \forall x \in M_{i+1}, i \geq 1 \tag{3.1.5}$$

By (3.1.3), to prove this equality we only need to show that $\tau(e_{i+2}\mathcal{E}_{i+1}^j(y)) = \lambda\tau(\mathcal{E}_{i+1}^j(y))$, $\forall y \in Q \vee A_{0,j}$, $\forall j \geq i+1$. But by [Po3], $Q' \cap M_\infty = A_{1,\infty}$ and if $i \geq 1$ then $\tau(e_{i+2}\mathcal{E}_{i+1}^j(y)) = \tau(ue_{i+2}\mathcal{E}_{i+1}^j(y)u^*) = \tau(e_{i+2}\mathcal{E}_{i+1}^j(uyu^*))$, $\forall u \in \mathcal{U}(Q)$.

By taking weak limits of convex combinations of uyu^*, $u \in \mathcal{U}(Q)$, and by using $Q' \cap M_\infty = A_{1,\infty}$ it thus follows that $\tau(e_{i+2}\mathcal{E}^j_{i+1}(y)) = \tau(e_{i+2}\mathcal{E}^j_{i+1}(E_{A_{1,\infty}}(y)))$. Similarly we get $\tau(\mathcal{E}^j_{i+1}(y)) = \tau(\mathcal{E}^j_{i+1}(E_{A_{1,\infty}}(y)))$. But $E_{A_{1,\infty}}(Q \vee A_{0,j}) = A_{1,j}$ and $\mathcal{E}^j_{i+1}|_{A_{1,j}} = E^{A_{1,j}}_{A_{1,i+1}}$. Since $\tau(e_{i+2}y') = \lambda\tau(y')$, $\forall y' \in A_{1,i+1}$, (3.1.5) follows.

We can now calculate the relative commutants of $M_0 \subset M_1 \subset \dots$.

(3.1.6) *If* (A_{ij}) *is standard then* $M'_k \cap M_i = A_{k,i}, \forall i \geq k \geq 1$.

Indeed, in the proof of (3.1.5) we already noted that $E_{Q' \cap M_i} = E_{A_{1,i}}, \forall i \geq 1$. Since $Q \subset M_k$, $e_2, \dots, e_k \in A_{0,k} \subset M_k$, it follows that $M'_k \cap M_i \subset \{e_2, \dots, e_k\}' \cap A_{1,i} = A_{k,i}$. Conversely, if $x \in A_{k,i}$ then clearly $x \in M_i$ by the definitions. Also let $y = \Phi_k \circ \dots \circ \Phi_j(y_0)$, for some $y_0 \in Q \vee A_{0,j}, j \geq i$. If $\{m_l\}_l$ is an orthonormal basis of P_k over P_j then $\Phi_k \circ \dots \circ \Phi_j(y_0) = \sum_l m_l y_0 m_l^* = \mathcal{E}^j_k(y_0)$. But $\{m_l\}_l$ is also an orthonormal basis of $A_{k,\infty}$ over $A_{j,\infty}$ so that we have

$$x \sum_l m_l y_0 m_l^* = \sum_{l,r} m_r E_{A_{j,\infty}}(m_r^* x m_l) y_0 m_l^*$$

But $[A_{j,\infty}, y_0] = 0$, so that the right hand term equals

$$\sum_{l,r} m_r y_0 E_{A_{j,\infty}}(m_r^* x m_l) m_l^* = (\sum_r m_r y_0 m_r^*) x,$$

proving (3.1.6).

We can now state the result:

3.1 Theorem. *Let* $(A_{ij})_{0 \leq i \leq j}$ *be a standard* λ*-lattice. Then there exists an extremal inclusion of factors* $N \subset M$ *of index* $[M : N] = \lambda^{-1}$ *such that* $M'_i \cap M_j = A_{ij}, 0 \leq i \leq j$*, where* $N \subset M \subset M_1 \subset \dots$ *is the Jones tower of factors associated to* $N \subset M$*. Moreover, if the graph* Γ *of* $(A_{ij})_{0 \leq i \leq j}$ *is strongly amenable then* N, M *can be taken hyperfinite.*

Proof. Assume first that Γ is strongly amenable, so that $A_{2,\infty} \subset A_{0,\infty}$ is an inclusion of factors. Since $A_{0,\infty} \supset A_{2,\infty} \supset A_{4,\infty} \supset \dots$ is a tunnel and $A'_{2k,\infty} \cap A_{0,\infty} \supset A_{0,2k}$, if follows that $||\Gamma_{A_{2,\infty}, A_{0,\infty}}|| \geq ||\Gamma\Gamma^t|| = \lambda^{-1}$. But $[A_{0,\infty} : A_{2,\infty}] = \lambda^{-2}$ so that $||\Gamma_{A_{2,\infty}, A_{0,\infty}}||^2 \leq \lambda^{-2}$. Thus $||\Gamma_{A_{2,\infty}, A_{0,\infty}}||^2 = \lambda^{-2} = [A_{0,\infty} : A_{2,\infty}]$ so that $A_{2,\infty} \subset A_{0,\infty}$ is extremal and strongly amenable. Let M_∞ be the enveloping algebra of $A_{2,\infty} \subset A_{0,\infty}$ and define $M = A'_{0,\infty} \cap M_\infty, M_2 = A'_{2,\infty} \cap M_\infty$ and more generally $M_k = A'_{k,\infty} \cap M_\infty, \forall k$. Thus $M \subset M_2 \subset M_4 \subset \dots$ is a Jones tower for $M \subset M_2$ and clearly $M' \cap M_{2k} = A'_{2k,\infty} \cap A_{0,\infty} \supset A_{0,2k}$, by the bicommutant relation $(M'_{2i} \cap M_\infty)' \cap M_\infty = M_{2i}$ for strongly amenable subfactors [Po1]. Before proving the opposite inclusion $M' \cap M_{2k} \subset A_{0,2k}$ note that if $e_2^4 = \lambda^{-1} e_3 e_2 e_4 e_3$ and more generally $e_{2k}^{2k+2} = \lambda^{-1} e_{2k+1} e_{2k} e_{2k+2} e_{2k+1}$ then $M_{2k-2} \subset M_{2k} \subset^{e_{2k}^{2k+2}} M_{2k+2}$ is a basic construction and so is $A_{2k+2,\infty} \subset A_{2k,\infty} \subset^{e_{2k}^{2k+2}} A_{2k-2,\infty}$. Thus, if $\{m_j\}_j$ is an orthonormal basis of $A_{2k,\infty}$ over $A_{2k+2,\infty}$ then $M_{2k} \ni x \mapsto \sum_j m_j e_{2k}^{2k+2} x e_{2k}^{2k+2} m_j^* = \sum m_j E^{M_{2k}}_{M_{2k-2}}(x) e_{2k}^{2k+2} m_j^* =$

$E^{M_{2k}}_{M_{2k-2}}(x) \in M_{2k-2}$. But if $x \in A_{0,2k}$ then $\sum_j m_j e^{2k+2}_{2k} x e^{2k+2}_{2k} m_j^* = E^{A_{0,2k}}_{A_{0,2k-2}}(x)$. Thus we have the commuting square relation $E_{M_{2k-2}}(A_{0,2k}) = A_{0,2k-2}$.

Now, if $x \in M' \cap M_{2k} \subset A_{0,\infty}$ then let $x_0 \in A_{0,2n}$, with $||x - x_0||_2 < \varepsilon$. By expecting on M_{2k} we then get $\varepsilon > ||x - E_{M_{2k}}(x_0)||_2 = ||x - E_{A_{0,2k}}(x_0)||_2$. By letting $\varepsilon \to 0$ we get $x \in A_{0,2k}$. Thus $M' \cap M_{2k} = A_{0,2k}$ and by expecting this relation onto $M'_{2i} \cap M_\infty = A_{2i,\infty}$ we get $M'_{2i} \cap M_{2k} = A_{2i,2k}$. Also, $M'_{2i+1} \cap M_{2k} = (M_{2i} \cup \{e_{2i+1}\})' \cap M_{2k} = \{e_{2i+1}\}' \cap A_{2i,2k} = A_{2i+1,2k}$, by (1.1.2). Similarly $A_{l,2k-1} = A_{l,2k} \cap \{e_{2k+1}\}' = M'_l \cap M_{2k} \cap \{e_{2k+1}\}'$, so that $A_{ij} = M'_i \cap M_j$, $\forall 0 \leq i \leq j$.

Since $\lambda^{-1} \leq 4$ implies Γ is strongly amenable, we only need to prove the rest of the statement for $\lambda^{-1} > 4$. Then let Q be a (separable) type II_1 factor and define $M_\infty = Q \otimes A_{1,\infty} *_{A_{1,\infty}} A_{0,\infty}$, like at the beginning of this section. Let also $M_0 \subset M_1 \subset M_2 \subset \dots$ be defined like in (3.1.3). By definitions, $Q \vee A_{0,j} \subset M_j$, so that $\overline{\cup_j M_j} \supset \overline{\cup_j (Q \vee A_{0,j})} = M_\infty$. By (3.1.6), M_j are factors, $\forall j \geq 1$. By (3.1.4), (3.1.5) we have $[M_2 : M_1] = \text{ Ind } E^{M_2}_{M_1} = \lambda^{-1}$ and $M_1 \subset M_2 \subset^{e_3} M_3 \subset^{e_4} \dots$ is the Jones tower for $M_1 \subset^{E^{M_2}_{M_1}} M_2$. Since $\lambda e_3 = e_3 e_2 e_3 = E^{M_2}_{M_1}(e_2) e_3$, we also have $E_{M_1}(e_2) = \lambda 1$ so that $e_2 \in M_2$ is a Jones projection for the inclusion of factors $M_1 \subset M_2$. Thus $M \overset{\text{def}}{=} \{e_2\}' \cap M_1$ is a factor and $M \subset M_1 \subset^{e_2} M_2$ is a basic construction [PiPo1]. But we also have $M e_2 = e_2 M_1 e_2 = \mathcal{E}^1_0(M_1) e_2 = M_0 e_2$, so that $M = M_0$.

We already showed that $M'_i \cap M_j = A_{i,j}$, if $j \geq i \geq 1$ in (3.1.6). Then $M'_0 \cap M_j = E_{M_j}(M' \cap M_\infty) = E_{M_j}(\text{ sp } (M'_1 \cap M_\infty e_2 M'_1 \cap M_\infty)) = E_{M_j}(\text{ sp } (A_{1,\infty} e_2 A_{1,\infty})) = E_{M_j}(A_{0,\infty}) = \overline{\cup_k E_{M_j}(A_{0,k})} = A_{0,j}$.

Finally, note that by the first part we also have that if Γ is strongly amenable, $M_\infty = Q *_{A} 1, \infty A_{0,\infty}$ as before and $M \overset{\text{def}}{=} A'_{0,\infty} \cap M_\infty$, $M_1 \overset{\text{def}}{=} A'_{1,\infty} \cap M_\infty$, then $(M'_i \cap M_j) = (A_{ij})$ $\square$

Note that the construction of hyperfinite $N \subset M$ with $M'_i \cap M_j = A_{ij}$, when (A_{ij}) has finite Γ, coincides with the one in [Oc]. The construction of $N \subset M$ from almalgamated free products coincides with the one in [Po3], when $A_{ij} = \text{ Alg } \{1, e_{i+2}, \dots, e_j\}$, i.e., when $\Gamma = A_n$, $n \leq \infty$, and with the one in [Ra] for Γ finite.

3.2. Corollary. *A system of finite dimensional algebras $(A_{ij})_{0 \leq i \leq j}$ with a trace τ is the lattice of higher relative commutants of an extremal inclusion of factors $N \subset M$ of index λ^{-1} if and only if it is a standard λ-lattice.*

Note that since by 2.4 sublattices of standard lattices are standard, Theorem 3.1 provides an alternative proof to Theorem 2.6 as well.

Like with the proof of 2.6, note that even if (A_{ij}) is a standard lattice coming from a hyperfinite inclusion of factors $N \subset M$ we may not be able to realise its sublattices (A^0_{ij}) as higher relative commutants of a hyperfinite inclusion (unless the graph of (A^0_{ij}) is strongly amenable). Indeed, in the construction of (3.1) the subfactors are non Γ by [Po3].

3.3. Corollary. *If $(A_{ij})_{i,j}$ is a standard lattice then there exists an extremal inclusion of non Γ factors $N \subset M$ such that (A_{ij}) is its lattice of higher relative commutants and such that if (A^0_{ij}) is a sublattice of (A_{ij}) then there exists $N^0 \subset M^0$ embedded in $N \subset M$ as a commuting square so that (A^0_{ij}) is the lattice of $N^0 \subset M^0$.*

3.4. Remark. It would be extremely interesting to show that if (A_{ij}) is the lattice of higher relative commutant of an inclusion of hyperfinite factors $N \subset M$ then any sublattice (A^0_{ij}) of (A_{ij}) is itself the lattice of an inclusion of hyperfinite factors. Note that, by [Po6], if (A_{ij}) comes from an amenable inclusion $N \subset M$ then for $(A^0_{ij}) \subset (A_{ij})$ to be realised as higher relative commutants of some $N^0 \subset M^0$ embedded in $N \subset M$ it is necessary that (A^0_{ij}) is itself amenable (so $A^0_{ij} = Alg\{1, e_{i+2}, \dots, e_j\}$ cannot be realised this way).

4. Constructing sublattices and calculating their graphs

A natural way to construct sublattices (A^0_{ij}) of a lattice (A_{ij}) is to take $A^0_{1,j} \subset A^0_{0,j}$ equal to $A_{1,j} \subset A_{0,j}, \forall j \leq n$, for some n, and then continue adding only the Jones projections to the previously chosen algebras, i.e. letting

$$A^0_{i,n+k+i} =< A_{i,n+1,e_{n+i+1}}, \dots, e_{n+i+k} > .$$

This is analogous to having a presentation of a finitely generated initial group G given by the generators $g_1, \dots, g_l$, and relations $R_1, R_2, \dots$ and then taking the group G^0 with the same generators $g_1, \dots, g_l$ but only the first n relations $R_1, \dots, R_n$. In the case of lattices though we still need the compatibility relation

$$E_{A_{1,\infty}}(< A_{0,n}, e_{n+1}, \dots, e_{n+k} >) \subset < A_{0,n}, e_{n+1}, \dots, e_{n+k} >$$

to be satisfied, in order for the above A^0_{ij} to be a sublattice. And this condition is not automatically fulfilled.

We do have however a sufficient condition which insures this compatibility:

4.1. Proposition. *Let (A_{ij}) be a λ-lattice. Assume that for some n we have:*

$$< A_{i,n+i}, e_{n+i+1} >=< A_{i+1,n+i+1}, e_{i+2} >, \forall i \geq 0. \tag{4.1.1}$$

Let $A^0_{i,j} = A_{i,j}$, if $0 \leq j - i \leq n$ and for $j - i \geq n + 1$ define recursively $A^0_{i,j} =$ sp $A^0_{i,j-1} e_j A^0_{i,j-1} + A^0_{i,j-1} =< A^0_{i,j-1}, e_j >$. Then (A^0_{ij}) is a sublattice of (A_{ij}).

Proof. We prove by induction over k that $A^0_{i,n+i+k} =< A_{i+1,n+i+k}, e_{i+2} >$, $\forall i \geq 0, \forall k \geq 1$. For $k = 1$ this is true by hypothesis. Assume that we have the equality up to some $k \geq 1$. Then $A^0_{i,n+i+k+1} =< A^0_{i,n+i+k}, e_{i,n+i+k+1} >$, by definition, and $e_{i+2} \in A^0_{i,n+i+k} \subset A^0_{i,n+i+k+1}$, $\{e_{i+2}, e_{n+i+k+1}\} \cup A^0_{i+1,n+i+k} \subset A^0_{i,n+i+k+1}$. Thus, $A^0_{i+1,n+i+k+1} =$ sp $(A^0_{i+1,n+i+k} e_{n+i+k+1} A^0_{i+1,n+i+k}) + A^0_{i+1,n+i+k} \subset A^0_{i,n+i+k+1}$, showing that $< A^0_{i+1,n+i+k+1}, e_{i+2} > \subset A^0_{i,n+i+k+1}$.

For the reverse inclusion we similarly have: $\{e_{i+2}\ e_{n+i+k+1}\} \cup A^0_{i+1,n+i+k} \subset\ < A^0_{i+1,n+i+k+1}, e_{i+2} >$ and since $A^0_{i,n+i+k+1} =<< A^0_{i+1,n+i+k}, e_{i+2} >, e_{n+i+k+1} >$ for $k \geq 2$, we are done. □

For (4.1.1) to hold true, $\forall i$, it is in fact sufficient that it is satisfied for $i = 0, 1$.

4.2. Proposition. *If* (A_{ij}) *is a standard lattice and it satisfies the condition:*

$$< A_{i,n+i}, e_{n+i+1} >=< A_{i+1,n+i+1}, e_{i+2} >, \tag{4.2.1}$$

for $i = 0, 1$ *then it satisfies this condition* $\forall i \geq 0$. *Moreover, for* (4.2.1) *to hold true for* $i = 0, 1$ *it is sufficient that:*

$$< A_{i,n+i}, e_{n+i+1} >= A_{i,n+i+1}, i = 0, 1. \tag{4.2.2}$$

Proof. Since $A_{ij} = M_i' \cap M_j$ for the tower of factors $M \subset M_1 \subset M_2 \subset \dots$ associated to some extremal inclusion $N \subset M$ it follows that there are cononical isomorphisms from $A_{i,j}$ to $A_{i+2,j+2}$ carrying $e_{i+2}, \dots, e_j$ onto $e_{i+4}, \dots, e_{j+2}$ respectively. Thus, if (4.2.1) holds true for $i = 0, 1$ then it holds true $\forall i$. Also, if (4.2.2) is satisfied for some odd n then there exist antiautomorphisms of $A_{0,n+1}$ (resp. $A_{1,n+2}$) carrying $A_{0,n}$ onto $A_{1,n+1}$ (resp. $A_{1,n+1}$ onto $A_{2,n+2}$) and e_{n+1} into e_2 (resp. e_{n+2} into e_3), showing that (4.2.1) holds true for $i = 0, 1$. If n is even then there exists an antiautomorphism of $A_{0,n+2}$ carrying $A_{1,n+1}$ onto itself, e_{n+2} into e_2 and $A_{1,n+2}$ onto $A_{0,n+1}$ and e_{n+1} into e_3. Thus (4.2.2) $\Rightarrow$ (4.2.1) $\square$

At this point we would like to be able to recognise the stability condition (4.2.2) by merely looking at the graphs of the lattice.

4.3. Proposition. *Let* (A_{ij}) *be a standard lattice with a pair of graphs* (Γ, Γ'). *Assume that, for some n, both* Γ *and* Γ' *satisfy the following stability condition:*

(4.3.1). There is a one to one correspondence, $j \leftrightarrow \bar{j}$, *given by single edges, between the vertices of* Γ *(resp.* Γ'*) at distance* n *from* $*$ *that are not end points and the vertices of* Γ *(resp.* Γ'*) at distance* $n+1$ *from* $*$, *i.e. there exists a unique edge exiting* j *and it goes to* $\bar{j}$, *and distinct such* j*'s give distinct* $\bar{j}$*'s.*

Then (A_{ij}) *satisfies the stability condition* (4.2.2) *for* $i = 0, 1$ *and thus* (4.2.1), $\forall i$.

Proof. Let $N \subset M$ be so that $M_i' \cap M_j = A_{ij}$. Let j be a vertex at distance n from $*$ on the graph Γ and let p_j be a minimal central projection in $M' \cap M_n$ of label j. Then j is an end point iff $(1 - z_{n+1})p_j = 0$, where $z_{n+1} = z_{M' \cap M_{n+1}}(e_{n+1}) = z_{<M' \cap M_n, e_{n+1}>}(e_{n+1})$. Also, there exists a unique edge exiting j with no other edge going to the same vertex at distance $n+1$, if and only if $(1 - z_{n+1})(M' \cap M_n)p_j = (1 - z_{n+1})(M' \cap M_{n+1})p_j$. Thus, if (4.2.1) is satisfied then $(1 - z_{n+1})M' \cap M_n = (1 - z_{n+1})M' \cap M_{n+1}$. But $z_{n+1}(M' \cap M_n e_{n+1} M' \cap M_n) = z_{n+1} M' \cap M_{n+1}$ always, so that $M' \cap M_{n+1} =< M' \cap M_n, e_{n+1} >$. $\square$

For the next result we denote by $\Gamma(n)$ (resp. $\Gamma'(n)$) the restriction of Γ (resp. Γ') to the vertices at distance $\leq n$ from $*$.

4.4. Corollary. *Let* (A_{ij}) *be a standard lattice and assume its graphs* (Γ, Γ') *satisfy the stability condition* (4.3.1) *for some n (so that* (A_{ij}) *satisfies* (4.1.1), *by 4.3). Let* $A_{ij}^0 \subset A_{ij}$ *be the sublattice obtained by truncating* (A_{ij}) *from the step n on like in 4.1 and let* $(\Gamma^0, \Gamma^{0'})$ *be its graphs which one calls the "truncation of* (Γ, Γ') *at step n". Then* Γ^0 *(resp.* $\Gamma^{0'}$*) is obtained from* Γ *(resp.* Γ'*) by adding to* *each boundary vertex j of $\Gamma(n)$ (resp. $\Gamma'(n)$) an A_{n_j} tail, for some $1 \leq n_j \leq \infty$, with $n_j = 1$ iff j is an end point.*

Proof. This is clear now by the proof of 4.3. Indeed, if j is not an end point in $\Gamma(n)$, p_j is the corresponding minimal central projection in $A_{0,n}$ like in the proof of 4.3 and we have $(1 - z_{n+1})p_j \neq 0$, then either $(1 - z_{n+2})p_j = 0$, meaning that $(1 - z_{n+1})p_j$ is a direct summand corresponding to an end point in $\Gamma^0(n + 1) = \Gamma(n + 1)$, or $(1 - z_{n+2})p_j \neq 0$ in which case $(1 - z_{n+2})p_j < A_{0,n+1}, e_{n+2} >= (1 - z_{n+2})p_j A_{0,n+1}$ (here $z_{n+1} = z_{<A_{0,n}, e_{n+1}>}(e_{n+1}) = z_{A_{0,n+1}}(e_{n+1})$).

By induction the above shows that Γ^0 will have an A_{n_j} graph departing from j, for each j. □

We can now deduce our main obstruction criteria for a pair of graphs (Γ, Γ') to be standard point. Thus, we show that if (Γ, Γ') is stable at some step n and is 'non-trivial' up to that level i.e. $\Gamma(n) \neq A_{n+1}$ (equivalently, the n'th relative commutant contains more that just the Jones projections), then the rest of the graphs MUST consist of A_{fin} tails only. So if either Γ or Γ' fail to continue with an A_{fin} tail form one of its vertices at distance n from $*$ then (Γ, Γ') is not standard.

4.5 Theorem. *If (Γ, Γ') is a standard pair of graphs corresponding to index $\lambda^{-1} > 4$ which is stable at distance n from $*$ then one of the following holds true:*

a) $\Gamma(n) = A_{n+1} = \Gamma'(n)$ and then the truncation at step n of (Γ, Γ') is $(\Gamma^0, \Gamma^{0'}) = (A_\infty, A_\infty)$.

b) From each vertex at distance n from $*$ both Γ and Γ' continue with A_{fin} tails.

Proof. By 4.4 it follows that $(\Gamma^0, \Gamma^{0'})$ is obtained from $(\Gamma(n), \Gamma'(n))$ as described in b). But then, if we get an A_∞ tail at some j, $(\Gamma^0, \Gamma^{0'})$ must be (A_∞, A_∞) cf. [Po5]. If we only get A_{fin} tails, then the weights at its vertices must be proportional to the weights at the vertices of an A_∞ standard graph with the $*$ point of A_∞ corresponding to the end of the A_{fin} tail, i.e. proportional to $(P_{n_j}(\lambda)/\lambda P_{n_{j-1}}(\lambda))^{1/2}$. But by 1.4.3 in [Sc] it then follows that $(\Gamma^0, \Gamma^{0'}) = (\Gamma, \Gamma')$, so (Γ, Γ') itself must continue with A_{fin} tails from the level n on. □

From the above, it follows that there are no standard pairs of infinite graphs (Γ, Γ'), which are stable at some step n for which $(\Gamma(n), \Gamma'(n)) \neq (A_n, A_n)$. Equivalently, we have:

4.6. Corollary. *If (Γ, Γ') is a standard pair of infinite graphs corresponding to index > 4 which is stable at step n then $(\Gamma(n), \Gamma'(n)) = (A_n, A_n)$ and the truncation at step n of (Γ, Γ') is (A_∞, A_∞).*

The above results show in particular that if $\Gamma = \Gamma'$ and Γ is stable at step n then Γ must be of a very particular form. In some situations, even if aprioricallly Γ, Γ' are not assumed equal, one can get some conclusions (e.g., exclude (Γ, Γ') as a standard pair) by looking at Γ only and by using 4.5.

4.7. Lemma. *Let (Γ, Γ') be a standard pair of graphs*

a) If n is an even number and Γ is stable at levels $n, n + 1, n + 2$ then (Γ, Γ') is stable at $n + 1$.

b) If $\Gamma(n - 1) = A_n$, then $\Gamma'(n - 1) = A_n$.

c) If $\Gamma(n-1) = A_n$*,* $\Gamma(n+1)$ *has just one edge more than* $\Gamma(n)$ *and the unique vertex at distance* $n-1$ *from* $*$ *is either a double, triple or quadruple point, then* $\Gamma(n+1) = \Gamma'(n+1)$ *and* (Γ, Γ') *is stable at* n*.*

d) If n *is odd,* $\Gamma(n-1) = A_n$ *and* $\Gamma(n+1)$ *has just one edge more than* $\Gamma(n)$*, with its unique end point at distance* n *from* $*$ *being related to the vertex* $n-1$ *by just one edge, then* $\Gamma(n+1) = \Gamma'(n+1)$ *and* (Γ, Γ') *is stable at* n*.*

Proof. a) Let $N \subset M$ be a subfactor with (Γ, Γ') as its standard pair of graphs. Then $M' \cap M_{n+3} = \text{ sp } M' \cap M_{n+2} e_{n+3} M' \cap M_{n+2} + M' \cap M_{n+2}$. But $M' \cap M_{n+2} = \text{ sp } M' \cap M_{n+1} e_{n+2} M' \cap M_{n+1} + M' \cap M_{n+1} = \text{ sp } M_1' \cap M_{n+2} e_2 M_1' \cap M_{n+2} + M_1' \cap M_{n+2}$, the last equality following from the parity of $n+2 (=$ parity of $n)$ and the existence of the antisymetry on $M' \cap M_{n+2}$. Thus we get $M' \cap M_{n+3} = \text{ sp } M_1' \cap M_{n+2} e_2 M_1' \cap M_{n+2} e_{n+3} M_1' \cap M_{n+2} e_2 M_1' \cap M_{n+2} + X$, where X is a set consisting of products of elements in $\{e_2, e_{n+3}\} \cup M_1' \cap M_{n+2}$, with e_2, e_{n+3} appearing at the most just one time each. But $e_2 M_1' \cap M_{n+2} e_{n+3} M_1' \cap M_{n+2} e_2 \subset (M_2' \cap M_{n+3}) e_2 = (\text{ sp } M_2' \cap M_{n+2} e_{n+3} M_2' \cap M_{n+2} + M_2' \cap M_{n+2}) e_2$. Thus, when expecting $M' \cap M_{n+3}$ onto $M_1' \cap M_{n+3}$ we get $M_1' \cap M_{n+3} = E_{M_1' \cap M_{n+3}}(M' \cap M_{n+3}) = E_{M_1' \cap M_{n+3}}(\text{ sp } M_1' \cap M_{n+2} e_{n+3} e_2 M_1' \cap M_{n+2} + X) = \text{ sp } M_1' \cap M_{n+2} e_{n+3} M_1' \cap M_{n+2} + M_1' \cap M_{n+2}$. But this shows that Γ' is stable at $n+1$.

b) If $\Gamma(n-1) = A_n$, it means that $M' \cap M_{n-1}$ is generated by the Jones projections $e_2, \dots, e_{n-1}$. Since $\dim M' \cap M_{n-1} = \dim M_1' \cap M_n$ and $M_1' \cap M_n \supset \text{ Alg } \{1, e_3, \dots, e_n\} \simeq \text{ Alg } \{1, e_2, \dots, e_{n-1}\} = M' \cap M_{n-1}$ [J1], one gets $M_1' \cap M_n = \text{ Alg } \{1, e_3, \dots, e_n\}$ so that $\Gamma'(n-1) = A_n$.

c) If Γ has a double, triple or quadruple point at the vertex $n-1$ then $M' \cap M_n \simeq \text{ Alg } \{1, e_2, \dots, e_n\} \oplus \mathbb{C}^i$ with $i = 0, 1, 2$. But since $\dim M' \cap M_n = \dim M_1' \cap M_{n+1}$ and $M_1' \cap M_{n+1} \supset \text{ Alg } \{1, e_3, \dots, e_{n+1}\} \oplus \mathbb{C}^i$ it then follows that $M_1' \cap M_{n+1} \simeq \text{ Alg } \{l, e_3, \dots, e_{n+1}\} + \mathbb{C}^i$ as well and $\Gamma(n) = \Gamma'(n)$. If $\Gamma(n+1)$ adds just one more edge to $\Gamma(n)$ then $\dim M' \cap M_{n+1} = \dim \text{ sp } M' \cap M_n e_{n+1} M' \cap M_n + 1 = \dim \text{ sp } M_1' \cap M_{n+1} e_{n+2} M_1' \cap M_{n+1} + 1$ so that $\Gamma'(n+1)$ adds just one more edge to $\Gamma'(n)$ too. Thus $\Gamma(n+1) = \Gamma'(n+1)$ and (Γ, Γ') is stable at n.

d) Similarly, if n is odd, $\Gamma(n-1) = A_n$ then $\Gamma'(n-1) = A_n$, $M_1' \cap M_{n+1} \simeq M' \cap M_n$ (via the antisymetry of $M' \cap M_{n+1}$) and all new summands of $M_1' \cap M_{n+1}$ will be related only to the vertex (summand) $n-1$. Thus $\Gamma(n) = \Gamma'(n)$. Since $\dim M_1' \cap M_{n+2} = \dim M' \cap M_{n+1} = \dim \text{ sp } M' \cap M_n e_{n+1} M' \cap M_n + 1$, from $\Gamma(n) = \Gamma'(n)$ it follows that $\dim M_1' \cap M_{n+2} = \dim \text{ sp } M_1' \cap M_{n+1} e_{n+2} M_1' \cap M_{n+1} + 1$, so that $\Gamma'(n+1)$ has just one more edge than $\Gamma'(n)$, related by a unique (i.e., multiplicity one) edge with the vertex $n-1$. Thus, $\Gamma(n+1) = \Gamma'(n+1)$ and (Γ, Γ') is stable at n. $\square$

By Theorem 4.5, the above observation yields:

4.8 Corollary. *Assume that a standard graph* Γ *satisfies one of the conditions 4.7.c) or 4.7 d). Then* $w(n)/w(n+1) = P_j(\lambda)/\lambda P_{j+1}(\lambda)$*,for some* j*, and if* Γ^0 *is the graph obtained by adding an* A_j *tail to* $\Gamma(n+1)$ *then* (Γ^0, Γ^0) *is a standard pair.* $\square$

The above Corollary shows that if $\lambda^{-1} > 4, \Gamma(n-1) = A_n$ and at the vertex $n-1$ one has only three edges one of which has an endpoint, then Γ can only be a $T_{n,m}$ type graph (with the notation of [GHJ]), with finite m. Thus, the above result covers part of the recent result in [J2]. By using the result of [H1], which shows that in fact $(T_{n,m}, T_{n,m})$ cannot be a standard pair, one actually recovers [J2] in its full generality.

4.9. Corallary. *Let* $N \subset M$ *be a type* II_1 *factor with index* $[M : N] > 4$ *and graph* $\Gamma = \Gamma_{N,M}$. *If* $\Gamma(n-1) = A_n$ *and* $\Gamma(n)$ *has a triple point then* $\Gamma(n+1)$ *has at least two edges more than* $\Gamma(n)$, *i.e.,*

References

[Bi] D. Bisch, *On the structure of finite depth subfactors*, preprint (1992).

[Bo] F. Boca, *On the method of constructing irreducible finite index subfactors of Popa*, to appear in Pac. J. of Math..

[GHJ] F. Goodman, P. de la Harpe, V.F.R. Jones, *Coxeter graphs and towers of algebras*, MSRI Publ. 14, Springer Verlag, Berlin, 1989.

[H1] U. Haagerup, *Principal graphs of subfactors in the index range* $4 < [M : N] < 3 + \sqrt{2}$.

[H2] U. Haagerup, *In preparation.*

[Iz] M. Izumi, *Applications of fusion rules to classification of subfactors*, Publ. RIMS Kyoto Univ. **27** (1991), 953-994.

[IzKa] M. Izumi, Y. Kawahigashi, *Classification of subfactors with principal graph* $D_n^{(1)}$.

[J1] V.F.R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1-25.

[J2] V.F.R. Jones, *In preparation.*

[J3] V.F.R. Jones, *An affine Hecke algebra quotient in the Brauer algebra*, preprint (1994).

[L] R. Longo, *Index of subfactors and statistics of quantum fields I*, Commn. Math. Phys. **130** (1989), 217-247.

[Oc] A. Ocneanu, *Quantized groups, string algebras and Galois theory for von Neumann algebras*, in:Op.Alg. and Apl. London Math. Soc. Lect. Notes Series **136** (1988), 119-172.

[PiPo1] M. Pimsner, S. Popa, *Entropy and index for subfactors*, Ann. Sci. Ec. Norm. Sup. **19** (1986), 57-106.

[PiPo2] M. Pimsner, S. Popa, *Iterating the basic construction*, Trans. Amer. Math Soc. **310** (1988), 127-133.

[Po1] S. Popa, *Classification of subfactors of type II*, Acta Math. **172 No.2** (1994), 252-345.

[Po2] S. Popa, *Classification of subfactors and of their endomorphisms*, CBMS **Lecture notes** (1994).

[Po3] S. Popa, *Markov traces on universal Jones algebras and subfactors of finite index*, Invent. Math. **111** (1993), 375-405.

[Po4] S. Popa, *Free independent sequences in type* II_1*, factors and related problems*, to appear in Asterique.

[Po5] S. Popa, *Some ergodic properties for infinite graphs associated to subfactors*, preprint 1994.

[Po6] S. Popa, *Summetric enveloping algebras amenability and AFD properties for subfactors.*

[Ra] F. Radulescu, *Random matrices, amalgamated free products and subfactors of the von Neumann algebra of the free group*, Invent. Math. (1994).

[Sc] J. Schon, *Commuting squares and index for subfactors*, thesis (1990).

[Su] S. Sunder, *Hypergroups.*

Sorin Popa, Department of Mathematics, UCLA, Los Angeles, CA 90024, USA
E-mail address: `popa@math.ucla.edu`