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for the Mass of the Susy Kink**

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# Heat kernels and zeta-function regularization for the mass of the susy kink

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## Abstract

We apply zeta-function regularization to the kink and susy kink and compute its quantum mass. We fix ambiguities by the renormalization condition that the quantum mass vanishes as one lets the mass gap tend to infinity while keeping scattering data fixed. As an alternative we write the regulated sum over zero point energies in terms of the heat kernel and apply standard heat kernel subtractions. Finally we discuss to what extent these procedures are equivalent to the usual renormalization conditions that tadpoles vanish.

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# 1 Introduction

In the 1970's and 1980's the problem of how to compute quantum corrections to the mass of 1+1 dimensional solitons received wide attention [1]<sup>5</sup>. These soliton models could be made supersymmetric (susy), and various methods to compute the quantum mass were developed (see [3] and references in [4], [5] and [6]). It turned out that if one repeated for the susy kink exactly the same steps as performed in [1] for the bosonic case, the results for the quantum mass of the susy kink depended both on the regularization method and the choice of boundary conditions. In 1997, in view of renewed interest in extended objects due to dualities in string theory, the issue of why different methods and different boundary conditions for the quantum fields gave different answers for the quantum mass  $M$ , and the central charge  $Z$  in the susy cases, was reopened [4]. It was found that two of the most prominent regularization schemes used for these problems, mode number cut-off and energy (=momentum) cut-off, gave different answers; imposing different boundary conditions yielded different answers as well. As we shall review below, the reasons for these differences are by now well-understood, and both schemes yield the same correct answers provided they are applied correctly.

In this paper we apply another prominent regularization scheme to this problem: zeta function regularization [7]. Our aim is not to settle the question what the correct mass of the kink or susy kink is, but rather to show in a clear and simple example how to use zeta function regularization in problems with boundaries. We start from the regulated sum  $\sum \omega_n^{1-2s}$  where  $s$  is a real positive number. To remove afterwards the pole  $1/s$  we use two subtraction schemes often employed in non-renormalizable field theories (note however that here we are dealing with a renormalizable theory). The first scheme comes from Casimir energy calculations [8]. One just subtracts the singularities (including the constant term) in the limit of large mass (to be discussed below). The second scheme, widely used in quantum calculations on curved backgrounds, requires subtraction of the contributions from the first several heat kernel coefficients [9]. The heat kernel formalism has been used in quantum theory since works by Fock [10] and Schwinger [11]. Given recent advances of the technique [12, 13], this scheme becomes a rather universal tool for quantum calculations in external fields. In this paper we demonstrate that the two schemes are equivalent: large mass asymptotics is the same as expansion in the proper time of the heat kernel. Moreover, in two-dimensional renormalizable field theories they are equivalent to the requirement that tadpoles are absent, as we prove in Appendix B.

Closer study of various regularization schemes for the kink and the susy kink in the period after 1997 revealed that the difference in answers was due

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<sup>5</sup>For general reviews of the quantization of solitons see [2].

first of all to the fact that by evaluating sums over zero point energies one included boundary energies for most of the boundary conditions considered [5]. For particular boundary conditions the boundary energy vanished, and for these conditions mode regularization gave the correct results. The realization that for the susy kink boundary energy should be subtracted suggested a refinement of this scheme: one should average over (quartets of) boundary conditions such that in the averaging the boundary energies would cancel [14]. For the energy cut-off scheme, on the other hand, it was realized that instead of an abrupt cut-off at some maximal energy one should use a smooth cut-off, and this resolved the problem with the quantum mass both of the ordinary and of the susy kink [15, 14]. The mass of the susy kink has also been calculated by scattering theory methods [6], which were applied to the bosonic case earlier in [16]. (In [16] even a renormalization prescription based on zeta function regularization was given. In this paper we shall justify this prescription.) Also some other schemes for evaluating the sums over zero-point energies were developed which remove boundary energies, in particular a method based on first evaluating the  $\partial/\partial m$  derivative of the sum (where  $m$  is the meson mass) and then integrating this result w.r.t.  $m$ , using the renormalization condition  $M^{(1)}(m = 0) = 0$  [17].

For the reader who may wonder whether there really is a problem in determining the quantum correction to the kink mass by evaluation of  $\sum \frac{1}{2}\hbar\omega_n$ , we give an example which should remove any doubt, and which also plays a role below. One can impose susy boundary conditions such that for every nonvanishing bosonic solution  $\omega_n^b$  there is a corresponding fermionic solution with the same eigenvalue,  $\omega_n^f = \omega_n^b$ . The one-loop corrections to the susy kink are given by

$$M^{(1)} = \left( \sum \frac{1}{2}\hbar\omega_n^b - \sum \frac{1}{2}\hbar\omega_n^f \right) - \left( \sum \frac{1}{2}\hbar\omega_n^{b(0)} - \sum \frac{1}{2}\hbar\omega_n^{f(0)} \right) + \Delta M \quad . \quad (1)$$

Namely, one subtracts the contributions from the trivial sector, denoted by  $(o)$ , from those of the topological sector, and one must add a mass counterterm  $\Delta M$ , the finite part of which is fixed by requiring the absence of tadpoles in flat space (or far from the kink). Clearly, all bosonic sums cancel corresponding fermionic sums, but as  $\Delta M$  is nonvanishing and divergent, even in the susy case, this procedure gives a divergent answer for  $M^{(1)}$ . One can explicitly evaluate the energy density near the boundary for susy boundary conditions, and finds then that it is also divergent, giving as the boundary contribution  $\Delta M - M^{(1)}$ , where  $M^{(1)}$  is the correct value of the mass of the susy kink [18]. Subtracting the boundary energy from the total energy then gives the correct result. Thus the global answer actually is correct for the entire system with susy boundary conditions, but it is insufficient for obtaining the mass of the isolated kink.

By directly evaluating local energy density one can obtain the correct value for the quantum mass by integrating this density around the kink. The first such method was developed in [5] where a higher-derivative action was used which preserves susy and is canonical (in the sense of no higher time derivatives). This scheme gave a result for the quantum mass which agreed with the results of Schonfeld [3], the  $\partial/\partial m$  method, and the phase shift method. For the susy sine-Gordon model, it also agrees with the results obtained from the Yang-Baxter equation. Another scheme to evaluate both the local energy density and the local central charge density employs the concept of local mode regularization [19]. This method also yields the correct result.

After this summary of previous work done on the calculations of the quantum mass of the susy kink, we come to a new development: the application of zeta-function regularization to this problem. One converts the sum  $\frac{1}{2} \sum \hbar \omega_n^{1-2s}$  into an integral by using a function which has poles in  $k$  corresponding to each solution  $\omega_n$ . One can also convert each term  $\omega_n^{1-2s}$  in the sum into an integral by using the gamma function  $\Gamma(s)$ . By writing the sum of these integrals as an integral over the sum, one encounters the heat kernel, about which an enormous literature exists. We shall give a simple self-contained account directly applied to the case of the kink and susy kink.

In section 2 we discuss the bosonic kink with Dirichlet boundary conditions and use the large mass subtraction scheme. In section 3 we consider the susy kink and use the heat kernel subtraction scheme. In section 4 we summarize the methods. In Appendix A we extend the large mass subtraction scheme of section 2 to Robin boundary conditions. As a by-product we derive a simple relation between the contributions to the vacuum energy from the boundaries and from the bound states. In Appendix B we show that these subtraction schemes are equivalent to absence of tadpoles.

## 2 Large mass method for the bosonic kink

The principle of zeta-function regularization of a sum  $\frac{1}{2} \sum \omega_n$  (we set  $\hbar = 1$ ) is to replace this sum by

$$E^{(0)} = \frac{1}{2} \sum_n (\omega_n^2 + \mu^2)^{1-2s} + \frac{1}{2} \sum_i (-\kappa_i^2 + m^2)^{\frac{1}{2}} \quad , \quad (2)$$

where  $\omega_n = \sqrt{k_n^2 + m^2}$  are the zero point energies for the continuum spectrum of the quantum fields in a kink background (discretized by putting the kink system in a box with length  $L$ ), and  $-\kappa_i^2 + m^2 = \omega_{B,i}^2$  are the squared energies of the discrete spectrum, namely the bound states and the zero mode. For the kink there is one bound state. The real number  $s$  is positive and large

enough to make the sum convergent, and  $\mu^2$  is a small mass temporarily introduced for the treatment of the zero mode (it will be used to move the pole due to the zero mode away from the starting point of a branch cut). Later we set  $\mu$  to zero, and consider the asymptotics of  $E^{(0)}$  for  $m \rightarrow \infty$  while keeping all scattering data  $\kappa_i$  and  $k_n$  fixed. Our renormalization prescription will be to subtract all non-negative powers of  $m$  in this limit (together with  $\ln m$  terms). We shall see that this indeed gives a finite result for  $E^{(0)}$ .

We first convert the continuum sum into an integral

$$E_{\text{cont}}^{(0)} = \frac{1}{2} \oint \frac{dk}{2\pi i} (k^2 + M^2 + \mu^2)^{\frac{1}{2}-s} \frac{\partial}{\partial k} \ln \phi(k, m) . \quad (3)$$

The function  $\phi(k, m)$  has zeroes at  $k = k_n$  such that  $k_n^2 + m^2 = \omega_n^2$ . The integration contour runs anti-clockwise and consists of one branch at  $k = \Re k + i\epsilon$ , a second branch at  $k = \Re k - i\epsilon$ , and a small segment  $-\epsilon \leq \Im k \leq \epsilon$  along the imaginary axis.

As the mass parameter  $m$  may also appear in  $\kappa_i$  and  $k_n$ , the function  $\phi(k, m)$  depends also on  $m$ . This  $m$  should be kept fixed in the asymptotic limit. Therefore, we have replaced  $m^2$  in  $\omega_n^2 = k_n^2 + m^2$  by  $M^2$ . We also replace  $m^2$  by  $M^2$  in the sum over bound states. The asymptotic limit we need is  $M \rightarrow \infty$ . The values of  $\omega_n$  depend on the boundary conditions, and the only place where boundary conditions enter in the calculations is in the choice of  $\phi$ .

We consider the supersymmetric Lagrangian

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu \Phi) (\partial^\mu \Phi) - \frac{1}{2} U^2(\Phi) - \frac{1}{2} \bar{\psi} (\not{\partial} + U') \psi \quad (4)$$

with polynomial superpotential

$$U = \sqrt{\frac{\lambda}{2}} \left( \Phi^2 - \frac{m^2}{2\lambda} \right) . \quad (5)$$

The kink solution reads  $\Phi = (m/\sqrt{2\lambda}) \tanh(mx/2)$ . Here  $m$  is the meson mass, and  $\lambda$  is the (dimensionful) coupling constant.

In this section we consider the bosonic case, and impose Dirichlet boundary conditions<sup>6</sup>  $\eta(-L/2) = \eta(L/2) = 0$  on the scalar field fluctuations  $\eta$ . Then  $\phi(k, m)$  is given by

$$\phi(k, m) = \sin(kL + \delta(k)) \quad (6)$$

where the phase shift  $\delta(k)$  is given by

$$\delta(k) = -2 \arctan \frac{3mk}{m^2 - 2k^2} .$$

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<sup>6</sup>Robin boundary conditions are considered in Appendix A.

Along the first part of the contour we can approximate  $\sin(kL + \delta(k))$  by  $-\frac{1}{2i} \exp(-ikL - i\delta(k))$  because the term with  $\exp(ikL + i\delta(k))$  vanishes as  $L \rightarrow \infty$ . Along the second part of the contour we retain  $-\frac{1}{2i} \exp(ikL + i\delta(k))$ . We obtain then

$$E_{cont}^{(0)I+II} = \frac{1}{2} \int_0^\infty \frac{dk}{2\pi i} (k^2 + M^2 + \mu^2)^{\frac{1}{2}-s} \frac{\partial}{\partial k} [(ikL - \ln e^{-i\delta(k)}) + (ikL + \ln e^{i\delta(k)})] .$$

The contribution from the third part of the contour is independent of the phase shift, and hence it is dropped. Although  $k = 0$  is formally a solution of the equation  $\phi(k, m) = 0$ , the corresponding mode is identically zero, and so the contribution from the pole at  $k = 0$  must be dropped; this is a peculiar property of Dirichlet boundary conditions. The terms proportional to  $L$  are independent of the phase shift and equal to the contribution from the trivial vacuum, and as usual we subtract them. For future use we replace  $\delta(k)$  by  $-\delta(-k)$  in the second term. In order to be able to estimate the limit  $M^2 \rightarrow \infty$  of the integral, it is useful to make a Wick rotation. The first part of the contour is rotated to the positive imaginary axis ( $k = i\kappa$  with real positive  $\kappa$ )

$$E_{cont}^{(0),I} - L \text{ term} = \frac{1}{2} \int_0^\infty \frac{d\kappa}{2\pi i} e^{i\pi(\frac{1}{2}-s)} [\kappa^2 - M^2 - \mu^2]^{\frac{1}{2}-s} \frac{\partial}{\partial \kappa} [-\ln e^{-i\delta(i\kappa)}] .$$

The second part of the contour is rotated downwards ( $k = -i\kappa$ , again with real positive  $\kappa$ )

$$E_{cont}^{(0),II} - L \text{ term} = \frac{1}{2} \int_0^\infty \frac{d\kappa}{2\pi i} e^{-i\pi(\frac{1}{2}-s)} [\kappa^2 - M^2 - \mu^2]^{\frac{1}{2}-s} \frac{\partial}{\partial \kappa} [\ln e^{-i\delta(i\kappa)}] .$$

For  $\kappa > M$  the sum of both terms contains the factor

$$-e^{i\pi(\frac{1}{2}-s)} + e^{-i\pi(\frac{1}{2}-s)} = -2i \cos \pi s .$$

For  $\kappa < M$  the two integrals (almost) cancel each other so that only contributions of the poles in  $\delta(k)$  at imaginary  $k$  need to be studied further.

$$E_{cont}^{(0),I+II} - L \text{ terms} = -\frac{1}{2} \int_{\sqrt{M^2+\mu^2}}^\infty \frac{d\kappa}{\pi} \cos \pi s (\kappa^2 - M^2 - \mu^2)^{\frac{1}{2}-s} \left[ -i \frac{\partial}{\partial \kappa} \delta(i\kappa) \right] + \text{pole contributions.} \quad (7)$$

We now use an alternative form of the phase shift<sup>7</sup>

$$\delta(k) = i \left[ \ln \frac{k - im/2}{k + im/2} + \ln \frac{k - im}{k + im} \right] .$$

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<sup>7</sup>This is a particular case of the more general formula (46) valid for all reflectionless potentials.

It is clear from this representation that the  $\kappa$  integral has poles at  $k = \pm im/2$  and  $k = \pm im$ . We now see the reason for adding  $\mu^2$  to  $m^2$  in (2): it avoids that the starting point of the cut at  $k = iM$  coincides with the pole at  $k = im$  (which is due to the zero mode). Each of the two contours contains a half-circle around these poles, but by combining these two  $\kappa$  contours from  $0 \leq \kappa \leq M$ , one obtains the two residues at the poles. These two residues precisely cancel the terms with bound states in the original sum. Hence we have arrived at an expression for the quantum mass which is expressed as an integral over only  $m \leq \kappa \leq \infty$ , and the bound state contributions no longer appear explicitly.<sup>8</sup> At this point we can put  $\mu = 0$ .

The expression for the quantum energy reduces to

$$E^{(0)} = -\frac{1}{2} \int_M^\infty \frac{d\kappa}{\pi} (\cos \pi s) (\kappa^2 - M^2)^{\frac{1}{2}-s} \frac{\partial}{\partial \kappa} \left[ \ln \frac{\kappa - m/2}{\kappa + m/2} + \ln \frac{\kappa - m}{\kappa + m} \right] .$$

We rewrite the expression in square brackets as a sum of the limit of this expression for  $\kappa \rightarrow \infty$  and a remainder

$$\ln \frac{\kappa - m/2}{\kappa + m/2} + \ln \frac{\kappa - m}{\kappa + m} = \left( -\frac{m}{\kappa} - \frac{2m}{\kappa} \right) + \left( \ln \frac{\kappa - m/2}{\kappa + m/2} + \ln \frac{\kappa - m}{\kappa + m} + \frac{3m}{\kappa} \right) .$$

With this decomposition inserted into the integral, the first term is explicitly evaluated for nonvanishing  $s$ , whereas in the second term we may set  $s = 0$ . This yields

$$\begin{aligned} E_{(1)}^{(0)} &= -\frac{1}{2} \int_M^\infty \frac{d\kappa}{\pi} \cos \pi s (\kappa^2 - M^2)^{\frac{1}{2}-s} \frac{3m}{\kappa^2} \\ &= -\frac{\cos \pi s}{4\pi} (3m) M^{-2s} \left[ \frac{\Gamma(3/2 - s)\Gamma(s)}{\Gamma(3/2)} \right] , \end{aligned}$$

$$E_{(2)}^{(0)} = -\frac{1}{2} \int_M^\infty \frac{d\kappa}{\pi} \sqrt{\kappa^2 - M^2} \left[ \frac{m}{\kappa^2 - m^2/4} + \frac{2m}{\kappa^2 - m^2} - \frac{3m}{\kappa^2} \right] .$$

For  $s \rightarrow 0$  the term  $E_{(1)}^{(0)}$  contains a pole  $1/s$  as well as finite contributions proportional to  $M^0$  and  $\ln M$ . According to our prescription this entire term must be discarded. The term  $E_{(2)}^{(0)}$  vanishes in the limit  $M \rightarrow \infty$  and, therefore, it must be kept in its entirety. A direct evaluation of this term<sup>9</sup> at

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<sup>8</sup>In other approaches one finds the quantum mass expressed in terms of only the energies of the bound states; see, e.g., [19] .

<sup>9</sup>It is convenient to represent the numerator of the 2nd term in square brackets as  $2m = -m + 3m$  and then combine these two parts with the 1st and the 3rd terms respectively. Each of the two integrals resulting from this procedure is convergent and can be easily calculated.

$M = m$  yields then the quantum mass of the bosonic kink

$$M_b = m \left( \frac{1}{4\sqrt{3}} - \frac{3}{2\pi} \right). \quad (8)$$

This is the correct result [4, 5, 6] first found in [1]. The term with  $-3/2\pi$  comes from the pole due to the zero mode, while the bound state contributes the term with  $1/4\sqrt{3}$ .

### 3 Heat kernel methods for the susy kink

We now consider the susy kink. The classical kink solution satisfies  $\partial_1 \phi_K + U' = 0$ , and this background solution is invariant under rigid susy with parameter  $\epsilon^-$ , as

$$\begin{aligned} \delta\psi^+ &= \partial_1 \phi \epsilon^+ - \partial_t \phi \epsilon^- - U \epsilon^+ \\ \delta\psi^- &= -\partial_1 \phi \epsilon^- + \partial_t \phi \epsilon^+ - U \epsilon^- . \end{aligned}$$

Note also  $\delta\phi = -i\epsilon^+ \psi^- + i\epsilon^- \psi^+$ . There are two sets of supersymmetric boundary conditions. The first set reads

$$\begin{aligned} \eta|_{\partial M} &= 0, & \psi_+|_{\partial M} &= 0, \\ (\partial_1 - U'(\phi))\psi_-|_{\partial M} &= 0 . \end{aligned} \quad (9)$$

The second set is

$$\begin{aligned} (\partial_1 + U'(\phi))\eta|_{\partial M} &= 0, & \psi_-|_{\partial M} &= 0, \\ (\partial_1 + U'(\phi))\psi_+|_{\partial M} &= 0 . \end{aligned} \quad (10)$$

For the spinor field it is sufficient to give the Dirichlet boundary conditions for one component. The Robin boundary conditions for the other component are then determined by the Dirac equation, which is equivalent to the following equations on the spinor field components

$$(\partial_1 + U')\psi_+ + \partial_0 \psi_- = 0 \quad , \quad (11)$$

$$(-\partial_1 + U')\psi_- - \partial_0 \psi_+ = 0 \quad . \quad (12)$$

If we require that the  $\psi_+$  component satisfy the Dirichlet boundary condition (the 2nd equation in (9)), equation (12) then yields the Robin boundary condition for  $\psi_-$  (2nd line in (9)). The same is also true for (10). We need these boundary condition for the other spinor component as well because the heat kernel method uses second-order differential operators, so that we must exclude spurious solutions.

To discuss the cancelation of the bosonic and fermionic contributions to the vacuum energy, it is convenient to introduce the operators

$$Q_{\pm} = \partial_x \pm U' \quad (13)$$

and build from them the second-order differential operators

$$\Delta_- = Q_+ Q_-, \quad \Delta_+ = Q_- Q_+ \quad . \quad (14)$$

The eigenfrequencies of the bosonic fluctuations and those of  $\psi_+$  are defined by the operator  $\Delta_+$ , but the eigenfrequencies for  $\psi_-$  are defined by  $\Delta_-$ . The operators  $\Delta_+$  and  $\Delta_-$  satisfy the following intertwining relations

$$\Delta_- Q_+ = Q_+ \Delta_+, \quad Q_- \Delta_- = \Delta_+ Q_- \quad . \quad (15)$$

Moreover, the boundary conditions (9) and (10) are compatible with these relations: if  $\psi_+$  is an eigenfunction of  $\Delta_+$  which vanishes at the boundary, then  $Q_+ \psi_+$  is an eigenfunction of  $\Delta_-$  which obeys the second condition in (9). Therefore, the two operators involved are isospectral (up to the zero modes which do not contribute to the vacuum energy). This shows a complete compensation of the regulated sum of the zero point energies in the bosonic and fermionic sector.

The property of isospectrality is crucial in our discussion. Nonsupersymmetric boundary conditions in principle also can be handled with heat kernel methods, but the treatment is more complicated.

Each operator  $\Delta_{\pm}$  can be represented as  $\Delta_{\pm} = \partial_1^2 - m^2 - V_{\pm}$ , where  $m^2$  is chosen in such a way that  $V_{\pm}$  vanishes as  $x^1 \rightarrow \pm\infty$ . The potentials then read  $V_{\pm} = U'^2 \pm U''U - m^2$ . From now on we drop the subscript  $\pm$  where this cannot lead to confusion. Let  $k_n^2 + m^2$  be the eigenvalues of  $\Delta$ . In zeta functional regularization the regularized vacuum energy has the form

$$E_0 = \frac{\tilde{\mu}^{2s}}{2} \sum_n (k_n^2 + M^2)^{\frac{1}{2}-s} \quad , \quad (16)$$

where  $\tilde{\mu}$  is an arbitrary parameter with the dimension of a mass which is needed to restore the correct dimension of  $E_0$ . Again, we have replaced the mass  $m$  in the spectrum by  $M$  to be able to consider later the limit  $M \rightarrow \infty$  without deforming the potential  $V$ . The mass parameter  $m$  which enters the susy boundary conditions (9) and (10) through the superpotential  $U$  also must be kept fixed in the limit  $M \rightarrow \infty$ . Values of  $\epsilon_n$  do not depend on  $M$  and are the same in the fermionic and bosonic sectors (up to the zero mode which must be excluded explicitly<sup>10</sup>). Therefore, the bosonic contribution to

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<sup>10</sup>Although for large  $M$  the term  $(k_n^2 + M^2)$  in the sum in (16) is always non-zero, keeping the zero modes in (16) would give an incorrect result for the susy kink mass.

the vacuum energy cancels the fermionic contribution for all values of  $M$ . Consequently, in the large  $M$  subtraction scheme both  $E_0$  and  $E_0^{\text{div}}$  are zero for the susy kink. This gives zero value for the total energy of the susy kink and the boundaries. As we shall see below, this is the correct result. In order to introduce the heat kernel subtraction scheme we first consider an individual contribution (bosonic or fermionic) to the regularized vacuum energy. We write

$$\sum_n (k_n^2 + M^2)^{\frac{1}{2}-s} = \int_0^\infty \frac{dt}{t} \frac{t^{-\frac{1}{2}+s}}{\Gamma(s - \frac{1}{2})} K(t) e^{-tM^2} \quad , \quad (17)$$

where

$$K(t) = \sum_n e^{-tk_n^2} \quad (18)$$

is the heat kernel for this problem. Its asymptotic expansion for  $t \rightarrow 0$ ,

$$K(t) \sim \sum_{n \geq 0} a_n t^{n-\frac{1}{2}} \quad (19)$$

is equivalent to the expansion of  $E_0$  for  $M \rightarrow \infty$ . The ultraviolet divergences are contained in the contributions with  $n = 0, \frac{1}{2}, 1$  which correspond to the nonnegative powers of  $M$ . From this we define the divergent part of the vacuum energy as

$$E_0^{\text{div}} \equiv \frac{\tilde{\mu}^{2s}}{2} \int_0^\infty \frac{dt}{t} \frac{t^{-\frac{1}{2}+s}}{\Gamma(s - \frac{1}{2})} \sum_{n=0}^1 a_n t^n e^{-tM^2} =$$

$$\frac{\tilde{\mu}^{2s}}{2\Gamma(s - \frac{1}{2})} \left\{ a_0 \Gamma(s - 1) M^{2-2s} + a_{\frac{1}{2}} \Gamma\left(s - \frac{1}{2}\right) M^{1-2s} + a_1 \Gamma(s) M^{-2s} \right\}. \quad (20)$$

We define the renormalized vacuum energy as

$$E_0^{\text{ren}} = E_0 - E_0^{\text{div}} \quad (21)$$

at  $s = 0$  and  $M = m$ . For  $n$  larger than unity, the coefficients  $a_n$  are multiplied by negative powers of  $M$ , so that the subtraction of the leading heat kernel asymptotics for small  $t$  is equivalent to the subtraction of the leading large mass asymptotics. In Appendix B it is demonstrated that for renormalizable theories these two procedures are equivalent to imposing the condition that tadpoles vanish.

To obtain the kink mass, we must determine the contributions to the heat kernel from the boundary. This requires a detailed discussion of the

heat kernel coefficients<sup>11</sup>. To write down these coefficients we need some new notations. Let us introduce the boundary operators  $\mathcal{B}^{R,D}$  acting on a field  $\varphi$  (either bosonic or fermionic)

$$\mathcal{B}^D \varphi = \varphi \quad , \quad \mathcal{B}^R \varphi = (\partial_N + S)\varphi \quad . \quad (22)$$

Here  $\partial_N$  denotes partial derivative with respect to an inward pointing unit vector. The field  $\varphi$  satisfies the field equation  $(-\partial_x^2 + V)\varphi_n = \epsilon_n^2 \varphi_n$ .

The asymptotic expansion of the heat kernel  $K(t)$  as  $t \rightarrow 0$  for the boundary condition  $\mathcal{B}\varphi|_{\partial M} = 0$  (where  $\mathcal{B}$  can be  $\mathcal{B}^D$  or  $\mathcal{B}^R$ ) reads

$$K(t) = \sum_{n=0, \frac{1}{2}, 1, \dots} t^{n-\frac{1}{2}} a_n(\mathcal{B}^{R,D}) \quad , \quad (23)$$

with [12]

$$\begin{aligned} a_0(\mathcal{B}^{R,D}) &= (4\pi)^{-\frac{1}{2}} \int_M dx \quad , \\ a_{\frac{1}{2}}(\mathcal{B}^R) &= \frac{1}{4} \int_{\partial M} dx \quad , \quad a_{\frac{1}{2}}(\mathcal{B}^D) = -\frac{1}{4} \int_{\partial M} dx \quad , \\ a_1(\mathcal{B}^R) &= (4\pi)^{-\frac{1}{2}} \left( -\int_M V dx + 2 \int_{\partial M} S dx \right) \quad , \\ a_1(\mathcal{B}^D) &= -(4\pi)^{-\frac{1}{2}} \int_M V dx \quad . \end{aligned} \quad (24)$$

In the present (one-dimensional) case the boundary integral becomes a sum over the two boundary points.

For the fermionic sector we sum the contributions from  $\psi_+$  and  $\psi_-$ , and divide by a factor 2. It is easy to see that in the “total” heat kernel

$$a_n^{tot} = a_n^{bos} - a_n^{ferm} \quad (25)$$

the contribution of  $\psi_+$  always cancels one half the contribution of  $\eta$ .

Let the kink be centered at the origin, but the box lie between  $x = L_1$  and  $x = L_2$ . In all supersymmetric theories we have  $a_0^{tot} = 0$  independently of the background and the boundary conditions. This is due to the equality of the numbers of bosonic and fermionic degrees of freedom. The coefficient  $a_{\frac{1}{2}}$  simply counts the number of Dirichlet minus Robin boundary conditions in each set (9) or (10). Obviously,

$$a_{\frac{1}{2}}^{tot} = -\frac{1}{2} \quad (26)$$

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<sup>11</sup>The expansion (19) has quite recently been used in numerical calculations of the kink mass [20].

for the boundary conditions (9), and

$$a_{\frac{1}{2}}^{tot} = \frac{1}{2} \quad (27)$$

for the boundary conditions (10). However, we must set  $a_{\frac{1}{2}} = 0$  for the following reason. The operator  $\Delta_+$  (resp.  $\Delta_-$ ) has a zero mode which satisfies the Robin boundary condition  $Q_+\psi_+ = 0$  (resp.  $Q_-\psi_- = 0$ ). Therefore, there is one zero mode in the fermionic sector for the boundary conditions (9). (There are of course no zero modes in the bosonic sector because we use Dirichlet boundary conditions there). This zero mode contributes  $-\frac{1}{2}$  to the heat kernel coefficient. Zero mode contributions must be explicitly subtracted from the vacuum energy. A similar mechanism works also for the boundary conditions (10).

Let us now turn to calculation of  $a_1$ . The volume terms in  $a_1$  combine to give

$$\begin{aligned} a_1^{tot}[\text{vol}] &= \frac{1}{2}(4\pi)^{-\frac{1}{2}} \int_M (-V_+ + V_-) dx \\ &= \frac{1}{2}(4\pi)^{-\frac{1}{2}} \int_{L_1}^{L_2} (-2UU'') dx \\ &= \frac{1}{2}(4\pi)^{-\frac{1}{2}} (-2U'(L_1) + 2U'(L_2)) \quad , \end{aligned} \quad (28)$$

where we have used the Bogomolny equation  $\partial_1\phi = -U(\phi)$ .

We next calculate contributions to  $a_1$  from  $\partial M$ . We have to cast the boundary conditions into the standard form (22). Note that  $\partial_1 = \partial_N$  at the left boundary, but  $\partial_1 = -\partial_N$  at the right boundary. Consequently, we have to set  $S = -U'$  at  $x = L_1$  and  $S = U'$  at  $x = L_2$  for  $\psi_-$  and the boundary conditions (9). For the boundary conditions (10)  $S$  assumes the opposite values, but now both  $\eta$  and  $\psi_+$  contribute. As a result, we obtain for (9) and (10) equal boundary contributions

$$a_1^{tot}[\text{boundary}] = \frac{1}{2}(4\pi)^{-\frac{1}{2}} (2U'(L_1) - 2U'(L_2)) \quad (29)$$

which exactly cancel the volume term (28) giving

$$a_1^{tot} = 0 \quad . \quad (30)$$

We conclude that all singularities as well as the finite part of the contributions to the vacuum energy cancel. Thus, according to (21), not only  $E_0$  but also the renormalized energy  $E_0^{\text{ren}}$  of the kink plus boundaries vanishes.

Qualitatively, it is clear what has happened. The vacuum energy associated with the kink has been cancelled by the vacuum energy associated with

the boundaries. Therefore, to obtain the correction to the mass of an isolated kink we have to subtract from the above vanishing result the Casimir energy of the boundaries as they are being moved to infinity.

Of course, the kink solution cannot be smoothly deformed to the trivial one. We can, however, replace all background fields in the wave operator for the scalar field fluctuations and in the squared Dirac operator by their asymptotic values ( $V = 0$ ), and replace  $U'$  in (10) by  $+m$  at  $x = L/2$  and  $-m$  at  $x = -L/2$ . In this way, we arrive at the problem of calculating the vacuum energy for free scalar and spinor fields with mass  $m$ . We consider the boundary conditions (10) because, as we shall later explain, the calculations with (9) are more complicated. The boundary conditions for the free fields read

$$\begin{aligned} (\partial_N - m)\eta|_{\partial M} &= 0, & \psi_-|_{\partial M} &= 0, \\ (\partial_N - m)\psi_+|_{\partial M} &= 0, \end{aligned} \quad (31)$$

where  $\partial_N = \partial_1$  on the left boundary  $x^1 = -L/2$ , and  $\partial_N = -\partial_1$  at  $x^1 = L/2$ . Obviously, the bosonic and fermionic contributions partially cancel each other. For (31) the vacuum energy  $E^{[B]}$  associated with the boundaries become

$$E^{[B]} = \frac{1}{2}(E^{[R]} - E^{[D]}), \quad (32)$$

where the superscripts  $[R]$  and  $[D]$  stand for Robin and Dirichlet boundary conditions respectively.

Let us start with the Robin sector. The eigenfunctions of the operator  $-\partial_1^2 + m^2$

$$\varphi_k = A_k \sin(2kx/L) + B_k \cos(2kx/L) \quad (33)$$

satisfy the Robin boundary conditions in (31) if

$$\frac{2k}{L} \cos k + m \sin k = 0 \quad \text{and} \quad B_k = 0 \quad (34)$$

or if

$$-\frac{2k}{L} \cos k + m \sin k = 0 \quad \text{and} \quad A_k = 0 \quad . \quad (35)$$

It is easy to see that these two equations on the wave numbers  $k$  in (34) and (35) are equivalent to a single equation,

$$f_R(k) = \sin(2(k + \delta_R(k))) = 0 \quad \text{with} \quad \delta_R(k) = \arctan(k/m) \quad . \quad (36)$$

We can use the integral representation (3) for the regularized vacuum energy

$$E_0^{[R]} = \frac{\tilde{\mu}^{2s}}{2} \oint \frac{dk}{2\pi i} \left( \frac{4k^2}{L^2} + m^2 \right)^{\frac{1}{2}-s} \frac{\partial}{\partial k} \ln f_R(k) \quad , \quad (37)$$

where the contour again encircles the real positive semi-axis.

Clearly, the same representation is also valid for  $E_0^{[D]}$  with  $\delta_D(k) = 0$ . The contour rotation proceeds in exactly the same way as before (although there are no bound states in this case).

After rather elementary (but somewhat lengthy) calculations we obtain in the limit  $L \rightarrow \infty$

$$E^{[B]}(s) = \frac{1}{2} \left( \frac{\tilde{\mu}^{2s} \Gamma(s) m^{1-2s}}{2\sqrt{\pi} \Gamma\left(s + \frac{1}{2}\right)} + \frac{1}{2} m \right) + \dots \quad (38)$$

Again we make the heat kernel expansion of the energy in the trivial sector (which is only located on the boundaries). Because the boundary conditions (which were susy in the kink sector) break susy in the trivial sector, both  $E_0$  and  $E_0^{\text{div}}$  are nonvanishing in this case. From the equations (24) we immediately obtain

$$a_0^{[B]} = 0, \quad a_{\frac{1}{2}}^{[B]} = \frac{1}{2}, \quad a_1^{[B]} = -\frac{m}{\sqrt{\pi}}. \quad (39)$$

These coefficients define the divergent part of the vacuum energy (20). The boundary energy is the difference of the total result in the boundary energy in (38) and the divergence in (20)

$$E^{[B]\text{ren}} = \frac{m}{2\pi}. \quad (40)$$

The one-loop correction  $M^{(1)}$  to the kink mass is the difference between the one-loop vacuum energy of the kink on a finite interval (which is zero) and the vacuum energy (40) associated with the boundaries. Thus we obtain

$$M^{(1)} = -\frac{m}{2\pi}. \quad (41)$$

This is the correct result.

Direct application of the same method to the other set of the boundary conditions (9) yields in addition to the eigenmodes (33) for  $\psi_-$  also the eigenmodes of the form  $A_k \sinh(2kx/L) + B_k \cosh(2kx/L)$  which correspond to negative eigenvalues of  $-\partial_1^2$ . Their absolute values are large enough to make  $-\partial_1^2 + m^2$  negative. In this case one cannot replace  $U'(\phi)$  in the boundary conditions by its asymptotic values  $\pm m$  before taking the limit  $L \rightarrow \infty$ . Keeping  $L$ -dependent boundary conditions considerably complicates the calculation. This case is considered from a little bit different point of view in Appendix A.

## 4 Conclusions

We have illustrated in a simple concrete physical model how one may use zeta function regularization and heat kernel methods. The problem was not totally trivial because there were boundaries present which played a role in the trivial and the topological sector.

In the zeta function approach, one converted the sum of the modes into a complex contour integral. The renormalization condition which fixes the finite part of the zero point energy is that one must discard all terms which do not vanish in the limit when the meson mass  $M$  tends to infinity while keeping the “scattering data” (namely  $k_i$  and  $\kappa_i$ ) fixed. In our case these terms were proportional to  $M^0$  and  $\ln M$ . The remainder gave the correct mass of the bosonic kink.

In the approach which uses heat kernels, the sum over modes was converted into an integral over the proper time  $t$ , and the asymptotic expansion in  $t$  yielded the heat kernel coefficients  $a_n$ . We first computed the total energy in the susy kink sector, and defined its renormalization as subtraction of the contributions from  $a_0$ ,  $a_{\frac{1}{2}}$  and  $a_1$ . The result was found to vanish. This was to be expected because we used susy boundary conditions in the kink sector, so that the sum over bosonic modes should exactly cancel the sum over fermionic modes. Then we repeated this calculation in the trivial sector, but with the same boundary conditions as for large  $x$  in the topological sector. These boundary conditions are not susy in the trivial sector. As a result we found that the total energy and the renormalized energy were nonvanishing. The difference yielded the correct result for the susy kink. A nontrivial technical aspect of this approach was the role of Robin boundary conditions and zero modes in the heat kernel expansion.

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## Appendix A: Large mass subtraction scheme for Robin boundary conditions

In this Appendix we demonstrate how the large  $M$  subtraction scheme works for generic Robin boundary conditions. All necessary information on one-dimensional scattering theory can be found in [21]. Consider a reflectionless

potential  $V(x)$  localized around the point  $x = 0$  (and vanishing for  $x \rightarrow \pm\infty$ ). Before imposing any boundary condition we have two independent eigenfunctions  $\varphi_{1,2}(k, x)$  of the operator  $\Delta = \partial_1^2 - m^2 - V(x)$  for each  $k$ . Their asymptotics as  $x \rightarrow \pm\infty$  are

$$\varphi_1(k, x) \rightarrow \cos(kx \pm \delta/2) \quad , \quad \varphi_2(k, x) \rightarrow \sin(kx \pm \delta/2) \quad , \quad (42)$$

where  $\delta(k)$  is the phase shift defined by the potential  $V$ . We put the system in the box  $-L/2 \leq x \leq L/2$  and suppose that  $L$  is large enough so that near the boundaries the eigenfunctions can be represented by their asymptotics.

Consider the Robin boundary conditions

$$(\partial_x + S_1)\varphi|_{(x=-L/2)} = (-\partial_x + S_2)\varphi|_{(x=L/2)} = 0 \quad (43)$$

with arbitrary  $S_1$  and  $S_2$ . The general eigenmode of the operator  $\Delta$  is a linear combination of  $\varphi_1$  and  $\varphi_2$ :  $\varphi(k, x) = A\varphi_1(k, x) + B\varphi_2(k, x)$ . Substituting the asymptotics (42) of this eigenmode in the boundary condition (43) we arrive at the following equation which defines the spectrum of  $k$ :

$$0 = \phi(k) = \sin(kL + \delta + \alpha_1 + \alpha_2) \quad , \quad (44)$$

where

$$\alpha_{1,2} = -\arctan(k/S_{1,2}) \quad . \quad (45)$$

The vacuum energy is now defined by the equation (3) where we have to substitute the function  $\phi(k)$  given in (44) instead of  $\phi(k, m)$ . If we now repeat the same steps as in sec. 2 we arrive at the equation (7) where  $\delta$  must be replaced by  $\delta + \alpha$ ,  $\alpha = \alpha_1 + \alpha_2$ . The important property of the equation (7) is the linearity in  $\delta$ . Therefore, the regularized vacuum energy for arbitrary auxiliary mass  $M$  can be represented as a sum of the contribution from the potential  $V$  (containing  $\delta$  under the integral) and the boundary contribution (containing  $\alpha$ ). To obtain the mass of the kink alone we have to subtract the whole boundary contribution, retaining the  $\delta$ -term only. Hence we arrive at the old expression for the kink mass obtained for the Dirichlet boundary conditions. We conclude, that if the boundary contributions are subtracted the large  $M$  scheme gives equivalent results for the kink mass independently of the boundary conditions.

We conclude this Appendix by a useful relation between the contributions to the vacuum energy from the bound states and from the boundaries. For any reflectionless potential the phase shift can be represented as a sum over the bound states [21]

$$\delta(k) = i \sum_n \ln \frac{k - i\kappa_n}{k + i\kappa_n} = \sum_n 2 \arctan \left( \frac{\kappa_n}{k} \right) \quad , \quad (46)$$

where  $\kappa_n$  are the bound state energies. We can rewrite (45) in a similar form

$$\alpha = \alpha_1 + \alpha_2 = \arctan\left(\frac{S_1}{k}\right) + \arctan\left(\frac{S_2}{k}\right) - \pi \quad . \quad (47)$$

Because of the derivative  $\partial_k$  in (7) the constant term  $-\pi$  in (47) does not contribute to the vacuum energy. By comparing (46) and (47) we see that the vacuum energy associated with a boundary with Robin boundary conditions<sup>12</sup> is, roughly speaking, one half of the contribution of a bound state with the energy  $\kappa = S$ .

As an example, consider the supersymmetric boundary conditions (9). The bosonic sector is described by the potential  $V_+$  and contains two bound states with the energies  $\kappa_1 = m$  and  $\kappa_2 = m/2$ . There is no boundary contribution to the vacuum energy. The same is true for  $\psi_+$ . The other spinor component  $\psi_-$  corresponds to the potential  $V_-$  and has one bound state with  $\kappa_2 = m/2$ . For large  $L$  the Robin boundary conditions have  $S_1 = S_2 = m$ . These two boundaries add the same contribution to the vacuum energy as would come from the bound state with  $\kappa_1 = m$ . This restores the balance between the bosonic and fermionic contributions and gives zero total energy for the susy kink and the boundaries. A similar mechanism works for the boundary conditions (10), where  $S_1 = S_2 = -m$  and the boundary contribution is minus the contribution of the bound state  $\kappa_1$ .

## Appendix B: Equivalence of the large mass and heat kernel subtraction schemes to the vanishing tadpole condition

In this Appendix we demonstrate that in a renormalizable theory in two dimensions the heat kernel subtraction procedure is equivalent to imposing the renormalization condition of vanishing tadpoles in the trivial vacuum. As the former scheme is equivalent to the large  $M$  subtraction (see sec. 3), this will also prove that all three schemes are equivalent. For simplicity, we restrict ourselves to a bosonic theory.

Consider first the case of a constant but arbitrary background field  $\Phi$ . The second-order differential operator acting on the fluctuations has the form  $-\partial_1^2 + M(\Phi)^2$ . For the action (4)  $M^2(\Phi) = U'(\Phi)^2 + U(\Phi)U''(\Phi)$ . The explicit

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<sup>12</sup>In the large  $M$  subtraction scheme the vacuum energy associated with the Dirichlet boundaries is zero. This is clear from the consideration of sec. 2. The reason is that the Dirichlet problem contains no dimensional parameter. Therefore, the boundary energy can only be proportional to  $M$ . In the large  $M$  scheme such contributions must be dropped.

form of  $M^2$  will play no role. The complete heat kernel in this case is given by the term with  $n = 0$  in (19). As this term is proportional to the volume of the manifold, it is more convenient to consider the energy density  $E_0(x)$  which can be read off from the first term of the divergent part of the energy (20):

$$E_0(x) = \frac{\tilde{\mu}^{2s}}{4\sqrt{\pi}\Gamma(s - \frac{1}{2})} M^{2-2s} \Gamma(s - 1) \quad . \quad (48)$$

Let us now expand  $E_0$  around  $s = 0$ :

$$E_0(x) \sim \left( \frac{1}{s} + \gamma_E + 1 - \ln \left( \frac{M^2}{\tilde{\mu}^2} \right) - \partial_s \ln \Gamma \left( s - \frac{1}{2} \right)_{s=0} \right) \frac{M^2}{4\pi} \quad . \quad (49)$$

In a renormalizable theory (49) is accompanied by a counterterm

$$\Delta E_0(x) = - \left( \frac{1}{s} + \gamma_E + 1 - \ln \left( \frac{\nu^2}{\tilde{\mu}^2} \right) - \partial_s \ln \Gamma \left( s - \frac{1}{2} \right)_{s=0} \right) \frac{M^2}{4\pi} \quad , \quad (50)$$

which repeats the pole structure of (49). Finite renormalization is encoded in an arbitrary parameter  $\nu^2$  in (50).

Vanishing of tadpoles means that the one-loop renormalized energy  $E_0(x) + \Delta E_0(x)$  has a minimum at the same value of  $\Phi = \Phi_0$  as the classical energy. Generically,  $\partial(M^2)/\partial\Phi \neq 0$  near  $\Phi = \Phi_0$ . We obtain:

$$\frac{\partial}{\partial M^2} (E_0(x) + \Delta E_0(x)) = -\frac{1}{4\pi} \left( \ln \frac{M^2}{\nu^2} + 1 \right) = 0 \quad . \quad (51)$$

This condition defines  $\nu^2$  in terms of  $M^2(\Phi_0)$ .

Consider now a non-trivial vacuum  $\Phi$ . The operator acting on the fluctuations is modified:  $-\partial_1^2 + M(\Phi_0)^2 + V(x)$ . It is essential that  $V(x) \rightarrow 0$  for  $x \rightarrow \pm\infty$ . Consider  $E_0^{\text{div}}$  (20). The first term containing  $a_0$  is now proportional to  $M(\Phi_0)^2$ , which is a constant. This term must be removed by renormalization of the ‘‘cosmological constant’’. This is always done in such a way that the whole infinite volume contribution coming from  $a_0$  is subtracted. The second term with  $a_{\frac{1}{2}}$  is simply absent on manifolds without boundary. We are left with the 3rd term. It requires a counterterm

$$\Delta E_0 = - \left( \frac{1}{s} + \gamma_E + 1 - \ln \left( \frac{\nu^2}{\tilde{\mu}^2} \right) - \partial_s \ln \Gamma \left( s - \frac{1}{2} \right)_{s=0} \right) \frac{a_1}{2\sqrt{\pi}} \quad . \quad (52)$$

The fact the such a counterterm is indeed present is guaranteed by renormalizability of the theory. As it is enough to do just one (mass) renormalization, the terms added to the pole  $\frac{1}{s}$  in the bracket in (52) are exactly the same as in (50). Finally we have

$$E_0^{\text{div}} + \Delta E_0 = -\frac{a_1}{2\sqrt{\pi}} \left( \ln \frac{M^2}{\nu^2} + 1 \right) \quad . \quad (53)$$

The right hand side of (53) vanishes due to the condition (51). We conclude that the renormalization condition that tadpoles vanish in the trivial vacuum is equivalent to dropping  $E_0^{\text{div}}$  in the kink vacuum.

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