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THE COHOMOLOGY OF THE COMPLEX OF G-INVARIANT FORMS ON G-MANIFOLDS

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Abstract. The cohomology of the complex of G-invariant forms for arbitrary G-manifolds and, especially for a certain class of G-manifolds, which are locally trivial fiber bundles over the orbit space, is considered. The transgression in the differential graded algebra of basic elements for tensor product of two identical Weil algebras of a group Lie G with a reductive Lie algebra is calculated and this is applied to the calculation of the transgression of the cross product of principal G-bundles over G. This allows to construct a convenient DG-algebra whose cohomology coincides with the cohomology of the complex of G-invariant forms on a G-manifold of the above class. In particular, for compact G the generalization of the Cartan theorem on the cohomology of homogeneous spaces is proved.

0. Introduction

In this paper we consider a certain class of G-manifolds formed by G-manifolds that are locally trivial fiber bundles over the orbit space M/G with a fiber G/H, where H is a closed subgroup of a Lie group G. For this class, the natural invariant, namely the cohomology $H(\Omega(M)^G)$ of the complex of G-invariant forms $\Omega(M)^G$, is studied. Note that, for a connected compact G, this cohomology coincides with the real cohomology of M and, therefore, in particular we obtain some information on the last one. For example, the generalization of the Cartan theorem on the cohomology of homogeneous spaces [4] to the above class is proved.

In Section 1 the complex $\Omega(M)^G$ is included as a subcomplex into a certain bicomplex for arbitrary G-manifolds. The first spectral sequence $E_r(M,G)$ of this bicomplex allows to give an interpretation, in terms of the continuous cohomology of G and the cohomology of M, for those classes cohomology of $\Omega(M)^G$, that vanish in the de Rham cohomology of M. The examples of the applications of the above interpretation for infinite-dimensional Lie groups were given in [10]. Here the interplay between $H(\Omega(M)^G)$ and $E_r(M,G)$ for finite-dimensional Lie groups is studied for arbitrary G-manifolds and, especially, for G-manifolds of the above class. In particular, in 1.2 it is shown (Theorem 1.2.4) that the spectral sequence $E_r(M,G)$ tends to $H(\Omega(M)^G)$ for the above class whenever H is compact.

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In Section 2 differential graded commutative associative algebras (DG-algebras) and DG-algebras with an action of a Lie group G with a Lie algebra $\mathfrak{g}$ ($G-DG$-algebras) are defined and the DG-algebra of basic elements for a tensor product of two $G-DG$-algebras is studied. The main result of Section 2 is Theorem 2.2.3 on the transgression in the DG-algebra of basic elements for a tensor product of two identical Weil algebras $W(\mathfrak{g})$ with a reductive $\mathfrak{g}$.

In Section 3 there are some applications of Theorem 2.2.3. Let P_i $(i = 1, 2)$ be a smooth principal K-bundle with a base B_i and $M = P_1 \times_K P_2$, where K is a connected Lie group with a Lie algebra $\mathfrak{k}$. Consider M as a fiber bundle over $B_1 \times B_2$. For compact K the transgression of this fiber bundle is calculated and the subalgebra of the DG-algebra $\Omega(M)$ with the same cohomology is indicated (Theorem 3.1.1).

Let G be a connected Lie group with a reductive Lie algebra, H and N its closed connected subgroups with reductive Lie algebras such that H is a normal subgroup of N, $K = N/H$, and P a smooth principal K-bundle. For G-manifold $M = P \times_K (G/H)$ the DG-algebra $C(M)$ quasiisomorphic to the DG-algebra $\Omega(M)^G$, i.e. has the minimal model isomorphic to one of $\Omega(M)^G$ in the sense of Sullivan [14], is indicated (Theorem 3.2.3). For a compact G it gives the generalization of the above Cartan theorem to the investigated class of G-manifolds when the normalizer $N = N_G(H)$ of subgroup H is connected or $M = P \times_K (G/H)$, where K is a component of the neutral element N/H (Corollary 3.2.4).

Throughout the paper all manifolds, differential forms, maps of manifolds, and so on belong to the class C^∞; a Lie group means a finite-dimensional real Lie group.

1. A class of G-manifolds

For G-manifolds a spectral sequence is constructed and studied. A certain class of G-manifolds is introduced and some its properties are obtained.

1.1. A spectral sequence and its aplications. Let G be a topological group. A topological cochain G-complex $K = \{K^q, d^q\}$ is a cochain complex such that K^q are topological real vector spaces and d^q are continuous linear maps with a continuous left action of G by topological automorphisms of K. Suppose that K^G is the complex of G-invariant cochains of K.

Suppose that $C^{p,q} = C^{p,q}(G, K)$ be the space of continuous maps from G^p to K^q for $p > 0$ and $C^{0,q} = K^q$. Define the maps $\delta' : C^{p,q} \to C^{p+1,q}$ and $\delta'' : C^{p,q} \to C^{p,q+1}$ as follows:

$$(\delta' c)(g_1, \ldots, g_{p+1}) = g_1 c(g_2, \ldots, g_{p+1}) + \sum_{i=1}^{i=p} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1}) + (-1)^{p+1} c(g_1, \ldots, g_p),$$

$$(\delta'' c)(g_1, \ldots, g_p) = (-1)^p d^q c(p_1, \ldots, g_p),$$

where $c \in C^{p,q}$, $g_1, \ldots, g_{p+1} \in G$. It is easily checked that $C(K, G) = \{C^{p,q}, \delta', \delta''\}$ is a cochain bicomplex and $C^q = \{C^{p,q}, \delta'\}$ is the complex of standard continuous cochains of G with values in the G-module K^q.

If K_1 and K_2 are topological cochain G-complexes, then each continuous equivariant complex homomorphism $h : K_1 \to K_2$ induces the homomorphism

$$C(K_1, G) \to C(K_2, G)$$

of bicomplexes.

Consider $C(K,G)$ as the complex with the total coboundary operator $\delta' + \delta''$ and its cohomology $H(K,G)$. It is easily seen that the inclusion $K^G \subset C^{0,*} \subset C(K,G)$ determines the cohomology homomorphism $H(K^G) \to H(C(K,G))$, which is nontrivial on the kernel of the homomorphism $H(K^G) \to H(K)$ induced by the inclusion $K^G \subset K$ in general.

Let $E_r(K,G) = \{E_r^{p,q}\}$ be the first spectral sequence of the bicomplex $C(K,G)$, i.e. the spectral sequence induced by the filtration

$$F_pC(K,G) = \oplus_{q\geq 0, n\geq p} C^{n,q} \qquad (p = 0, 1, \ldots)$$

of $C(K,G)$. It is easily proved that $E_2^{p,q} = H_c^p(G; H^q(K))$, where the index c means the continuous cohomology in the sense of Mostov [12] and the action of G on $H^q(K)$ is determined by the action of G on K. Evidently every continuous equivariant homomorphism of topological cochain G-complexes induces the homomorphism of the corresponding spectral sequences.

Proposition 1.1.1. *If K_i $(i = 1, 2)$ is a topological cochain G-complex and $h : K_1 \to K_2$ is an equivariant continuous homomorphism of complexes inducing a cohomology isomorphism $H(K_1) \to H(K_2)$, then the homomorphism of the corresponding spectral sequences is an isomorphism beginning with the terms E_2 and, hence, the cohomologies $H(C(K_i, G))$ are isomorphic.*

The proof is immediately follows from the above representation of the term $E_2(K,G)$.

Proposition 1.1.2. *If K_1, K_2 are topological cochain G-complexes and $h_1, h_2 : K_1 \to K_2$ are equivariant cochain homotopic equivariant complex homomorphisms, then the homomorphisms of the cohomologies $H(C(K_1,G)) \to H(C(K_2,G))$ induced by h_1 and h_2 coincide.*

The proof is obvious.

Denote by H_p^q the image of q-dimensional cohomology of the complex $F_pC(K,G)$ in $H(K,G)$ under the inclusion $F_pC(K,G) \subset C(K,G)$. It is known that $E_\infty^{p,q} = H_p^q/H_{p+1}^q$, i.e. $E_\infty^{p,q}$ $(p = 0, 1 \ldots,)$ is the graduated vector space associated with the filtration of the cohomology $H^q(K,G)$ by H_p^q. Hence, the inclusion $K^G \subset C(K,G)$ induces the filtration $H_p^q(K^G)$ of $H^q(K^G)$ and the associated graduated vector space $H_p^q(K^G)/H_{p+1}^q(K^G)$.

Consider now the category $GTop$ of topological spaces with a continuous left action of a topological group G and equivariant maps as morphisms. Let X be an object of this category and $\Delta^*(X)$ the complex of singular cochains of X with real coefficients. $\Delta^*(X)$ is a topological cochain G-complex under the usual function topology and the natural left action of G. Applying the above constructions to this complex we obtain the bicomplex $C(X,G)$, its cohomology $H(X,G)$, and the corresponding spectral sequence $E_r(X,G)$.

Now let G be a Lie group and M be a G-manifold, i.e. a manifold M with a left smooth action of G. Consider the complexes $\Delta^*(M)$ and Δ^*_{diff} of singular cochains with real coefficients and differentiable singular cochains with the same coefficients, respectively, as topological cochain G-complexes with the usual function space topologies and the left action of G. Let $\Omega(M) = \{\Omega^q(M), d\}$ be the de Rham complex of M with the natural structure of a topological cochain G-complex. Let us apply the above constructions to these complexes. Clearly the bicomplex homomorphisms

$$\Omega^*(M) \to \Delta^*_{diff}(M) \to \Delta^*(M)$$

induced by the inclusions of the corresponding chain complexes produce the isomorphisms of the first spectral sequences of these bicomplexes beginning with the terms E_1 and cohomologies of the associated total complexes. Denote by $C_{diff}(\Omega(M), G)$ the subcomplex of the bicomplex $C(\Omega(M)$ formed by the differentiable maps from G^p to $\Omega^q(M)$. Note that the cohomology of the complex $(C_{diff}(\Omega(M), G), \delta')$ is the van Est continuous cohomology of the Lie group G with coefficients in $\Omega(M)$ [15]. Clearly $C_{diff}(\Omega(M), G)$ is a bicomplex and from [12] it follows that that the inclusion $C_{diff}(\Omega(M), G) \subset C(\Omega(M), G)$ induces the isomorphism of the corresponding first spectral sequences of these bicomplexes beginning with the terms E_2 and, then, the isomorphism of the corresponding total complexes. Further we denote by $C(M, G)$ the bicomplex $C_{diff}(\Omega(M), G)$, by $E_r(M, G)$ its first spectral sequence, and by $H(M, G)$ the cohomology of $C(M, G)$ as a complex with a total coboundary operator.

Finally, we obtain that the spectral sequence $E_r(M, G)$ and the cohomology $H(M, G)$ are depend only on the topologies of M and G but the cohomology $H(\Omega(M)^G)$ of the complex $\Omega(M)^G$ of G-invariant forms on M depends on the smooth structure of M and the topology of G.

Remark. 1. If G is a connected Lie group $E_2^{p,q}(M, G) = H_c^p(G; \mathbf{R}) \otimes H^q(M, \mathbf{R})$ Then, for a connected manifold M, the homomorphism

$$H_c^p(G; \mathbf{R}) = E_2^{p,0} \to E_\infty^{p,0} \to H(M, G)$$

defines the structure of a $H_c(G; \mathbf{R})$-module on $H(M, G)$. Moreover, there are the filtrations of $H(M, G)$ and $H(\Omega(M)^G)$ defined above in the general case for $H(K)$ and $H(K^G)$.

2. If G is a connected compact Lie group $E_2^{p,q}(M, G) = 0$ for $p > 0$ and $E_2^{0,q}(M, G) = H^q(M; \mathbf{R}) = H^q(\Omega(M)^G)$ [15].

In [10] we used $E_r(M, G)$ and, particularly, $E_\infty^{p,q}$ for q equal to the highest dimension of the nontrivial cohomology $H(M; \mathbf{R})$. In this case $E_2^{p,q} \subset E_\infty^{p,q}$ and one can interpret elements of $H_p^q(\Omega(M)^G)/H_{p+1}^q(\Omega(M)^G)$ as classes of $H_c^p(G; H^q(M; \mathbf{R}))$.

1.2. A class of G-manifolds. Consider now the full subcategory $GHTop$ of $GTop$ formed by those objects X for which the natural projection $X \to X/G$ defines a locally trivial fiber bundle over X/G with a fiber G/H, where H is a closed subgroup of G. Note that, if G is a compact Lie group and X is completely regular, these conditions are satisfied whenever all orbits have the type G/H as it follows from the existence of a slice [11].

Let us fix the subgroup H and consider the set X^H of points of X with the stabilizer H. Let $N = N_G(H)$ be the normalizer of the subgroup H. Clearly the action of N on X^H induces the action of N/H on X^H. One can consider the following right action of N on $X^H \times G$: $(x, g)n = (n^{-1}x, gn)$ $(x \in X^H, g \in G, n \in N)$. This action of N on $G \times X^H$ induces the action of N/H on $X^H \times (G/H)$. The following proposition, due to Borel [2], holds.

Proposition 1.2.1. *The action of N/H on X^H defines the structure of a principal N/H-bundle over X/G on X^H, $X^H \times_N G = X^H \times_{N/H} (G/H)$ is canonically equivariant isomorphic to X, and X with the natural projection $X = X^H \times_N G \to G/N$ is a locally trivial fiber bundle with the fiber X^H.*

Proof. If X is a trivial fiber bundle, i.e. $X = (X/G) \times (G/H)$, all statements of the theorem are obvious. In the general case these statements follow from the local triviality of $X \to X/G$. The isomorphism $X^H \times_N G \cong X$ is induced by the assignment $(x, g) \to gx$, where $x \in X^H$ and $g \in G$. □

Let $E(N/H) \to B(N/H)$ be an universal N/H-bundle. The following proposition is the evident consequence of the known properties of an universal N/H-bundle.

Proposition 1.2.2. *There is a one-to-one correspondence between the set of isomorphism classes of objects for the category $GHTop$ with paracompact orbit spaces and the set of homotopy classes of continuous maps from their orbit spaces to the classifying space $B(N/H)$.*

Proof. By Proposition 1.2.1 for each object X of the category $GHTop$ there is the canonical representation $X = X^H \times_{N/H} (G/H)$ and, then, a one-to-one correspondence between the set of isomorphism classes of objects for the category $GHTop$ and the set of isomorphism classes of principal G/H-bundles. Therefore, the proposition follows from the known properties of the universal N/H-bundle $E(G/H)$. □

Let G be a Lie group and H a closed subgroup of G. Consider a smooth object of $GHTop$, i.e. a G-manifold M such that all orbits of G have the type G/H, M/G is a manifold and, under the projection $M \to M/G$, M is a smooth locally trivial fiber bundle. Denote by M^H the set of points of M with the stabilizer H and by N the normalizer of the subgroup H. Then one has the following smooth version of Proposition 1.2.1.

Proposition 1.2.3. *M^H is a smooth subbundle of the bundle $M \to M/G$ and a smooth principal N/H-bundle under the action of N/H on M^H. $M^H \times_N G = M^H \times_{N/H} (G/H)$ is a G-manifold canonically equivariant isomorphic to M and M with the natural projection $M = M^H \times_N G \to G/N$ is a smooth locally trivial fiber bundle with the fiber M^H.*

The proof is analogous to one of Proposition 1.2.1.

Now we prove the following generalization of the van Est theorem [16] for $M = G/H$ by a slight generalization of the proof of this theorem in [6].

Theorem 1.2.4. *If H is compact, for the above G-manifold M, the spectral sequence $E_r(M, G)$ converges to the cohomology $H(\Omega(M)^G)$.*

Proof. It is convenient to suppose that G acts on M on the right and consider the corresponding left action $g \to R_g^*$ of G on $\Omega(M)$. Consider the second spectral sequence E_r of the bicomplex $C(M,G)$. If one identifies E_0 with $C(M,G)$ the differential operator $d_0 : E_0 \to E_0$ coincides with δ'. Let the fiber bundle $M \to M/G$ be trivial, i.e. $M = (M/G) \times (H \backslash G)$. Define the linear map $D^{p,q} : C^{p,q} \to C^{p-1,q}$ for $p > 0$ as follows:

$$D^{p,q}c(g_1,\ldots,g_{p-1})(x,Hg) = \int_{Hg} R^*_{g^{-1}}(c(g,g_1,\ldots,g_{q-1})(x,H))dg,$$

where $c \in C^{p,q}$, $x \in M/G$, $g, g_1, \ldots, g_{p-1} \in G$, dg is the measure on Hg induced by the normed Haar measure on H. For $p = 0$ put $D^{0,q} = 0$.

It can be easily proved that $\delta' D^{p,q} + D^{p+1,q}\delta'$ is equal to the identity map on $C^{p,q}$ for $p > 0$. For $p = 0$ and $c \in C^{0,q}$ one has

$$D^{1,q}\delta' c(\,)(x,Hg) = c(\,)(x,Hg) - \int_{Hg} R^*_{g^{-1}}(c(\,)(x,H))dg,$$

It follows from this that $D^{1,q}d' - I$, where I is the identity map on $C^{0,q}$, is the projection of $C^{0,q} = \Omega^q(M)$ onto the space $\Omega^q(M)^G$ of G-invariant q-forms on M.

Since $D^{p,q}$ commutes with the product on functions on M/G the linear maps $D^{p,q}$ can be extended to $C(M,G)$ for an arbitrary fiber bundle $M \to M/G$ by means of a partition of unity on M/G; the above properties of $D^{p,q}$ are valid in this case. Therefore, $E_1^{p,q} = 0$, for $p > 0$, and $E_1^{0,q} = \Omega(M)^G$ in the general case. Since d_1 is induced by δ'' $E_2^{p,q} = 0$, for $p > 0$, and $E_2^{0,q} = H(\Omega(M)^G)$. □

Corollary 1.2.5. *If M is connected and H is compact there is a canonical homomorphism of cohomology algebras $H_c(G;\mathbf{R}) \to H(\Omega(M)^G)$.*

Proof. As it was noticed above there is the canonical homomorphism $H_c(G;\mathbf{R}) \to H(C(M,G))$. Then the statement of the corollary follows from Theorem 1.2.4. □

2. DG- and $G - DG$-algebras

In this Section differential graded associative commutative $\mathbf{R}$-algebras, more concisely a GD-algebras, and DG-algebras with an action of a group Lie G called $G - DG$-algebras are considered.

2.1. Some properties of DG- and $G - DG$-algebras. Let $U = \oplus_{n \geq 0} U^n$ be a differential graded associative commutative $\mathbf{R}$-algebra. Recall that the multiplication and differential operator d in U satisfies the rules

$$ba = (-1)^{deg\, a \cdot deg\, b}ab, \quad d(ab) = (da)b + (-1)^{deg\, a}a(db),$$

where a and b are homogeneous elements of U.

Let G be a Lie group with a Lie algebra $\mathfrak{g}$ and U a DG-algebra U with a given left action of G by means of automorphisms of U. Denote by $\theta_x : U \to U$ $(x \in \mathfrak{g})$ the derivation of degree 0 of U for which $g \mapsto \theta_x$ is the representation of $\mathfrak{g}$ in U associated with the action of G.

Definition. A DG-algebra U is called a $G - DG$-algebra if, for each $x \in \mathfrak{g}$, an antiderivation i_x of degree -1 of U satisfying the following Cartan conditions:

$$i_{[x,y]} = \theta_x\, i_y - i_y\, \theta_x, \quad \theta_x = i_x\, d + d\, i_x, \tag{2.1.1}$$

is given [3].

Introduce the following notations: $IU = U^G$ is the DG-subalgebra of U formed by $G - invariant$ elements of U. (Obviously, if G is connected $IU = \cap_{x \in \mathfrak{g}} Ker\, \theta_x$);

$SBU = \cap_{x \in \mathfrak{g}} Ker\, i_x$ is a graded subalgebra of U formed by *semibasic* elements of U;

$BU = IU \cap SBU$ is the DG-subalgebra of U formed by *basic* elements of U.

If H is a closed subgroup of G with the Lie algebra $\mathfrak{h}$, one can consider any $G - DG$-algebra U as a $H - DG$-algebra relative to the restriction of the action of G to H, the derivations θ_x, and the antiderivations i_x to elements $x \in \mathfrak{h}$; denote by $I_H U$, $SB_H U$, and $B_H U$ the subalgebras of U corresponding to this restriction.

Examples. 1. Let P be a smooth principal G-bundle. The action of G on $\Omega(P)$ defines on the graded algebra $\Omega(P)$ a structure of $G - DG$-algebra.

2. The Weil algebra. Let $\mathfrak{g}'$ be the dual space of $\mathfrak{g}$, $\Lambda(\mathfrak{g}')$ the exterior, and $S(\mathfrak{g}')$ the symmetric algebra of $\mathfrak{g}'$. If $x \in \mathfrak{g}' \subset \Lambda(\mathfrak{g}')$, then the corresponding element of $\mathfrak{g}' \subset S(\mathfrak{g}')$ is denoted by $\overline{x}$. We consider $\Lambda(\mathfrak{g})$ and $S(\mathfrak{g})$ as graded algebras supposing that, for each nonzero $x \in \mathfrak{g}'$, $deg\, x = 1$ and $deg\, \overline{x} = 2$. Consider the graded algebra $W(\mathfrak{g}) = \Lambda(\mathfrak{g}') \otimes S(\mathfrak{g}')$. Let e_i $(i = 1, \ldots, r = dim\, \mathfrak{g})$ be a basis of $\mathfrak{g}$, $c^i_{j\,k}$ $(i, j, k = 1, \ldots, r)$ the corresponding structure constants, and e^i the dual basis. Define the differential operator $d : W(\mathfrak{g}) \to W(\mathfrak{g})$ by its values on the generators $e^i \in \Lambda(\mathfrak{g}')$ and $\overline{e^i} \in S(\mathfrak{g}')$ as follows:

$$de^i = -\frac{1}{2} \sum_{j,k=1}^{r} c^i_{jk} e^j e^k + \overline{e^i}, \quad d\overline{e^i} = \sum_{j,k=1}^{r} c^i_{ji} \overline{e^j} e^k. \tag{2.1.2}$$

The adjoint representation of $\mathfrak{g}$ induces the action of $\mathfrak{g}$ on $W(\mathfrak{g})$ and the corresponding operators θ_x $(x \in \mathfrak{g})$. Define the antiderivation i_x of $W(\mathfrak{g})$ putting it equal to the inner product of a vector $x \in \mathfrak{g}$ and a form on $\Lambda(\mathfrak{g}')$ and equal to zero on $S(\mathfrak{g}')$. It is easily checked that $W(\mathfrak{g}) = \{W(\mathfrak{g}), d, \theta_x, i_x\}$ is a $G - DG$-algebra called the Weil algebra [3].

Let U be a $G - DG$-algebra, $P(x_1, \ldots, x_p)$ a p-linear form on $\mathfrak{g}$ with values in a vector space L, and $\omega_i \in U^{k_i} \times \mathfrak{g}$ $(i = 1, \ldots, p)$. Denote by $P(\omega_1, \ldots, \omega_p)$ the image of $\omega_1 \otimes \cdots \otimes \omega_p$ under the following composition of maps:

$$(U^{k_1} \otimes \mathfrak{g}) \otimes \cdots \otimes (U^{k_p} \otimes \mathfrak{g}) \to U^{k_1} \otimes \cdots \otimes U^{k_p} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g} \xrightarrow{m \otimes P} U^k \otimes L,$$

where $k = k_1 + \cdots + k_p$ and m is the multiplication $U^{k_1} \otimes \cdots \otimes U^{k_p} \to U^k$. It is easily seen that $P(\omega_1, \ldots, \omega_p) \in SBU$ whenever $\omega_i \in SBU^{k_i} \times \mathfrak{g}$ $(i = 1, \ldots, p)$. In the sequel we use $L = \mathbf{R}$, $\mathfrak{g}$, dropping $\mathbf{R}$ in the first case.

Denote the operators $d \otimes 1$, $\theta_x \otimes 1$, $i_x \otimes 1 : U \otimes \mathfrak{g} \to U \otimes \mathfrak{g}$, and the operator $1 \otimes ad\, x : U \otimes \mathfrak{g} \to U \otimes \mathfrak{g}$ just by d, θ_x, i_x and $ad\, x$.

Definition. Let G be a connected Lie group and U a $G{-}DG$-algebra. A connection ω on U is an element of $U^1 \otimes \mathfrak{g}$ satisfying the following conditions:

(1) $i_x \omega = x \quad (x \in \mathfrak{g})$;
(2) $\theta_x \omega = -ad\, x\, \omega \quad (x \in \mathfrak{g})$.

Let ω be a connection on a $G-DG$-algebra U. From the definition of connection and conditions (2.1.1) it easily follows that

$$d\omega = -\frac{1}{2}[\omega,\omega] + \Omega, \quad d\Omega = [\Omega,\omega], \tag{2.1.3}$$

where $\Omega \in SBU \otimes \mathfrak{g}$. Ω is called the *curvature* of the connection ω. It is easily proved that $\theta_x \Omega = -ad\, x\, \Omega$.

Examples. 1. Let P be a smooth principal G-bundle and $\Omega(P)$ the corresponding $G-DG$-algebra. For a connected Lie group G a connection ω on $G-DG$-algebra $\Omega(P)$ is a connection form ω on P and the curvature Ω of ω is the curvature form.

2. Consider the Weil algebra $W(\mathfrak{g})$. It is easily proved that the form $\omega = \sum_{i=1}^r e^i e_i$, i.e. identity form on $\mathfrak{g}$ with values in $\mathfrak{g}$, is a connection on $W(\mathfrak{g})$, called the canonical connection of $W(\mathfrak{g})$, and $\Omega = \sum_{i=1}^r \overline{e^i} e_i$ is a curvature of this connection.

Definition. Let U_i $(i = 1, 2)$ be a GD-algebra. A morphism from U_1 to U_2 is a homomorphism $f : U_1 \to U_2$ of graded algebras which commutes with d. If U_i is a $G-DG$-algebra a morphism $f : U_1 \to U_2$ of DG-algebras commuting with the operators θ_x and i_x is called a morphism of $G-DG$-algebras.

It is evident that a morphism $f : U_1 \to U_2$ of $G-DG$-algebras induces the morphisms $IU_1 \to IU_2$, $SBU_1 \to SBU_2$, and $BU_1 \to BU_2$ of the corresponding subalgebras of G-invariant elements, semibasic elements, and basic elements.

Example. Let U be $G-DG$-algebra with a connection ω_U. It is known [3] that there is a morphism of $G-DG$-algebras $f : W(\mathfrak{g}) \to U$ uniquely defined by the correspondence $\omega \mapsto \omega_U$, where ω is the canonical connection on $W(\mathfrak{g})$.

Let $f : U_1 \to U_2$ be a morphism of DG-algebras, $U_1^0 = \mathbf{R}$, and $U_{1,p} = \otimes_{q \geq p} U_1^q$. It is clear that $F_p = f(U_{1,p})U_2$ is an ideal of U_2 and $F_p = f(U_{1,p})U_2$ $(p = 0, 1, \dots)$ is a decreasing filtration of U_2 defining the corresponding spectral sequence. For example, if U is a $G-DG$-algebra of forms of a connected smooth principal fiber bundle the inclusion $BU \subset U$ induces the spectral sequence which is the Leray-Serre spectral sequence of this fiber bundle.

Let U_k $(k = 1, 2)$ be a $G-DG$-algebra with the operators $\theta_{k,x}$, $i_{k,x}$, and the differential operator d_k. Further we consider the tensor product of graded algebras $U_1 \otimes U_2$ as a $G-DG$-algebra under the diagonal action of G and the operators θ_x and i_x that coincide with the corresponding operators on the factors U_1 and U_2 of $U_1 \otimes U_2$. It is clear that, if $f_i : U_i \to V_i$ $(i = 1, 2)$ is a morphism of $G{-}DG$-algebras, then $f_1 \otimes f_2 : U_1 \otimes U_2 \to V_1 \otimes V_2$ is a morphism of the correspoding $G{-}DG$-algebras.

A p-linear form P on $\mathfrak{g}$ is called *invariant* if it is invariant under the action of G induced by the adjoint representation of G. For connected G P is invariant iff the following condition is satisfied:

$$\sum_{i=1}^{p} P(x_1,\dots,[x,x_i],\dots,x_p)=0, \tag{2.1.4}$$

where $x, x_1,\dots,x_p \in \mathfrak{g}$. If $\omega \in U^q \otimes \mathfrak{g}$ and $\omega_i \in U^{k_i}\otimes \mathfrak{g}$, (2.1.4) implies the equality

$$\sum_{i=1}^{p}(-1)^{k_1+\cdots+k_{i-1}+q}P(\omega_1,\dots,[\omega,\omega_i],\dots,\omega_p)=0. \tag{2.1.5}$$

Let G be a connected Lie group, P a p-linear invariant form on $\mathfrak{g}$, and let ω_i satisfy the condition $\theta_x\omega_i=-ad\,x\,\omega_i$. From (2.1.4) it follows that $P(\omega_1,\dots,\omega_p)$ is a basic element of U.

Suppose that U is a $G-DG$-algebra with a connection ω and $\Lambda(\mathfrak{g}')$ is the exterior algebra of the dual space $\mathfrak{g}'$ of $\mathfrak{g}$. Define the morphism of graded algebras $\Lambda(\mathfrak{g}')\to U$ assigning to $P\in\Lambda^p(\mathfrak{g}')$ $P(\omega,\dots,\omega)\in U^p$. A $G-DG$-algebra U such that the following composition of morphisms of graded algebras

$$\Lambda(\mathfrak{g}')\otimes SBU \to U\otimes SBU \xrightarrow{m} U$$

is an isomorphism is called a $W-algebra$. Further we identify W-algebra U with $\Lambda(\mathfrak{g}')\otimes SBU$. Note that the Weil algebra and the algebra of differential forms on a principal G-bundle are W-algebras.

Consider the DG-algebra $B(W(\mathfrak{g})\otimes U)$, where U is a W-algebra. Denote by ω_1 the canonical connection on $W(\mathfrak{g})$ and by ω_2 the connection on U. Identifying ω_i and its cuvatures Ω_i with their images under the natural inclusions

$$W(\mathfrak{g})\otimes\mathfrak{g}\to W(\mathfrak{g})\otimes U\otimes\mathfrak{g} \text{ and } U\otimes\mathfrak{g}\to W(\mathfrak{g})\otimes U\otimes\mathfrak{g}$$

put

$$\phi=\frac{1}{2}(\omega_1-\omega_2),\quad \psi=\frac{1}{2}(\omega_1+\omega_2),\quad \Phi_i=\frac{1}{2}\Omega_i.$$

Then,

$$\begin{aligned} d\phi &= \Phi_1-\Phi_2-[\psi,\phi],\\ d\Phi_1 &= [\Phi_1,\phi]-[\psi,\Phi_1],\\ d\Phi_2 &= -[\Phi_2,\phi]-[\psi,\Phi_2]. \end{aligned} \tag{2.1.6}$$

It is obvious that $\phi,\Phi_i\in SB(W(\mathfrak{g})\otimes U)\otimes\mathfrak{g}$ and the following equations hold:

$$\theta_x\phi=-ad\,x\,\phi,\quad \theta_x\Phi_i=-ad\,x\,\Phi_i.$$

Hence, for an invariant $(a+b+c)$-linear form P on $\mathfrak{g}$,

$$P(\underbrace{\phi,\dots,\phi}_{a\text{ times}},\underbrace{\Phi_1,\dots,\Phi_1}_{b\text{ times}},\underbrace{\Phi_2,\dots,\Phi_2}_{c\text{ times}}) \tag{2.1.7}$$

is an element of $B(W(\mathfrak{g})\otimes U)$ of degree $a+2b+2c$. Note that, if $U=W(\mathfrak{g})$, each element of $B(W(\mathfrak{g})\otimes U)$ is a sum of elements of the type (2.1.7). For an element of $B(W(\mathfrak{g})\otimes U)$ of the type (2.1.7), from (2.1.5) and (2.1.6) it follows that

$$\begin{aligned} &dP(\phi,\dots,\phi,\Phi_1,\dots,\Phi_1,\Phi_2,\dots,\Phi_2)\\ &=\sum_{i=1}^{a}(-1)^{i-1}P(\phi,\dots,\Phi_1-\Phi_2,\dots,\phi,\Phi_1,\dots,\Phi_1,\Phi_2,\dots,\Phi_2)\\ &+(-1)^a\sum_{j=1}^{b}P(\phi,\dots,\phi,\Phi_1,\dots,[\Phi_1,\phi],\dots,\Phi_1,\Phi_2,\dots,\Phi_2)\\ &-(-1)^a\sum_{k=1}^{c}P(\phi,\dots,\phi,\Phi_1,\dots,\Phi_1,\Phi_2,\dots,[\Phi_2,\phi],\dots,\Phi_2). \end{aligned} \tag{2.1.8}$$

Thus, one can use in the calculation of differential operator of $B(W(\mathfrak{g})\otimes U)$ the following formulas:

$$d\phi=\Phi_1-\Phi_2,\quad d\Phi_1=[\Phi_1,\phi]\quad d\Phi_2=-[\Phi_2,\phi]. \tag{2.1.9}$$

Theorem 2.1.1. *The inclusion $BU\subset B(W(\mathfrak{g})\otimes U)$ induces a cohomology isomorphism.*

Proof. It is easily proved that identifying $P\in\Lambda^p(\mathfrak{g}')$ with $P(\phi,\dots,\phi)$ one obtains the natural isomorphism of graded algebras $SB(W(\mathfrak{g})\otimes U)=\Lambda(\mathfrak{g}')\otimes S(\mathfrak{g}')\otimes SBU$. Put

$$F_p=\otimes_{q\geq p}(\Lambda(\mathfrak{g}')\otimes S(\mathfrak{g}')\otimes SBU^q)\cap B(W(\mathfrak{g})\otimes U).$$

¿From (2.1.1) it follows that $d(SBU)\subset SBU\otimes\mathfrak{g}'$. Therefore F_p is an ideal *DG*-algebra $B(W(\mathfrak{g})\otimes U)$ and $\{F_p\}$ is a decreasing filtration of $B(W(\mathfrak{g})\otimes U)$. Consider the spectral sequence $\{E_r^{p,q},d_r\}$ induced by this filtration. It is easily proved that to calculate d_0 on $E_0=B(W(\mathfrak{g})\otimes U)$ one can use the following equations:

$$d_0\phi=\Phi_1,\quad d_0\Phi_1=[\Phi_1,\phi].$$

Take the antiderivation k of a graded algebra $SB(W(\mathfrak{g})\otimes U)$ of degree -1 which vanishes on SBU and is determined on $\Lambda(\mathfrak{g}')\otimes S(\mathfrak{g}')$ in the following way:

$$k(\phi^i)=0,\quad k(\Phi_1^i)=\phi^i.$$

It is clear that $k(F_p)\subset F_p$ and, then, one can consider k as an antiderivation of $E_0=B(W(\mathfrak{g})\otimes U)$. For $c\in B(W(\mathfrak{g})\otimes U)$ denote by $|c|_1$ the highest number p such that c has nonzero component in $\Lambda^p(\mathfrak{g}')\otimes S(\mathfrak{g}')\otimes SBU$ and by $|c|_2$ the highest number q such that c has nonzero component in $\Lambda(\mathfrak{g}')\otimes S^q(\mathfrak{g}')\otimes SBU$. It is easily checked that, if $c\in E_0^{p,q}$, $|c|_1=m$, $|c|_2=n$, $m+n\neq 0$, and

$$c_1=c-\frac{1}{m+n}(d_0k+kd_0)(c),$$

then $|c_1|_2<n$. Iterating this procedure one obtains the finite sequence $c,c_1,\dots,c_s\in B(W(\mathfrak{g})\otimes U)$ such that $|c_s|_2=0$. If $|c_s|_1\neq 0$, applying the above procedure to c_s, one obtains an element $b\in BU$. Note that, if $d_0c=0$, $c_1,\dots,c_s,b$ are cohomologous to c. Since $BU^p\subset E_0^{p,0}$ this implies that $E_1^{p,q}=0$, if $q>0$, and $E_1^{p,0}=BU^p$. Hence, $E_2^{p,q}=E_\infty^{p,q}=0$, for $q>0$, and $E_2^{p,0}=E_\infty^{p,0}=H^p(BU)$. Therefore, the statement of the theorem follows from the known properties of spectral sequences. □

2.2. The algebra $C(\mathfrak{g})$. Denote by $C(\mathfrak{g})$ the DG-algebra

$$B(W(\mathfrak{g}) \otimes W(\mathfrak{g})) = (S(\mathfrak{g}_1') \otimes \Lambda(\mathfrak{g}') \otimes S(\mathfrak{g}_2'))^G,$$

where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two copies of $\mathfrak{g}$. Since the differential operator d vanishes on $BW(\mathfrak{g})$ we obtain the following corollary of Theorem 2.1.1.

Proposition 2.2.1. $H(C(\mathfrak{g})) = S(\mathfrak{g})^G$.

Suppose now that $\mathfrak{g}$ is a reductive Lie algebra. It is known [4] $S(\mathfrak{g}')^G$ is the polinomial algebra with generators having even degrees. Then, Proposition 2.2.1 implies

Proposition 2.2.2. *If G is a connected Lie group with a reductive Lie algebra $\mathfrak{g}$ the DG-algebra $S(\mathfrak{g})$ is a minimal model of the GD-algebra $C(\mathfrak{g})$ in the sense of Sullivan [14].*

Now let G be a connected Lie group with a reductive Lie algebra $\mathfrak{g}$. It is known [9] that the cohomology $H(\mathfrak{g}; \mathbf{R}) = \Lambda(\mathfrak{g}')^G$ is the exterior algebra of the vector space $P_\mathfrak{g}$ of primitive elements of the algebra $H(\mathfrak{g}; \mathbf{R})$ and $P_\mathfrak{g}$ has a basis consisting of the elements of odd degrees. All elements of this basis are universally transgressive, i.e. if $a \in \Lambda(\mathfrak{g}')^G$ is an element of this basis of $P_\mathfrak{g}$, there exists an element $c' \in BW(\mathfrak{g})$, called a transgression cochain, such that the component of c' in $\Lambda(\mathfrak{g}')^G$ coincides with a and $d'c' = b \in S(\mathfrak{g}')^G = BW(\mathfrak{g})$ [4].

Theorem 2.2.3. *Let G be a connected Lie group with a reductive Lie algebra $\mathfrak{g}$ and a an element of the basis of $\Lambda(P_\mathfrak{g}) = \Lambda(\mathfrak{g}')^G$ of degree 2p-1. Then, there exists $c \in C(\mathfrak{g})$ such that its component in $\Lambda(\mathfrak{g}')^G$ coincides with a and $dc = b_1 + b_2$, where $b_i \in S(\mathfrak{g}_i')^G$ $(i = 1, 2)$.*

Proof. Let a, c', and b be as above and P an invariant p-form such that

$$P(\Omega, \ldots, \Omega) = b.$$

Further, for the sake of brevity, put $P(x) = P(x, \ldots, x)$, $P(x, x') = P(x, x', \ldots, x')$, and $P(x, x', x'') = P(x, x', x'', \ldots, x'')$. Suppose that $\Omega_t = t\Omega + \frac{1}{2}(t^2 - t)[\omega, \omega]$ $(t \in [0, 1])$. By [5] one can take $p \int_0^1 P(\omega, \Omega_t)\, dt$ for c'. Thus, a is equal to $P(\phi, [\phi, \phi])$ up to a rational factor.

Now put

$$\Phi_t = t\Phi_1 + (1 - t)\Phi_2 + (t^2 - t)[\phi, \phi]$$

It is evident that

$$\int_0^1 \frac{d}{dt} P(\Phi_t)\, dt = P(\Phi_1) - P(\Phi_2).$$

On the other hand, by (2.1.5) and (2.1.9)

$$\begin{aligned} \int_0^1 \frac{d}{dt} P(\Phi_t)\, dt &= p \int_0^1 P(\Phi_1 - \Phi_2 + (2t - 1)[\phi, \phi], \Omega_t)\, dt \\ &= p \int_0^1 (P(\Phi_1 - \Phi_2, \Phi_t) - (p - 1)(2t - 1) P(\phi, [\Phi_1, \phi], \Phi_t) \\ &\quad + (p - 1)(2t - 1)(t - 1) P(\phi, [\Phi_2, \phi], \Phi_t))\, dt = pd \int_0^1 P(\phi, \Phi_t)\, dt. \end{aligned}$$

Since $P(\Phi_1) \in S(\mathfrak{g}_1')^G$ and $P(\Phi_2) \in S(\mathfrak{g}_2')^G$ and the component of $\int_0^1 P(\phi, \Phi_t)\, dt$ in $\Lambda(\mathfrak{g})^G$ is equal to $P(\phi, [\phi, \phi])$ up to a rational factor the required cochain c is equal to $P(\phi, \Phi_t)$ up to a rational factor. $\square$

By analogy with the Weil complex we call the cochain c from Theorem 2.2.3 the transgression cochain of the element $a \in P_{\mathfrak{g}}$ in the DG-algebra $C(\mathfrak{g})$.

3. Some applications of the complex $C(\mathfrak{k})$

In this Section we use the transgression in the DG-algebra of forms on the cross product of two smooth principal K-bundles to prove some theorems on the cohomology of this cross product and the cohomology of DG-algebra of G-invariant forms, where G is a Lie group of automorphisms of the second factor. In particular, the generalization of the Cartan theorem on the cohomology of homogeneous spaces [4] is proved. Note that, since a group Lie G will play another role in this Section, we use, instead of G, a group Lie K with a Lie algebra $\mathfrak{k}$.

3.1. The transgression in a cross product of principal fiber bundles and its applications. Let P_i $(i = 1, 2)$ be a smooth principal K-bundle, where K is a connected Lie group, ω_i a connection form on P_i, and Ω_i its curvature form. Consider the natural morphism of $K - DG$-algebras $h_i : W(\mathfrak{k}) \to \Omega(P_i)$ defined as above by means of ω_i, the canonical connection of $W(\mathfrak{k})$, and the corresponding morphism of DG-algebras

$$h : C(\mathfrak{k}) = B(W(\mathfrak{k}) \otimes W(\mathfrak{k})) \to B(\Omega(P_1) \otimes \Omega(P_2)).$$

Define now the GD-algebra $C(P_1, P_2; \mathfrak{k})$ in the following way. As a graded algebra

$$C(P_1, P_2; \mathfrak{k}) = \Omega(B_1) \otimes \Lambda(P_{\mathfrak{k}}) \otimes \Omega(B_2),$$

where B_i is the base of P_i, and the differential operator d is equal to the exterior derivative on $\Omega(B_i)$ and, for each basic element $a \in P_{\mathfrak{k}}$ of degree $2p - 1$, $da = h(dc) = h(b_1) + h(b_2)$, where c is the transgression cochain of a defined in Theorem 2.2.3. Since, by Theorem 2.2.3, $h(b_i) \in \Omega(B_i)$ $da \in \Omega(B_1) \otimes \Omega(B_2)$.

Consider the manifold $M = P_1 \times_K P_2$ and the DG-algebra $\Omega(M)$. The composition of the natural inclusions

$$C(P_1, P_2; \mathfrak{k}) \to B(\Omega(P_1) \otimes \Omega(P_2)) \to \Omega(M)$$

is the inclusion of $C(P_1, P_2; \mathfrak{k})$ into $\Omega(M)$.

Consider now $M = P_1 \times_K P_2$ as a locally trivial fiber bundle over $B_1 \times B_2$. This fiber bundle is not principal but its standard fiber F is diffeomorphic to $K \times_K K = K$.

Theorem 3.1.1. *Let K be a connected compact Lie group, P_i $(i = 1, 2)$ a smooth principal K-bundle, $M = P_1 \times_K P_2$, and ω_i a connection form on P_i. Then*

1. *Each homogeneous element of $P_{\mathfrak{k}}$ is a transgressive element of the fiber bundle $M \to B_1 \times B_2$;*
2. *The inclusion of $C(P_1, P_2; \mathfrak{k})$ into $\Omega(M)$ induces a cohomology isomorphism.*

Proof. Since the Lie algebra of a compact Lie group is reductive the first statement of the theorem follows from the inclusion $C(P_1, P_2; \mathfrak{k})$ into $\Omega(M)$ and Theorem 2.2.3. The second statement of the theorem follows from the Leray-Serre spectral sequence of the fiber bundle $M \to B_1 \times B_2$ in the usual way [4, 8]. □

Remark. If the complex $\Omega(B_i)$ contains a subalgebra H_i consisting of cocycles, which have the only representative of each cohomology class, one can replace $\Omega(B_i)$ in the definition of $C(P_1, P_2; \mathfrak{k})$ with $H(B_i; \mathbf{R})$ and revise the values of differential operator d on the homogeneous elements of $P_{\mathfrak{k}}$ in the obvious way. This condition is satisfied, for example, if B_i is diffeomorphic to an orientable compact locally symmetric Riemannian space. In this case one can take, for the required subalgebra, the subalgebra of harmonic differential forms.

Let now K be a connected Lie group with a reductive Lie algebra $\mathfrak{k}$, P_1 be a smooth principal K-bundle and P_2 a smooth principal K-bundle with an automorphism group G. It is clear that the action of the group G is naturally extended to the action of G on $M = P_1 \times_K P_2$. Suppose that ω_1 is a connection form on P_1 and ω_2 is a G-invariant connection form on P_2.

Put $C(P_1, P_2; \mathfrak{k})^G = \Omega(B_1) \otimes \Lambda(P_{\mathfrak{k}}) \otimes \Omega(B_2)^G$. It is easily seen that the image of the composition of morphisms

$$C(\mathfrak{k}) \xrightarrow{h} B(\Omega(P_1) \otimes \Omega(P_2)) \subset \Omega(M)$$

consists of the G-invariant forms on M. Thus, the composition of inclusions

$$C(P_1, P_2; \mathfrak{k})^G \subset C(P_1, P_2; \mathfrak{k}) \to \Omega(M)$$

factors through the inclusion of $C(P_1, P_2; \mathfrak{k})^G$ into $\Omega(M)^G$.

Proposition 3.1.2. *Let K be a connected Lie group with a reductive Lie algebra $\mathfrak{k}$, P_i $(i = 1, 2)$ a smooth principal K-bundle, G a transitive group of automorphisms of P_2, ω_i a connection form on P_i, and $M = P_1 \times_K P_2$. If ω_2 is a G-invariant connection form on P_2 the inclusion of $C(P_1, P_2; \mathfrak{k})^G$ into $\Omega(M)^G$ is a complex isomorphism.*

Proof. Since $\Omega(P_i)$ is a W-algebra and G is a transitive group of automorphisms of P_2

$$\Omega(M)^G = \Omega(B_1) \otimes \Lambda(P_{\mathfrak{k}}) \otimes \Omega(B_2)^G.$$

Therefore, by definition of the inclusion of $C(P_1, P_2; \mathfrak{k})^G$ into $\Omega(M)$, it induces the complex isomorphism $C(P_1, P_2; \mathfrak{k})^G = \Omega(M)^G$. □

Remarks. 1. The necessary and sufficient conditions of the existence of invariant connections on a principal K-bundle P with an automorphism Lie group G inducing the transitive action of G on the base B of P is given in [17]. The simplest sufficient condition is the existence of a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m}$, where $\mathfrak{n}$ is the Lie algebra of the stabilizer N of a point of B and $\mathfrak{m}$ is invariant under the adjoint representation of N on $\mathfrak{g}$ (see for detailed information [7]). This condition is satisfied, for example, if N is compact or N is connected and $\mathfrak{n}$ is a reductive subalgebra of $\mathfrak{g}$.

2. For a connected compact Lie group G $H(\Omega(M)^G) = H(M; \mathbf{R})$ and, then, $H(C(P_1, P_2; \mathfrak{k})^G) = H(M; \mathbf{R})$.

3.2. A generalization of the Cartan theorem on the cohomology of homogeneous spaces. In this Section we consider a Lie group G with a reductive Lie algebra $\mathfrak{g}$ and its connected closed subgroups H and N with Lie algebras $\mathfrak{h}$ and $\mathfrak{n}$, such that H is a normal subgroup of N and $\mathfrak{h}$ and $\mathfrak{n}$ are reductive subalgebras of $\mathfrak{g}$. Denote by K the quotient group N/H and by $\mathfrak{k}$ the quotient algebra $\mathfrak{n}/\mathfrak{h}$. Let P be a smooth principal K-bundle with a base B and $M = P \times_K (G/H)$, where G/H is considered as a principal K-bundle. It is clear that M is a G-manifold and the G-manifolds of Section 1 are obtained in the same way.

Since there exists a G-invariant connection on the principal K-bundle G/H and $\Omega(G/N)^G = C(\mathfrak{g}, \mathfrak{n})$ [9], where $C(\mathfrak{g}, \mathfrak{n})$ is a complex of cochains of $\mathfrak{g}$ relative to subalgebra $\mathfrak{n}$, by Proposition 3.1.2

$$\Omega(M)^G = C(P, G/H; \mathfrak{k})^G = \Omega(B)^G \otimes \Lambda(P_\mathfrak{k}) \otimes C(\mathfrak{g}, \mathfrak{n}).$$

We want to replace the algebra $C(\mathfrak{g}, \mathfrak{n})$ in $C(P, G/H; \mathfrak{k})^G$ with a more simple algebra.

It is known [9] that the projection $\mathfrak{n} \to \mathfrak{k}$ and the inclusion $\mathfrak{h} \subset \mathfrak{n}$ induce the linear maps $i : P_\mathfrak{k} \to P_\mathfrak{n}$ and $j : P_\mathfrak{n} \to P_\mathfrak{h}$.

Proposition 3.2.1. *The sequence*

$$0 \to P_\mathfrak{k} \xrightarrow{i} P_\mathfrak{n} \xrightarrow{j} P_\mathfrak{h} \to 0$$

is exact.

Proof. Since the algebra $H(\mathfrak{k}; \mathbf{R}) = \Lambda(P_\mathfrak{k})$ has not generators of even degrees the subalgebra $\mathfrak{h}$ of $\mathfrak{n}$ non homologous to zero [9]. Consider the ideals $\mathfrak{H}^q$ of $H(\mathfrak{n}; \mathbf{R})$ generated by the images of elements of $H(\mathfrak{k}; \mathbf{R})$ with degrees $\geq q$ under the homomorphism $H(\mathfrak{k}, \mathbf{R}) \to H(\mathfrak{n}; \mathbf{R})$. Then the graded algebra associated with the filtration of $H(\mathfrak{n}; \mathbf{R})$ by these ideals $\mathfrak{H}^q$ is isomorphic to $H(\mathfrak{h}; \mathbf{R}) \otimes H(\mathfrak{k}; \mathbf{R})$ [9]. This implies the statement of the proposition immediately. □

Consider the graded algebra

$$C(G, H, N) = \Lambda(P_\mathfrak{k}) \otimes \Lambda(P_\mathfrak{g}) \otimes S(P_\mathfrak{n})$$

with the differential operator ∂ defined as follows:

$$\partial x = \overline{i(x)}, \quad \partial y = \overline{l(y)}, \quad \partial z = 0,$$

where x and y are homogeneous elements of $P_\mathfrak{k}$ and $P_\mathfrak{g}$, respectively, $l : P_\mathfrak{g} \to P_\mathfrak{n}$ is the linear map induced by the inclusion $\mathfrak{n} \subset \mathfrak{g}$, $z \in S(P_\mathfrak{n})$, and, for a primitive element a, $\overline{a}$ is equal to a but belongs to the corresponding symmetric algebra. We shall show that $C(G, H, N)$ is the appropriate candidate for the above replacement.

Analogously, let $C(G/H) = \Lambda(P_\mathfrak{g}) \otimes S(\mathfrak{h})$ be the D-algebra with the differential operator δ defined as follows:

$$\delta y = \overline{j \circ l(y)}, \quad \delta z = 0,$$

where y is a homogeneous element of $P_\mathfrak{g}$ and $z \in S(P_\mathfrak{h})$. The Cartan theorem [1,4] asserts that, for compact G and H, $H(C(G/H))$ is canonically isomorphic to $H(G/H; \mathbf{R})$.

Proposition 3.2.2. *The morphism of GD-algebras $C(G,H,N) \to C(G/H)$, which vanishes on $\Lambda(P_{\mathfrak{k}})$, is identical on $\Lambda(P_{\mathfrak{g}})$, and is induced by j on $S(P_{\mathfrak{n}})$, induces the cohomology isomorphism.*

Proof. The homomorphism i defines the structure of $S(P_{\mathfrak{k}})$-module on $S(P_{\mathfrak{n}})$. Put

$$F_p = \oplus_{q+2r \geq p} \Lambda^q(P_{\mathfrak{k}}) \otimes \Lambda(P_{\mathfrak{g}}) \otimes S^r(P_{\mathfrak{k}}) S(P_{\mathfrak{n}}).$$

It is clear that F_p is an ideal of $C(G,H,N)$ and $\{F_p\}$ is a decreasing filtration of $C(G,H,N)$. Consider the spectral sequence $\{E_r, d_r\}$ associated with this filtration. It is easily proved that

$$E_0 = \Lambda(P_{\mathfrak{k}}) \otimes S(P_{\mathfrak{k}}) \otimes C(G/H),$$

with the differential operator d_0 induced by the differential operator ∂ of $C(G,H)$. Therefore,

$$E_1 = \Lambda(P_{\mathfrak{k}}) \otimes S(P_{\mathfrak{k}}) \otimes H(C(G/H)),$$

with the differential operator d_1 defined as follows: $d_1 x = \overline{x}$, where $x \in P_{\mathfrak{k}}$, and d_1 vanishes on $S(P_{\mathfrak{k}})$ and $H(C(G,H))$. Then $E_2 = E_\infty = H(C(G/H))$ and the statement of the proposition follows from the known properties of spectral sequences. □

Consider the $G-DG$-algebra $\Lambda(\mathfrak{g}')$ of cochains of the Lie algebra $\mathfrak{g}$ with coefficients in $\mathbf{R}$ as a $N-DG$-algebra. It is obvious that $B_N(\Lambda(\mathfrak{g}')) = C(\mathfrak{g},\mathfrak{n})$.

Proposition 3.2.3. *The natural inclusion $B_N(\Lambda(\mathfrak{g}')) \subset B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}'))$ induces a cohomology isomorphism.*

Proof. Since $\mathfrak{n}$ is a reductive subalgebra of $\mathfrak{g}$ there exists a following representation of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m}$, where $\mathfrak{n}$ is the invariant under the adjoint representation of $\mathfrak{g}$ subspace of $\mathfrak{g}$. Consider the projection $\mathfrak{g} \to \mathfrak{n}$ associated with this representation as an element ω of $\mathfrak{g}' \otimes \mathfrak{n}$. Clearly ω is a connection on the $N-DG$-algebra $\Lambda(\mathfrak{g}')$ and relative to this connection $\Lambda(\mathfrak{g}')$ is a W-algebra. Then the statement of the proposition is the immediate consequence of theorem 2.1.1. □

Consider the graded algebra

$$C(M) = \Omega(B) \otimes \Lambda(P_{\mathfrak{k}}) \otimes \Lambda(P_{\mathfrak{g}}) \otimes S(P_{\mathfrak{n}})$$

and define the differential operator d on it as an antiderivation of degree -1 in the following way. d is equal to the exterior derivative on $\Omega(B)$ and coincides with the differential operator ∂ of $C(G,H,N)$ on $\Lambda(P_{\mathfrak{g}}) \otimes S(P_{\mathfrak{n}})$. Let a be a homogeneous basic element of $P_{\mathfrak{g}}$ and b the component of da in the factor $\Omega(B)$ of $C(P, G/H; \mathfrak{k})$. Then, put $da = b + \partial a$. Clearly $(C(M), d)$ is a DG-algebra.

Theorem 3.2.3. *The cohomologies of DG-algebras $\Omega(M)^G$ and $C(M)$ are isomorphic.*

Proof. Consider the DG-algebra

$$C_1(M) = \Omega(B) \otimes \Lambda(P_{\mathfrak{k}}) \otimes B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}'))$$

and change the differential operator d on $\Lambda(P_{\mathfrak{k}})$ so that the morphism of graded algebras

$$f_1 : C(P, G/H, \mathfrak{k})^G \to \Omega(B) \otimes \Lambda(P_{\mathfrak{k}}) \otimes B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}')),$$

induced by the inclusion $C(\mathfrak{g}, \mathfrak{n}) \subset B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}'))$ will be a morphism of DG-algebras. Further we consider $C_1(M)$ as a DG-algebra with this new differential operator.

On the other hand, there is a natural morphism of DG-algebras $C(G/N) \to B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}'))$, which induces a cohomology isomorphism [4] (see the detailed proof of this statement for compact G and N in [13]; this proof is valid also in the general case). Let $f_2 : C(M) \to C_1(M)$ be the morphism of graded algebras induced by the morphism

$$C(G/N) = \Lambda(P_{\mathfrak{g}}) \otimes S(P_{\mathfrak{n}}) \to B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}')).$$

Clearly f_2 is a morphism of DG-algebras.

Consider the filtrations and the spectral sequences associated with these filtrations induced by the morphisms of DG-algebras

$$\Omega(B) \to C(P, G/H, \mathfrak{k})^G, \quad \Omega(B) \to C_1(M), \quad \Omega(B) \to C(M).$$

¿From Proposition 3.2.2 and the above property of the morphism

$$C(G/N) \to B_N(W(\mathfrak{g}) \otimes \Lambda(\mathfrak{g}'))$$

it follows that the morphisms f_1 and f_2 induce the isomorphisms of these spectral sequences beginning with the terms E_1. Thus, the statement of the theorem follows from theorem 3.1.2. □

Theorem 3.2.3 implies the following immediate corollary.

Corollary 3.2.4. *Let G be a connected compact Lie group, H and N its connected closed subgroups such that H is a normal subgroup of N, and P a smooth principal K-bundle, where $K = N/H$. Then the real cohomology of $M = P \times_K (G/H)$ is isomorphic to the cohomology of the complex $C(M)$.*

One can consider Corollary 3.2.4 as the generalization of the Cartan theorem on the cohomology of homogeneous spaces; this theorem follows from Corollary 3.2.4 and Proposition 3.2.2 whenever B is a one-point space.

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